EXERCISE 5.1 (NCERT)

Q No 1. Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$ at $x = -3$ and $x = 5$

Sol:

$f(x) = 5x - 3$

(i) At $x = 0$ $\quad f(0) = 5 \times 0 - 3 = -3$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} 5x - 3 = 5(0) - 3 = -3$

$\therefore \lim_{x \to 0} f(x) = f(0)$ $\therefore$ f is continuous at $x = 0$

(ii) At $x = -3$

$f(-3) = 5(-3) - 3 = -15 - 3 = -18$

$\lim_{x \to -3} f(x) = \lim_{x \to -3} 5x - 3 = 5(-3) - 3 = -18 = f(-3)$

$\therefore$ f is continuous at $x = -3$

(iii) At $x = 5$

$f(5) = 5(5) - 3 = 25 - 3 = 22$

$\lim_{x \to 5} f(x) = \lim_{x \to 5} 5x - 3 = 5(5) - 3 = 22 = f(5)$

$\therefore$ f is Cont. at $x = 5$

Q No. 2: Examine the continuity of function $f(x) = 2x^2 - 1$ at $x = 3$

Sol. We know that $f(x) = 2x^2 - 1$ is a polynomial function.

$\therefore \lim_{x \to 3} f(x) = \lim_{x \to 3} (2x^2 - 1) = 2(3)^2 - 1 = 17 = f(3)$

Hence f is cont. at $x = 3$

Q No. 3. Examine the following functions for Continuity:

(a) $f(x) = x - 5$

Sol. Given function $f(x) = x - 5$ is polynomial function with domain R.

Let $a \in R$

Then $\lim_{x \to a} f(x) = \lim_{x \to a} x - 5 = a - 5 = f(a)$

$\therefore$ f is continuous at $x = a$

ie f is Cont. at every point of its domain i.e. f is Cont. function.

(b) $f(x) = \frac{1}{x-5}$, $x \neq 5$

Sol. Given function is $f(x) = \frac{1}{x-5}$ which is Rational function.

and Domain of $f = D_f = R - \{x;\ x \in R \text{ and } x-5=0\} = R - \{5\}$

Let any $a \in D_f$, then $a \neq 5$

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{1}{x-5} = \frac{1}{a-5} = f(a)$$

$\therefore$ f is cont. at every point of its domain.

Hence f is Continuous function.

(c) $f(x) = \frac{x^2-25}{x+5}$; $x \neq -5$

Sol. Given function is $f(x) = \frac{x^2-25}{x+5}$, which is Rational function

and $D_f = R - \{x \in R;\ x+5=0\} = R - \{-5\}$

$\Rightarrow x \neq -5 \Rightarrow x+5 \neq 0$

$$\therefore f(x) = \frac{x^2-25}{x+5} = \frac{(x-5)(x+5)}{(x+5)} = x-5$$

Let any $a \in D_f$, then $a \neq -5$

then $\lim_{x \to a} f(x) = \lim_{x \to a} x-5 = a-5 = f(a)$

$\therefore$ f is Cont. at every point of its Domain

$\Rightarrow$ f is Continuous function.

(d) $f(x) = |x-5|$

Sol. Given function is $f(x) = |x-5|$ which is defined $\forall x \in R$

$\therefore D_f = R$.

Also $f(x) = \begin{cases} x-5\ ; & x-5 \geq 0 \text{ or } x \geq 5 \\ -(x-5); & x-5 < 0 \text{ or } x < 5 \end{cases}$

Let any $a \in D_f$ ie $a \in R$.

Three Cases arise :

Case I When $a < 5$

Then $\lim_{x \to a} f(x) = \lim_{x \to a} -(x-5) = -(a-5) = f(a)$

$\Rightarrow$ f is Cont. at $x = a < 5$.

CASE II When $a > 5$

$$\underset{x \to a}{Lt} f(x) = \underset{x \to a}{Lt} x-5 = a-5 = f(a)$$

$\therefore$ $f(x)$ is Cont at $x = a > 5$

CASE III : When $a = 5$

Then every neighbourhood of a contains points which are less than 5 and also points which are greater than 5. So we have to find Left and Right limit separately.

$$\therefore \underset{x \to a^-}{Lt} f(x) = \underset{x \to 5^-}{Lt} -(x-5) = -(5-5) = 0 \quad [\because f(x) = -(x-5) \text{ for } x<5]$$

and $$\underset{x \to a^+}{Lt} f(x) = \underset{x \to 5^+}{Lt} (x-5) = 5-5 = 0 \quad [\because f(x) = x-5 \text{ for } x>5]$$

Also $f(5) = 0$

$$\therefore \underset{x \to 5^+}{Lt} f(x) = \underset{x \to 5^-}{Lt} f(x) = f(5)$$

$\therefore$ $f(x)$ is cont. at $x = 5$

$\therefore$ f is continuous $\forall\ a \in R$

$\therefore$ f is cont function.

QNo 4: Prove that the function $f(x) = x^n$ is continuous at $x = n$ where n is +ve integer.

Sol. Given function is $f(x) = x^n$; $n \in Z^+$

$\therefore$ $f(x)$ is polynomial function with domain R.

$$\underset{x \to n}{Lt} f(x) = \underset{x \to n}{Lt} x^n = n^n = f(n)$$

$\Rightarrow$ f is continuous for $n \in Z^+$.

QNo 5 : Is the function defined by $f(x) = \begin{cases} x ; & x \le 1 \\ 5 ; & x > 1 \end{cases}$ is continuous at $x = 0$? At $x = 1$? At $x = 2$?

Soln : At $x = 0$: Since $f(x) = x$ for $x < 1$

$\therefore$ There exist a nbd of 0 every point of which is less than 1.

$\therefore \lim_{x\to 0} f(x) = \lim_{x\to 0} x = 0 = f(0)$

$\therefore$ f is continuous at $x=0$

At $x=1$: Since f is defined differently on left and Right side of 1 ie for $x>1$ and $x<1$

$\therefore$ We find Left and Right Limits at $x=1$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$$

and $$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 5 = 5$$

$\therefore \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$ $\therefore$ Limit does not exists.

$\therefore$ f is discontinuous at $x=1$.

At $x=2$. Since $f(x) = 5 \ \forall\ x>1$

$\therefore \exists$ a nbd of 2 every point of which is greater than 1

$\therefore \lim_{x\to 2} f(x) = \lim_{x\to 2} 5 = 5 = f(2)$

$\therefore$ f is Continuous at $x=2$.

* Find all points of discontinuity of f where f is defined by.

QNo. 6:
$$f(x) = \begin{cases} 2x+3 & \text{if } x \leq 2 \\ 2x-3 & \text{if } x > 2 \end{cases}$$

Sol: Here $D_f = \mathbb{R}$.

Let any $a \in \mathbb{R}$.

Three cases arise

Case I : If $a<2$

$\exists$ a nbd of 2 every point of which is less than 2

$\therefore \lim_{x\to a} f(x) = \lim_{x\to a} 2x+3 = 2a+3 = f(a)$

$\therefore$ f is Cont. at $a<2$.

Case II : If $a>2$, $\exists$ a nhd of a, all point of which are greater than 2.

$\therefore \lim_{x \to a} f(x) = \lim_{x \to a} 2x-3 = 2a-3 = f(a)$

$\therefore f$ is cont. $\forall a>2$

Case III : If $a=2$.

$f(2) = 2(2)+3 = 4+3 = 7$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x+3) = 2\times2+3 = 7$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x-3) = 2\times2-3 = 1$

$\therefore \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x) \Rightarrow$ Limit does not exist

$\Rightarrow f(x)$ is discontinuous at $x=2$

So only point of discontinuity is 2.

Q No. 7 :

$$f(x) = \begin{cases} |x|+3 & ; x \leq -3 \\ -2x & ; -3<x<3 \\ 6x+2 & ; x \geq 3 \end{cases}$$

Sol : Here $D_f = \mathbb{R}$ and let any $a \in D_f$.

Five case arise.

Case I : If $a<-3$

$\lim_{x \to a} f(x) = \lim_{x \to a} (|x|+3) = |a|+3 = f(a)$

$\Rightarrow f$ is continuous at $a<-3$

Case II : $f(-3) = |-3|+3 = 3+3 = 6$

$\lim_{x \to -3^-} f(x) = \lim_{x \to 3^-} (|x|+3) = \lim_{x \to -3^-} (-x+3)$.

$= -(-3)+3 = 6$ $[\because |x| = -x \text{ for } x<-3]$

$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (-2x) = -2(-3) = 6$ $[\because f(x) = -2x \text{ for } -3<x<3]$

$\Rightarrow \lim_{x\to -3^-} f(x) = \lim_{x\to -3^+} f(x) = f(-3) = 6$

$\therefore$ f is continuous at $x=-3$

Case III : If $-3<a<3$

$f(x) = -2x$ which being polynomial function is continuous.

$\Rightarrow f(x)$ is continuous at $a \in (-3,3)$

Case IV : If $a=3$

$f(3) = 6\times 3+2 = 18+2 = 20.$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (-2x) = -2\times 3 = -6$ $\left[\because f(x) = -2x \text{ for } -3<x<3\right]$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (6x+2) = 6\times 3+2 = 18+2 = 20$

$\therefore \lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x) \Rightarrow$ Limit does not exist at $x=3$

$\therefore$ f is discontinuous at $x=3$.

Case V : If $a>3$

$\therefore \lim_{x\to a} f(x) = \lim_{x\to a} 6x+2 = 6a+2 = f(a)$.

$\Rightarrow$ f is continuous at $x=a>3$

$\therefore$ The only point of discontinuity is $x=3$

Q No. 8 $f(x) = \begin{cases} \frac{|x|}{x} ; & x\neq 0 \\ 0 ; & x=0 \end{cases}$

Sol : Given function is $f(x) = \begin{cases} \frac{|x|}{x} ; & x\neq 0 \\ 0 ; & x=0 \end{cases}$

$\Rightarrow f(x) = \begin{cases} \frac{x}{x} ; & x>0 \\ -\frac{x}{x} ; & x<0 \\ 0 ; & x=0 \end{cases} = \begin{cases} -1 ; & x<0 \\ 1 ; & x>0 \\ 0 ; & x=0 \end{cases}$

Clearly $D_f = R$. Let $a \in R$ be any arbitrary real no. (7)

Three cases arise.

Case I: If $a<0$

$$\lim_{x\to a} f(x) = \lim_{x\to a}(-1) = -1 = f(a)$$

$\Rightarrow$ f is continuous at $x = a < 0$

Case II: If $a = 0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-1) = -1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(1) = 1$$

$\Rightarrow \lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x) \Rightarrow$ Limit of f does not exist at $x=0$

$\therefore$ f is discontinuous at $x = 0$

Case III: If $a>0$.

$$\therefore \lim_{x\to a} f(x) = \lim_{x\to a}(1) = 1 = f(a)$$

$\therefore$ f is cont. for $x = a > 0$

$\therefore$ Only point of discontinuity is $x = 0$.

Q No 9: $f(x) = \begin{cases} \frac{x}{|x|} & \text{if } x<0 \\ -1 & \text{if } x \geq 0 \end{cases}$

Sol: Given $f(x) = \begin{cases} \frac{x}{|x|} ; & x<0 \\ -1 ; & x \geq 0 \end{cases} = \begin{cases} \frac{x}{-x} ; & x<0 \\ -1 ; & x \geq 0 \end{cases}$

$$= \begin{cases} -1 ; & x<0 \\ -1 ; & x \geq 0 \end{cases}$$

$\Rightarrow f(x) = -1 \quad \forall x \in R$

$$\therefore \lim_{x\to a} f(x) = \lim_{x\to a}(-1) = -1 = f(a) \quad \forall a \in R.$$

$\therefore$ f is continuous for all $x \in R$.

So No point of discontinuity.

Q No 10

$$f(x) = \begin{cases} x+1 & \text{if } x \geq 1 \\ x^2+1 & \text{if } x < 1 \end{cases}$$

Sol.: Since f is defined $\forall\ x \in R$ $\therefore D_f = R$.

Let any arbitrary $a \in R$

Three case arise:

Case I: When $a < 1$

$$\lim_{x \to a} f(x) = \lim_{x \to a} (x^2+1) = a^2+1 = f(a)$$

$\therefore$ $f(x)$ is cont. for $x = a < 1$

Case II: When $a = 1$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2+1) = (1)^2+1 = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x+1) = 1+1 = 2.$$

Also $f(1) = 1+1 = 2$.

$$\therefore \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$\therefore$ f is cont. at $x = 1$

Case III: When $a > 1$

$$\lim_{x \to a} f(x) = \lim_{x \to a} (x+1) = a+1 = f(a)$$

$\therefore$ f is cont. for $x = a > 1$.

$\therefore$ f is cont $\forall\ x \in R$

$\Rightarrow$ f has no point of discontinuity.

Q No 11:

$$f(x) = \begin{cases} x^2-3 & \text{if } x \leq 2 \\ x^2+1 & \text{if } x > 2 \end{cases}$$

Sol. Clearly $D_f = R$

Let $a \in R$ be any arbitrary real No.

Three Case arise.

Case I : If $a<2$

$\therefore \lim_{x\to a} f(x) = \lim_{x\to a}(x^3-3) = a^3-3 = f(a)$

$\therefore$ $f(x)$ is cont. for $a<2$

Case II : If $a=2$

$f(2) = 2^3-3 = 8-3=5$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^3-3 = 2^3-3=5$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2} x^2+1 = 4+1 = 5$

$\therefore \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$.

$\therefore$ f is cont. at $x=2$.

Case III : When $a>2$

$\lim_{x\to a} f(x) = \lim_{x\to a} x^2+1 = a^2+1 = f(a)$

$\Rightarrow$ $f(x)$ is Cont. at $x \leq a > 2$

$\therefore$ f is continuous at every point of its domain.

$\therefore$ No point of discontinuity.

Q No. 12 :

$$f(x) = \begin{cases} x^{10}-1 & ; x \leq 1 \\ x^2 & ; x>1 \end{cases}$$

Sol : Here $D_f = R$ and let $a\in R$ be any arbitrary no.

Three cases arise

Case I : If $a<1$

$\lim_{x\to a} f(x) = \lim_{x\to a} x^{10}-1 = a^{10}-1 = f(a)$.

$\therefore$ f is cont. at $a<1$

Case II When $a = 1$

Now $f(1) = 1^{10} - 1 = 1 - 1 = 0$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^{10} - 1 = 1^{10} - 1 = 0$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 = 1^2 = 1$$

$$\therefore \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

$\Rightarrow$ f is discontinuous at $x = 1$

Case III When $a > 1$

$$\lim_{x \to a} f(x) = \lim_{x \to a} x^2 = a^2 = f(a)$$

$\Rightarrow$ f is cont. at $a > 1$

Thus, only point of discontinuity is $x = 1$.

QNo 13: Is the function defined by $f(x) = \begin{cases} x+5 \; ; & x \leq 1 \\ x-5 \; ; & x > 1 \end{cases}$ a continuous function.

Sol: A function is said to be continuous function if it is continuous at every point of its domain.

Here $D_f = \mathbb{R}$.

Now $$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+5) = 1 + 5 = 6$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x - 5 = 1 - 5 = -4$$

$$\therefore \lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x)$$

$\Rightarrow$ f is discontinuous at $x = 1 \in \mathbb{R}$.

$\therefore$ f is not a continuous function.

Discuss the continuity of function f where f is defined by

Q No. 14 $f(x) = \begin{cases} 3 & \text{if } 0 \le x \le 1 \\ 4 & \text{if } 1 < x < 3 \\ 5 & \text{if } 3 \le x \le 10 \end{cases}$

Sol: Clearly f is defined at all points of its domain.

Now, When $x \in [0, 1)$ i.e $0 \le x < 1$

$f(x) = 3$ which is constant function

So is continuous.

When $x = 1$. $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 3 = 3$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 4 = 4.$

$\therefore \lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x)$

$\therefore$ f is discontinuous at $x = 1$

When $1 < x < 3$ $f(x) = 4$ being constant, continuous.

When $x = 3$ $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} 4 = 4$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 5 = 5$

$\therefore \lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$

$\therefore$ f is discontinuous at $x = 3$

When $3 < x \le 10$ $f(x) = 5$ which is constant

So is continuous.

$\therefore$ f is not continuous at $x = 1$ and $x = 3$.

Q No 15: $f(x) = \begin{cases} 2x & ; x < 0 \\ 0 & ; 0 \le x \le 1 \\ 4x & ; x > 1 \end{cases}$

Sol. Clearly $D_f = R$.

Now D_f can be splitted into intervals

$R = (-\infty, 0) \cup [0, 1] \cup (1, \infty)$

When $x \in (-\infty, 0)$ ie $x < 0$, $f(x) = 2x$, which is linear. polynomial So, $f(x)$ is Continuous for $x < 0$

At $x = 0$

$$\underset{x \to 0^-}{Lt} f(x) = \underset{x \to 0^-}{Lt} 2x = 0$$

$$\underset{x \to 0^+}{Lt} f(x) = \underset{x \to 0^+}{Lt} 0 = 0$$

Also $f(0) = 0$

$\therefore$ f is continuous at $x = 0$

When $x \in (0, 1)$ ie $0 < x < 1$ $f(x) = 0$ which is Constant

So is Continuous.

At $x = 1$

$f(1) = 0$

$$\underset{x \to 1^-}{Lt} f(x) = \underset{x \to 1^-}{Lt} 0 = 0$$

$$\underset{x \to 1^+}{Lt} f(x) = \underset{x \to 1^+}{Lt} 4x = 4 \times 1 = 4$$

$$\therefore \underset{x \to 1^-}{Lt} f(x) \neq \underset{x \to 1^+}{Lt} f(x)$$

$\therefore$ f is discontinuous at $x = 1$

When $x \in (0, 1)$ ie $x > 1$, $f(x) = 4x$ which being linear polynomial is continuous.

$\therefore$ f is discontinuous at only $x = 1$.

Q No 16

$$f(x) = \begin{cases} -2 \,; & x \le -1 \\ 2x \,; & -1 < x \le 1 \\ 2 \,; & x > 1 \end{cases}$$

Sol: Clearly here $D_f = R$

When $x < -1$ $\quad f(x) = -2$, which is constant and so is continuous.

At $x = -1$ $\quad \displaystyle \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} -2 = -2$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} 2x = 2(-1) = -2$$

Also $f(-1) = -2$

$$\therefore \lim_{x \to -1^+} f(x) = \lim_{x \to -1^-} f(x) = f(-1)$$

$\therefore$ f is Continuous at $x = -1$

When $-1 < x < 1$ $\quad f(x) = 2x$, which being a Linear polynomial it is continuous

At $x = 1$ $\quad f(1) = 2(1) = 2$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x = 2(1) = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 = 2$$

$$\therefore \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$$

$\therefore$ f is cont. at $x = 1$

When $x > 1$ $\quad f(x) = 2$, which being const. is cont.

$\therefore$ f is cont. $\forall x \in R$

$\therefore$ f is continuous function.

QNo 17: Find the relationship between a and b so that the function defined by $f(x) = \begin{cases} ax+1 ; & x \le 3 \\ bx+3 ; & x > 3 \end{cases}$ is cont. at $x=3$.

Sol. Since f is cont. at $x=3$

$\therefore \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$

ie if $\lim_{x\to 3^-}(ax+1) = \lim_{x\to 3^+}(bx+3) = f(3)$

ie if $3a+1 = 3b+3 = a(3)+1$

ie if $3a+1 = 3b+3 = 3a+1$

ie if $3a = 3b+3-1$

ie if $a = b+\frac{2}{3}$

QNo 18: For what value of λ is the function defined by $f(x) = \begin{cases} \lambda(x^2-2x) ; & x \le 0 \\ 4x+1 ; & x > 0 \end{cases}$ is cont. at $x=0$? What about continuity at $x=1$.

Sol: f is cont. at $x=0$

if $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

i.e. if $\lim_{x\to 0^-} \lambda(x^2-2x) = \lim_{x\to 0^+}(4x+1) = \lambda(0^2-2\times 0)$

ie $\lambda(0-0) = 4(0)+1 = \lambda(0)$

ie. $0\lambda = 1$ which is not possible for any real λ.

$\therefore f$ is discont. at $x=0$ whatever λ may be.

At $x=1$ $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$

ie if $\lim_{x\to 1^+}(4x+1) = \lim_{x\to 1^+} f(x) = f(1)$

ie $\lim_{x\to 1^+}(4x+1) = \lim_{x\to 1^-}(4x+1) = f(1) = 5$.

$\therefore f$ is continuous at $x=1$.

Q No. 19: Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x.

Sol. Given function is $g(x) = x - [x]$

Let c be any integer.

For $\underset{x \to c^-}{Lt}\, g(x)$ Put $x = c - h$; $h > 0$

$\therefore$ As $x \to c$, $h \to 0$.

$$\therefore \underset{x \to c^-}{Lt}\, g(x) = \underset{x \to c^-}{Lt}\, x - [x] = \underset{h \to 0}{Lt}\, (c-h) - [c-h]$$

$$= \underset{h \to 0}{Lt}\, [c - h - (c-1)] = c - c + 1 = 1$$

For $\underset{x \to c^+}{Lt}\, g(x)$ Put $x = c + h$; $h > 0$

$\therefore$ As $x \to c$, $h \to 0$

$$\therefore \underset{x \to c^+}{Lt}\, g(x) = \underset{x \to c^+}{Lt}\, (x - [x]) = \underset{h \to 0}{Lt}\, [(c+h) - [c+h]]$$

$$= \underset{h \to 0}{Lt}\, [c + h - c] = c + 0 - c = 0$$

$$\therefore \underset{x \to c^+}{Lt}\, g(x) \neq \underset{x \to c^-}{Lt}\, g(x)$$

$\therefore$ $g(x)$ is discontinuous at $x = c$

$\therefore$ $g(x)$ is discontinuous at every integer.

Q No 20: Is the function defined by $f(x) = x^2 - \sin x + 5$ is continuous at $x = \pi$.

Sol. Here $f(\pi) = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$

and $\underset{x \to \pi}{Lt}\, f(x) = \underset{x \to \pi}{Lt}\, (x^2 - \sin x + 5) = \pi^2 - \sin \pi + 5 = \pi^2 + 5$

$\therefore \underset{x \to \pi}{Lt}\, f(x) = f(\pi)$ $\therefore$ f is Cont. at π.

Q No. 21 Discuss the continuity of following functions:
(a) $f(x) = \sin x + \cos x$ (b) $\sin x - \cos x$ (c) $f(x) = \sin x \cdot \cos x$.

Sol. (a) Here $f(x) = \sin x + \cos x$.
Clearly $D_f = R$
Let a be any arbitrary real no.
then $\underset{x \to a}{Lt} f(x) = \underset{x \to a}{Lt} (\sin x + \cos x) = \sin a + \cos a = f(a)$
$\therefore$ $f(x)$ is cont. at $x = a \in R$.
$\therefore$ $f(x)$ is cont. function.

(b) Same as above. (c) Same as above.

Q No. 22 : Discuss the continuity of cosine, cosecant, Secant and Cotangent function.

Sol : (i) Let $f(x) = \cos x$, then $D_f = R$.
Let $a \in D_f$
Now $\underset{x \to a}{Lt} f(x) = \underset{x \to a}{Lt} \cos x$
Put $x = a + h$ so that as $x \to a$, $h \to 0$
$\therefore \underset{x \to a}{Lt} \cos x = \underset{h \to 0}{Lt} \cos(a+h) = \underset{h \to 0}{Lt} (\cos a \cosh - \sin a \sinh)$
$= \cos a \underset{h \to 0}{Lt} \cosh - \sin a \underset{h \to 0}{Lt} \sinh$
$= \cos a \times 1 - \sin a \times 0 = \cos a = f(a)$
$\therefore$ f is cont. at $x = a$
$\therefore$ f is cont. at every point of its domain.

(ii) Let $f(x) = \operatorname{cosec} x = \dfrac{1}{\sin x}$
$\therefore D_f = R - \{x \in R;\ \sin x = 0\}$
$= R - \{x \in R;\ x = n\pi;\ n \in Z\}$
Let a be any arbitrary real no. $\in D_f$.

$$\underset{x\to a}{Lt} f(x) = \underset{x\to a}{Lt} \frac{1}{\sin x}$$ Put $x = a+h$, so that $h \to 0$ as $x \to a$

$$= \underset{h\to 0}{Lt} \frac{1}{\sin(a+h)} = \underset{h\to 0}{Lt} \left[\frac{1}{\sin a \cos h + \cos a \sin h}\right]$$

$$= \frac{1}{\sin a \times 1 + \cos a \times 0} = \frac{1}{\sin a} = f(a)$$

$\therefore \underset{x\to a}{Lt} f(x) = f(a) \quad \forall \; a \in D_f$

$\therefore$ $f(x) = \operatorname{cosec} x$ is cont. at every point of its domain

(iii) Let $f(x) = \sec x = \frac{1}{\cos x}$

$D_f = R - \{x ; \cos x = 0\}$

$= R - \{x = (2n+1)\pi/2 \; ; n \in Z\}$

$=$ R except odd multiples of $\pi/2$.

Let $a \in D_f$.

$$\underset{x\to a}{Lt} f(x) = \underset{x\to a}{Lt} \frac{1}{\cos x}$$ [Put $x = a+h$; $h>0$ so as $x \to a$ $h \to 0$.]

$$= \underset{h\to 0}{Lt} \frac{1}{\cos(a+h)} = \underset{h\to 0}{Lt} \left(\frac{1}{\cos a \cos h - \sin a \sinh}\right)$$

$$= \frac{1}{\cos a \times 1 - \sin a \times 0} = \frac{1}{\cos a}.$$

Also $f(a) = \frac{1}{\cos a}$.

$\therefore$ $f(x)$ is cont at $x \in a$

But a is any arbitrary real no. $\in D_f$.

$\therefore$ f is cont. at every point of its domain.

(iv) Let $f(x) = \cot x = \frac{\cos x}{\sin x}$

Here $D_f = R - \{x \in R : \sin x = 0\}$

$= R - \{x = n\pi \; ; n \in Z\}$

Let a be any real No $\in D_f$.

$$\underset{x\to a}{Lt}\, f(x) = \underset{x\to a}{Lt}\, \frac{\cos x}{\sin x}$$

Put $x = a+h$, $h>0$ so that as $x\to a$, $h\to 0$

$$= \underset{h\to 0}{Lt}\, \frac{\cos(a+h)}{\sin(a+h)} = \underset{h\to 0}{Lt}\left(\frac{\cos a\cosh - \sin a \sin h}{\sin a \cos h - \cos a \sin h}\right)$$

$$= \frac{\cos a \times 1 - \sin a \times 0}{\sin a \times 1 - \cos a \times 0} = \frac{\cos a}{\sin a}.$$

Also $f(a) = \dfrac{\cos a}{\sin a}$ $\therefore$ f is cont. at $x = a$.

But a is any real no. $\in D_f$

$\therefore$ f is continuous at every point of its domain.

QNo 23: Find all points of discontinuity of f where

$$f(x) = \begin{cases} \frac{\sin x}{x} & ; x<0 \\ x+1 & ; x\geq 0 \end{cases}$$

Sol: Clearly $D_f = R$.

Let $a\in R$ be arbitrary, then three cases arise;

CASE I: If $a<0$, $\underset{x\to a}{Lt}\, f(x) = \underset{x\to a}{Lt}\, \frac{\sin x}{x} = \frac{\sin a}{a} = f(a)$

$\Rightarrow$ f is continuous at $a<0$

CASE II: If $a=0$, $f(0) = 0+1 = 1$.

$$\underset{x\to 0^-}{Lt}\, f(x) = \underset{x\to 0^-}{Lt}\, \frac{\sin x}{x} = 1$$

$$\underset{x\to 0^+}{Lt}\, f(x) = \underset{x\to 0^+}{Lt}\, (x+1) = 0+1 = 1$$

$$\therefore \underset{x\to 0^+}{Lt}\, f(x) = \underset{x\to 0^-}{Lt}\, f(x) = f(0)$$

$\therefore$ $f(x)$ is continuous at $x=0$

CASE III: When $a>0$

$\underset{x\to a}{Lt}\, f(x) = \underset{x\to a}{Lt}\, (x+1) = a+1 = f(a) \Rightarrow f(x)$ is cont. at $a>0$

$\therefore$ No point of discontinuity.

Q No 24. Determine if f defined by
$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$ is a continuous function.

Sol: Clearly $D_f = R$.
Let a be any Real No.
Two cases arise

CASE I When $a \neq 0$

$$\lim_{x\to a} f(x) = \lim_{x\to a} x^2 \sin\frac{1}{x} = a^2 \sin\frac{1}{a} = f(a).$$

$\therefore$ $f(x)$ is Continuous at $x = a \neq 0$

CASE II When $a = 0$

$f(a) = f(0) = 0$

and $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$
$= f(0)$

[$\because$ $\lim_{x\to 0} x^2 = 0$ and $\left|\sin\frac{1}{x}\right| \leq 1 \ \forall x \neq 0$
$\therefore$ $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$
$\because$ If $\lim_{x\to 0} f(x) = 0$ and $g(x)$ is bounded fn. then $\lim_{x\to 0} f(x)g(x) = 0$.]

$\therefore$ f is continuous at $x = a$

$\therefore$ f is cont. at every point of its domain
$\Rightarrow$ f is cont. function.

Q No 25: Examine the continuity of f where f is defined by
$$f(x) = \begin{cases} \sin x - \cos x & ; x \neq 0 \\ -1 & ; x = 0 \end{cases}$$

Sol: Clearly $D_f = R$. Let $a \in R$ be any arbitrary no.

CASE I: When $a \neq 0$.

$$\therefore \lim_{x\to a} f(x) = \lim_{x\to a} (\sin x - \cos x) = \sin a - \cos a = f(a)$$

$\therefore$ f is cont. at $x = a \neq 0$.

CASE II: When $a = 0$, then $f(a) = f(0) = -1$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} (\sin x - \cos x) = 0 - 1 = -1.$$

$\therefore$ $\lim_{x\to 0} f(x) = f(0)$

$\therefore$ $f(x)$ is Cont at $x = 0$ Hence f is Cont at every pt of D_f

* Find the values of k so that the function is continuous at indicated point in Exercise 26 to 29.

Q No 26 :

$$f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x} & ; x \neq \pi/2 \\ 3 & ; x = \pi/2 \end{cases} \quad \text{at } x = \pi/2.$$

Sol. Here $f(\pi/2) = 3$.

$$\lim_{x \to \pi/2} f(x) = \lim_{x \to \pi/2} \frac{k\cos x}{\pi - 2x}.$$

Put $x = \pi/2 + h$, $h > 0$ so that $h \to 0$ as $x \to \pi/2$

$$= \lim_{h \to 0} \frac{k \cdot \cos(\pi/2 + h)}{\pi - 2(\pi/2 + h)} = \lim_{h \to 0} \frac{k(-\sin h)}{\pi - \pi - 2h}$$

$$= \lim_{h \to 0} \frac{-k\sin h}{-2h} = \frac{k}{2} \lim_{h \to 0} \frac{\sin h}{h} = \frac{k}{2}.$$

Since f is cont. at $x = \pi/2$.

$$\therefore \lim_{x \to \pi/2} f(x) = f(\pi/2)$$

$$\Rightarrow \frac{k}{2} = 3 \Rightarrow k = 6$$

Q No 27.

$$f(x) = \begin{cases} kx^2 & ; x \le 2 \\ 3 & ; x > 2 \end{cases} \quad \text{at } x = 2.$$

Sol : Since f is Cont. at $x = 2$

$$\therefore \lim_{x \to 2^-} f(x) = f(2) = \lim_{x \to 2^+} f(x)$$

$$\Rightarrow \lim_{x \to 2^-} kx^2 = k(2)^2 = \lim_{x \to 2^+} 3$$

$$\Rightarrow k(2)^2 = 4k = 3$$

$$\Rightarrow 4k = 4k = 3$$

$$\Rightarrow k = \frac{3}{4}.$$

QNo 28. $f(x) = \begin{cases} kx+1 & ; x \leq \pi \\ \cos x & ; x > \pi \end{cases}$ at $x = \pi$.

Sol: Here $f(\pi) = k\pi + 1$

$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi} kx + 1 = k\pi + 1.$

$\lim_{x \to \pi^+} f(x) = \lim_{x \to \pi} \cos x = \cos \pi = -1$

Since f is cont. function

$\therefore \quad k\pi + 1 = -1$

$\Rightarrow k\pi = -2 \Rightarrow k = -\frac{2}{\pi}.$

($\because$ f is cont. at π iff $f(\pi) = \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x)$

QNo 29: $f(x) = \begin{cases} kx+1 & ; x \leq 5 \\ 3x-5 & ; x > 5 \end{cases}$ at $x = 5$

Sol. Here $f(5) = k(5)+1 = 5k+1.$

$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (kx+1) = 5k+1.$

$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (3x-5) = 3(5)-5 = 15-5 = 10$

Since f is cont. at $x = 5$ iff.

$\therefore \lim_{x \to 5^-} f(x) = f(5) = \lim_{x \to 5^+} f(x)$

ie $5k+1 = 5k+1 = 10$

$\Rightarrow k = 9/5.$

QNo 30: Find the values of a and b such that the function defined by $f(x) = \begin{cases} 5 & ; x \leq 2 \\ ax+b & ; 2 < x < 10 \\ 21 & ; x \geq 10 \end{cases}$ is continuous function

Sol.: Clearly $D_f = R$.

Let $c \in R$ be arbitrary.

$\therefore$ f will be continuous if f is cont. at all $c \in R$.

When $c < 2$ $\quad \lim_{x \to c} f(x) = \lim_{x \to c} 5 = 5 = f(c)$

$\Rightarrow$ f is cont at $c < 2$

When $c=2$ $\quad f(2)=5$

$$\underset{x\to 2^-}{Lt}\, f(x) = \underset{x\to 2^-}{Lt}\, 5 = 5$$

$$\underset{x\to 2^+}{Lt}\, f(x) = \underset{x\to 2}{Lt}\,(ax+b) = 2a+b.$$

$\therefore$ f is cont. at $c=2$ iff $2a+b=5$ ----(1)

When $2<c<10$ $\quad \underset{x\to c}{Lt}\, f(x) = \underset{x\to c}{Lt}\,(ax+b) = ac+b = f(c)$

$\therefore$ $f(x)$ is cont.

When $c=10$ $\quad f(10)=21$

and $\underset{x\to 10^-}{Lt}\, f(x) = \underset{x\to 10}{Lt}\,(ax+b) = 10a+b$

$$\underset{x\to 10^+}{Lt}\, f(x) = \underset{x\to 10^+}{Lt}\, 10 = 10$$

$\therefore$ f is cont. at $x=10$ iff $10a+b=21$ ----(2)

When $c>10$. $\quad \underset{x\to c}{Lt}\, f(x) = \underset{x\to c}{Lt}\, 21 = 21 = f(c)$

$\therefore$ f is cont.

$\therefore$ f is cont. if (1) and (2) are satisfied.

Solving (1) and (2) for a and b we get

$8a = 16 \Rightarrow a=2$

and $2\times 2+b=5 \Rightarrow b=1$

Hence f is cont. when $a=2$ and $b=1$.

Q No. 31: Show that the function $f(x)=\cos(x^2)$ is a continuous function.

Sol: Given $f(x)=\cos(x^2)$, $D_f=R$.

Let $g(x)=\cos x$ and $h(x)=x^2$

Then both g and h are cont. on R.

Also $goh(x) = g[h(x)] = g(x^2) = \cos(x^2) = f(x)$

$\therefore$ $\cos(x^2)$ is cont. fn. on R

$\because$ Continuous function of a continuous function is continuous.

Q No 32: Show that function defined by $f(x) = |\cos x|$ is cont. function.

Sol: Given $f(x) = |\cos x|$; $D_f = R$.

Let $g(x) = |x|$ and $h(x) = \cos x$.

Then both g, h are cont. functions on R.

Also $goh(x) = g[h(x)] = g[\cos x] = |\cos x| = f(x)$

$\therefore$ $f(x)$ is continuous fn.

($\because$ Cont. function of a Cont. fn. is cont.)

Q No 33: Examine that $\sin |x|$ is Cont. fn.

Sol. Given $f(x) = \sin |x|$, $D_f = R$.

Let $g(x) = \sin x$ and $h(x) = |x|$, then both g and h are cont. fns. on R.

Also $goh(x) = g(h(x)) = g(|x|) = \sin |x| = f(x)$

$\therefore$ $f(x)$ is cont. fn.

($\because$ Cont. fn. of a Cont. fn. is cont.)

Q No 34: Find the points of discontinuity of $f(x) = |x| - |x+1|$

Sol: Let $g(x) = |x|$ and $h(x) = x+1$

Then both $g(x)$ and $h(x)$ are cont. on R.

Also $goh(x) = g(h(x)) = g(x+1) = |x+1|$

$\therefore$ $|x+1|$ is cont. on R [$\because$ Cont. fn. of a cont. fn. is cont.]

Also $g(x) = |x|$ is cont. on R.

$\Rightarrow$ $g(x) - |x+1|$ i.e. $|x| - |x+1|$ is Cont. on R.

$\Rightarrow$ $f(x)$ is cont. on R.

$\Rightarrow$ $f(x)$ has no point of discontinuity.

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