Chapter 8

Conservation Laws

8.1 Charge and Energy

8.1.1 The Continuity Equation

In this chapter we study conservation of energy, momentum, and angular momentum, in electrodynamics. But I want to begin by reviewing the conservation of *charge*, because it is the paradigm for all conservation laws. What precisely does conservation of charge tell us? That the total charge in the universe is constant? Well, sure—that's **global** conservation of charge; but **local** conservation of charge is a much stronger statement: If the total charge in some volume changes, then exactly that amount of charge must have passed in or out through the surface. The tiger can't simply rematerialize outside the cage; if it got from inside to outside it must have found a hole in the fence.

Formally, the charge in a volume $\mathcal{V}$ is

$$Q(t) = \int_{\mathcal{V}} \rho(\mathbf{r}, t)\, d\tau, \tag{8.1}$$

and the current flowing out through the boundary $\mathcal{S}$ is $\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}$, so local conservation of charge says

$$\frac{dQ}{dt} = -\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}. \tag{8.2}$$

Using Eq. 8.1 to rewrite the left side, and invoking the divergence theorem on the right, we have

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, d\tau = -\int_{\mathcal{V}} \nabla \cdot \mathbf{J}\, d\tau, \tag{8.3}$$

and since this is true for *any* volume, it follows that

$$\boxed{\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}.} \tag{8.4}$$

This is, of course, the continuity equation—the precise mathematical statement of local conservation of charge. As I indicated earlier, it can be derived from Maxwell's equations—conservation of charge is not an *independent* assumption, but a *consequence* of the laws of electrodynamics.

The purpose of this chapter is to construct the corresponding equations for conservation of energy and conservation of momentum. In the process (and perhaps more important) we will learn how to express the energy density and the momentum density (the analogs to ρ), as well as the energy "current" and the momentum "current" (analogous to $\mathbf{J}$).

8.1.2 Poynting's Theorem

In Chapter 2, we found that the work necessary to assemble a static charge distribution (against the Coulomb repulsion of like charges) is (Eq. 2.45)

$$W_{\mathrm{e}} = \frac{\epsilon_0}{2}\int E^2\, d\tau,$$

where $\mathbf{E}$ is the resulting electric field. Likewise, the work required to get currents going (against the back emf) is (Eq. 7.34)

$$W_{\mathrm{m}} = \frac{1}{2\mu_0}\int B^2\, d\tau,$$

where $\mathbf{B}$ is the resulting magnetic field. This suggests that the total energy stored in electromagnetic fields is

$$U_{\mathrm{em}} = \frac{1}{2}\int\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau. \tag{8.5}$$

I propose to derive Eq. 8.5 more generally, now, in the context of the energy conservation law for electrodynamics.

Suppose we have some charge and current configuration which, at time t, produces fields $\mathbf{E}$ and $\mathbf{B}$. In the next instant, dt, the charges move around a bit. *Question*: How much work, dW, is done by the electromagnetic forces acting on these charges in the interval dt? According to the Lorentz force law, the work done on a charge q is

$$\mathbf{F}\cdot d\mathbf{l} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\mathbf{v}\,dt = q\mathbf{E}\cdot\mathbf{v}\,dt.$$

Now, $q = \rho\, d\tau$ and $\rho\mathbf{v} = \mathbf{J}$, so the rate at which work is done on all the charges in a volume $\mathcal{V}$ is

$$\frac{dW}{dt} = \int_{\mathcal{V}}(\mathbf{E}\cdot\mathbf{J})\,d\tau. \tag{8.6}$$

Evidently $\mathbf{E}\cdot\mathbf{J}$ is the work done per unit time, per unit volume—which is to say, the *power* delivered per unit volume. We can express this quantity in terms of the fields alone, using the Ampère-Maxwell law to eliminate $\mathbf{J}$:

$$\mathbf{E}\cdot\mathbf{J} = \frac{1}{\mu_0}\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) - \epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}.$$

From product rule 6,

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B}).$$

Invoking Faraday's law ($\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$), it follows that

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) = -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{B}).$$

Meanwhile,

$$\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(B^2), \quad \text{and} \quad \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E^2), \tag{8.7}$$

so

$$\mathbf{E} \cdot \mathbf{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B}). \tag{8.8}$$

Putting this into Eq. 8.6, and applying the divergence theorem to the second term, we have

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a}, \tag{8.9}$$

where $\mathcal{S}$ is the surface bounding $\mathcal{V}$. This is **Poynting's theorem**; it is the "work-energy theorem" of electrodynamics. The first integral on the right is the total energy stored in the fields, U_{em} (Eq. 8.5). The second term evidently represents the rate at which energy is carried out of $\mathcal{V}$, across its boundary surface, by the electromagnetic fields. Poynting's theorem says, then, that *the work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy that flowed out through the surface.*

The *energy per unit time, per unit area,* transported by the fields is called the **Poynting vector**:

$$\boxed{\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}).} \tag{8.10}$$

Specifically, $\mathbf{S} \cdot d\mathbf{a}$ is the energy per unit time crossing the infinitesimal surface $d\mathbf{a}$—the energy *flux*, if you like (so $\mathbf{S}$ is the **energy flux density**).[1] We will see many applications of the Poynting vector in Chapters 9 and 11, but for the moment I am mainly interested in using it to express Poynting's theorem more compactly:

$$\frac{dW}{dt} = -\frac{dU_{\text{em}}}{dt} - \oint_{\mathcal{S}} \mathbf{S} \cdot d\mathbf{a}. \tag{8.11}$$

[1] If you're very fastidious, you'll notice a small gap in the logic here: We know from Eq. 8.9 that $\oint \mathbf{S} \cdot d\mathbf{a}$ is the total power passing through a *closed* surface, but this does not prove that $\int \mathbf{S} \cdot d\mathbf{a}$ is the power passing through any *open* surface (there could be an extra term that integrates to zero over all closed surfaces). This is, however, the obvious and natural interpretation; as always, the precise *location* of energy is not really determined in electrodynamics (see Sect. 2.4.4).

Of course, the work W done on the charges will increase their mechanical energy (kinetic, potential, or whatever). If we let u_{mech} denote the mechanical energy density, so that

$$\frac{dW}{dt} = \frac{d}{dt}\int_{\mathcal{V}} u_{\text{mech}}\, d\tau, \tag{8.12}$$

and use u_{em} for the energy density of the fields,

$$\boxed{u_{\text{em}} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right),} \tag{8.13}$$

then

$$\frac{d}{dt}\int_{\mathcal{V}} (u_{\text{mech}} + u_{\text{em}})\, d\tau = -\oint_{\mathcal{S}} \mathbf{S}\cdot d\mathbf{a} = -\int_{\mathcal{V}} (\nabla\cdot\mathbf{S})\, d\tau,$$

and hence

$$\boxed{\frac{\partial}{\partial t}(u_{\text{mech}} + u_{\text{em}}) = -\nabla\cdot\mathbf{S}.} \tag{8.14}$$

This is the differential version of Poynting's theorem. Compare it with the continuity equation, expressing conservation of charge (Eq. 8.4):

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{J};$$

the charge density is replaced by the energy density (mechanical plus electromagnetic), and the current density is replaced by the Poynting vector. The latter represents the flow of *energy* in exactly the same way that $\mathbf{J}$ describes the flow of *charge*.[2]

Example 8.1

When current flows down a wire, work is done, which shows up as Joule heating of the wire (Eq. 7.7). Though there are certainly *easier* ways to do it, the energy per unit time delivered to the wire can be calculated using the Poynting vector. Assuming it's uniform, the electric field parallel to the wire is

$$E = \frac{V}{L},$$

where V is the potential difference between the ends and L is the length of the wire (Fig. 8.1). The magnetic field is "circumferential"; at the surface (radius a) it has the value

$$B = \frac{\mu_0 I}{2\pi a}.$$

Accordingly, the magnitude of the Poynting vector is

$$S = -\frac{1}{\mu_0}\frac{V}{L}\frac{\mu_0 I}{2\pi a} = \frac{VI}{2\pi a L},$$

[2]In the presence of linear media, one is typically interested only in the work done on *free* charges and currents (see Sect. 4.4.3). In that case the appropriate energy density is $\frac{1}{2}(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H})$, and the Poynting vector becomes $(\mathbf{E}\times\mathbf{H})$. See J. D. Jackson, *Classical Electrodynamics*, 3rd. ed., Sect. 6.7 (New York: John Wiley, 1999).

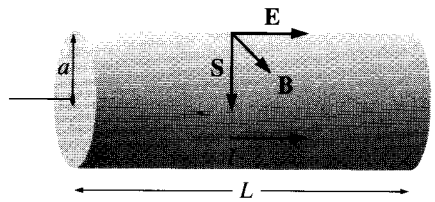

Figure 8.1

and it points radially inward. The energy per unit time passing in through the surface of the wire is therefore

$$\int \mathbf{S} \cdot d\mathbf{a} = S(2\pi a L) = VI,$$

which is exactly what we concluded, on much more direct grounds, in Sect. 7.1.1.

Problem 8.1 Calculate the power (energy per unit time) transported down the cables of Ex. 7.13 and Prob. 7.58, assuming the two conductors are held at potential difference V, and carry current I (down one and back up the other).

Problem 8.2 Consider the charging capacitor in Prob. 7.31.

(a) Find the electric and magnetic fields in the gap, as functions of the distance s from the axis and the time t. (Assume the charge is zero at $t = 0$.)

(b) Find the energy density u_{em} and the Poynting vector $\mathbf{S}$ in the gap. Note especially the *direction* of $\mathbf{S}$. Check that Eq. 8.14 is satisfied.

(c) Determine the total energy in the gap, as a function of time. Calculate the total power flowing into the gap, by integrating the Poynting vector over the appropriate surface. Check that the power input is equal to the rate of increase of energy in the gap (Eq. 8.9—in this case $W = 0$, because there is no charge in the gap). [If you're worried about the fringing fields, do it for a volume of radius $b < a$ well inside the gap.]

8.2 Momentum

8.2.1 Newton's Third Law in Electrodynamics

Imagine a point charge q traveling in along the x axis at a constant speed v. Because it is moving, its electric field is *not* given by Coulomb's law; nevertheless, $\mathbf{E}$ still points radially outward from the instantaneous position of the charge (Fig. 8.2a), as we'll see in Chapter 10. Since, moreover, a moving point charge does not constitute a steady current, its magnetic

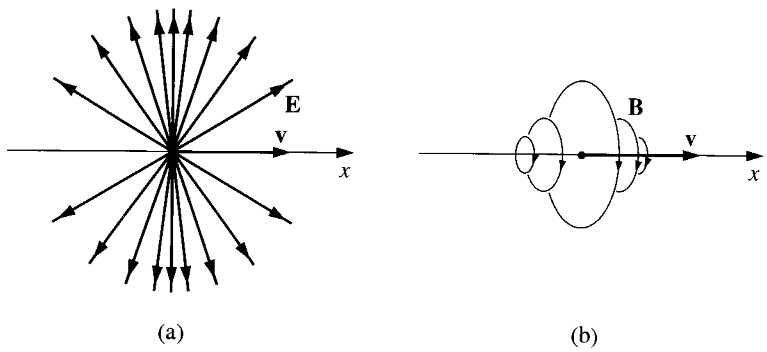

Figure 8.2

field is *not* given by the Biot-Savart law. Nevertheless, it's a fact that **B** still circles around the axis in a manner suggested by the right-hand rule (Fig. 8.2b); again, the proof will come in Chapter 10.

Now suppose this charge encounters an identical one, proceeding in at the same speed along the y axis. Of course, the electromagnetic force between them would tend to drive them off the axes, but let's assume that they're mounted on tracks, or something, so they're forced to maintain the same direction and the same speed (Fig. 8.3). The electric force between them is repulsive, but how about the magnetic force? Well, the magnetic field of q_1 points into the page (at the position of q_2), so the magnetic force on q_2 is toward the *right*, whereas the magnetic field of q_2 is *out* of the page (at the position of q_1), and the magnetic force on q_1 is *upward*. *The electromagnetic force of q_1 on q_2 is equal but not opposite to the force of q_2 on q_1, in violation of Newton's third law.* In electro*statics* and magneto*statics* the third law holds, but in electro*dynamics* it does *not*.

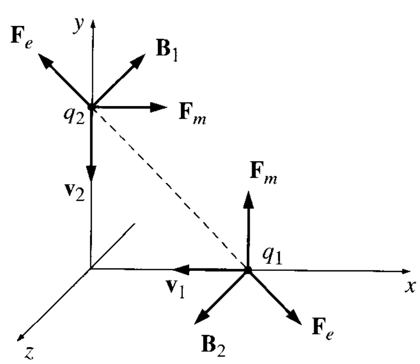

Figure 8.3

Well, that's an interesting curiosity, but then, how often does one actually use the third law, in practice? *Answer:* All the time! For the proof of conservation of momentum rests on the cancellation of internal forces, which follows from the third law. When you tamper with the third law, you are placing conservation of momentum in jeopardy, and there is no principle in physics more sacred than *that*.

Momentum conservation is rescued in electrodynamics by the realization that *the fields themselves carry momentum*. This is not so surprising when you consider that we have already attributed *energy* to the fields. In the case of the two point charges in Fig. 8.3, whatever momentum is lost to the particles is gained by the fields. Only when the field momentum is added to the mechanical momentum of the charges is momentum conservation restored. You'll see how this works out quantitatively in the following sections.

8.2.2 Maxwell's Stress Tensor

Let's calculate the total electromagnetic force on the charges in volume $\mathcal{V}$:

$$\mathbf{F} = \int_{\mathcal{V}} (\mathbf{E} + \mathbf{v} \times \mathbf{B})\rho \, d\tau = \int_{\mathcal{V}} (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) \, d\tau. \tag{8.15}$$

The *force per unit volume* is evidently

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}. \tag{8.16}$$

As before, I propose to write this in terms of fields alone, eliminating ρ and $\mathbf{J}$ by using Maxwell's equations (i) and (iv):

$$\mathbf{f} = \epsilon_0(\nabla \cdot \mathbf{E})\mathbf{E} + \left(\frac{1}{\mu_0}\nabla \times \mathbf{B} - \epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{B}.$$

Now

$$\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) = \left(\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}\right) + \left(\mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t}\right),$$

and Faraday's law says

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E},$$

so

$$\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} = \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times (\nabla \times \mathbf{E}).$$

Thus

$$\mathbf{f} = \epsilon_0[(\nabla \cdot \mathbf{E})\mathbf{E} - \mathbf{E} \times (\nabla \times \mathbf{E})] - \frac{1}{\mu_0}[\mathbf{B} \times (\nabla \times \mathbf{B})] - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}). \tag{8.17}$$

Just to make things look more symmetrical, let's throw in a term $(\nabla \cdot \mathbf{B})\mathbf{B}$; since $\nabla \cdot \mathbf{B} = 0$, this costs us nothing. Meanwhile, product rule 4 says

$$\nabla(E^2) = 2(\mathbf{E} \cdot \nabla)\mathbf{E} + 2\mathbf{E} \times (\nabla \times \mathbf{E}),$$

so

$$\mathbf{E} \times (\nabla \times \mathbf{E}) = \frac{1}{2}\nabla(E^2) - (\mathbf{E} \cdot \nabla)\mathbf{E},$$

and the same goes for **B**. Therefore,

$$\begin{aligned} \mathbf{f} = \; & \epsilon_0[(\nabla \cdot \mathbf{E})\mathbf{E} + (\mathbf{E} \cdot \nabla)\mathbf{E}] + \frac{1}{\mu_0}[(\nabla \cdot \mathbf{B})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{B}] \\ & -\frac{1}{2}\nabla\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}). \end{aligned} \tag{8.18}$$

Ugly! But it can be simplified by introducing the **Maxwell stress tensor**,

$$T_{ij} \equiv \epsilon_0\left(E_i E_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}B^2\right). \tag{8.19}$$

The indices i and j refer to the coordinates x, y, and z, so the stress tensor has a total of nine components (T_{xx}, T_{yy}, T_{xz}, T_{yx}, and so on). The **Kronecker delta**, δ_{ij}, is 1 if the indices are the same ($\delta_{xx} = \delta_{yy} = \delta_{zz} = 1$) and zero otherwise ($\delta_{xy} = \delta_{xz} = \delta_{yz} = 0$). Thus

$$\begin{aligned} T_{xx} = \; & \frac{1}{2}\epsilon_0(E_x^2 - E_y^2 - E_z^2) + \frac{1}{2\mu_0}(B_x^2 - B_y^2 - B_z^2), \\ T_{xy} = \; & \epsilon_0(E_x E_y) + \frac{1}{\mu_0}(B_x B_y), \end{aligned}$$

and so on. Because it carries *two* indices, where a vector has only one, T_{ij} is sometimes written with a double arrow: $\overleftrightarrow{\mathbf{T}}$. One can form the dot product of $\overleftrightarrow{\mathbf{T}}$ with a vector **a**:

$$(\mathbf{a} \cdot \overleftrightarrow{\mathbf{T}})_j = \sum_{i=x,y,z} a_i T_{ij}; \tag{8.20}$$

the resulting object, which has one remaining index, is itself a vector. In particular, the divergence of $\overleftrightarrow{\mathbf{T}}$ has as its jth component

$$\begin{aligned} (\nabla \cdot \overleftrightarrow{\mathbf{T}})_j \; = \; & \epsilon_0\left[(\nabla \cdot \mathbf{E})E_j + (\mathbf{E} \cdot \nabla)E_j - \frac{1}{2}\nabla_j E^2\right] \\ & + \frac{1}{\mu_0}\left[(\nabla \cdot \mathbf{B})B_j + (\mathbf{B} \cdot \nabla)B_j - \frac{1}{2}\nabla_j B^2\right]. \end{aligned}$$

Thus the force per unit volume (Eq. 8.18) can be written in the much simpler form

$$\mathbf{f} = \nabla \cdot \overleftrightarrow{\mathbf{T}} - \epsilon_0\mu_0\frac{\partial \mathbf{S}}{\partial t}, \tag{8.21}$$

where **S** is the Poynting vector (Eq. 8.10).

The *total* force on the charges in $\mathcal{V}$ (Eq. 8.15) is evidently

$$\mathbf{F} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \epsilon_0 \mu_0 \frac{d}{dt} \int_{\mathcal{V}} \mathbf{S}\, d\tau. \tag{8.22}$$

(I used the divergence theorem to convert the first term to a surface integral.) In the *static* case (or, more generally, whenever $\int \mathbf{S}\, d\tau$ is independent of time), the second term drops out, and the electromagnetic force on the charge configuration can be expressed entirely in terms of the stress tensor at the boundary. Physically, $\overleftrightarrow{\mathbf{T}}$ is the force per unit area (or **stress**) acting on the surface. More precisely, T_{ij} is the force (per unit area) in the ith direction acting on an element of surface oriented in the jth direction—"diagonal" elements (T_{xx}, T_{yy}, T_{zz}) represent *pressures*, and "off-diagonal" elements (T_{xy}, T_{xz}, etc.) are *shears*.

Example 8.2

Determine the net force on the "northern" hemisphere of a uniformly charged solid sphere of radius R and charge Q (the same as Prob. 2.43, only this time we'll use the Maxwell stress tensor and Eq. 8.22).

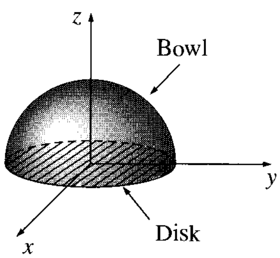

Figure 8.4

Solution: The boundary surface consists of two parts—a hemispherical "bowl" at radius R, and a circular disk at $\theta = \pi/2$ (Fig. 8.4). For the bowl,

$$d\mathbf{a} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{r}}$$

and

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \hat{\mathbf{r}}.$$

In Cartesian components,

$$\hat{\mathbf{r}} = \sin\theta \cos\phi\, \hat{\mathbf{x}} + \sin\theta \sin\phi\, \hat{\mathbf{y}} + \cos\theta\, \hat{\mathbf{z}},$$

so

$$T_{zx} = \epsilon_0 E_z E_x = \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 R^2} \right)^2 \sin\theta \cos\theta \cos\phi,$$

$$T_{zy} = \epsilon_0 E_z E_y = \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 R^2} \right)^2 \sin\theta \cos\theta \sin\phi,$$

$$T_{zz} = \frac{\epsilon_0}{2} (E_z^2 - E_x^2 - E_y^2) = \frac{\epsilon_0}{2} \left(\frac{Q}{4\pi\epsilon_0 R^2} \right)^2 (\cos^2\theta - \sin^2\theta). \tag{8.23}$$

The net force is obviously in the z-direction, so it suffices to calculate

$$(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z = T_{zx}\, da_x + T_{zy}\, da_y + T_{zz}\, da_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 \sin\theta\cos\theta\, d\theta\, d\phi.$$

The force on the "bowl" is therefore

$$F_{\text{bowl}} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 2\pi \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{8R^2}. \tag{8.24}$$

Meanwhile, for the equatorial disk,

$$d\mathbf{a} = -r\, dr\, d\phi\, \hat{\mathbf{z}}, \tag{8.25}$$

and (since we are now *inside* the sphere)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\mathbf{r} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3} r(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}).$$

Thus

$$T_{zz} = \frac{\epsilon_0}{2}(E_z^2 - E_x^2 - E_y^2) = -\frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 r^2,$$

and hence

$$(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 r^3\, dr\, d\phi.$$

The force on the disk is therefore

$$F_{\text{disk}} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 2\pi \int_0^R r^3 dr = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{16R^2}. \tag{8.26}$$

Combining Eqs. 8.24 and 8.26, I conclude that the net force on the northern hemisphere is

$$F = \frac{1}{4\pi\epsilon_0}\frac{3Q^2}{16R^2}. \tag{8.27}$$

Incidentally, in applying Eq. 8.22, *any* volume that encloses all of the charge in question (and no *other* charge) will do the job. For example, in the present case we could use the whole region $z > 0$. In that case the boundary surface consists of the entire xy plane (plus a hemisphere at $r = \infty$, but $E = 0$ out there anyway, so it contributes nothing). In place of the "bowl," we now have the outer portion of the plane ($r > R$). Here

$$T_{zz} = -\frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 \frac{1}{r^4}$$

(Eq. 8.23 with $\theta = \pi/2$ and $R \to r$), and $d\mathbf{a}$ is given by Eq. 8.25, so

$$(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 \frac{1}{r^3}\, dr\, d\phi,$$

and the contribution from the plane for $r > R$ is

$$\frac{\epsilon_0}{2}\left(\frac{Q}{2\pi\epsilon_0}\right)^2 2\pi \int_R^\infty \frac{1}{r^3}\, dr = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{8R^2},$$

the same as for the bowl (Eq. 8.24).

I hope you didn't get too bogged down in the details of Ex. 8.2. If so, take a moment to appreciate what happened. We were calculating the force on a solid object, but instead of doing a *volume* integral, as you might expect, Eq. 8.22 allowed us to set it up as a *surface* integral; somehow the stress tensor sniffs out what is going on inside.

! **Problem 8.3** Calculate the force of magnetic attraction between the northern and southern hemispheres of a uniformly charged spinning spherical shell, with radius R, angular velocity ω, and surface charge density σ. [This is the same as Prob. 5.42, but this time use the Maxwell stress tensor and Eq. 8.22.]

Problem 8.4

(a) Consider two equal point charges q, separated by a distance $2a$. Construct the plane equidistant from the two charges. By integrating Maxwell's stress tensor over this plane, determine the force of one charge on the other.

(b) Do the same for charges that are opposite in sign.

8.2.3 Conservation of Momentum

According to Newton's second law, the force on an object is equal to the rate of change of its momentum:

$$\mathbf{F} = \frac{d\mathbf{p}_{\text{mech}}}{dt}.$$

Equation 8.22 can therefore be written in the form

$$\frac{d\mathbf{p}_{\text{mech}}}{dt} = -\epsilon_0\mu_0\frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\,d\tau + \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}, \tag{8.28}$$

where $\mathbf{p}_{\text{mech}}$ is the total (mechanical) momentum of the particles contained in the volume $\mathcal{V}$. This expression is similar in structure to Poynting's theorem (Eq. 8.9), and it invites an analogous interpretation: The first integral represents *momentum stored in the electromagnetic fields themselves*:

$$\mathbf{p}_{\text{em}} = \mu_0\epsilon_0 \int_{\mathcal{V}} \mathbf{S}\,d\tau, \tag{8.29}$$

while the second integral is the *momentum per unit time flowing in through the surface*. Equation 8.28 is the general statement of *conservation of momentum* in electrodynamics: Any increase in the *total* momentum (mechanical plus electromagnetic) is equal to the momentum brought in by the fields. (If $\mathcal{V}$ is *all* of space, then *no* momentum flows in or out, and $\mathbf{p}_{\text{mech}} + \mathbf{p}_{\text{em}}$ is constant.)

As in the case of conservation of charge and conservation of energy, conservation of momentum can be given a differential formulation. Let $\wp_{\text{mech}}$ be the density of *mechanical* momentum, and $\wp_{\text{em}}$ the density of momentum in the fields:

$$\boxed{\wp_{\text{em}} = \mu_0\epsilon_0\mathbf{S}.} \tag{8.30}$$

Then Eq. 8.28, in differential form, says

$$\boxed{\frac{\partial}{\partial t}(\wp_{\text{mech}} + \wp_{\text{em}}) = \nabla \cdot \overleftrightarrow{\mathbf{T}}.} \tag{8.31}$$

Evidently $-\overleftrightarrow{\mathbf{T}}$ is the **momentum flux density**, playing the role of **J** (current density) in the continuity equation, or **S** (energy flux density) in Poynting's theorem. Specifically, $-T_{ij}$ is the momentum in the i direction crossing a surface oriented in the j direction, per unit area, per unit time. Notice that the Poynting vector has appeared in two quite different roles: **S** itself is the energy per unit area, per unit time, transported by the electromagnetic fields, while $\mu_0\epsilon_0\mathbf{S}$ is the momentum per unit volume stored in those fields. Similarly, $\overleftrightarrow{\mathbf{T}}$ plays a dual role: $\overleftrightarrow{\mathbf{T}}$ itself is the electromagnetic stress (force per unit area) acting on a surface, and $-\overleftrightarrow{\mathbf{T}}$ describes the flow of momentum (the momentum current density) transported by the fields.

Example 8.3

A long coaxial cable, of length l, consists of an inner conductor (radius a) and an outer conductor (radius b). It is connected to a battery at one end and a resistor at the other (Fig. 8.5). The inner conductor carries a uniform charge per unit length λ, and a steady current I to the right; the outer conductor has the opposite charge and current. What is the electromagnetic momentum stored in the fields?

Solution: The fields are

$$\mathbf{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\,\hat{\mathbf{s}}, \qquad \mathbf{B} = \frac{\mu_0}{2\pi}\frac{I}{s}\,\hat{\boldsymbol{\phi}}.$$

The Poynting vector is therefore

$$\mathbf{S} = \frac{\lambda I}{4\pi^2\epsilon_0 s^2}\,\hat{\mathbf{z}}.$$

Evidently energy is flowing down the line, from the battery to the resistor. In fact, the power transported is

$$P = \int \mathbf{S}\cdot d\mathbf{a} = \frac{\lambda I}{4\pi^2\epsilon_0}\int_a^b \frac{1}{s^2}2\pi s\,ds = \frac{\lambda I}{2\pi\epsilon_0}\ln(b/a) = IV,$$

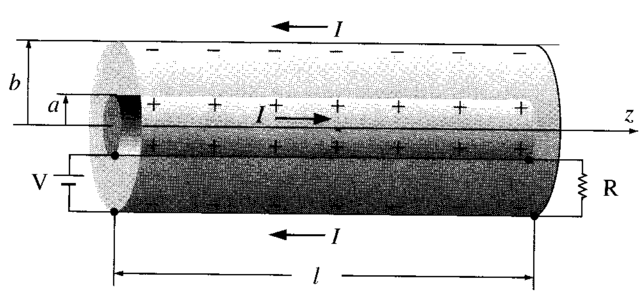

Figure 8.5

as it should be. But that's not what we're interested in right now. The *momentum* in the fields is

$$\mathbf{p}_{\text{em}} = \mu_0\epsilon_0 \int \mathbf{S}\, d\tau = \frac{\mu_0\lambda I}{4\pi^2}\,\hat{\mathbf{z}} \int_a^b \frac{1}{s^2} l 2\pi s\, ds = \frac{\mu_0\lambda I l}{2\pi} \ln(b/a)\,\hat{\mathbf{z}}.$$

This is an astonishing result. The cable is not moving, and the fields are static, and yet we are asked to believe that there is momentum in the system. If something tells you this cannot be the whole story, you have sound intuitions. In fact, if the center of mass of a localized system is at rest, its total momentum *must* be zero. In this case it turns out that there is "hidden" mechanical momentum associated with the flow of current, and this exactly cancels the momentum in the fields. But *locating* the **hidden momentum** is not easy, and it is actually a relativistic effect, so I shall save it for Chapter 12 (Ex. 12.12).

Suppose now that we turn up the resistance, so the current decreases. The changing magnetic field will induce an electric field (Eq. 7.19):

$$\mathbf{E} = \left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln s + K\right]\hat{\mathbf{z}}.$$

This field exerts a force on $\pm\lambda$:

$$\mathbf{F} = \lambda l\left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln a + K\right]\hat{\mathbf{z}} - \lambda l\left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln b + K\right]\hat{\mathbf{z}} = -\frac{\mu_0\lambda l}{2\pi}\frac{dI}{dt}\ln(b/a)\,\hat{\mathbf{z}}.$$

The total momentum imparted to the cable, as the current drops from I to 0, is therefore

$$\mathbf{p}_{\text{mech}} = \int \mathbf{F}\, dt = \frac{\mu_0\lambda I l}{2\pi}\ln(b/a)\,\hat{\mathbf{z}},$$

which is precisely the momentum originally stored in the fields. (The cable will not recoil, however, because an equal and opposite impulse is delivered by the simultaneous disappearance of the hidden momentum.)

Problem 8.5 Consider an infinite parallel-plate capacitor, with the lower plate (at $z = -d/2$) carrying the charge density $-\sigma$, and the upper plate (at $z = +d/2$) carrying the charge density $+\sigma$.

(a) Determine all nine elements of the stress tensor, in the region between the plates. Display your answer as a 3×3 matrix:

$$\begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

(b) Use Eq. 8.22 to determine the force per unit area on the top plate. Compare Eq. 2.51.

(c) What is the momentum per unit area, per unit time, crossing the xy plane (or any other plane parallel to that one, between the plates)?

(d) At the plates this momentum is absorbed, and the plates recoil (unless there is some nonelectrical force holding them in position). Find the recoil force per unit area on the top

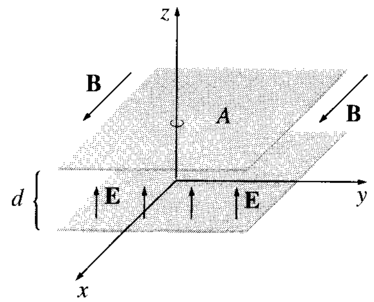

Figure 8.6

plate, and compare your answer to (b). [*Note:* This is not an *additional* force, but rather an alternative way of calculating the *same* force—in (b) we got it from the force law, and in (d) we did it by conservation of momentum.]

Problem 8.6 A charged parallel-plate capacitor (with uniform electric field $\mathbf{E} = E\,\hat{\mathbf{z}}$) is placed in a uniform magnetic field $\mathbf{B} = B\,\hat{\mathbf{x}}$, as shown in Fig. 8.6.[3]

(a) Find the electromagnetic momentum in the space between the plates.

(b) Now a resistive wire is connected between the plates, along the z axis, so that the capacitor slowly discharges. The current through the wire will experience a magnetic force; what is the total impulse delivered to the system, during the discharge?

(c) Instead of turning off the *electric* field (as in (b)), suppose we slowly reduce the *magnetic* field. This will induce a Faraday electric field, which in turn exerts a force on the plates. Show that the total impulse is (again) equal to the momentum originally stored in the fields.

8.2.4 Angular Momentum

By now the electromagnetic fields (which started out as mediators of forces between charges) have taken on a life of their own. They carry *energy* (Eq. 8.13)

$$u_{\text{em}} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right), \tag{8.32}$$

and *momentum* (Eq. 8.30)

$$\boldsymbol{\wp}_{\text{em}} = \mu_0\epsilon_0\mathbf{S} = \epsilon_0(\mathbf{E}\times\mathbf{B}), \tag{8.33}$$

and, for that matter, *angular* momentum:

$$\boldsymbol{\ell}_{\text{em}} = \mathbf{r}\times\boldsymbol{\wp}_{\text{em}} = \epsilon_0\left[\mathbf{r}\times(\mathbf{E}\times\mathbf{B})\right]. \tag{8.34}$$

[3]See F. S. Johnson, B. L. Cragin, and R. R. Hodges, *Am. J. Phys.* **62**, 33 (1994).

Even perfectly static fields can harbor momentum and angular momentum, as long as $\mathbf{E} \times \mathbf{B}$ is nonzero, and it is only when these field contributions are included that the classical conservation laws hold.

Example 8.4

Imagine a very long solenoid with radius R, n turns per unit length, and current I. Coaxial with the solenoid are two long cylindrical shells of length l—one, *inside* the solenoid at radius a, carries a charge $+Q$, uniformly distributed over its surface; the other, *outside* the solenoid at radius b, carries charge $-Q$ (see Fig. 8.7; l is supposed to be much greater than b). When the current in the solenoid is gradually reduced, the cylinders begin to rotate, as we found in Ex. 7.8. *Question*: Where does the angular momentum come from?[4]

Solution: It was initially stored in the fields. Before the current was switched off, there was an electric field,

$$\mathbf{E} = \frac{Q}{2\pi\epsilon_0 l}\frac{1}{s}\hat{\mathbf{s}} \quad (a < s < b).$$

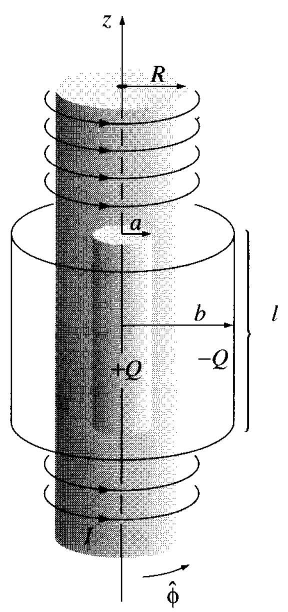

Figure 8.7

[4]This is a variation on the "Feynman disk paradox" (R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures*, vol 2, pp. 17-5 (Reading, Mass.: Addison-Wesley, 1964) suggested by F. L. Boos, Jr. (*Am. J. Phys.* **52**, 756 (1984)). A similar model was proposed earlier by R. H. Romer (*Am. J. Phys.* **34**, 772 (1966)). For further references, see T.-C. E. Ma, *Am. J. Phys.* **54**, 949 (1986).

in the region between the cylinders, and a magnetic field,

$$\mathbf{B} = \mu_0 n I \,\hat{\mathbf{z}} (s < R),$$

inside the solenoid. The momentum density (Eq. 8.33) was therefore

$$\boldsymbol{\wp}_{\text{em}} = -\frac{\mu_0 n I Q}{2\pi l s}\,\hat{\boldsymbol{\phi}},$$

in the region $a < s < R$. The *angular* momentum density was

$$\boldsymbol{\ell}_{\text{em}} = \mathbf{r} \times \boldsymbol{\wp}_{\text{em}} = -\frac{\mu_0 n I Q}{2\pi l}\,\hat{\mathbf{z}},$$

which is *constant*, as it turns out; to get the *total* angular momentum in the fields, we simply multiply by the volume, $\pi(R^2 - a^2)l$:

$$\mathbf{L}_{\text{em}} = -\frac{1}{2}\mu_0 n I Q (R^2 - a^2)\,\hat{\mathbf{z}}. \tag{8.35}$$

When the current is turned off, the changing magnetic field induces a circumferential electric field, given by Faraday's law:

$$\mathbf{E} = \begin{cases} -\dfrac{1}{2}\mu_0 n \dfrac{dI}{dt}\dfrac{R^2}{s}\,\hat{\boldsymbol{\phi}}, & (s > R), \\[2ex] -\dfrac{1}{2}\mu_0 n \dfrac{dI}{dt} s\,\hat{\boldsymbol{\phi}}, & (s < R). \end{cases}$$

Thus the torque on the outer cylinder is

$$\mathbf{N}_b = \mathbf{r} \times (-Q\mathbf{E}) = \frac{1}{2}\mu_0 n Q R^2 \frac{dI}{dt}\,\hat{\mathbf{z}},$$

and it picks up an angular momentum

$$\mathbf{L}_b = \frac{1}{2}\mu_0 n Q R^2\,\hat{\mathbf{z}} \int_I^0 \frac{dI}{dt}\,dt = -\frac{1}{2}\mu_0 n I Q R^2\,\hat{\mathbf{z}}.$$

Similarly, the torque on the inner cylinder is

$$\mathbf{N}_a = -\frac{1}{2}\mu_0 n Q a^2 \frac{dI}{dt}\,\hat{\mathbf{z}},$$

and its angular momentum increase is

$$\mathbf{L}_a = \frac{1}{2}\mu_0 n I Q a^2\,\hat{\mathbf{z}}.$$

So it all works out: $\mathbf{L}_{\text{em}} = \mathbf{L}_a + \mathbf{L}_b$. The angular momentum *lost* by the fields is precisely equal to the angular momentum *gained* by the cylinders, and the *total* angular momentum (fields plus matter) is conserved.

Incidentally, the angular case is in some respects *cleaner* than the linear analog (Ex. 8.3), because there is no "hidden" angular momentum to compensate for the angular momentum in the fields, and the cylinders really *do* rotate when the magnetic field is turned off. If a localized system is not moving, its total *linear* momentum *has* to be zero,[5] but there is no corresponding theorem for angular momentum, and in Prob. 8.12 you will see a beautiful example in which nothing at *all* is moving—not even currents—and yet the angular momentum is nonzero.

Problem 8.7 In Ex. 8.4, suppose that instead of turning off the *magnetic* field (by reducing I) we turn off the *electric* field, by connecting a weakly[6] conducting radial spoke between the cylinders. (We'll have to cut a slot in the solenoid, so the cylinders can still rotate freely.) From the magnetic force on the current in the spoke, determine the total angular momentum delivered to the cylinders, as they discharge (they are now rigidly connected, so they rotate together). Compare the initial angular momentum stored in the fields (Eq. 8.35). (Notice that the *mechanism* by which angular momentum is transferred from the fields to the cylinders is entirely different in the two cases: in Ex. 8.4 it was Faraday's law, but here it is the Lorentz force law.)

! **Problem 8.8**[7] Imagine an iron sphere of radius R that carries a charge Q and a uniform magnetization $\mathbf{M} = M\hat{\mathbf{z}}$. The sphere is initially at rest.

(a) Compute the angular momentum stored in the electromagnetic fields.

(b) Suppose the sphere is gradually (and uniformly) demagnetized (perhaps by heating it up past the Curie point). Use Faraday's law to determine the induced electric field, find the torque this field exerts on the sphere, and calculate the total angular momentum imparted to the sphere in the course of the demagnetization.

(c) Suppose instead of *demagnetizing* the sphere we *discharge* it, by connecting a grounding wire to the north pole. Assume the current flows over the surface in such a way that the charge density remains uniform. Use the Lorentz force law to determine the torque on the sphere, and calculate the total angular momentum imparted to the sphere in the course of the discharge. (The magnetic field is discontinuous at the surface ... does this matter?) [*Answer:* $\frac{2}{9}\mu_0 M Q R^2$]

More Problems on Chapter 8

Problem 8.9[8] A very long solenoid of radius a, with n turns per unit length, carries a current I_s. Coaxial with the solenoid, at radius $b \gg a$, is a circular ring of wire, with resistance R. When the current in the solenoid is (gradually) decreased, a current I_r is induced in the ring.

---

[5] S. Coleman and J. H. van Vleck, *Phys. Rev.* **171**, 1370 (1968).

[6] In Ex. 8.4 we turned the current off slowly, to keep things quasistatic; here we reduce the electric field slowly to keep the displacement current negligible.

[7] This version of the Feynman disk paradox was proposed by N. L. Sharma (*Am. J. Phys.* **56**, 420 (1988)); similar models were analyzed by E. M. Pugh and G. E. Pugh, *Am. J. Phys.* **35**, 153 (1967) and by R. H. Romer, *Am. J. Phys.* **35**, 445 (1967).

[8] For extensive discussion, see M. A. Heald, *Am. J. Phys.* **56**, 540 (1988).

(a) Calculate I_r, in terms of dI_s/dt.

(b) The power ($I_r^2 R$) delivered to the ring must have come from the solenoid. Confirm this by calculating the Poynting vector just outside the solenoid (the *electric* field is due to the changing flux in the solenoid; the *magnetic* field is due to the current in the ring). Integrate over the entire surface of the solenoid, and check that you recover the correct total power.

Problem 8.10[9] A sphere of radius R carries a uniform polarization $\mathbf{P}$ and a uniform magnetization $\mathbf{M}$ (not necessarily in the same direction). Find the electromagnetic momentum of this configuration. [*Answer:* $(4/9)\pi\mu_0 R^3(\mathbf{M}\times\mathbf{P})$]

Problem 8.11[10] Picture the electron as a uniformly charged spherical shell, with charge e and radius R, spinning at angular velocity ω.

(a) Calculate the total energy contained in the electromagnetic fields.

(b) Calculate the total angular momentum contained in the fields.

(c) According to the Einstein formula ($E = mc^2$), the energy in the fields should contribute to the mass of the electron. Lorentz and others speculated that the *entire* mass of the electron might be accounted for in this way: $U_{\text{em}} = m_e c^2$. Suppose, moreover, that the electron's spin angular momentum is entirely attributable to the electromagnetic fields: $L_{\text{em}} = \hbar/2$. On these two assumptions, determine the radius and angular velocity of the electron. What is their product, ωR? Does this classical model make sense?

! **Problem 8.12**[11] Suppose you had an electric charge q_e and a magnetic monopole q_m. The field of the electric charge is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q_e}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}},$$

of course, and the field of the magnetic monopole is

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q_m}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}.$$

Find the total angular momentum stored in the fields, if the two charges are separated by a distance d. [*Answer:* $(\mu_0/4\pi)q_e q_m$.][12]

Problem 8.13 Paul DeYoung, of Hope College, points out that because the cylinders in Ex. 8.4 are left rotating (at angular velocities ω_a and ω_b, say), there is actually a residual magnetic field, and hence angular momentum in the fields, even after the current in the solenoid has been extinguished. If the cylinders are heavy, this correction will be negligible, but it is interesting to do the problem *without* making that assumption.

---

[9] For an interesting discussion and references, see R. H. Romer, *Am. J. Phys.* **63**, 777 (1995).

[10] See J. Higbie, *Am. J. Phys.* **56**, 378 (1988).

[11] This system is known as **Thomson's dipole**. See I. Adawi, *Am. J. Phys.* **44**, 762 (1976) and *Phys. Rev.* **D31**, 3301 (1985), and K. R. Brownstein, *Am. J. Phys.* **57**, 420 (1989), for discussion and references.

[12] Note that this result is *independent of the separation distance d* (!); it points from q_e toward q_m. In quantum mechanics angular momentum comes in half-integer multiples of $\hbar$, so this result suggests that if magnetic monopoles exist, electric and magnetic charge must be quantized: $\mu_0 q_e q_m/4\pi = n\hbar/2$, for $n = 1, 2, 3, \ldots$, an idea first proposed by Dirac in 1931. If even *one* monopole exists somewhere in the universe, this would "explain" why electric charge comes in discrete units.

(a) Calculate (in terms of ω_a and ω_b) the final angular momentum in the fields.

(b) As the cylinders begin to rotate, their changing magnetic field induces an extra azimuthal electric field, which, in turn, will make an additional contribution to the torques. Find the resulting extra angular momentum, and compare it to your result in (a). [*Answer:* $\mu_0 Q^2 \omega_b (b^2 - a^2)/4\pi l$]

Problem 8.14[13] A point charge q is a distance $a > R$ from the axis of an infinite solenoid (radius R, n turns per unit length, current I). Find the linear momentum and the angular momentum in the fields. (Put q on the x axis, with the solenoid along z; treat the solenoid as a nonconductor, so you don't need to worry about induced charges on its surface.) [*Answer:* $\mathbf{p}_{\text{em}} = (\mu_0 q n I R^2/2a)\,\hat{\mathbf{y}}$; $\mathbf{L}_{\text{em}} = 0$]

Problem 8.15[14] (a) Carry through the argument in Sect. 8.1.2, starting with Eq. 8.6, but using $\mathbf{J}_f$ in place of $\mathbf{J}$. Show that the Poynting vector becomes

$$\mathbf{S} = \mathbf{E} \times \mathbf{H},$$

and the rate of change of the energy density in the fields is

$$\frac{\partial u_{\text{em}}}{\partial t} = \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}.$$

For *linear* media, show that

$$u_{\text{em}} = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}).$$

(b) In the same spirit, reproduce the argument in Sect. 8.2.2, starting with Eq. 8.15, with ρ_f and $\mathbf{J}_f$ in place of ρ and $\mathbf{J}$. Don't bother to construct the Maxwell stress tensor, but do show that the momentum density is

$$\boldsymbol{\wp} = \mathbf{D} \times \mathbf{B}.$$

---

[13] See F. S. Johnson, B. L. Cragin, and R. R. Hodges, *Am. J. Phys.* **62**, 33 (1994), for a discussion of this and related problems.

[14] This problem was suggested by David Thouless of the University of Washington. Refer to Sect. 4.4.3 for the meaning of "energy" in this context.