Continuity and Differentiability

Continuity of a Function

- Suppose a function *f* is a real valued function defined on a subset of real numbers. Let *c* be a point in the domain of *f*. Then, *f* is continuous at *c* if

  $$\lim_{x \to c} f(x) = f(c)$$.

- A function is continuous at *x* = *c* if the function is defined at *x* = *c* and if the value of the function at *x* = *c* equals the limit of the function at *x* = *c*

  i.e., $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$.

- If *f* is not continuous at point *c*, then *f* is said to be discontinuous at *c* and '*c*' is called the point of discontinuity of *f*.

- Consider the function $f(x) = \begin{cases} x^2 + 2, & x = 0 \\ 1, & x \neq 0 \end{cases}$

  At point *x* = 0,

  $$\text{Left hand limit} = \lim_{x \to 0^-} f(x) = 1; \text{ Right hand limit} = \lim_{x \to 0^+} f(x) = 1$$
  $$f(0) = 0^2 + 2 = 2$$

  $$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \neq f(0)$$

  Therefore, *f* is not continuous at *x* = 0.

  At point *x* = 1,

  $$\text{Left hand limit} = \lim_{x \to 1^-} f(x) = 1; \text{ Right hand limit} = \lim_{x \to 1^+} f(x) = 1$$
  $$f(1) = 1$$

  $$\therefore \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

  Therefore, *f* is continuous at *x* = 1.

- A real function *f* is said to be continuous if it is continuous at every point in the domain of *f*.

- The continuity of function can be written as:
- Suppose *f* is a function defined on close interval [*a*, *b*]. Then, for *f* to be continuous, it needs to be continuous at every point of [*a*, *b*] including *a* and *b*.

  Thus, the continuity of *f* at *a* means $\lim_{x \to a^+} f(x) = f(a)$.
  The continuity of *f* at *b* means $\lim_{x \to b^-} f(x) = f(b)$.

- Suppose a function *f* is defined at a point *c*. Then, *f* is continuous in the domain {*c*}.
- To understand how to check the continuity of a function at a point,

Concept of Infinity

- Analyse the function $f(x) = \frac{1}{x}$ near $x = 0$. Here, *x* can approach 0 either from the left of 0 or from the right of 0.
- When *x* approaches 0 from the right, then we have the following values:

x	1	0.5	0.25	$0.01 = 10^{-2}$	$0.0001 = 10^{-4}$	10^{-n}
$f(x)$	1	2	4	10^2	10^4	10^n

  As *x* gets closer to 0 from the right, the value of *f* (*x*) keeps on increasing.
  When *f* (*x*) keeps on increasing, it is said to approach infinity(∞).
  Mathematically, it is written as $\lim_{x \to 0^+} f(x) = \infty$, where $f(x) = \frac{1}{x}, x \neq 0$.

- When *x* approaches 0 from the left, then we have the following values:

x	-1	-0.5	-0.25	$-0.01 = -10^{-2}$	$-0.0001 = 10^{-4}$	-10^{-n}

$f(x)$	-1	-2	-4	-100	-10^4	-10^n

- As x gets closer to 0 from the left, the value of $f(x)$ keeps on decreasing.
  When $f(x)$ keeps on decreasing, it is said to approach infinity ($-\infty$). Mathematically, it is written as $\lim_{x \to 0^-} f(x) = -\infty$, where $f(x) = \frac{1}{x}, x \neq 0$.

Solved Examples

Example 1

Check whether the given function is continuous or not.

$$f(x) = \begin{cases} \frac{1}{4}x + 1, & x \leq 4 \\ \frac{8}{x}, & x > 4 \end{cases}$$

Solution:

Let us first check the continuity of f at $x = 4$.

$$\lim_{x \to 4^-} f(x) = \frac{4}{4} + 1 = 2,\ \lim_{x \to 4^+} f(x) = \frac{8}{4} = 2,\ f(4) = \frac{4}{4} + 1 = 2$$
$$\therefore \lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x) = f(4)$$

Hence, f is continuous at $x = 4$.

Let c be a real number such that $c < 4$.

Accordingly, $\lim_{x \to c} f(x) = \frac{c}{4} + 1 = f(c)$.

Hence, f is continuous at all real numbers less than 4.

Let c be a real number such that $c > 4$.

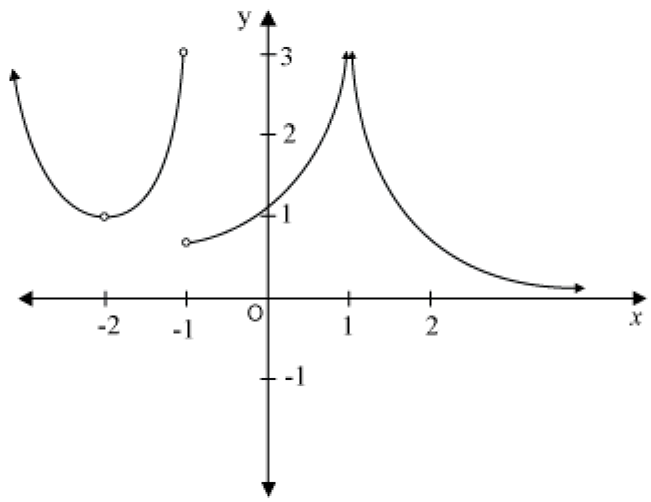
Accordingly, . $\lim_{x \to c} f(x) = \frac{8}{c} = f(c)$

Hence, f is continuous at all real numbers greater than 4.

Thus, f is continuous at all points. Hence, f is a continuous function.

Example 2

Consider the graph of $f(x)$ given below.



State the conditions of continuity that are not met at each discontinuous point within the interval [−2, 2].

Solution:

Consider the given function at point $x = -2$.

f is not defined at point = −2.

Therefore, f is discontinuous at $x = -2$.

Consider the given function at point $x = -1$.

Function f is not defined at $x = -1$. Hence, it is discontinuous at $x = -1$.

Consider the point $x = 1$.

The function f is not defined at $x = 1$. Hence, it is discontinuous at $x = 1$.

Algebra of Continuous Functions

- Suppose f and g are two real functions continuous at a real number c, then
- $f + g$ is continuous at $x = c$

- $f - g$ is continuous at $x = c$
- $f.g$ is continuous at $x = c$
- $\left(\frac{f}{g}\right)$ is continuous at $x = c$ (provided $g(c) \neq 0$)
- If f is a constant function, i.e. $f(x) = k$ for some real number k, then
- The function $f.g = (k.g)$ defined by $(k.g)(x) = k.g(x)$ is also continuous, provided g is continuous.
- The function $\left(\frac{k}{g}\right)$ defined by $\frac{k}{g}(x) = \frac{k}{g(x)}$ is also continuous, provided g is continuous and $g(x) \neq 0$.

  All polynomial functions, sine function, and cosine function are continuous.
- Suppose f and g are real-valued functions such that (fog) is defined at c. If g is continuous at c and if f is continuous at $g(c)$, then (fog) is continuous at c.

Solved Examples

Example 1:

For what values of x is the function $f(x) = \frac{|\sin x|}{4 - \sqrt{x^2 - 9}}$ continuous?

Solution:

Let $g(x) = \sin x$, $h(x) = |x|$

Then, numerator of $f(x) = |\sin x| = h(g(x))$

Since g and h are continuous functions, the numeration of $f(x)$ is also continuous for all real x.

[Functional composition of 2 continuous functions is also continuous]

Now, consider the denominator of $f(x)$, which is $4 - \sqrt{x^2 - 9}$.

Let $g(x) = 4$, $h(x) = x^2 - 9$, and $k(x) = \sqrt{x}$

Functions g and h are continuous for all values of x since both are polynomials.

Function k is continuous for all $x \geq 0$

Now, $h(x) = x^2 - 9 = (x + 3)(x - 3)$

$\Rightarrow h(x) = 0$, when $x = 3$ or $x = -3$

$\therefore h(x) \geq 0$ for $x \geq 3$ and $x \leq -3$

$\Rightarrow k(h(x)) = \sqrt{x^2 - 9}$ is continuous for $x \geq 3$ and $x \leq -3$

Thus, the denominator of $f(x) = 4 - \sqrt{x^2 - 9}$ is continuous for $x \geq 3$ and $x \leq -3$.

However, for the function f to be defined, denominator should never be 0.

If $4 - \sqrt{x^2 - 9} = 0$,

then $x^2 - 9 = 16$

$\Rightarrow x = \pm 5$

Thus, denominator is zero, if $x = 5$ or $x = -5$.

$\therefore f(x) = \dfrac{|\sin x|}{4 - \sqrt{x^2 - 9}} \; (x \neq 5,\ x \neq -5)$ is continuous for $x \geq 3$ and $x \leq 3$.

Example 2:

For what values of x is the function $f(x) = \dfrac{x^2}{2} + \dfrac{\sin x}{\cos x}$ continuous?

Solution:

Let $h(x) = \dfrac{x^2}{2}$, $g(x) = \sin x$, $k(x) = \cos x$

Now, h, being a polynomial function, is a continuous function.

g and k are continuous functions.

$\Rightarrow \left(\dfrac{g}{k}\right)(x) = \dfrac{g(x)}{k(x)} = \dfrac{\sin x}{\cos x}$ is continuous, provided $\cos x \neq 0$.

Now, cos $x = 0$ for $x = (2n + 1)\frac{\pi}{2}$, $n \in \mathbf{Z}$

Thus, $\left(\frac{g}{k}\right)$ is not continuous for $x = (2n + 1)\frac{\pi}{2}$, $n \in \mathbf{Z}$

$\Rightarrow h + \left(\frac{g}{k}\right)$ is not continuous for $x = (2n + 1)\frac{\pi}{2}$, $n \in \mathbf{Z}$

Thus, $f(x)$ is continuous everywhere except at the points $x = (2n + 1)\frac{\pi}{2}$, $n \in \mathbf{Z}$

Differentiability of a Function

- The derivative of a real function f at a point c of its domain is defined by

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

- A real function f is said to be differentiable in an interval $[a, b]$ if it is differentiable at every point of $[a, b]$. Similarly, f is differentiable in interval (a, b) if it is differentiable at every point of (a, b).
- If a function f is differentiable at a point c, then it is also continuous at that point.
- Every differentiable function is continuous. However, every continuous function may or may not be differentiable.

Solved Examples

Example 1

Check the continuity and differentiability of the function $f(x) = |x|$ at $x = 0$.

Solution:

The given function is $f(x) = |x|$.

At $x = 0$, it can be observed that

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} |x| = \lim_{x\to 0}(-x) = 0, \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} |x| = \lim_{x\to 0}(x) = 0$$

$$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

Therefore, *f* is continuous at point *x* = 0.

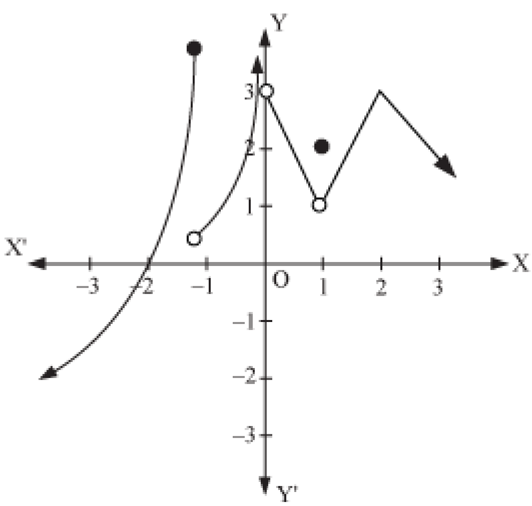
$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|h|-0}{h} = \lim_{h\to 0} \frac{-h}{h} = -1$$

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|h|-0}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

Therefore, $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ does not exist. Hence, *f* is not differentiable at *x* = 0.

Example 2

The graph of a function *f* given as



At what points is the function not necessarily differentiable?

Solution:

Consider the function at *x* = −1. It is seen that $\lim_{x\to -1^-} f(x) = f(-1) \neq \lim_{x\to -1^+} f(x)$.

∴ *f* is not continuous at *x* = −1. Hence, it is not differentiable at *x* = −1.

Consider the function at *x* = 0. It is seen that *f* is not defined at *x* = 0.

∴ *f is* not continuous. Hence, it is not differentiable at *x* = 0.

Consider the function at $x = 1$. It is seen that $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \neq f(1)$
because $\lim_{x \to 1^-} f(x) = 1, \ \lim_{x \to 1^+} f(x) = 1, f(1) = 2.$

∴ f is not continuous. Hence, it is not differentiable at $x = 1$.

Thus, the function is not necessarily differentiable at $x = -1$, $x = 0$ and $x = 1$.

Chain Rule to Find the Derivatives

Chain Rule

- Let f be a real-valued function, which is a composite of 2 functions u and v. i.e., $f = vou$.

  Suppose $t = u(x)$ and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ = exist, then
  $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$

  The above rule is called chain rule.

  Suppose f is a real-valued function, which is a composite of 3 functions u, v, and w i.e., $f = (wou)\, ov$. If $t = v(x)$ and $s = u(t)$, then

  $$\frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Derivatives of Implicit Functions

- When a relationship between x and y can be expressed in such a way that $y = f(x)$, then y is said to an explicit function of x.

  Example:

  $x^2 + 1 - y = 0$

  $\Rightarrow y = x^2 + 1$, which is an explicit function of x.

- When a function between two variables x and y is represented by an equation such that x and y are neither the subject of the equation, the function is said to be an implicit function of x and y.

For example:

$x^5 + 4x^2y^3 - 3y^4 = -6$ is an implicit function of x and y.

- Implicit function can be differentiated using chain rule.

Consider the equation $x^5 + 4x^2y^3 - 3y^4 = -6$

We can find $\frac{dy}{dx}$ by differentiating both sides as

$$\frac{d}{dx}\left(x^5\right)+4\frac{d}{dx}\left(x^2y^3\right)-3\frac{d}{dx}\left(y^4\right)=\frac{d}{dx}(-6)$$

$$\Rightarrow 5x^4+4x^2\frac{d}{dx}\left(y^3\right)+4y^3\frac{d}{dx}\left(x^2\right)-3\left(4y^3\right)\cdot\frac{dy}{dx}=0 \qquad \text{[Using product rule and chain rule]}$$

$$\Rightarrow 5x^4+4x^2\cdot 3y^2\cdot\frac{dy}{dx}+8xy^3-12y^3\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}\left(12x^2y^2-12y^3\right)=-8xy^3-5x^4$$

$$\Rightarrow \frac{dy}{dx}=-\frac{8xy^3+5x^4}{12x^2y^2-12y^3}$$

Derivatives of Inverse Trigonometric Functions

- Inverse trigonometric functions are continuous functions and hence, differentiable.
- Inverse trigonometric functions can be differentiated using chain rule.
- Differentiation of various inverse trigonometric functions is as follows:

$f(x)$	$\sin^{-1} x$	$\cos^{-1} x$	$\tan^{-1} x$	$\cot^{-1} x$	$\sec^{-1} x$	$\text{cosec}^{-1} x$
$f'(x)$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{-1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$	$\frac{-1}{1+x^2}$	$\frac{1}{x\sqrt{x^2-1}}$	$\frac{-1}{x\sqrt{x^2-1}}$
Domain of f'	$(-1, 1)$	$(-1, 1)$	**R**	**R**	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, -1) \cup (1, \infty)$

Solved Examples

Example 1:

Find $\frac{dy}{dx}$ from the equation $y\cos x + x\sin y = 0$

Solution:

Differentiating both sides of the given equation,

$$y\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}\cdot y + x\frac{d}{dx}\cdot\sin y + \sin y\cdot\frac{d}{dx}\cdot x = 0$$
$$\Rightarrow y(-\sin x) + \cos x\cdot\frac{dy}{dx} + x\cos y\frac{dy}{dx} + \sin y = 0$$
$$\Rightarrow -y\sin x + (\cos x + x\cos y)\frac{dy}{dx} + \sin y = 0$$
$$\Rightarrow \frac{dy}{dx} = \frac{y\sin x - \sin y}{\cos x + x\cos y}$$

Example 2:

Differentiate: $y = \left(\sin^{-1}\left(x^3\right)\right)^4$

Solution:

Applying chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\left(\sin^{-1}\left(x^3\right)\right)^4\right]$$
$$= 4\cdot\left(\sin^{-1}\left(x^3\right)\right)^3\cdot\frac{d}{dx}\left(\sin^{-1}\left(x^3\right)\right)$$
$$= 4\left(\sin^{-1}\left(x^3\right)\right)^3\cdot\frac{1}{\sqrt{1-\left(x^3\right)^2}}\cdot\frac{d}{dx}\left(x^3\right)$$
$$= 4\left(\sin^{-1}\left(x^3\right)\right)^3\cdot\frac{3x^2}{\sqrt{1-x^6}}$$
$$= \frac{12x^2}{\sqrt{1-x^6}}\cdot\left(\sin^{-1}\left(x^3\right)\right)^3$$

Example 3:

Find the derivative of $y = \sec\left(\tan\left(x^3\right)\right)$

Solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left[\sec\left(\tan\left(x^3\right)\right)\right] \\ &= \sec\left(\tan\left(x^3\right)\right)\cdot\tan\left(\tan\left(x^3\right)\right)\cdot\frac{d}{dx}\left(\tan\left(x^3\right)\right) \\ &= \sec\left(\tan\left(x^3\right)\right)\cdot\tan\left(\tan\left(x^3\right)\right)\cdot\sec^2\left(x^3\right)\frac{d}{dx}\left(x^3\right) \\ &= 3x^2\sec\left(\tan\left(x^3\right)\right)\cdot\tan\left(\tan\left(x^3\right)\right)\cdot\sec^2\left(x^3\right)\end{aligned}$$

Exponential Function

- The exponential function with positive base *b* > 1 is the function $y = f(x) = b^x$
- Some properties of exponential functions are:
- The domain of an exponential function is R.
- The range of an exponential function is R^+.
- Point (0, 1) always lies on the graph of exponential function.
- Exponential function is always increasing.
- For very large negative values of *x*, exponential function tends to 0.
- The exponential function with base 10 is known as common exponential function.
- The number that lies between 2 and 3 and is equal to the sum of the series $1 + \frac{1}{1!} + \frac{1}{2!} + \ldots$ is denoted by e. The exponential function with base e is known as a natural exponential function.
- The derivative of $e^x$with respect to *x* is $\frac{d}{dx}\left(e^x\right) = e^x$

Solved Examples

Example 1

Find the derivative of $y = \cos 2x\left(e^{x^2-1}\right)$.

Solution:

On applying chain rule, we obtain

$$\frac{dy}{dx} = \cos 2x \frac{d}{dx}\left(e^{x^2-1}\right) + \left(e^{x^2-1}\right).\frac{d}{dx}(\cos 2x)$$
$$\frac{dy}{dx} = \cos 2x \cdot e^{x^2-1} \cdot 2x + \left(e^{x^2-1}\right).(-2\sin 2x)$$
$$\frac{dy}{dx} = 2e^{\left(x^2-1\right)}\left[x\cos 2x - \sin 2x\right]$$

Example 2

Show that $y = e^{-x}\sin x$ satisfies the equation $\frac{dy}{dx} + y\left(1 - \frac{1}{\tan x}\right) = 0$.

Solution:

$$y = e^{-x}\sin x$$
$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(e^{-x}\right)\cdot \sin x + e^{-x}\frac{d}{dx}(\sin x)$$
$$\Rightarrow \frac{dy}{dx} = -e^{-x}\sin x + e^{-x}\cos x$$
$$\Rightarrow \frac{dy}{dx} = e^{-x}\cos x - y \qquad \ldots (1)$$

We have to prove: $\frac{dy}{dx} + y\left(1 - \frac{1}{\tan x}\right) = 0$

Consider L.H.S. as $\frac{dy}{dx} + y\left(1 - \frac{1}{\tan x}\right)$.

$$\frac{dy}{dx} + y - \frac{y}{\tan x}$$

On using (1), we obtain

$$e^{-x}\cos x - \frac{y}{\tan x}$$
$$= e^{-x}\cos x - \frac{e^{-x}\sin x}{\tan x}$$
$$= e^{-x}\cos x - e^{-x}\cos x$$

= 0

= R.H.S.

Thus, the given result is proved.

Logarithmic Functions

Logarithmic Functions

- Let $b > 1$ be a real number. Then, the logarithm of a to base b is x if $b^x = a$. It is denoted by $\log_b a$ i.e., $\log_b a = x$ if $b^x = a$.

- In other words, if $b > 1$, then logarithmic functions are defined from positive real numbers to all real numbers such that

  $\log_b : \boldsymbol{R}^+ \to \boldsymbol{R}$
  $x \to \log_b x = y$ if $b^y = x$

- Based on their bases, logarithmic functions are of two types.

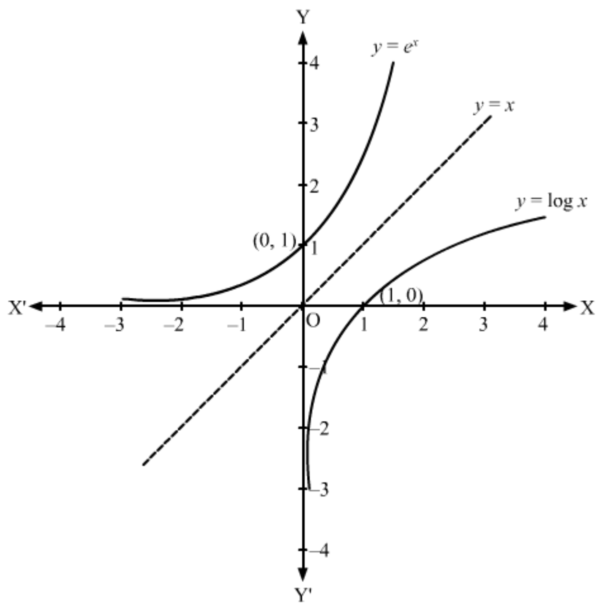
Base of logarithmic function	**Name**
Base 10	Common logarithmic
Base e	Natural logarithmic

- Some observations about logarithmic functions are:

- Domain of log function is $\boldsymbol{R}^+$.

- Range of log function is ***R***.
- Point (1, 0) always lies on the graph of log function.
- Log function is ever increasing.
- For x very near to zero, the value of log x can be made lesser than any given real number i.e., when x approaches 0, log x approaches the y-axis.

- The graphs of $y = e^x$ and $y = \log x$ are mirror images of each other, reflected in the line $y = x$.



- Some properties of log functions are:

- $\log_a p = \dfrac{\log_b p}{\log_b a}$
- $\log_b pq = \log_b p + \log_b q$
- $\log_b p^2 = \log_b p + \log_b p = 2 \log_b p$
- $\log_b p^n = n \log_b p$
- $\log_b \left(\dfrac{x}{y}\right) = \log_b (x) - \log_b (y)$

- $x = e^{\log x}$ for all positive values of x.
- The derivative of log x with respect to x is $\dfrac{1}{x}$ i.e., $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$

Solved Examples

Example 1

Find the derivative of $y = \log(\cos^3 2x)$.

Solution:

On using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(\log\left(\cos^3(2x)\right)\right)$$
$$\frac{dy}{dx} = \frac{1}{\cos^3(2x)} \cdot \frac{d}{dx}\left(\cos^3(2x)\right)$$
$$\frac{dy}{dx} = \frac{1}{\cos^3(2x)} \cdot 3 \cdot \cos^2(2x)\left(-\sin(2x)\right)2$$
$$\frac{dy}{dx} = -6\tan 2x$$

Example 2

Find the derivative of $y = \log(\log(\log x))$.

Solution:

$y = \log(\log(\log x))$

$$\frac{dy}{dx} = \frac{1}{\log(\log x)} \cdot \frac{d}{dx}\left(\log(\log x)\right)$$
$$\frac{dy}{dx} = \frac{1}{\log(\log x)} \cdot \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)$$
$$\frac{dy}{dx} = \frac{1}{\log(\log x)} \cdot \frac{1}{\log x} \cdot \frac{1}{x}$$
$$\therefore \frac{dy}{dx} = \frac{1}{x \cdot \log x \log(\log x)}$$

Example 3

If $y = (\log x)^2$, then prove that $x^2 (y')^2 - 4y = 0$.

Solution:

$$y = (\log x)^2$$
$$\Rightarrow \frac{dy}{dx} = 2\log x.\frac{d}{dx}(\log x) = 2.\log x.\frac{1}{x} = \frac{2}{x}\log x$$

We have to prove that $x^2(y') - 4y = 0$

Consider L.H.S.
x2(2xlog x)2 − 4(log x)2x22xlog x2 - 4log x2
$$\Rightarrow 4(\log x)^2 - 4(\log x)^2 = 0 = \text{R.H.S.}$$

Thus, the given result is proved.

Logarithmic Differentiation

- If $y = f(x) = [u(x)]^{v(x)}$, then by taking logarithm (to base e) on both sides and then differentiating using chain rule, we obtain

$$\log y = v(x) \log u(x)$$
$$\frac{1}{y}\frac{dy}{dx} = v(x)\frac{1}{u(x)}u'(x) + v'(x)\log u(x)$$
$$\frac{dy}{dx} = y\left[\frac{v(x)}{u(x)} \cdot u'(x) + v'(x)\log u(x)\right] \text{ provided } f(x) > 0 \text{ and } u(x) > 0$$

This process is called logarithms differentiation.

- If $y = a^x$, where $a > 0$, then $\frac{dy}{dx} = a^x \log a$

Solved Examples

Example 1:

If $y = 10^{x^2}$, find $\frac{dy}{dx}$.

Solution:

Taking logarithm on both sides, we obtain

$$\log y = x^2 \log 10$$

Differentiating on both sides with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = 2x\log 10$$

$$\frac{dy}{dx} = 2(\log 10)xy = 2(\log 10)x10^{x^2}$$

Example 2:

Find the derivative of y with respect to x, if $y = (\sin x)^x$

Solution:

Taking logarithm on both sides, we obtain

$$\log y = \log(\sin x)^x$$
$$= x\log(\sin x) \qquad \left[\log(a^n) = n\log a\right]$$

Differentiating both sides with respect to x,

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{\sin x}\cdot\cos x + \log(\sin x)$$
$$\Rightarrow \frac{dy}{dx} = (\sin x)^x\left[\log(\sin x) + x\cot x\right]$$

Example 3:

Find the derivative y' of the function y given by $y = \sqrt{\frac{(x-2)(x+4)}{(x+1)(x+5)}}$

Solution:

Taking logarithms on both sides, we obtain

$$\log y = \frac{1}{2}\log\left[\frac{(x-2)(x+4)}{(x+1)(x+5)}\right]$$

Now, we know that $\log\frac{a}{b} = \log a - \log b$ and $\log ab = \log a + \log b$

$$\therefore \log y = \frac{1}{2}\left[\log(x-2) + \log(x+4) - \log(x+1) - \log(x+5)\right]$$

Differentiating both sides with respect to *x*,

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{x-2}+\frac{1}{x+4}-\frac{1}{x+1}-\frac{1}{x+5}\right]$$

$$\therefore\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{x-2}+\frac{1}{x+4}-\frac{1}{x+1}-\frac{1}{x+5}\right]\sqrt{\frac{(x-2)(x+4)}{(x+1)(x+5)}}$$

Derivatives of Functions in Parametric Form

Differentiation of Functions in Parametric Form

- Parametric equations are of the form $x = f(t)$ and $y = g(t)$, where t is called a parameter. These equations are used for establishing a relationship between two variables with the help of a third variable.
- The functions in parametric form can be differentiated using chain rule.
- If $x = f(t)$ and $y = g(t)$, then

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \qquad \left[\frac{dx}{dt}\neq 0\right]$$

$$\Rightarrow\frac{dy}{dx}=\frac{g'(t)}{f'(t)} \qquad \left[f'(t)\neq 0\right]$$

Solved Examples

Example 1

Find $\frac{dy}{dx}$ in terms of the parameter $x=\frac{1}{s}, y=\frac{s+1}{s-1}$.

Solution:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{ds}\right)}{\left(\frac{dx}{ds}\right)}$$

$$\frac{dy}{ds} = \frac{d}{ds}\left(\frac{s+1}{s-1}\right) = \frac{(s-1)(1)-(s+1)(1)}{(s-1)^2} = \frac{s-1-s-1}{(s-1)^2} = -\frac{2}{(s-1)^2}$$

$$\frac{dx}{ds} = -\frac{1}{s^2}$$

$$\therefore \frac{dy}{dx} = \frac{-\frac{2}{(s-1)^2}}{-\frac{1}{s^2}} = \frac{2s^2}{(s-1)^2}$$

$$\therefore \frac{dy}{dx} = 2\left(\frac{s}{s-1}\right)^2$$

Example 2

What is the value of $\frac{dy}{dx}$ in terms of *x* and *y* if $y = e^u - e^{-u}$ and $x = e^u + e^{-u}$?

Solution:

$$\frac{dy}{du} = e^u - \left(-e^{-u}\right) = e^u + e^{-u}$$

$$\frac{dx}{du} = e^u + \left(-e^{-u}\right) = e^u - e^{-u}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{du}\right)}{\left(\frac{dx}{du}\right)} = \frac{e^u + e^{-u}}{e^u - e^{-u}} = \frac{x}{y}$$

$$\therefore \frac{dy}{dx} = \frac{x}{y}$$

Example 3

What is the value of $\frac{dy}{dx}$ at $\alpha = 60°$ if $x = \cos\alpha$ and $y = \sin 2\alpha$?

Solution:

$$\frac{dy}{d\alpha} = 2\cos 2\alpha$$
$$\frac{dx}{d\alpha} = -\sin\alpha$$
$$\Rightarrow \frac{dy}{dx} = \frac{2\cos 2\alpha}{-\sin\alpha}$$
$$\text{At } \alpha = 60°, \frac{dy}{dx} = \frac{2\cos 120°}{-\sin 60°} = -\frac{2\cos 60°}{-\frac{\sqrt{3}}{2}} = \frac{-2\times\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

Second Order Derivatives

Second Order Derivatives

- If $y = f(x)$, then $\frac{dy}{dx} = f'(x)$

  If $f'(x)$ is a differentiable function of x, then the derivative of $f'(x)$ exists and is called the second order derivative of $y = f(x)$ with respect to x.

- Notation of the second order derivatives depends on the notations of the original functions and first order derivative.

Original	First order derivative	Second order derivative
y	y'	y''
	Dy	D^2y
	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$

	y_1	y_2
$f(x)$	$f'(x)$	$f''(x)$
	$\frac{df}{dx}$	$\frac{d^2 f}{dx^2}$

Solved Examples

Example 1:

If $y = \frac{a}{x} + b$, then prove that $\frac{d^2 y}{dx^2} + \frac{2}{x}\left(\frac{dy}{dx}\right) = 0$

Solution:

$$y = \frac{a}{x} + b$$
$$\Rightarrow \frac{dy}{dx} = -\frac{a}{x^2}$$
$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{2a}{x^3}$$

To prove: $\frac{d^2 y}{dx^2} + \frac{2}{x}\left(\frac{dy}{dx}\right) = 0$

L.H.S. = $\frac{2a}{x^3} + \frac{2}{x}\left(\frac{-a}{x^2}\right) = \frac{2a}{x^3} - \frac{2a}{x^3} = 0 = \text{R.H.S.}$

Hence proved.

Example 2:

Find the second order derivative of $y = \cos^{-1}(x^2)$

Solution:

$y = \cos^{-1}(x^2)$

$$\frac{dy}{dx} = \frac{-1(2x)}{\sqrt{1-x^4}} = \frac{-2x}{\sqrt{1-x^4}}$$

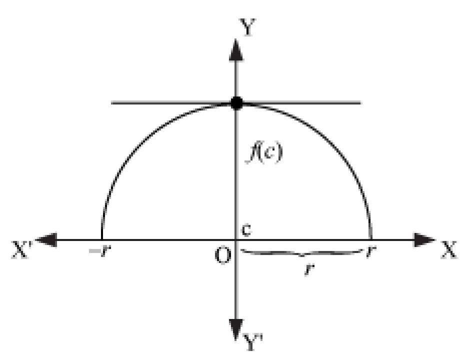
$$\frac{d^2y}{dx^2} = \frac{\sqrt{1-x^4}(-2) - (-2x)\left(-\frac{1}{2}\right)\frac{(-4x^3)}{(1-x^4)^{\frac{3}{2}}}}{(1-x^4)} = \frac{-2\sqrt{1-x^4} + \frac{4x^4}{(1-x^4)^{\frac{3}{2}}}}{(1-x^4)}$$

$$= \frac{-2(1-x^4)^2 + 4x^4}{(1-x^4)^{\frac{5}{2}}} = \frac{(-2-2x^8+4x^4)+4x^4}{(1-x^4)^{\frac{5}{2}}} = \frac{-2(1+x^8-4x^4)}{(1-x^4)^{\frac{5}{2}}}$$

Rolle's Theorem

- Let f: $[a, b] \rightarrow \mathbf{R}$ be continuous in $[a, b]$ and differentiable in (a, b) such that $f(a) = f(b)$, where a and b are some real numbers.

  Then, there exists some c in (a, b) such that $f'(c) = 0$.
- The tangent to graph of f where the slope becomes zero at any point on the graph of graph $y = f(x)$ is the claim of Rolle's theorem.
- Example:

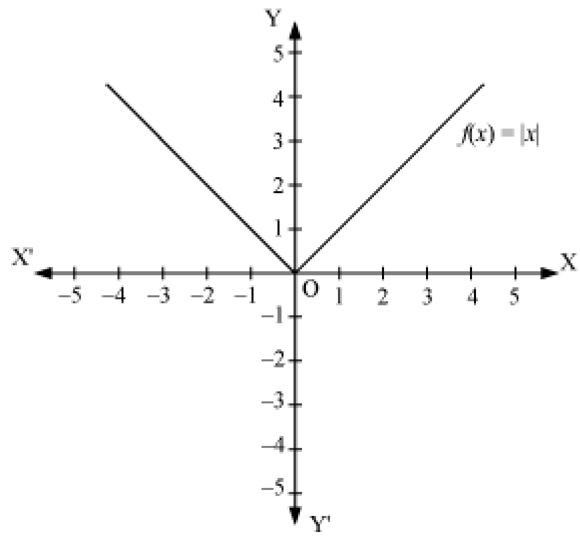
$f(x) = \sqrt{r^2 - x^2}, r > 0,\ x \in [-r,\ r]$ $f(x)$ is continuous in $[-r, r]$ and differentiable in $(-r, r)$. Also, $f(-r) = f(r)$ $\therefore$ Rolle's theorem is applicable for $c = 0$. Here, $f'(c) = f'(0) = 0$	

$f(x) = |x|,\ x \in [-1, 1]$

f is continuous in $[-1, 1]$ and $f(-1) = f(1)$.

However, *f* is not differentiable in $(-1, 1)$ as it is not differentiable at $x = 0$.

∴ Rolle's theorem is not applicable.



Solved Examples

Example 1

Verify Rolle's theorem for the function $f(x) = x^2 - 5x + 4$ in the interval $[1, 4]$.

Solution:

The given function is $f(x) = x^2 - 5x + 4$.

(i) $f(x)$ is continuous in $[1, 4]$ as *f* is a polynomial function.

(ii) $f(x)$ is differentiable in $(1, 4)$ as *f* is a polynomial function.

(iii) $f(1) = 1^2 - 5(1) + 4 = 0$, $f(4) = 4^2 - 5(4) + 4 = 0$

$\therefore f(1) = f(4)$

∴The hypothesis of Rolle's theorem is satisfied.

Thus, there exists $c \in (1, 4)$ such that $f'(c) = 0$.

$$f'(x) = 2x - 5$$
$$f'(c) = 0$$
$$\Rightarrow 2c - 5 = 0$$
$$\Rightarrow c = \frac{5}{2} \in (1, 4)$$

Thus, Rolle's theorem is verified.

Example 2

Verify Rolle's theorem for the function $f(x) = 1 + \sin^2 x$ in the interval $[0, \pi]$.

Solution:

The given function is $f(x) = 1 + \sin^2 x$, as f is a trignometric function.

(i) f is continuous in $[0, \pi]$ as f is a trignometric function.

(ii) f is differentiable in $(0, \pi)$ as f is a trignometric function.

(iii) $f(0) = 1 + \sin^2 0 = 1$, $f(\pi) = 1 + \sin^2 \pi = 1$

$\therefore f(0) = f(\pi)$

∴ The hypothesis of Rolle's theorem is satisfied.

Thus, there exists $c \in (0, \pi)$ such that $f'(c) = 0$.

$$f'(c) = 0$$
$$\Rightarrow 2 \sin c \cos c = 0 \qquad [f'(x) = 2 \sin x \cos x]$$
$$\Rightarrow \sin 2c = 0$$
$$\Rightarrow 2c = n\pi, \ n \in \mathbf{Z}$$
$$\Rightarrow c = \frac{n\pi}{2}, \ n \in \mathbf{Z}$$
$$\text{For } n = 1, \ c = \frac{\pi}{2} \in (0, \pi)$$
$$\therefore \ f'(c) = 0 \text{ for } c = \frac{\pi}{2} \in (0, \pi)$$

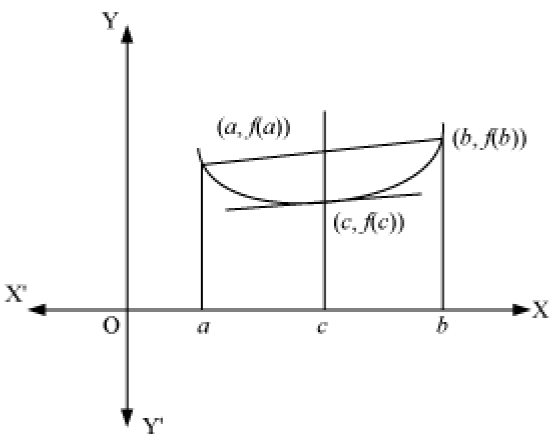
Thus, Rolle's theorem is verified.

Mean Value Theorem

- Let f: $[a, b] \to \mathbf{R}$ be a continuous function in $[a, b]$ and differentiable in (a, b). Then, there exists some c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- Some observations about mean value theorem:
- Mean Value Theorem (MVT) is an extension of Rolle's theorem.
- $f'(c)$ is the slope of the tangent to the curve $y = f(x)$ at point $(c, f(c))$.
- $\frac{f(b) - f(a)}{b - a}$ is the slope of the secant drawn between $(a, f(a)$ and $(b, f(b)$. This can be diagrammatically represented as



- Mean Value Theorem can be described by an example as

Function	**Graph**

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Consider $f(x) = x^3$, $x \in [3, 6]$.

f is continuous in [3, 6] and differentiable in (3, 6).

Thus, there exists $c \in (3, 6)$ such

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \ b = 6, \ a = 3$$

that $f'(x) = 3x^2$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow 3c^2 = \frac{6^3 - 3^3}{3}$$

$$\Rightarrow c^2 = \frac{189}{9} = 21$$

$$\Rightarrow c = \sqrt{21} \in (3, 6)$$

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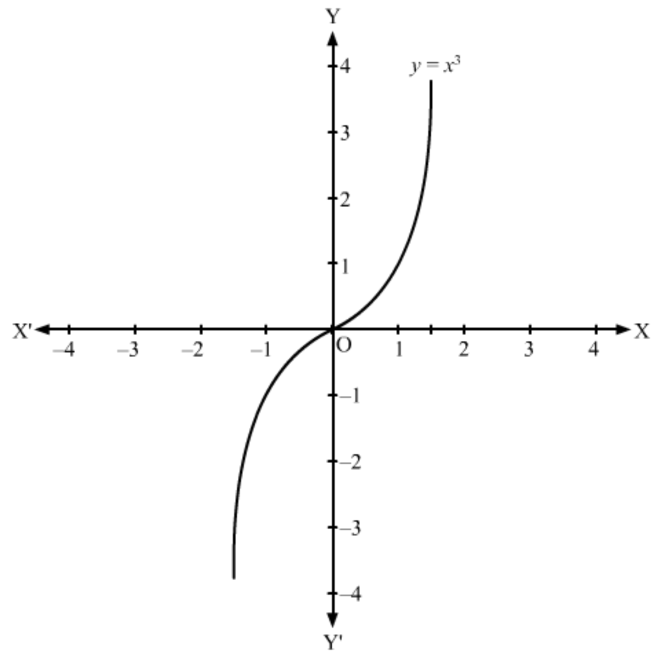

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Consider $f(x) = \frac{1}{x}$, $x \in [-1, 1]$. The function f does not exist at $x = 0$.

Thus, it is not continuous in [−1, 1]. Hence, the hypothesis of mean value theorem is not satisfied.

Thus, mean value theorem cannot be applied to the given function.

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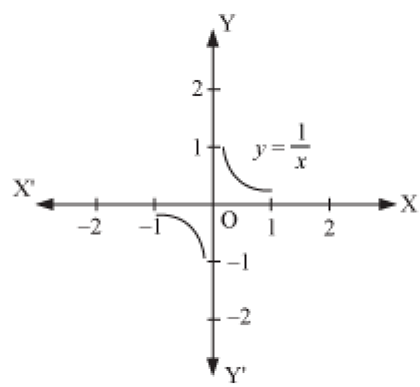

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Solved Examples

Example 1

Discuss the applicability of mean value theorem for the function $f(x) = |\sin x|$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Solution:

$f(x) = |\sin x|, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Let $h(x) = \sin x$, $g(x) = |x|$

$(goh)(x) = g(h(x)) = g(\sin x) = |\sin x| = f(x)$

Since h and g are continuous in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, f is continuous in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Now, $h(x) = \sin x$ is differentiable in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

However, $g(x) = |x|$ is not differentiable in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Thus, $f(x)$ is not differentiable in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

∴ The hypothesis of mean value theorem is not satisfied.

Thus, mean value theorem is not applicable for the given function.

Example 2

Use the mean value theorem to prove that for any two real numbers a and b $|\cos a - \cos b| \le |a - b|$.

Solution:

Let $f(x) = \cos x$ in $[a, b]$

It is clear that $f(x) = \cos x$ is continuous in $[a, b]$ and differentiable in (a, b).

Therefore, the hypothesis of mean value theorem is satisfied.

Thus, by mean value theorem, there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow -\sin c = \frac{\cos b - \cos a}{b - a} \qquad [f'(x) = -\sin x]$$

$$\Rightarrow |-\sin c| = \left|\frac{\cos b - \cos a}{b - a}\right|$$

We know that $|-\sin c| \leq 1$.

$$\therefore \left|\frac{\cos b - \cos a}{b - a}\right| \leq 1$$

$$|\cos b - \cos a| \leq |b - a|$$

$$\text{or, } |\cos a - \cos b| \leq |a - b|$$

Hence proved.