

## DAY THIRTY NINE

# Mock Test 2

(Based on Complete Syllabus)

### Instructions ●

1. This question paper contains of 30 Questions of Mathematic, divided into two Sections :  
**Section A** Objective Type Questions and **Section B** Numerical Type Questions.
2. Section A contains 20 Objective questions and all Questions are compulsory (**Marking Scheme** : Correct +4, Incorrect -1).
3. Section B contains 10 Numerical value questions out of which only 5 questions are to be attempted (**Marking Scheme** : Correct +4, Incorrect 0).

### Section A : Objective Type Questions

- 1 The average weight of students in a class of 35 students is 40 kg. If the weight of the teacher be included, then average rises by  $\frac{1}{2}$  kg; the weight of the teacher is

(a) 40.5 kg (b) 50 kg (c) 41 kg (d) 58 kg

- 2 Let  $f(x) = \frac{1}{[\sin x]}$ ,  $[\cdot]$  being the greatest integer function, then

(a)  $f(x)$  is not continuous, where  $x \in (2n\pi, 2n\pi + \pi), n \in I$

(b)  $f(x)$  is differentiable at  $x = \frac{\pi}{4}$

(c)  $f(x)$  is differentiable at  $x = \frac{\pi}{2}$

(d) None of the above

- 3 If  $\int f(x) \sin x \cos x \, dx = \frac{1}{2(b^2 - a^2)} \log [f(x)] + C$ , then

$f(x)$  is equal to

(a)  $\frac{1}{a^2 \sin^2 x + b^2 \cos^2 x}$  (b)  $\frac{1}{a^2 \sin^2 x - b^2 \cos^2 x}$

(c)  $\frac{1}{a^2 \cos^2 x - b^2 \sin^2 x}$  (d)  $\frac{1}{a^2 \cos^2 x + b^2 \sin^2 x}$

- 4 The solution of the differential equation  $\frac{dy}{dx} = \frac{y}{x} + \frac{\phi\left(\frac{y}{x}\right)}{\phi'\left(\frac{y}{x}\right)}$  is

(a)  $x\phi\left(\frac{y}{x}\right) = k$

(b)  $\phi\left(\frac{y}{x}\right) = kx$

(c)  $y\phi\left(\frac{y}{x}\right) = k$

(d)  $\phi\left(\frac{y}{x}\right) = ky$

- 5 If  $a \leq \tan^{-1} x + \cot^{-1} x + \sin^{-1} x \leq b$ , then  $a + b$  is equal to

(a) 0

(b)  $\frac{\pi}{2}$

(c)  $\frac{\pi}{4}$

(d)  $\pi$

- 6 The two consecutive terms in the expansion of  $(3 + 2x)^{74}$  whose coefficients are equal, are

(a) 11, 12

(b) 7, 8

(c) 30, 31

(d) None of these

- 7 A parabola is drawn with its focus at (3, 4) and vertex at the focus of the parabola  $y^2 - 12x - 4y + 4 = 0$ . The equation of the parabola is

(a)  $y^2 - 8x - 6y + 25 = 0$

(b)  $y^2 - 6x + 8y - 25 = 0$

(c)  $x^2 - 6x - 8y + 25 = 0$

(d)  $x^2 + 6x - 8y - 25 = 0$

- 8** If  $p, p'$  denote the lengths of the perpendiculars from the focus and the centre of an ellipse with semi-major axis of length  $a$  respectively on a tangent to the ellipse and  $r$  denotes the focal distance of the point, then
- (a)  $ap' = rp + 1$  (b)  $rp = ap'$   
(c)  $ap = rp' + \frac{1}{\sqrt{3}}$  (d)  $ap = rp'$
- 9** The equation of the locus of the pole with respect to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , of any tangent line to the auxiliary circle, is the curve  $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \lambda^2$ , where
- (a)  $\lambda^2 = a^2$  (b)  $\lambda^2 = \frac{1}{a^2}$   
(c)  $\lambda^2 = b^2$  (d)  $\lambda^2 = \frac{1}{b^2}$
- 10** The solution set for  $\frac{(2x+3)(4-3x)^3(x-4)}{(x-2)^2x^5} \leq 0$
- (a)  $\left(-\infty, -\frac{3}{2}\right) \cup \left(0, \frac{4}{3}\right) \cup (4, \infty)$   
(b)  $\left(-\frac{3}{2}, 0\right) \cup \left(\frac{4}{3}, 4\right)$   
(c)  $(-\infty, 0) \cup (2, \infty)$   
(d) None of the above
- 11**  $f(x) = x - \cot^{-1} x - \log(x + \sqrt{1+x^2})$  is increasing in
- (a)  $(-\infty, \infty)$  (b)  $(-\infty, 2)$   
(c)  $(2, 5)$  (d)  $(-\infty, 10)$
- 12** The contrapositive of the statement, "if  $x$  is a prime number and  $x$  divides  $ab$ , then  $x$  divides  $a$  or  $x$  divides  $b$ ", can be symbolically represented using logical connectives on appropriately defined statements  $p, q, r$  and  $s$  as
- (a)  $(\sim r \vee \sim s) \rightarrow (\sim p \wedge \sim q)$   
(b)  $(r \wedge s) \rightarrow (\sim p \wedge \sim q)$   
(c)  $(\sim r \wedge \sim s) \rightarrow (\sim p \vee \sim q)$   
(d)  $(r \vee s) \rightarrow (\sim p \vee \sim q)$
- 13** The area bounded by the lines  $y = 2, x = 1, x = a$  and the curve  $y = f(x)$ , which cuts the last two lines above the first line for all  $a > 1$ , is equal to  $\frac{2}{3}[(2a)^{3/2} - 3a + 3 - 2\sqrt{2}]$ . Then,  $f(x)$  equals
- (a)  $2\sqrt{2x}, x > 1$  (b)  $\sqrt{2x}, x > 1$   
(c)  $2\sqrt{x}, x > 1$  (d) None of these
- 14** Area of a triangle with vertices  $(a, b), (x_1, y_1)$  and  $(x_2, y_2)$ , where  $a, x_1$  and  $x_2$  are in GP with common ratio  $r$  and  $b, y_1$  and  $y_2$  are in GP with common ratio  $s$ , is given by
- (a)  $\frac{1}{2}ab(r-1)(s-1)(s-r)$   
(b)  $\frac{1}{2}ab(r+1)(s+1)(s-r)$   
(c)  $ab(r-1)(s-1)(s-r)$   
(d) None of the above
- 15** The equation of perpendicular bisectors of sides  $AB$  and  $AC$  of a  $\Delta ABC$  are  $x - y + 5 = 0$  and  $x + 2y = 0$ , respectively. If the coordinates of vertex  $A$  are  $(1, -2)$ , then equation of  $BC$  is
- (a)  $14x + 23y - 40 = 0$  (b)  $14x - 23y + 40 = 0$   
(c)  $23x + 14y - 40 = 0$  (d)  $23x - 23y + 40 = 0$
- 16** If  $|z - 25i| \leq 15$ , then  $|\max \operatorname{amp}(z) - \min \operatorname{amp}(z)|$  is equal to
- (a)  $\cos^{-1}\left(\frac{3}{5}\right)$  (b)  $\pi - 2\cos^{-1}\left(\frac{3}{5}\right)$   
(c)  $\frac{\pi}{2} + \cos^{-1}\left(\frac{3}{5}\right)$  (d)  $\sin^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{3}{5}\right)$
- 17** If  $ABCD$  is a cyclic quadrilateral such that  $12 \tan A - 5 = 0$  and  $5 \cos B + 3 = 0$ , then the quadratic equation whose roots are  $\cos C$  and  $\tan D$  is
- (a)  $39x^2 + 88x + 48 = 0$   
(b)  $39x^2 - 16x - 48 = 0$   
(c)  $39x^2 - 88x + 48 = 0$   
(d) None of the above
- 18** A mirror and a source of light are situated at the origin  $O$  and a point on  $OX$ , respectively. A ray of light from the source strikes the mirror and is reflected. If the DR's of the normal to the plane of mirror are  $1, -1, 1$ , then DC's for the reflected ray are
- (a)  $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$  (b)  $-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$   
(c)  $-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}$  (d)  $-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$
- 19** If both roots of the equation  $4x^2 - 2x + a = 0, a \in R$ , lies in the interval  $(-1, 1)$ , then
- (a)  $-2 < a \leq \frac{1}{4}$  (b)  $-6 \leq a \leq \frac{1}{4}$   
(c)  $a > \frac{1}{4}$  (d)  $a < -2$
- 20** Let  $\lambda$  and  $\theta$  be real numbers. Then, the set of all values of  $\lambda$  for which the system of linear equations
- $$\begin{aligned} \lambda x + (\sin \theta)y + (\cos \theta)z &= 0 \\ x + (\cos \theta)y + (\sin \theta)z &= 0 \\ -x + (\sin \theta)y - (\cos \theta)z &= 0 \end{aligned}$$
- has a non-trivial solution, is
- (a)  $[0, \sqrt{2}]$  (b)  $[-\sqrt{2}, 0]$   
(c)  $[-\sqrt{2}, \sqrt{2}]$  (d) None of these
- ### Section B : Numerical Type Questions
- 21** If the value of integral  $\left| \int_0^\pi [2 \sin x] dx \right|$ , where  $[x]$  denotes greatest integer function, is  $\frac{2\pi}{T}$ , then value of  $T$  is equal to
- 22** Suppose  $A_1, A_2, \dots, A_{30}$  are thirty sets, each having 5 elements and  $B_1, B_2, \dots, B_n$  are  $n$  sets each with 3 elements, let  $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^n B_j = S$  and each element of

$S$  belongs to exactly 10 of the  $A_i$ 's and exactly 9 of the  $B_j$ 's. Then,  $n$  is equal to

- 23** An experiment succeeds twice as often as it fails. Then, the probability that in the next 4 trials there will be atleast 2 successes, is  $p/q$ , then value of  $pq$  is equal to

- 24** A variable plane at a constant distance  $P$ , from origin cuts axes at  $A, B$  and  $C$ , the locus of centroid of tetrahedron  $OABC$  is  $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{k^2}{p^2}$ , then value of  $k^2$  is equal to

- 25** A parallelogram is constructed on the vectors  $\mathbf{a} = 3\alpha - \beta$ ,  $\mathbf{b} = \alpha + 3\beta$ , if  $|\alpha| = |\beta| = 2$  and angle between  $\alpha$  and  $\beta$  is  $\frac{\pi}{3}$ , length of a diagonal of the parallelogram is  $p\sqrt{3}$ , then value of  $p$  is equal to .....

- 26** Let  $A = \begin{bmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{bmatrix}$ , where  $0 \leq \theta \leq 2\pi$ .

Then, largest value of  $|A|$  is

- 27** The value of the expression  $\frac{\sin^3 x}{1 + \cos x} + \frac{\cos^3 x}{1 - \sin x}$  is  $\frac{1}{\sqrt{a}} \cos\left(\frac{\pi}{4} - x\right)$ , then value of  $a$  is equal to .....

- 28** Number of ways in which 5 boys and 4 girls can be arranged on a circular table such that no two girls sit together and two particular boys are always together is .....

- 29** If  $S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}$  and  $2 - S_n < \frac{1}{100}$ , then the least value of  $n$  must be .....

- 30** If  $X$  and  $Y$  are independent binomial variates  $B\left(5, \frac{1}{2}\right)$  and  $B\left(7, \frac{1}{2}\right)$ . If the value of  $P(X + Y = 3)$  is  $\lambda$ , then the value of  $4096\lambda$  must be .....

# Hints and Explanations

- 1 (d) Let the weight of teacher be  $x$  kg, then

$$40 + \frac{1}{2} = \frac{35 \times 40 + x}{35 + 1}$$

$$\Rightarrow \frac{81}{2} \times 36 = 35 \times 40 + x$$

$$\Rightarrow 81 \times 18 = 1400 + x$$

$$\Rightarrow 1458 = 1400 + x \Rightarrow x = 58$$

Hence, the weight of teacher is 58 kg.

- 2 (a) We have,  $f(x) = \frac{1}{[\sin x]}$

Clearly,  $\sin x \notin [0, 1]$

$[\because \text{if } 0 \leq \sin x < 1, [\sin x] = 0]$

$$\Rightarrow x \notin [2n\pi, (2n+1)\pi] - (4n+1)\frac{\pi}{2}, n \in \mathbb{I}$$

Thus,  $f(x)$  is not continuous if

$$x \in (2n\pi, 2n\pi + \pi), n \in \mathbb{I}.$$

- 3 (a) Given that,  $\int f(x) \sin x \cos x \, dx$

$$= \frac{1}{2(b^2 - a^2)} \log [f(x)] + C$$

On differentiating both sides, we get

$$f(x) \sin x \cos x = \frac{d}{dx} \left\{ \frac{\log[f(x)]}{2(b^2 - a^2)} + C \right\}$$

$$\Rightarrow f(x) \sin x \cos x = \frac{1}{2(b^2 - a^2)}$$

$$\frac{1}{f(x)} f'(x)$$

$$\Rightarrow 2(b^2 - a^2) \sin x \cos x = \frac{f'(x)}{[f(x)]^2}$$

On integrating both sides, we get

$$-b^2 \cos^2 x - a^2 \sin^2 x = -\frac{1}{f(x)}$$

$$\therefore f(x) = \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x}$$

- 4 (b) Given,  $\frac{dy}{dx} - \frac{y}{x} = \phi\left(\frac{y}{x}\right)$

$$\Rightarrow \frac{\phi\left(\frac{y}{x}\right) \left(\frac{xdy - ydx}{x^2}\right)}{\phi\left(\frac{y}{x}\right)} = \frac{1}{x} dx$$

$$\Rightarrow \int \frac{\phi\left(\frac{y}{x}\right) d\left(\frac{y}{x}\right)}{\phi\left(\frac{y}{x}\right)} = \int \frac{1}{x} dx + \log k$$

$$\Rightarrow \log \phi\left(\frac{y}{x}\right) = \log x + \log k$$

$$\Rightarrow \phi\left(\frac{y}{x}\right) = kx$$

$\therefore$  Sum of two

- 5 (d) We know,  $-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$

$$\therefore 0 \leq \frac{\pi}{2} + \sin^{-1} x \leq \pi$$

$$\Rightarrow 0 \leq \tan^{-1} x + \cot^{-1} x + \sin^{-1} x \leq \pi$$

$$\Rightarrow a = 0 \text{ and } b = \pi$$

$$\text{Hence, } a + b = \pi$$

- 6 (c) General term of  $(3 + 2x)^{74}$  is

$$T_{r+1} = {}^{74}C_r (3)^{74-r} (2)^r x^r$$

Let two consecutive terms be  $T_{r+1}$  th and  $T_{r+2}$  th terms.

According to the question

Coefficient of  $T_{r+1}$  = Coefficient of  $T_{r+2}$

$$\Rightarrow {}^{74}C_r 3^{74-r} 2^r = {}^{74}C_{r+1} 3^{74-(r+1)} 2^{r+1}$$

$$\Rightarrow \frac{{}^{74}C_{r+1}}{{}^{74}C_r} = \frac{3}{2}$$

$$\Rightarrow \frac{74-r}{r+1} = \frac{3}{2}$$

$$\Rightarrow 148 - 2r = 3r + 3$$

$$\therefore r = 29$$

Hence, two consecutive terms are 30 and 31.

- 7 (c) Given, equation can be rewritten as  $(y-2)^2 = 12x$

Here, vertex and focus are (0, 2) and (3, 2).

$\therefore$  Vertex of the required parabola is (3, 2) and focus is (3, 4).

The axis of symmetry is  $x = 3$  and

$$\text{latusrectum} = 4 \cdot 2 = 8$$

Hence, required equation is

$$(x-3)^2 = 8(y-2)$$

$$\Rightarrow x^2 - 6x - 8y + 25 = 0$$

- 8 (d) Tangent to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

at  $P(a \cos \theta, b \sin \theta)$  is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1 \quad \dots(i)$$

Now,  $p$  = perpendicular distance from focus  $S(ae, 0)$  to the line (i)

$$\begin{aligned} &= \frac{\left| \frac{ae}{a} \cos \theta + 0 - 1 \right|}{\sqrt{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}} \\ &= \frac{1 - e \cos \theta}{\sqrt{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}} \quad \dots(ii) \end{aligned}$$

Also,  $p'$  = perpendicular distance from centre (0, 0) to the line (i) .

$$= \frac{1}{\sqrt{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}} \quad \dots(iii)$$

Again,  $r = SP = a(1 - e \cos \theta)$

$$\therefore ap = \frac{a - ae \cos \theta}{\sqrt{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}} = rp'$$

[using Eqs. (ii) and (iii)]

- 9 (b) The equation of the auxiliary circle is  $x^2 + y^2 = a^2$ .

Let  $(h, k)$  be the pole, then equation of the polar of  $(h, k)$  with respect to the given ellipse is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1$$

Since, this is tangent to the circle.

$$\Rightarrow \frac{|0 + 0 - 1|}{\sqrt{\left(\frac{h}{a^2}\right)^2 + \left(\frac{k}{b^2}\right)^2}} = \pm a$$

$$\Rightarrow \frac{h^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2}$$

Hence, locus of  $(h, k)$  is

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2}$$

- 10 (d) Given inequality can be rewritten as

$$\frac{(2x+3)(3x-4)^3(x-4)}{(x-2)^2 x^5} \geq 0$$

$$\Rightarrow (2x+3)(3x-4)^3(x-4)(x-2)^2 x^5 \geq 0, x \neq 0, 2$$

$$\Rightarrow (2x+3)(3x-4)^3(x-4)x^5 \geq 0;$$

$$x \neq 0, 2 \quad [\because (x-2)^2 > 0]$$

$$x = -\frac{3}{2}, \frac{4}{3}, 4, 0$$

$$\begin{array}{c} + \quad - \quad + \quad - \quad + \\ -3/2 \quad 0 \quad 4/3 \quad 4 \end{array}$$

$$\Rightarrow x \in (-\infty, -\frac{3}{2}] \cup (0, \frac{4}{3}] \cup [4, \infty)$$

- 11 (a) Given that,

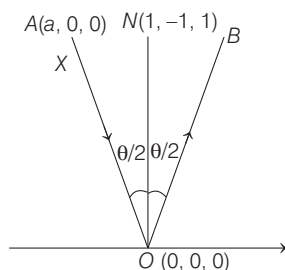
$$f(x) = x - \cot^{-1} x - \log(x + \sqrt{1+x^2})$$

On differentiating w.r.t.  $x$ , we get

$$\begin{aligned} f'(x) &= 1 + \frac{1}{1+x^2} - \frac{1}{(x + \sqrt{1+x^2})} \\ &\quad \left( 1 + \frac{x}{\sqrt{1+x^2}} \right) \end{aligned}$$

$$= 1 + \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}$$

Let **AO** be the incident ray and **OB** be the reflected ray, **ON** is the normal to the mirror at **O**.



Then,  $\angle AON = \angle NOB = \frac{\theta}{2}$  (say)

DR's of  $\vec{OA}$  are  $(a, 0, 0)$  and so its DC's are  $(1, 0, 0)$ .

DC's of  $\vec{ON}$  are  $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

$$\therefore \cos \frac{\theta}{2} = \frac{1}{\sqrt{3}}$$

Let  $\vec{l}, \vec{m}$  and  $\vec{n}$  be the DC's of the reflected ray  $\vec{OB}$ .

$$\text{Then, } \frac{\vec{l} + \vec{m}}{2 \cos \frac{\theta}{2}} = \frac{1}{\sqrt{3}}, \quad \frac{\vec{m} + \vec{n}}{2 \cos \frac{\theta}{2}} = -\frac{1}{\sqrt{3}}$$

$$\text{and } \frac{\vec{n} + \vec{o}}{2 \cos \frac{\theta}{2}} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \vec{l} = \frac{2}{3} - 1,$$

$$\vec{m} = -\frac{2}{3}, \vec{n} = \frac{2}{3}$$

$$\Rightarrow \vec{l} = -\frac{1}{3}, \vec{m} = -\frac{2}{3}, \vec{n} = \frac{2}{3}$$

Hence, DC's of the reflected ray are

$$-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$$

**19 (a)** Let  $f(x) = 4x^2 - 2x + a$

Since, both roots of  $f(x) = 0$  lie in the interval  $(-1, 1)$ , we can take

$$D \geq 0, f(-1) > 0 \text{ and } f(1) > 0$$

1. Consider  $D \geq 0$ ,

$$\Rightarrow (-2)^2 - 4 \cdot 4 \cdot a \geq 0$$

$$\Rightarrow a \leq \frac{1}{4} \quad \dots(i)$$

2. Consider  $f(-1) > 0$ ,

$$4(-1)^2 - 2(-1) + a > 0$$

$$\Rightarrow a > -6 \quad \dots(ii)$$

3. Consider  $f(1) > 0$ ,

$$4(1)^2 - 2(1) + a > 0$$

$$\Rightarrow a > -2 \quad \dots(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$-2 < a \leq \frac{1}{4}$$

**20 (c)** Since the system of given equation has a non-trivial solution, therefore

$$\begin{vmatrix} \lambda & \sin \theta & \cos \theta \\ 1 & \cos \theta & \sin \theta \\ -1 & \sin \theta & -\cos \theta \end{vmatrix} = 0$$

$$\Rightarrow \lambda[-\cos^2 \theta - \sin^2 \theta] - \sin \theta[-\cos \theta + \sin \theta] + \cos \theta[\sin \theta + \cos \theta] = 0$$

$$\Rightarrow -\lambda + \sin \theta \cos \theta - \sin^2 \theta + \sin \theta \cos \theta + \cos^2 \theta = 0$$

$$\Rightarrow -\lambda + 2 \sin \theta \cos \theta + \cos^2 \theta - \sin^2 \theta = 0$$

$$\Rightarrow -\lambda + \sin 2\theta + \cos 2\theta = 0$$

$$\Rightarrow \lambda = \sqrt{2} \cos \left( 2\theta - \frac{\pi}{4} \right) \quad \dots(i)$$

$$\therefore -1 \leq \cos \left( 2\theta - \frac{\pi}{4} \right) \leq 1 \quad \forall \theta \in \mathbb{R}$$

$$\therefore -\sqrt{2} \leq \sqrt{2} \cos \left( 2\theta - \frac{\pi}{4} \right) \leq \sqrt{2} \quad \forall \theta \in \mathbb{R}$$

$$\Rightarrow -\sqrt{2} \leq \lambda \leq \sqrt{2} \quad [\text{using Eq. (i)}]$$

$$\text{Hence, } \lambda \in [-\sqrt{2}, \sqrt{2}].$$

**21 (3)** Let  $I = \int_0^\pi [2 \sin x] dx$

$$= \int_0^{\pi/6} [2 \sin x] dx + \int_{\pi/6}^{\pi/2} [2 \sin x] dx + \int_{\pi/2}^\pi [2 \sin x] dx$$

$$= 0 + \int_{\pi/6}^{\pi/2} 1 dx + \int_{\pi/2}^\pi 1 dx + 0$$

$$= [x]_{\pi/6}^{\pi/2} + [x]_{\pi/2}^\pi = \frac{\pi}{2} - \frac{\pi}{6} + \frac{5\pi}{6} - \frac{\pi}{2} = \frac{2\pi}{3} \quad \therefore T = 3$$

**22 (45)** Given,  $A_i$ 's are 30 sets with five elements each, so

$$\sum_{i=1}^{30} n(A_i) = 5 \times 30 = 150 \quad \dots(i)$$

If there are  $m$  distinct elements in  $S$  and each element of  $S$  belongs to exactly 10 of the  $A_i$ 's, then

$$\sum_{i=1}^{30} n(A_i) = 10m \quad \dots(ii)$$

From Eqs. (i) and (ii), we get  $m = 15$

$$\text{Similarly, } \sum_{j=1}^n n(B_j) = 3n$$

$$\text{and } \sum_{j=1}^n n(B_j) = 9m$$

$$\therefore 3n = 9m$$

$$\Rightarrow n = \frac{9m}{3} = 3 \times 15 = 45$$

**23 (72)** Given,  $p = 2q$

$$\therefore p + q = 1 \Rightarrow p = \frac{2}{3} \text{ and } q = \frac{1}{3}$$

Now, required probability

$$= {}^4C_2 \left( \frac{2}{3} \right)^2 \left( \frac{1}{3} \right)^2 + {}^4C_3 \left( \frac{2}{3} \right)^3 \left( \frac{1}{3} \right)^1 + {}^4C_4 \left( \frac{2}{3} \right)^4$$

$$= 6 \times \frac{4}{81} + \frac{4 \times 8}{81} + \frac{1 \times 16}{81} = \frac{72}{81} = \frac{8}{9}$$

$$\therefore p = 8, q = 9$$

$$\text{Hence, } pq = 8 \times 9 = 72$$

**24 (16)** Let the equation of plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Let the centroid of tetrahedron  $OABC$  is  $(\alpha, \beta, \gamma)$ , then

$$\alpha = \frac{0+a+0+0}{4}, \quad \beta = \frac{0+0+b+0}{4},$$

$$\gamma = \frac{0+0+0+c}{4}$$

$$\Rightarrow a = 4\alpha, b = 4\beta, c = 4\gamma$$

Since, distance of plane from origin is  $P$ ,

$$P = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

$$\Rightarrow \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{P^2}$$

On putting the values of  $a, b$  and  $c$ , then

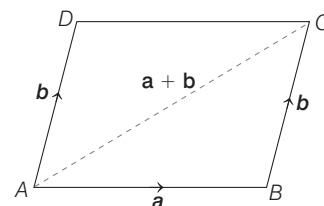
$$\frac{1}{16\alpha^2} + \frac{1}{16\beta^2} + \frac{1}{16\gamma^2} = \frac{1}{P^2}$$

Hence, the locus of centroid is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{16}{P^2}$$

$$\therefore k^2 = 16$$

**25 (4)** Clearly,  $AC = a + b$



$$\Rightarrow |AC|^2 = |a|^2 + |b|^2 + 2a \cdot b$$

$$\Rightarrow |AC|^2 = \{ |3\alpha - \beta|^2 + |\alpha + 3\beta|^2$$

$$+ 2(3\alpha - \beta)(\alpha + 3\beta) \}$$

$$= 9\alpha^2 + \beta^2 - 6\alpha\beta + \alpha^2 + 9\beta^2 + 6\alpha \cdot \beta$$

$$+ 6\alpha^2 - 6\beta^2 + 16\alpha \cdot \beta$$

$$\Rightarrow |AC|^2 = 16\alpha^2 + 4\beta^2 + 16\alpha \cdot \beta$$

$$\Rightarrow |AC|^2 = 64 + 16 + 16 |\alpha| |\beta| \cos \frac{\pi}{3}$$

$$\Rightarrow |AC|^2 = 64 + 16 + 16 \times 2 \times 2 \times \frac{1}{2}$$

$$\Rightarrow |AC| = 4\sqrt{7}$$

$$\text{Similarly, } |BD| = |a - b| = 4\sqrt{3}$$

$$\therefore p = 4$$

**26** (4) Clearly,

$$|A| = \begin{vmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{vmatrix}$$

$$\Rightarrow |A| = 2(1 + \sin^2 \theta)$$

Now,  $0 \leq \sin^2 \theta \leq 1$ , for all  $\theta \in [0, 2\pi]$ .

$$\Rightarrow 2 \leq 2 + 2\sin^2 \theta \leq 4, \text{ for all } \theta \in [0, 2\pi]$$

$\therefore$  Largest value of  $|A|$  is 4.

**27** (2) Let

$$A = \frac{(\sin^3 x + \cos^3 x) + (\cos^4 x - \sin^4 x)}{(1 + \cos x)(1 - \sin x)}$$

$$= \frac{(\sin^3 x + \cos^3 x) + (\cos x + \sin x)(\cos^2 x + \sin^2 x)(\cos x - \sin x)}{(1 + \cos x)(1 - \sin x)}$$

$$= \frac{(\sin x + \cos x)\{(1 - \sin x \cos x) + (\cos x - \sin x)\}}{1 + \cos x - \sin x - \sin x \cos x}$$

$$\Rightarrow A = \sin x + \cos x \quad \dots(i)$$

$$= \frac{1}{\sqrt{2}} \sin \left( \frac{\pi}{4} + x \right)$$

Again from Eq. (i), we get

$$A = \frac{1}{\sqrt{2}} \cos \left( \frac{\pi}{4} - x \right)$$

$$\therefore a = 2$$

**28** (288) Let two particular boys as one boy, we have only four boys.

$\therefore$  5 boys can be seated at a round table when two particular boys are always together =  $3!2!$

$\Rightarrow$  4 girls have are 4 places.

$\therefore$  4 girls can be arranged in  $4!$  ways.

$$\therefore \text{Required number} = 3!2!4! = 6 \cdot 2 \cdot 24 = 288$$

$$\mathbf{29} \quad (8) \therefore S_n = \frac{1 \left( 1 - \left( \frac{1}{2} \right)^n \right)}{\left( 1 - \frac{1}{2} \right)} = 2 - \frac{1}{2^{n-1}}$$

$$\text{or } 2 - S_n = \frac{1}{2^{n-1}} < \frac{1}{100}$$

$$\left[ \because 2 - S_n < \frac{1}{100} \right]$$

$$\text{or } 2^{n-1} > 100 > 2^6$$

$$\Rightarrow 2^{n-1} > 2^6$$

$$\Rightarrow n - 1 > 6$$

$$\therefore n > 7$$

Hence, least value of  $n$  is 8.

**30** (220) We have,

$$\begin{aligned} P(X + Y = 3) &= P(X = 0, Y = 3) \\ &+ P(X = 1, Y = 2) + P(X = 2, Y = 1) \\ &+ P(X = 3, Y = 0) \\ P(X = 0)P(Y = 3) &+ P(X = 1)P(Y = 2) \\ &+ P(X = 2)P(Y = 1) + P(X = 3)P(Y = 0) \\ [\because X \text{ and } Y \text{ are independent}] \\ &= {}^5C_0 \left( \frac{1}{2} \right)^5 \cdot {}^7C_3 \left( \frac{1}{2} \right)^7 + {}^5C_1 \left( \frac{1}{2} \right)^5 \cdot {}^7C_2 \left( \frac{1}{2} \right)^7 \end{aligned}$$

$$\begin{aligned} &+ {}^5C_2 \left( \frac{1}{2} \right)^5 \cdot {}^7C_1 \left( \frac{1}{2} \right)^7 + {}^5C_3 \left( \frac{1}{2} \right)^5 \cdot {}^7C_0 \left( \frac{1}{2} \right)^7 \\ &= \left( \frac{1}{2} \right)^{12} [(1)(35) + (5)(21) + (10)(7) + (10)(1)] \\ &= \frac{220}{2^{12}} = \frac{55}{1024} = \lambda \quad (\text{given}) \\ \therefore 4096 \lambda &= 4096 \times \frac{55}{1024} = 4 \times 55 = 220 \end{aligned}$$