Inverse Trigonometric Functions

Short Answer Type Questions

1. **Find the principal value of cos⁻¹x, for** $x=\frac{\sqrt{3}}{2}$ **.**

Sol. If $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\theta$, then $\cos\theta=\frac{\sqrt{3}}{2}$.

Since we are considering principal branch, $\theta\in[0,\pi]$. Also, since $\frac{\sqrt{3}}{2}>0$, θ being in the first quadrant, hence $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$.

2. **Evaluate** $\tan^{-1}\left(\sin\left(\frac{-\pi}{2}\right)\right)$.

Sol. $\tan^{-1}\left(\sin\left(\frac{-\pi}{2}\right)\right).=\tan^{-1}\left(-\sin\left(\frac{\pi}{2}\right)\right)=\tan^{-1}(-1)=-\frac{\pi}{4}$

3. **Find the value of** $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

Sol. $\cos^{-1}\left(\cos\frac{13\pi}{6}\right).=\cos^{-1}\left(\cos\left(2\pi+\frac{\pi}{6}\right)\right)=\cos^{-1}\left(\cos\frac{\pi}{6}\right)$

$=\frac{\pi}{6}$.

4. **Find the value of** $\tan^{-1}\left(\tan\frac{9\pi}{8}\right)$.

Sol. $\tan^{-1}\left(\tan\frac{9\pi}{8}\right).=\tan^{-1}\tan\left(\pi+\frac{\pi}{8}\right)$

$=\tan^{-1}\left(\tan\left(\frac{\pi}{8}\right)\right)=\frac{\pi}{8}$.

5. **Evaluate** $tan\left(tan^{-1}(-4)\right)$ **.**

Sol. Since $tan\left(tan^{-1}x\right)=x,\forall x\in R, tan\left(tan^{-1}(-4)\right)=-4$

6. **Evaluate:** $tan^{-1}\sqrt{3}-sec^{-1}(-2)$ **.**

Sol. $tan^{-1}\sqrt{3}-sec^{-1}(-2)=tan^{-1}\sqrt{3}-\left[\pi-sec^{-1}2\right]$

$=\frac{\pi}{3}-\pi+\cos^{-1}\left(\frac{1}{2}\right)$

$=-\frac{2\pi}{3}+\frac{\pi}{3}=-\frac{\pi}{3}$

7. **Evaluate:** $\sin^{-1}\left[\cos\left(sin^{-1}\frac{\sqrt{3}}{2}\right)\right]$**.**

Sol. $\sin^{-1}\left[\cos\left(sin^{-1}\frac{\sqrt{3}}{2}\right)\right]$

$=\sin^{-1}\left[cos\left(\frac{\pi}{3}\right)\right]$

$=\sin^{-1}\left[\frac{1}{2}\right]=\frac{\pi}{6}.$

8. **Prove that** $tan\left(cot^{-1}x\right)=cot\left(tan^{-1}x\right)$**. State with reason whether the equality is valid for all values of x.**

Sol. Let $\cot^{-1}x=\theta. Then \cot\theta=x$

Or, $\tan\left(\frac{\pi}{2}-\theta\right)=x\Rightarrow\tan^{-1}x=\frac{\pi}{2}-\theta$

So $tan\left(cot^{-1}x\right)=\tan\theta=\cot\left(\frac{\pi}{2}-\theta\right)$

$=\cot\left(\frac{\pi}{2}-\cot^{-1}x\right)=cot\left(tan^{-1}x\right)$

The equality is valid for all values of x since $\tan^{-1}x$ and $\cot^{-1}x$ are true for $x\in$ R.

9. **Find the value of** $\sec\left(\tan^{-1}\frac{y}{2}\right)$

Sol. Let $\tan^{-1}\frac{y}{2}=\theta, where\ \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. So, $\tan\theta=\frac{y}{2}$, which gives $\sec\theta=\frac{\sqrt{4+y^2}}{2}$.

Therefore, $\sec\left(\tan^{-1}\frac{y}{2}\right)=\sec\theta=\frac{\sqrt{4+y^2}}{2}$.

10. **Find value of tan ($\cos^{-1}$x) and hence evaluate** $\tan\left(\cos^{-1}\frac{8}{17}\right)$

Sol. Let $\cos^{-1}x=\theta$, then $\cos\theta=x$, where $\theta\in[0,\pi]$

Therefore, $\tan\left(\cos^{-1}x\right)=\tan\theta=\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}=\frac{\sqrt{1-x^2}}{x}$

Hence, $\tan\left(\cos^{-1}\frac{8}{17}\right)=\frac{\sqrt{1-\left(\frac{8}{17}\right)^2}}{\frac{8}{17}}=\frac{15}{8}$

11. **Find the value of** $\sin\left[2\cot^{-1}\left(\frac{-5}{12}\right)\right]$

Sol. Let $cot^{-1}\left(\frac{-5}{12}\right)=y$. Then $\cot y=\frac{-5}{12}$.

Now $\sin\left[2\cot^{-1}\left(\frac{-5}{12}\right)\right]=\sin 2y$

$=2\sin y\cos y=2\left(\frac{12}{13}\right)\left(\frac{-5}{13}\right)$ $\left[\text{since } \cot y<0,\ so\ y\in\left(\frac{\pi}{2},\pi\right)\right]$

$=\frac{-120}{169}$

12. Evaluate $\cos\left[\sin^{-1}\frac{1}{4}\sec^{-1}\frac{4}{3}\right]$

Sol. $\cos\left[\sin^{-1}\frac{1}{4}\sec^{-1}\frac{4}{3}\right]=\cos\left[\sin^{-1}\frac{1}{4}+\cos^{-1}\frac{3}{4}\right]$

$\cos\left(\sin^{-1}\frac{1}{4}\right)\cos\left(\cos^{-1}\frac{3}{4}\right)-\sin\left(\sin^{-1}\frac{1}{4}\right)\sin\left(\cos^{-1}\frac{3}{4}\right)$

$=\frac{3}{4}\sqrt{1-\frac{1}{4}^2}-\frac{1}{4}\sqrt{1-\frac{3}{4}^2}$

$=\frac{3}{4}\frac{\sqrt{15}}{4}-\frac{1}{4}\frac{\sqrt{7}}{4}=\frac{3\sqrt{15}-\sqrt{7}}{16}$

Long Answer Type Questions

13. Prove that $2\sin^{-1}\frac{3}{5}-\tan^{-1}\frac{17}{31}=\frac{\pi}{4}$

Sol. Let $\sin^{-1}\frac{3}{5}=\theta$, then $sin\theta=\frac{3}{5}$, where $\theta\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

Thus $\tan\theta=\frac{3}{4}$, *which gives* $\theta=\tan^{-1}\frac{3}{4}$.

Therefore, $2\sin^{-1}\frac{3}{5}-\tan^{-1}\frac{17}{31}$

$=2\theta-\tan^{-1}\frac{17}{31}=2\tan^{-1}\frac{3}{4}-\tan^{-1}\frac{17}{31}$

$\tan^{-1}\left(\frac{2.\frac{3}{4}}{1-\frac{9}{16}}\right)-\tan^{-1}\frac{17}{31}=\tan^{-1}\frac{24}{7}-\tan^{-1}\frac{17}{31}$

$=\tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}.\frac{17}{31}}\right)=\frac{\pi}{4}$

14. Prove that $\cot^{-1}7+\cot^{-1}8+\cot^{-1}18=\cot^{-1}3$

Sol. We have $\cot^{-1}7+\cot^{-1}8+\cot^{-1}18$

$$=\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{18}\quad\left(\text{since } \cot^{-1}x=\tan^{-1}\frac{1}{x}, \text{if } x>0\right)$$

$$=\tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}\right)+\tan^{-1}\frac{1}{18}\quad\left(\text{since } x.y=\frac{1}{7}.\frac{1}{8}<1\right)$$

$$=\tan^{-1}\frac{3}{11}+\tan^{-1}\frac{1}{18}=\tan^{-1}\left(\frac{\frac{3}{11}+\frac{1}{18}}{1-\frac{3}{11}\times\frac{1}{18}}\right)\quad(\text{since } xy<1)$$

$$=\tan^{-1}\frac{65}{195}=\tan^{-1}\frac{1}{3}=\cot^{-1}3$$

15. Which is greater, tan 1 or tan^{-1} 1?

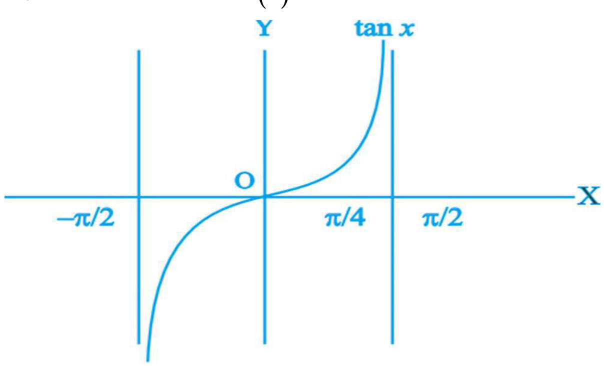
Sol. From Fig. we note that tan x is an increasing function in the interval $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$,

since $1>\frac{\pi}{4}\Rightarrow \tan 1>\tan\frac{\pi}{4}$. This gives

tan 1>1

$\Rightarrow$ $\tan 1>1>\frac{\pi}{4}$

$\Rightarrow$ $\tan 1>1>\tan^{-1}(1)$



16. Find the value of $\sin\left(2\tan^{-1}\frac{2}{3}\right)+\cos\left(\tan^{-1}\sqrt{3}\right)$

Sol. Let $\tan^{-1}\frac{2}{3}=x$ and $\tan^{-1}\sqrt{3}=y$ so that $\tan x=\frac{2}{3}$ and $\tan y=\sqrt{3}$

Therefore, $\sin\left(2\tan^{-1}\frac{2}{3}\right)+\cos\left(\tan^{-1}\sqrt{3}\right)$

$$=\sin(2x)+\cos y$$

$$=\frac{2\tan x}{1+\tan^2 x}+\frac{1}{\sqrt{1+\tan^2 y}}$$

$$=\frac{2.\frac{2}{3}}{1+\frac{4}{9}}+\frac{1}{1+\sqrt{\left(\sqrt{3}\right)^2}}$$

$$=\frac{12}{13}+\frac{1}{2}=\frac{37}{26}$$

17. **Solve for x** $\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x, x>0$

Sol. From given equation, we have $2\tan^{-1}\left(\frac{1-x}{1+x}\right)=\tan^{-1}x$

$$\Rightarrow\ 2\left[\tan^{-1}1-\tan^{-1}x\right]=\tan^{-1}x$$

$$\Rightarrow\ 2\left(\frac{\pi}{4}\right)=3\tan^{-1}x\Rightarrow\frac{\pi}{6}=\tan^{-1}x$$

$$\Rightarrow\ \text{x}=\frac{1}{\sqrt{3}}$$

18. **Find the values of x which satisfy the equation** $\sin^{-1}x+\sin^{-1}(1-x) = cos^{-1}x.$

Sol. From the given equation, we have

$$\sin\left(\sin^{-1}x+\sin^{-1}(1-x)\right)=\sin\left(cos^{-1}x\right)$$

$$\Rightarrow\ \sin\left(\sin^{-1}x\right)\cos\left(\sin^{-1}(1-x)\right)+\cos\left(\sin^{-1}x\right)\sin\left(\sin^{-1}(1-x)\right)=\sin\left(\cos^{-1}x\right)$$

$$\Rightarrow x\sqrt{1-(1-x)^2}+(1-x)\sqrt{1-x^2}=\sqrt{1-x^2}$$

$$\Rightarrow x\sqrt{2x-x^2}+\sqrt{1-x^2}\,(1-x-1)=0$$

$$\Rightarrow x\left(\sqrt{2x-x^2}-\sqrt{1-x^2}\right)=0$$

$$\Rightarrow x=0\quad or\quad 2x-x^2=1-x^2$$

$$\Rightarrow x=0\quad \text{or}\quad \text{x}=\frac{1}{2}$$

19. **Solve the equation** $\sin^{-1}6x+\sin^{-1}6\sqrt{3}x=-\frac{\pi}{2}$

Sol. From the given equation, we have $\sin^{-1}6x=-\frac{\pi}{2}-\sin^{-1}6\sqrt{3}x$

$$\Rightarrow\sin\left(\sin^{-1}6x\right)=\sin\left(-\frac{\pi}{2}-\sin^{-1}6\sqrt{3}x\right)$$

$$\Rightarrow 6x=-\cos\left(\sin^{-1}6\sqrt{3}x\right)$$

$\Rightarrow 6x=-\sqrt{1-108x^2}$. Squaring, we get

$$36x^2=1-108x^2$$

$$\Rightarrow 144x^2 = 1 \qquad \Rightarrow x \pm \frac{1}{12}$$

Note that $x = -\frac{1}{12}$ is the only root of the equation as $x = \frac{1}{12}$ does not satisfy it.

20. Show that

$$2\tan^{-1}\left\{\tan\frac{\alpha}{2}.\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right\} = \tan^{-1}\frac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}$$

Sol. L.H.S. = $\tan^{-1}\dfrac{2\tan\frac{\alpha}{2}.\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}{1-\tan^2\frac{\alpha}{2}\tan^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}$ $\left(\text{since } 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right)$

$$= \tan^{-1}\frac{2\tan\frac{\alpha}{2}\dfrac{1-\tan\frac{\beta}{2}}{1+\tan\frac{\beta}{2}}}{1-\tan^2\frac{\alpha}{2}\left(\dfrac{1-\tan\frac{\beta}{2}}{1+\tan\frac{\beta}{2}}\right)^2}$$

$$= \tan^{-1}\frac{2\tan\frac{\alpha}{2}.\left(1-\tan^2\frac{\beta}{2}\right)}{\left(1+\tan\frac{\beta}{2}\right)^2-\tan^2\frac{\alpha}{2}\left(1-\tan\frac{\beta}{2}\right)^2}$$

$$= \tan^{-1}\frac{2\tan\frac{\alpha}{2}.\left(1-\tan^2\frac{\beta}{2}\right)}{\left(1+\tan^2\frac{\beta}{2}\right)\left(1-\tan^2\frac{\alpha}{2}\right)+2\tan\frac{\beta}{2}\left(1+\tan^2\frac{\alpha}{2}\right)}$$

$$= \tan^{-1}\frac{\dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}\dfrac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}{\dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}+\dfrac{2\tan\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}$$

$$= \tan^{-1}\left(\frac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}\right) = R.H.S$$

Objective Type Questions

Choose the correct answer from the given four options in each of 21 to 41.

21. Which of the following corresponds to the principal value branch of $\tan^{-1}$?

(A) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(B) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

(C) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)-\{0\}$

(D) $(0,\ \pi)$

Sol. (A) is the correct answer.

22. The principal value branch of sec^{-1} is

(A) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)-\{0\}$

(B) $[0,\pi]-\left\{\frac{\pi}{2}\right\}$

(C) (0, π)

(D) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Sol. (B) is the correct answer.

23. One branch of cos–1 other than the principal value branch corresponds to

(A) $\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$

(B) $[\pi,2\pi]-\left\{\frac{3\pi}{2}\right\}$

(C) (0, π)

(D) [2π, 3π]

Sol. (D) is the correct answer.

24. The value of $\sin^{-1}\left(\cos\left(\frac{43\pi}{5}\right)\right) is$

(A) $\frac{3\pi}{5}$

(B) $\frac{-7\pi}{5}$

(C) $\frac{\pi}{10}$

(D) $-\frac{\pi}{10}$

Sol. (D) is the correct answer. $\sin^{-1}\left(\cos\frac{40\pi+3\pi}{5}\right)=\sin^{-1}\cos\left(8\pi+\frac{3\pi}{5}\right)$

$=\sin^{-1}\left(\cos\frac{3\pi}{5}\right)=\sin^{-1}\left(\sin\left(\frac{\pi}{2}-\frac{3\pi}{5}\right)\right)$

$$\sin^{-1}\left(\sin\left(-\frac{\pi}{10}\right)\right)=-\frac{\pi}{10}.$$

25. **The principal value of the expression cos**

(A) $\frac{2\pi}{9}$

(B) $\frac{-2\pi}{9}$

(C) $\frac{34\pi}{9}$

(D) $\frac{\pi}{9}$

Sol. (A) is the correct answer. Cos^{-1}(cos (680°)) = cos^{-1} [cos (720° – 40°)]

$$=\cos^{-1}\left[\cos(-40°)\right]=\cos^{-1}\left[\cos\left(40^0\right)\right]=40^0=\frac{2\pi}{9}$$

26. **The value of** $cot\ (sin^{-1}x)$ **is**

(A) $\frac{\sqrt{1+x^2}}{x}$

(B) $\frac{x}{\sqrt{1+x^2}}$

(C) $\frac{1}{x}$

(D) $\frac{\sqrt{1-x^2}}{x}$

Sol. (D) is the correct answer. Let $sin^{-1}x=\theta,$ then $sin\theta=x$

$$\cos ec\theta=\frac{1}{x}\Rightarrow\cos ec^2\theta=\frac{1}{x^2}$$

$$1+\cot^2\theta=\frac{1}{x^2}\Rightarrow\cot\theta=\frac{\sqrt{1+x^2}}{x}$$

27. **If** $\tan^{-1}x=\frac{\pi}{10}$ **for some** $x\in R,$ **then the value of** $\cot^{-1}x$ **is**

(A) $\frac{\pi}{5}$

(B) $\frac{2\pi}{5}$

(C) $\frac{3\pi}{5}$

(D) $\frac{4\pi}{5}$

Sol. (B) is the correct answer. We know $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$. Therefore

$$\cot^{-1}x=\frac{\pi}{2}-\frac{\pi}{10}$$

$$\Rightarrow \cot^{-1}x=\frac{\pi}{2}-\frac{\pi}{10}=\frac{2\pi}{5}$$

28. The domain of $sin^{-1}2x$ is

(A) [0, 1]

(B) [– 1, 1]

(C) $\left[-\frac{1}{2},\frac{1}{2}\right]$

(d) $[-2,2]$

Sol. (C) is the correct answer. Let $sin^{-1}2x=\theta$ so that $2x=sin\ \theta$.

Now $-1\le sin\ \theta\le 1$, i.e., $-1\le 2x\le 1$ which gives $-\frac{1}{2}\le x\le\frac{1}{2}$.

29. The principal value of $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ is

(A) $-\frac{2\pi}{3}$

(B) $-\frac{\pi}{3}$

(C) $\frac{4\pi}{3}$

(D) $\frac{5\pi}{3}$

Sol. (B) is the correct answer.

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\sin^{-1}\left(-\sin\frac{\pi}{3}\right)=\sin^{-1}\left(\sin\frac{\pi}{3}\right)=-\frac{\pi}{3}$$

30. The greatest and least values of $\left(\sin^{-1}x\right)^2+\left(\cos^{-1}x\right)^2$ are respectively

(A) $\frac{5\pi^2}{4}\ and\ \frac{\pi^2}{8}$

(B) $\frac{\pi}{2}\ and\ \frac{-\pi}{2}$

(C) $\frac{\pi^2}{4}\ and\ \frac{-\pi^2}{4}$

(D) $\frac{\pi^2}{4}\ and\ 0$

Sol. (A) is the correct answer. We have

$$\left(\sin^{-1}x\right)^2+\left(\cos^{-1}x\right)^2=\left(\sin^{-1}x+\cos^{-1}x\right)^2-2\sin^{-1}x\cos^{-1}x$$

$$=\frac{\pi^2}{4}-2\sin^{-1}x\left(\frac{\pi}{2}-\sin^{-1}x\right)$$

$$=\frac{\pi^2}{4}-\pi\sin^{-1}x+2\left(\sin^{-1}x\right)^2$$

$$=2\left[\left(\sin^{-1}x\right)^2-\frac{\pi}{2}\sin^{-1}x+\frac{\pi^2}{8}\right]$$

$$=2\left[\left(\sin^{-1}x-\frac{\pi}{4}\right)^2+\frac{\pi^2}{16}\right]$$

Thus, the least value is $2\left(\frac{\pi^2}{16}\right)$ *i.e.* $\frac{\pi^2}{8}$ and the Greatest value is

$$2\left[\left(\frac{-\pi}{2}-\frac{\pi}{4}\right)^2+\frac{\pi^2}{16}\right]\text{, i.e.}\frac{5\pi^2}{4}$$

31. Let $\theta = sin^{-1}(sin\ (-\ 600°)$, then value of θ is

(A) $\frac{\pi}{3}$

(B) $\frac{\pi}{2}$

(C) $\frac{2\pi}{3}$

(D) $\frac{-2\pi}{3}$

Sol. (A) is the correct answer.

$$\sin^{-1}\sin\left(-600\times\frac{\pi}{180}\right)=\sin^{-1}\sin\left(\frac{-10\pi}{3}\right)$$

$$=\sin^{-1}\left[-\sin\left(4\pi-\frac{2\pi}{3}\right)\right]=\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$$

$$=\sin^{-1}\left(\sin\left(\pi-\frac{\pi}{3}\right)\right)=\sin^{-1}\left(\sin\frac{\pi}{3}\right)=\frac{\pi}{3}$$

32. The domain of the function $y = sin^{-1}\left(-\ x^2\right)$ is

(A) [0, 1]
(B) (0, 1)
(C) [–1, 1]
(D) ϕ

Sol. (C) is the correct answer. $y = sin^{-1}\left(-\ x^2\right)\Rightarrow siny=-x^2$

i.e. $-1 \le -x^2 \le 1$ (since $-1 \le \sin y \le 1$)

$\Rightarrow 1 \ge x^2 \ge -1$

$\Rightarrow 0 \le x^2 \le 1$

$\Rightarrow |x| \le$ *i.e.* $-1 \le x \le 1$

33. The domain of $y = \cos^{-1}\left(x^2 - 4\right)$ is

(A) [3, 5]

(B) [0, π]

(C) $\left[-\sqrt{5}, -\sqrt{3}\right] \cap \left[-\sqrt{5}, \sqrt{3}\right]$

(D) $\left[-\sqrt{5}, -\sqrt{3}\right] \cup \left[\sqrt{3}, 5\right]$

Sol. (D) is the correct answer. $y = cos^{-1}\left(x^2 - 4\right) \Rightarrow cosy = x^2 - 4$

i.e. $-1 \le x^2 - 4 \le 1 \ (since -1 \le cosy \le 1)$

$\Rightarrow 3 \le x^2 \le 5$

$\Rightarrow \sqrt{3} \le |x| \le \sqrt{5}$

$\Rightarrow x \in \left[-\sqrt{5}, -\sqrt{3}\right] \cup \left[\sqrt{3}, \sqrt{5}\right]$

34. The domain of the function defined by $f(x) = sin^{-1}x + cosx$ is

(A)[–1, 1]

(B) [–1, π + 1]

(C) $(-\infty, \infty)$

(D) ϕ

Sol. (A) is the correct answer. The domain of cos is R and the domain of $\sin^{-1}$ is [–1, 1]. Therefore, the domain of $cos\ x + sin^{-1}x\ is\ R \cap [-1,1]$ i.e. $[-1,1]$

35. The value of sin $\left(2\ sin^{-1}(.6)\right)$ is

(A) .48

(B) .96

(C) 1.2

(D) sin 1.2

Sol. (B) is the correct answer. Let $\sin^{-1}(.6) = \theta$, i.e., $\sin\theta = .6$.

Now $\sin(2\theta) = 2\sin\theta\cos\theta = 2(.6)(.8) = .96$.

36. If $sin^{-1}x + sin^{-1}y = \frac{\pi}{2}$, then value of $cos^{-1}x + cos^{-1}y$ is

(A) $\frac{\pi}{2}$

(B) π

(C) 0

(D) $\frac{2\pi}{3}$

Sol. (A) is the correct answer. Given that $sin^{-1}x+ sin^{-1}y =\frac{\pi}{2}$

Therefore, $\left(\frac{\pi}{2}-\cos^{-1}x\right)+\left(\frac{\pi}{2}-\cos^{-1}y\right)=\frac{\pi}{2}$

$\Rightarrow \cos^{-1}x+\cos^{-1}y=\frac{\pi}{2}.$

37. The value of $\tan\left(\cos^{-1}\frac{3}{5}+\tan^{-1}\frac{1}{4}\right)$ **is**

(A) $\frac{19}{8}$

(B) $\frac{8}{19}$

(C) $\frac{19}{12}$

(D) $\frac{3}{4}$

Sol. (A) is the correct answer. $\tan\left(\cos^{-1}\frac{3}{5}+\tan^{-1}\frac{1}{4}\right)=\tan\left(\tan^{-1}\frac{4}{3}+\tan^{-1}\frac{1}{4}\right)$

$=\tan\tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{4}}{1-\frac{4}{3}\times\frac{1}{4}}\right)=\tan\tan^{-1}\left(\frac{19}{8}\right)=\frac{19}{8}$

38. The value of the expression $sin\left[cot^{-1}\left(cos\left(tan^{-1}1\right)\right)\right]$ **is**

(A) 0

(B) 1

(C) $\frac{1}{\sqrt{3}}$

(D) $\sqrt{\frac{2}{3}}$

Sol. (D) is the correct answer.

$\sin\left[\cot^{-1}\left(\cos\frac{\pi}{4}\right)\right]=\sin\left[\cot^{-1}\frac{1}{\sqrt{2}}\right]=\sin\left[\sin^{-1}\sqrt{\frac{2}{3}}\right]=\sqrt{\frac{2}{3}}$

39 The equation $\tan^{-1}x-\cot^{-1}x=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ **has**

(A) no Solution

(B) unique Solution

(C) infinite number of Solutions

(D) two Solutions

Sol. (B) is the correct answer. We have $\tan^{-1}x-\cot^{-1}x=\frac{\pi}{6}$ *and* $\tan^{-1}+\cot^{-1}x=\frac{\pi}{2}$

Adding them, we get $2tan^{-1}x=\frac{2\pi}{3}$

$\Rightarrow \tan^{-1}x=\frac{\pi}{3}$ *i.e.* $x=\sqrt{3}$

40. If $\alpha\le\ 2sin^{-1}x+\ cos^{-1}x\le\beta$, then

(A) $\alpha=\frac{-\pi}{2},\beta=\frac{\pi}{2}$

(B) $\alpha=0,\ \beta=\pi$

(C) $\alpha=\frac{-\pi}{2},\beta=\frac{3\pi}{2}$

(D) $\alpha=0,\ \beta=2\pi$

Sol. (B) is the correct answer. We have $\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2}$

$\Rightarrow\frac{-\pi}{2}+\frac{\pi}{2}\le\sin^{-1}x+\frac{\pi}{2}\le\frac{\pi}{2}+\frac{\pi}{2}$

$\Rightarrow 0\le\sin^{-1}x+\left(\sin^{-1}x+\cos^{-1}x\right)\le\pi$

$\Rightarrow 0\le 2\sin^{-1}x+\cos^{-1}x\le\pi$

41. The value of tan $\tan^2\left(\sec^{-1}2\right)+\cot^2\left(\cos ec^{-1}3\right)$ **is**

(A) 5

(B) 11

(C) 13

(D) 15

Sol. (B) is the correct answer.

$\tan^2\left(\sec^{-1}2\right)+\cot^2\left(\cos ec^{-1}3\right)=\sec^2\left(\sec^{-1}2\right)-1+\cos ec^2\left(\cos ec^{-1}3\right)-1$

$=2^2\times1+3^3-2=11.$

Inverse Trigonometric Functions

Objective Type Questions

Choose the correct answers from the given four options in each of the Exercises from 20 to 37 (M.C.Q.)

20. Which of the following is the principal value branch of $cos^{-1}x$ **?**

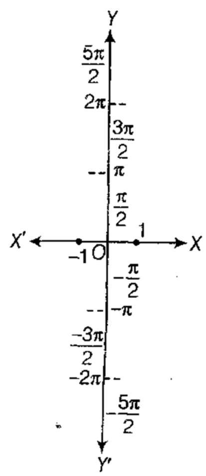
(a) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

(b) $(0,\pi)$

(c) $[0,\pi]$

(d) $(0,\pi)-\left\{\frac{\pi}{2}\right\}$

Sol. (c) We know that, the principal value branch of $cos^{-1}x$ is $[0,\pi]$



$y = cos^{-1}x$

21. Which of the following is the principal value branch of $cosec^{-1}\, x$ **?**

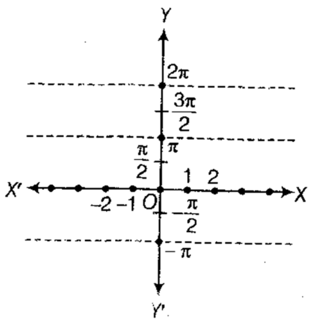
(a) $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

(b) $[o,\pi]-\left\{\frac{\pi}{2}\right\}$

(c) $\left[\frac{\pi}{2},\frac{\pi}{2}\right]$

(d) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-[0]$

Sol. (d) We know that, the principal value branch of $cosec^{-1}\, x$ is $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-[0]$



$y = \text{cosec}^{-1} x$

22. **If** $3\, tan^{-1}x + \, cot^{-1}x = \pi$ **, then x equals to**

(A) 0

(B) 1

(C) –1

(D) $\frac{1}{2}$

Sol. (B) Given that, $3\, tan^{-1}x + \, cot^{-1}x = \pi \quad (i)$

$\Rightarrow 2\tan^{-1} x + \tan^{-1} x + \cot^{-1} x = \pi$

$\Rightarrow 2\tan^{-1} x = \pi - \frac{\pi}{2} \quad \left[\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right]$

$\Rightarrow 2\tan^{-1} x = \frac{\pi}{2}$

$= \tan^{-1} \frac{2x}{1-x^2} = \frac{\pi}{2} \quad \left[\because 2\tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}, \forall x \in (-1,1)\right]$

$\Rightarrow \frac{2x}{1-x^2} = \tan \frac{\pi}{2}$

$\Rightarrow \frac{2x}{1-x^2} = \frac{1}{0} \Rightarrow 1 - x^2 = 0$

$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow x = 1$

Hence, only x=1 Satisfies the given equation.

Note *Here, putting x=-1 in the given equation, we get*

$3\tan^{-1}(-1) + \cot^{-1}(-1) = \pi$

$\Rightarrow 3\tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right] + \cot^{-1}\left[\cot\left(\frac{-\pi}{4}\right)\right] = \pi$

$\Rightarrow 3\tan^{-1}\left[-\tan\frac{\pi}{4}\right] + \cot^{-1}\left(-\cot\frac{\pi}{4}\right) = \pi$

$\Rightarrow 3\tan^{-1}\left(\tan\frac{\pi}{4}\right) + \pi - \cot^{-1}\left(\cot\frac{\pi}{4}\right) = \pi$

$\Rightarrow -3\frac{\pi}{4} + \pi - \frac{\pi}{4} = \pi$

$\Rightarrow -\pi + \pi = \pi \Rightarrow 0 \neq \pi$

Hence, x=-1 does not satisfy the given equation.

23. **The value of** $\sin^{-1}\left[\cos\left(\frac{33\pi}{5}\right)\right]$ **is**

(a) $\frac{3\pi}{5}$

(b) $\frac{-7\pi}{5}$

(c) $\frac{\pi}{10}$

(d) $\frac{-\pi}{10}$

Sol. (d) We have

$$\sin^{-1}\left[\cos\left(\frac{33\pi}{5}\right)\right]=\sin^{-1}\left[\cos\left(6\pi+\frac{33\pi}{5}\right)\right]$$

$$=\sin^{-1}\left[\cos\left(\frac{3\pi}{5}\right)\right]\left[\because \cos(2n\pi+\theta)=\cos\theta\right]$$

$$=\sin^{-1}\left[\cos\left(\frac{\pi}{2}+\frac{\pi}{10}\right)\right]=\sin^{-1}\left(-\sin\frac{\pi}{10}\right)$$

$$=-\sin^{-1}\left(\sin\frac{\pi}{10}\right)\ \left[\because \sin^{-1}(-x)=-\sin^{-1}x\right]$$

$$=-\frac{\pi}{10}\quad\left[\because \sin^{-1}(\sin x)=x, x\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)\right]$$

24. **The domain of the function** $\cos^{-1}(2x-1)$ **is**

(A) [0, 1]
(B) [−1, 1]
(C) (−1, 1)
(D) [0, π]

Sol. (A) we have, $f(x)=\cos^{-1}(2x-1)$

$\because -1\le 2x-1\le 1$

$\Rightarrow\ 0\le 2x\le 2$

$\Rightarrow\ 0\le x\le 1$

$\therefore\ x\in[0,1]$

25. **The domain of the function defined by** $f(x)\ =\ sin^{-1}\sqrt{x-1}$ **is**

(A) [1, 2]
(B) [−1, 1]
(C) [0, 1]
(D) none of these

Sol. (A) $\because f(x)=\sin^{-1}\sqrt{x-1}$

$\Rightarrow\ 0 \le x\text{-}1 \le 1\ [\because\ \sqrt{x-1} \ge 0\ and\ \ -1 \le \sqrt{x-1} \le 1]$

$\Rightarrow\ 1 \le x \le 2$

$\therefore\ x \in [1,2]$

26. **If** $\cos\left(\sin^{-1}\frac{2}{5}+\cos^{-1}x\right)=0$ **then x is equal to**

(a) $\frac{1}{5}$

(b) $\frac{2}{5}$

(c) 0

(d) 1

Sol. (b) We have $\cos\left(\sin^{-1}\frac{2}{5}+\cos^{-1}x\right)=0$

$\Rightarrow \sin^{-1}\frac{2}{5}+\cos^{-1}x=\cos^{-1}0$

$\Rightarrow \sin^{-1}\frac{2}{5}+\cos^{-1}x=\cos^{-1}\cos\frac{\pi}{2}$

$\Rightarrow \sin^{-1}\frac{2}{5}+\cos^{-1}x=\frac{\pi}{2}$

$\Rightarrow \cos^{-1}x=\frac{\pi}{2}-\sin^{-1}\frac{2}{5}$

$\Rightarrow \cos^{-1}x=\cos^{-1}\frac{2}{5}\quad \left[\because \cos^{-1}x+\sin^{-1}x=\frac{\pi}{2}\right]$

$\therefore x=\frac{2}{5}$

27. **The value of** $sin\left[2\ tan^{-1}(.75)\right]$ **is equal to**

(A) 0.75

(B) 1.5

(C) 0.96

(D) sin 1.5

Sol. (C) We have, $sin\left[2\ tan^{-1}(.75)\right]=\sin\left(2\sin^{-1}\frac{3}{4}\right)\quad \left[\because 0.75=\frac{75}{100}=\frac{3}{4}\right]$

$=\sin\left(\sin^{-1}\frac{2.\frac{3}{4}}{1+\frac{9}{16}}\right)=\sin\left[\sin^{-1}\frac{3/2}{25/16}\right]$

$=\sin\left[\sin^{-1}\left(\frac{48}{50}\right)\right]=\sin\left[\sin^{-1}\left(\frac{24}{25}\right)\right]=\frac{24}{25}=0.96$

28. **The value of** $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$ **is equal to**

(A) $\frac{\pi}{2}$

(B) $\frac{3\pi}{2}$

(C) $\frac{5\pi}{2}$

(D) $\frac{7\pi}{2}$

Sol. (A) We have, $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$

$=\cos^{-1}\cos\left(2\pi-\frac{\pi}{2}\right)\left[\because \cos\left(2\pi-\frac{\pi}{2}\right)=\cos\frac{\pi}{2}\right]$

$=\cos^{-1}\cos\left(\frac{\pi}{2}\right)=\frac{\pi}{2}$ $\quad \{\because \cos^{-1}(\cos x)=x, x\in[0,\pi]\}$

Note *Remember that,* $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)\neq\frac{3\pi}{2}$

$\because \frac{3\pi}{2}\notin(0,\pi)$

29. The value of the expression $2\sec^{-1}2+\sin^{-1}\left(\frac{1}{2}\right)is$

(A) $\frac{\pi}{6}$

(B) $\frac{5\pi}{6}$

(C) $\frac{7\pi}{6}$

(D) 1

Sol. (B) We have, $2\sec^{-1}2+\sin^{-1}\left(\frac{1}{2}\right)=2\sec^{-1}\sec\frac{\pi}{3}+\sin^{-1}\sin\frac{\pi}{6}$

$=2.\frac{\pi}{3}+\frac{\pi}{6}$ $\quad \left[\because \sec^{-1}(\sec x)=x \, and \, \sin^{-1}(\sin x)=x\right]$

$=\frac{4\pi+\pi}{6}=\frac{5\pi}{6}$

30. If $\tan^{-1}x+\tan^{-1}y=\frac{4\pi}{5}$, **then** $\cot^{-1}x+\cot^{-1}y$ **equals to**

(A) $\frac{\pi}{5}$

(B) $\frac{2\pi}{5}$

(C) $\frac{3\pi}{5}$

(D) π

Sol. (A) We have, $\tan^{-1}x+\tan^{-1}y=\frac{4\pi}{5}$,

$\Rightarrow \frac{\pi}{2}-\cot^{-1}x+\frac{\pi}{2}-\cot^{-1}y=\frac{4\pi}{5}$

$\Rightarrow$ -$\left(\cot^{-1}x+\cot^{-1}y\right)=\frac{4\pi}{5}-\pi\left[\because \tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}\right]$

$\Rightarrow \cot^{-1}x+\cot^{-1}y=-\left(-\frac{\pi}{5}\right)$

$\Rightarrow \cot^{-1}x+\cot^{-1}y=\frac{\pi}{5}$

31. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, **where** $a,x\in[0,1]$ **then the value of x is**

(A) 0

(B) $\frac{a}{2}$

(C) a

(D) $\frac{2a}{1-a^2}$

Sol. (D) We have, $\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Let $a=\tan\theta \Rightarrow \theta=\tan^{-1}a$

$\therefore \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)+\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\tan^{-1}\frac{2x}{1-x^2}$

$\Rightarrow \sin^{-1}\sin 2\theta+\cos^{-1}\cos 2\theta=\tan^{-1}\frac{2x}{1-x^2}$

$\Rightarrow 2\theta+2\theta=\tan^{-1}\frac{2x}{1-x^2}$

$\Rightarrow 4\tan^{-1}a=\tan^{-1}\frac{2x}{1-x^2}$

$\Rightarrow 2.2\tan^{-1}a=\tan^{-1}\frac{2x}{1-x^2}$

$\Rightarrow 2.\tan^{-1}\frac{2a}{1-a^2}=\tan^{-1}\frac{2x}{1-x^2}\left[\because 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}\right]$

$$\Rightarrow \tan^{-1}\frac{2.\left(\frac{2a}{1-a^2}\right)}{1-\left(\frac{2a}{1-a^2}\right)} = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$\therefore\ x = \frac{2a}{1-a^2}$$

32. **The value of** $\cot\left[\cos^{-1}\left(\frac{7}{25}\right)\right]$ **is**

(a) $\frac{25}{24}$

(b) $\frac{25}{7}$

(c) $\frac{24}{25}$

(d) $\frac{7}{24}$

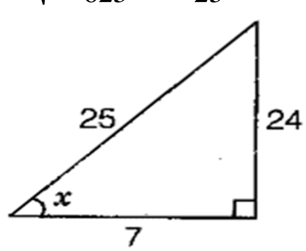
Sol. (d) We have, $\cot\left[\cos^{-1}\left(\frac{7}{25}\right)\right]$

Let $\cos^{-1}\frac{7}{25} = x$

$\Rightarrow\ \cos x = \frac{7}{25}$

$$\therefore\ \sin x = \sqrt{1-\cos^2 x} = \sqrt{1-\left(\frac{7}{25}\right)^2}$$

$$= \sqrt{\frac{625-49}{625}} = \frac{24}{25}$$



$$\therefore\ \cot x = \frac{\cos x}{\sin x} = \frac{\frac{7}{25}}{\frac{24}{25}} = \frac{7}{24}$$

$$\Rightarrow\ x = \cot^{-1}\left(\frac{7}{24}\right) = \cos^{-1}\left(\frac{7}{25}\right)$$

$$\therefore \cot\left(\cos^{-1}\frac{7}{25}\right) = \cot\left(\cot^{-1}\frac{7}{24}\right) = \frac{7}{24}\quad\left[\because \cot^{-1}\frac{7}{24} = \cos^{-1}\frac{7}{25}\right]$$

33. The value of the expression $\tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right)$ **is**

(a) $2+\sqrt{5}$

(b) $\sqrt{5}-2$

(c) $\frac{\sqrt{5}+2}{2}$

(d) $5+\sqrt{2}$

Sol. (b) We have, $\tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right)$

$$\Rightarrow Let \quad \frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}=\theta$$

$$\Rightarrow \cos^{-1}\frac{2}{\sqrt{5}}=2\theta \Rightarrow \cos 2\theta=\frac{2}{\sqrt{5}}$$

$$\therefore \left(1-2\sin^2\theta\right)=\frac{2}{\sqrt{5}}$$

$$\Rightarrow 2\sin^2\theta=1-\frac{2}{\sqrt{5}}$$

$$\Rightarrow \sin^2\theta=\frac{1}{2}-\frac{1}{\sqrt{5}}$$

$$\Rightarrow \sin\theta=\sqrt{\frac{1}{2}-\frac{1}{\sqrt{5}}}$$

$$\therefore \cos^2\theta=1-\sin^2\theta$$

$$=1-\frac{1}{2}+\frac{1}{\sqrt{5}}=\frac{1}{2}+\frac{1}{\sqrt{5}}$$

$$\Rightarrow \cos\theta=\sqrt{\frac{1}{2}+\frac{1}{\sqrt{5}}}$$

$$\therefore \tan\theta=\sqrt{\frac{\frac{1}{2}-\frac{1}{\sqrt{5}}}{\frac{1}{2}+\frac{1}{\sqrt{5}}}}=\sqrt{\frac{\sqrt{5}-2}{\sqrt{5}+2}} \quad \left[\because \tan\theta=\frac{\sin\theta}{\cos\theta}\right]$$

$$\Rightarrow \theta=\tan^{-1}\sqrt{\frac{\sqrt{5}-2}{\sqrt{5}+2}}=\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}$$

$$\therefore \quad \tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right)=\tan\tan^{-1}\sqrt{\frac{\sqrt{5}-2}{\sqrt{5}+2}}$$

$$=\sqrt{\frac{\sqrt{5}-2}{\sqrt{5}+2}.\frac{\sqrt{5}-2}{\sqrt{5}-2}}$$

$$= \sqrt{\frac{\left(\sqrt{5}-2\right)^2}{5-4}} = \sqrt{5}-2$$

34. **If | x| ≤ 1, then** $2\ tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ **is equal to**

(a) $4\tan^{-1}x$

(b) 0

(c) $\frac{\pi}{2}$

(d) π

Sol. (a) We have, $2\ tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Let $x = \tan\theta$

$\therefore 2\tan^{-1}\tan\theta + \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}$ $\left[\because \tan^{-1}(\tan x) = x\right]$

$= 2\theta + \sin^{-1}\sin 2\theta \left[\because \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right]$

$= 2\theta + 2\theta$ $\left[\because \sin^{-1}(\sin x) = x\right]$

$= 4\theta$ $\left[\because \theta = \tan^{-1}x\right]$

$= 4\tan^{-1}x$

35. **If** $cos^{-1}\alpha + cos^{-1}\beta + cos^{-1}\gamma = 3\pi$, **then** $\alpha(\beta+\gamma) + \beta(\gamma+\alpha) + \gamma(\alpha+\beta)$ **equals**

(A) 0

(B) 1

(C) 6

(D) 12

Sol. (C) We have $cos^{-1}\alpha + cos^{-1}\beta + cos^{-1}\gamma = 3\pi$,

We know that, $0 \le \cos^{-1}x \le \pi$

$\Rightarrow \cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$

If and only if, $\cos^{-1}\alpha = \cos^{-1}\beta = \cos^{-1}\gamma = \pi$

$\Rightarrow \cos\pi = \alpha = \beta = \gamma$

$\Rightarrow -1 = \alpha = \beta = \gamma$

$\Rightarrow \alpha = \beta = \gamma = -1$

$\therefore \quad \alpha(\beta+\gamma) + \beta(\gamma+\alpha) + \gamma(\alpha+\beta)$

$= -1(-1-1) - 1(1-1) - 1(-1-1)$

$= 2+2+2 = 6$

36. **The number of real Solution of the equation**

$\sqrt{1+\cos 2x} = \sqrt{2}\cos^{-1}(\cos x)\, in\left[\frac{\pi}{2},\pi\right]$ **is**

(A) 0
(B) 1
(C) 2
(D) ∞

Sol. (a) We have, $\sqrt{1+\cos 2x} = \sqrt{2}\cos^{-1}(\cos x), \left[\frac{\pi}{2},\pi\right]$

$\Rightarrow \sqrt{1+2\cos^2 x - 1} = \sqrt{2}\cos^{-1}(\cos x)$

$\Rightarrow \sqrt{2}\cos = \sqrt{2}\cos^{-1}(\cos x)$

$\Rightarrow \cos x = \cos^{-1}(\cos x)$

$\Rightarrow \cos x = x \qquad \left[\because \cos^{-1}(\cos x) = x\right]$

Which is not true for any real value of x.
Hence, there is no solution possible for the given equation.

37. If $\cos^{-1} x > \sin^{-1} x$, then

(a) $\frac{1}{\sqrt{2}} < x \le 1$

(b) $0 \le x < \frac{1}{\sqrt{2}}$

(c) $-1 \le x < \frac{1}{\sqrt{2}}$

(d) $x > 0$

Sol. (c) We have, $\cos^{-1} x > \sin^{-1} x$ where $x \in [-1,1]$

$\Rightarrow x < \cos\left(\sin^{-1} x\right)$

$\Rightarrow x < \cos\left[\cos^{-1}\sqrt{1-x^2}\right] \quad \left[let \sin^{-1} x = \theta \Rightarrow \sin\theta = \frac{x}{1}\right]$

$\left[\because \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2} \Rightarrow \theta = \cos^{-1}\sqrt{1-x^2}\right]$

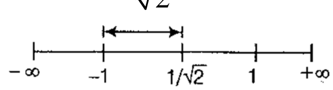
$\Rightarrow x < \sqrt{1-x^2}$

$\Rightarrow x^2 < 1 - x^2 \Rightarrow 2x^2 < 1$

$\Rightarrow x^2 < \frac{1}{2} \Rightarrow x < \pm\left(\frac{1}{\sqrt{2}}\right) \quad ...(i)$

Also $-1 \le x \le 1 \quad ...(ii)$

$\therefore -1 \le x \le \frac{1}{\sqrt{2}}$



Alternate Method

$\frac{\pi}{2}-\sin^{-1}x>\sin^{-1}x$

$\frac{\pi}{2}>2\sin^{-1}x \Rightarrow \frac{\pi}{4}>\sin^{-1}x$

$\frac{1}{\sqrt{2}}>x \Rightarrow \frac{1}{\sqrt{2}}<x\leq 1$

We know that, $\sin^{-1}x\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

Fill in the blanks in each of the Exercises 38 to 48.

38. The principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is________.

Sol. $\because 0\leq\cos^{-1}x\leq\pi$

$\cos^{-1}\left(-\frac{1}{2}\right)=\cos^{-1}\left(\cos\frac{2\pi}{3}\right)=\frac{2\pi}{3}$

39. The value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$ is ……………….

Sol. $\because\ -\frac{\pi}{2}\leq\sin^{-1}x\leq\frac{\pi}{2}$

$\therefore \sin^{-1}\left(\sin\frac{3\pi}{5}\right)=\sin^{-1}\sin\left(\pi-\frac{2\pi}{5}\right)$

$=\sin^{-1}\left(\sin\frac{2\pi}{5}\right)=\frac{2\pi}{5}$

40. If $\cos\left(\tan^{-1}x+\cot^{-1}\sqrt{3}\right)=0,$ then the value of x is …………

Sol. We have, $\cos\left(\tan^{-1}x+\cot^{-1}\sqrt{3}\right)=0,$

$\Rightarrow \tan^{-1}x+\cot^{-1}\sqrt{3}=\cos^{-1}0$

$\Rightarrow \tan^{-1}x+\cot^{-1}\sqrt{3}=\cos^{-1}\cos\frac{\pi}{2}$

$\Rightarrow \tan^{-1}x+\cot^{-1}\sqrt{3}=\frac{\pi}{2}$

$\Rightarrow \tan^{-1}x=\frac{\pi}{2}-\cot^{-1}\sqrt{3}$

$\Rightarrow \tan^{-1}x=\tan^{-1}\sqrt{3}\quad\left[\because \tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}\right]$

$\therefore\ x=\sqrt{3}$

41. The set of values of $\sec^{-1}\frac{1}{2}$ is________.

Sol. Since, domain of $\sec^{-1}\frac{1}{2}$ is $R-(-1,1)$

$\Rightarrow\ (-\infty,-1)\ \cup[1,\infty)$

So, there is no set of values exist for $\sec^{-1}\frac{1}{2}$.

So, ϕ is the answer.

42. **The principal value of** $\tan^{-1}\sqrt{3}$ **is________.**

Sol. $\because \quad \tan^{-1}\sqrt{3}=\tan^{-1}\tan\left(\frac{\pi}{3}\right)$

$\left[\because \tan^{-1}(\tan x)=x, x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$

$=\left(\frac{\pi}{3}\right)$

43. **The value of** $\cos^{-1}\left(\cos\frac{14\pi}{3}\right)$ **is________.**

Sol. We have $\cos^{-1}\left(\cos\frac{14\pi}{3}\right)=\cos^{-1}\cos\left(4\pi+\frac{2\pi}{3}\right)$

$=\cos^{-1}\cos\frac{2\pi}{3}\left[\because \cos(2n\pi+\pi)=\cos\theta\right]$

$=\frac{2\pi}{3}\{\because \cos^{-1}(\cos x)=x, x\in[0,\pi]\}$

Note *Remember that,* $\cos^{-1}\left(\cos\frac{14\pi}{3}\right)\neq\frac{14\pi}{3}$

Since, $\frac{14\pi}{3}\notin[0,\pi]$

44. **The value of** $\cos\left(\sin^{-1}x+\cos^{-1}x\right), \textit{where } |x|\le 1,$ **is ……………………**

Sol. $\cos\left(\sin^{-1}x+\cos^{-1}\right)$

$=\cos\frac{\pi}{2}=0 \quad \left[\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right]$

45. **The value of** $\tan\left(\frac{\sin^{-1}x+\cos^{-1}x}{2}\right), \textit{when } x=\frac{\sqrt{3}}{2},$ **is ……………..**

Sol. $\because\ \tan\left(\frac{\sin^{-1}x+\cos^{-1}x}{2}\right)=\tan\left(\frac{\pi/2}{2}\right)\ \left[\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right]$

$=\tan\frac{\pi}{4}=1$

46. **If** $y=2\tan^{-1}x+\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, **then …… <** y **< ……….**

Sol. We have, $y=2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}$

$\therefore\ y = 2\tan^{-1}\tan\theta + \sin^{-1}\dfrac{2\tan\theta}{1+\tan^2\theta}$ $[let\ x = \tan\theta]$

$\Rightarrow y = 2\theta + \sin^{-1}\sin 2\theta$ $\left[\because \sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}\right]$

$\Rightarrow y = 2\theta + 2\theta = 4\theta$ $\left[\because \theta = \tan^{-1} x\right]$

$\Rightarrow y = 4\tan^{-1} x$

$\because -\pi/2 < \tan^{-1} x < 4\pi/2$

$\therefore\ -\dfrac{4\pi}{2} < 4\tan^{-1} x < 2\pi$

$\Rightarrow -2\pi < 4\tan^{-1} x < 2\pi$

$\Rightarrow -2\pi < y < 2\pi$ $\left[\because y = 4\tan^{-1} x\right]$

47. The result $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$ **is true when value of xy is ____**

Sol. We know that $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$

Where, $xy > -1$

48. The value of $\cot^{-1}(-x)\, x \in R$ **in terms of** $\cot^{-1} x$ **is ……….**

Sol. We know that

$\cot^{-1}(-x) = \pi - \cot^{-1} x,\ \ x \in R$

State True or False for the statement in each of the Exercises 49 to 55.

49. All trigonometric functions have inverse over their respective domains.

Sol. False

We know that, all trigonometric functions have inverse over their restricted domains.

50. The value of the expression $\left(\cos^{-1} x\right)^2$ **is equal to** $\sec^2 x.$

Sol. False

$\because\ \left[\cos^{-1} x\right]^2 = \left[\sec^{-1}\dfrac{1}{x}\right]^2 \neq \sec^2 x$

51. The domain of trigonometric functions can be restricted to any one of their branch (not necessarily principal value) in order to obtain their inverse functions.

Sol. True

We know that, the domain of trigonometric functions are restricted in their domain to obtain their inverse functions.

52. The least numerical value, either positive or negative of angle θ is called principal value of the inverse trigonometric function.

Sol. True

We know that, the smallest numerical value, either positive or negative of θ is called the principal value of the function.

53. The graph of inverse trigonometric function can be obtained from the graph of their corresponding trigonometric function by interchanging x and y-axes.

Sol. True

We know that, the graph of an inverse function can be obtained from the corresponding graph of original function as a mirror image (i.e. reflection) along the line y=x.

54. The minimum value of n for which $\tan^{-1}\frac{n}{\pi} > \frac{\pi}{4}, n \in N$ **is valid is 5.**

Sol. False

$\because \tan^{-1}\frac{n}{\pi} > \frac{\pi}{4} \Rightarrow \frac{n}{\pi} > \tan\frac{\pi}{4}$

$\Rightarrow \frac{n}{\pi} > 1 \left[\because \tan\frac{\pi}{4} = 1\right]$

$\Rightarrow n > \pi$

So, the minimum value of n is 4. $[\because n \in N \ \ and \ \pi = 3.14......]$

55. The principal value of $\sin^{-1}\left[\cos\left(\sin^{-1}\frac{1}{2}\right)\right] \ is \ \frac{\pi}{3}$**.**

Sol. True

Given that, $\sin^{-1}\left[\cos\left(\sin^{-1}\frac{1}{2}\right)\right] = \sin^{-1}\left[\cos\sin^{-1}\left(\sin\frac{\pi}{6}\right)\right]$

$= \sin^{-1}\left[\cos\frac{\pi}{6}\right] \quad \left[\because \sin^{-1}(\sin x) = x\right]$

$= \sin^{-1}\frac{\sqrt{3}}{2}$

$= \sin^{-1}\sin\frac{\pi}{3} = \frac{\pi}{3}$

Inverse Trigonometric Functions

Short Answer Type Questions

1. **Find the value of** $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

Sol. We know that, $\tan^{-1}\tan x = x; x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\cos^{-1}\cos x = x; x\in[0,\pi]$

$$\therefore \tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$$

$$=\tan^{-1}\left[\tan\left(\pi-\frac{\pi}{6}\right)\right]+\cos^{-1}\left[\cos\left(\pi+\frac{7\pi}{6}\right)\right]$$

$$=\tan^{-1}\left(\tan\frac{\pi}{6}\right)+\cos^{-1}\left(-\cos\frac{7\pi}{6}\right)\ \left[\because \cos(\pi+\theta)=-\cos\theta\right]$$

$$=-\tan^{-1}\left(\tan\frac{\pi}{6}\right)+\pi-\left[\cos^{-1}\cos\left(\frac{7\pi}{6}\right)\right]$$

$$\left\{\because \tan^{-1}(-x)=-\tan^{-1}x; x\in R \textit{ and } \cos^{-1}(-x)=\pi-\cos^{-1}x; x\in[-1,1]\right\}$$

$$=-\tan^{-1}\left(\tan\frac{\pi}{6}\right)+\pi-\cos^{-1}\left[\cos\left(\pi+\frac{\pi}{6}\right)\right]$$

$$=-\tan^{-1}\left(\tan\frac{\pi}{6}\right)+\pi-\left[\cos^{-1}\left(-\cos\frac{\pi}{6}\right)\right]\left[\because \cos(\pi+\theta)=-\cos\theta\right]$$

$$=-\tan^{-1}\left(\tan\frac{\pi}{6}\right)+\pi-\pi+\cos^{-1}\left(\cos\frac{\pi}{6}\right)\left[\because \cos^{-1}(-x)=\pi-\cos^{-1}x\right]$$

$$=-\frac{\pi}{6}+0+\frac{\pi}{6}=0$$

Note *Remember that,* $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)\neq\frac{5\pi}{6}$ *and* $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)\neq\frac{13\pi}{6}$

Since, $\frac{5\pi}{6}\notin\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ *and* $\frac{13\pi}{6}\notin[0,\pi]$

2. **Evaluate** $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$

Sol. We have, $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$

$$=\cos\left[\cos^{-1}\left(\cos\frac{5\pi}{6}\right)+\frac{\pi}{6}\right]\quad\left[\because \cos\frac{5\pi}{6}=\frac{-\sqrt{3}}{2}\right]$$

$$=\cos\left(\frac{5\pi}{6}+\frac{\pi}{6}\right)\ \left\{\because \cos^{-1}\cos x = x;\ x\in[0,\pi]\right\}$$

$= \cos\left(\frac{6\pi}{6}\right)$

$= \cos(\pi) = -1$

3. **Prove that** $\cot\left(\frac{\pi}{4} - 2\cot^{-1} 3\right) = 7.$

Sol. We have to prove, $\cot\left(\frac{\pi}{4} - 2\cot^{-1} 3\right) = 7$

$\Rightarrow \left(\frac{\pi}{4} - 2\cot^{-1} 3\right) = \cot^{-1} 7$

$\Rightarrow \left(2\cot^{-1} 3\right) = \frac{\pi}{4} - \cot^{-1} 7$

$\Rightarrow 2\tan^{-1}\frac{1}{3} = \frac{\pi}{4} - \tan^{-1}\frac{1}{7}$

$\Rightarrow 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\frac{2/3}{1+(1/3)^2} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\frac{2/3}{8/9} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}.\frac{1}{7}} = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\frac{(21+4)/28}{(28-3)/28} = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\frac{25}{25} = \frac{\pi}{4}$

$\Rightarrow 1 = \tan\frac{\pi}{4}$

$1 = 1$

$\Rightarrow$ LHS=RHS Hence Proved.

4. **Find the value of** $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right].$

Sol. We have, $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right].$

$= \tan^{-1}\left(\tan\frac{5\pi}{6}\right) + \cot^{-1}\left(\cot\frac{\pi}{3}\right) + \tan^{-1}(-1).$

$$= \tan^{-1}\left[\tan\left(\pi - \frac{\pi}{6}\right)\right] + \cot^{-1}\left[\cot\left(\frac{\pi}{3}\right)\right] + \tan^{-1}\left[\tan\left(\pi - \frac{\pi}{4}\right)\right]$$

$$= \tan^{-1}\left(-\tan\frac{\pi}{6}\right) + \cot^{-1}\left(\cot\frac{\pi}{3}\right) + \tan^{-1}\left(-\tan\frac{\pi}{4}\right)$$

$$\left[\begin{array}{l} \because\ \tan^{-1}(\tan x) = x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ \cot^{-1}(\cot x) = x, x \in (0, \pi) \\ \textit{and}\ \ \tan^{-1}(-x) = -\tan^{-1} x \end{array}\right]$$

$$= -\frac{\pi}{6} + \frac{\pi}{3} - \frac{\pi}{4} = \frac{-2\pi + 4\pi - 3\pi}{12}$$

$$= \frac{-5\pi + 4\pi}{12} = \frac{-\pi}{12}$$

5. **Find the value of** $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$.

Sol. We have, $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \tan^{-1}\tan\left(\pi - \frac{\pi}{3}\right)$

$$= \tan^{-1}\left(-\tan\frac{\pi}{3}\right) \qquad \left[\because \tan^{-1}(-x) = -\tan^{-1} x\right]$$

$$= \tan^{-1}\tan\frac{\pi}{3} = -\frac{\pi}{3} \qquad \left[\because \tan^{-1}(\tan x) = x, x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\right]$$

Note *Remember that,* $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) \neq \frac{2\pi}{3}$

Since, $\tan^{-1}(\tan x) = x,\ \textit{if}\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *and* $\frac{2\pi}{3} \notin \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

6. **Show that** $2\tan^{-1}(-3) = \frac{-\pi}{2} + \tan^{-1}\left(\frac{-4}{3}\right)$

Sol. $LHS = 2\tan^{-1}(-3) = -2\tan^{-1} 3 \quad \left[\because\ \tan^{-1}(-x) = -\tan^{-1} x, x \in R\right]$

$$= -\left[\cos^{-1}\frac{1-3^2}{1+3^2}\right] \qquad \left[\because\ 2\tan^{-1} x = \cos^{-1}\frac{1-x^2}{1+x^2} x \geq 0\right]$$

$$= -\left[\cos^{-1}\left(\frac{-8}{10}\right)\right] = -\left[\cos^{-1}\left(\frac{-4}{5}\right)\right]$$

$$= -\left[\pi - \cos^{-1}\left(\frac{4}{5}\right)\right] \left\{\because\ \cos^{-1}(-x) = \pi - \cos^{-1} x, x \in [-1,1]\right\}$$

$$= -\pi + \cos^{-1}\left(\frac{4}{5}\right)\left[\textit{let}\ \cos^{-1}\left(\frac{4}{5}\right) = \theta \Rightarrow \cos\theta = \frac{4}{5} \Rightarrow \tan\theta = \frac{3}{4} \Rightarrow \theta = \tan^{-1}\frac{3}{4}\right]$$

$$= -\pi + \tan^{-1}\left(\frac{3}{4}\right) = -\pi + \left[\frac{\pi}{2} - \cot^{-1}\left(\frac{3}{4}\right)\right]$$

$$= -\frac{\pi}{2} - \cot^{-1}\frac{3}{4} = -\frac{\pi}{2} - \tan^{-1}\frac{4}{3}$$

$$= -\frac{\pi}{2} + \tan^{-1}\left(\frac{-4}{3}\right) \quad \left[\because \tan^{-1}(-x) = -\tan^{-1}x\right]$$

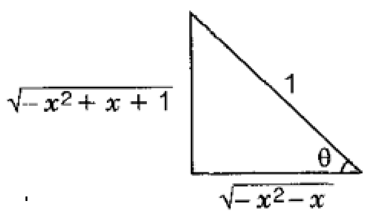
= RHS (Hence Proved)

7. Find the real Solution of the equation

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+x} = \frac{\pi}{2}$$

Sol. We have, $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$ ...(*i*)

Let $\sin^{-1}\sqrt{x^2+x+1} = \theta$



$$\Rightarrow \sin\theta\sqrt{\frac{x^2+x+1}{1}}$$

$$\Rightarrow \tan\theta = \frac{\sqrt{x^2+x+1}}{\sqrt{-x^2-x}} \quad \left[\because \tan\theta = \frac{\sin\theta}{\cos\theta}\right]$$

$$\because \theta = \tan^{-1}\frac{\sqrt{x^2+x+1}}{\sqrt{-x^2-x}}$$

$$= \sin^{-1}\sqrt{x^2+x+1}$$

On putting the value of θ in Eq. (i), we get

$$\tan^{-1}\sqrt{x(x+1)} + \tan^{-1}\frac{\sqrt{x^2+x+1}}{\sqrt{-x^2-x}} = \frac{\pi}{2}$$

We know that, $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, $xy < 1$

$$\therefore \tan^{-1}\left[\frac{\sqrt{x(x+1)}\sqrt{\frac{x^2+x+1}{-x^2-x}}}{1-\sqrt{x(x+1)}\sqrt{\frac{x^2+x+1}{-x^2-x}}}\right] = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\sqrt{x^2+x}\sqrt{\dfrac{x^2+x+1}{-1(x^2+x)}}}{1-(\sqrt{x^2+x).\dfrac{(x^2+x+1)}{-1(x^2+x)}}}\right]=\frac{\pi}{2}$$

$$\Rightarrow \frac{x^2+x+\sqrt{-(x^2+x+1)}}{\left[1-\sqrt{-(x^2+x+1)}\sqrt{(x^2+x)}\right]}=\tan\frac{\pi}{2}=\frac{1}{0}$$

$$\Rightarrow \left[1-\sqrt{-(x^2+x+1)}\right]\sqrt{(x^2+x)}=0$$

$$\Rightarrow \quad -(x^2+x+1)=1 \;\; or \; x^2+x=0$$

$$\Rightarrow \quad -x^2-x-1=1 \;\; or \; x(x+1)=0$$

$$\Rightarrow \quad x^2+x+2=0 \quad or \; x(x+1)=0$$

$$\therefore \; x=\frac{-1\pm\sqrt{1-4\times 2}}{2}$$

$\Rightarrow$ x=0 or x=-1

For real solution, we have x= 0, -1.

8. Find the value of $\sin\left(2\tan^{-1}\frac{1}{3}\right)+\cos\left(\tan^{-1}2\sqrt{2}\right)$

Sol. We have, $\sin\left(2\tan^{-1}\frac{1}{3}\right)+\cos\left(\tan^{-1}2\sqrt{2}\right)$

$$=\sin\left[\sin^{-1}\left\{\frac{2\times\frac{1}{3}}{1+\left(\frac{1}{3}\right)^2}\right\}\right]+\cos\left(\cos^{-1}\frac{1}{3}\right)\;\; \left[\because\; \tan^{-1}x=\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right]$$

$$\left[\because 2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2},-1\le x\le 1 \;\; and \;\; \tan^{-1}\left(2\sqrt{2}\right)=\cos^{-1}\frac{1}{3}\right]$$

$$=\sin\left[\sin^{-1}\left(\frac{\frac{2}{3}}{1+\frac{1}{9}}\right)\right]+\frac{1}{3} \qquad \left\{\because \cos\left(\cos^{-1}x\right)=x; x\in[-1,1]\right\}$$

$$=\sin\left[\sin^{-1}\left(\frac{2\times 9}{3\times 10}\right)\right]+\frac{1}{3}=sin\left[\sin^{-1}\left(\frac{3}{5}\right)\right]+\frac{1}{3}\;\;\left[\because \sin\left(\sin^{-1}x=x\right)\right]$$

$$=\frac{3}{5}+\frac{1}{3}=\frac{9+5}{15}=\frac{14}{15}$$

9. If $2\tan^{-1}(\cos\theta)=\tan^{-1}(2\operatorname{cosec}\theta)$, then show that $\theta=\frac{\pi}{4}$, where n is any integer.

Sol. We have, $2\tan^{-1}(\cos\theta)=\tan^{-1}(2\operatorname{cosec}\theta)$,

$$\Rightarrow \tan^{-1}\left(\frac{2\cos\theta}{1-\cos^2\theta}\right)=\tan^{-1}(2\operatorname{cosec}\theta)$$

$$\left[\because 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\Rightarrow \left(\frac{2\cos\theta}{\sin^2\theta}\right)=(2\operatorname{cosec}\theta)$$

$$\Rightarrow (\cot\theta.2\operatorname{cosec}\theta)=(2\operatorname{cosec}\theta)\Rightarrow \cot\theta=1$$

$$\Rightarrow \cot\theta=\cot\frac{\pi}{4}\Rightarrow\theta=\frac{\pi}{4}$$

10. Show that $\cos\left(2\tan^{-1}\frac{1}{7}\right)=\sin\left(4\tan^{-1}\frac{1}{3}\right)$

Sol. We have, $\cos\left(2\tan^{-1}\frac{1}{7}\right)=\sin\left(4\tan^{-1}\frac{1}{3}\right)$

$$\Rightarrow \cos\left[\cos^{-1}\left(\frac{1-\left(\frac{1}{7}\right)^2}{1+\left(\frac{1}{7}\right)^2}\right)\right]=\sin\left(2.2\tan^{-1}\frac{1}{3}\right)\quad\left[\because\ 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$\Rightarrow \cos\left[\cos^{-1}\left(\frac{\frac{48}{49}}{\frac{50}{49}}\right)\right]=\sin\left[2.\left(\tan^{-1}\frac{\frac{2}{3}}{1-\left(\frac{1}{3}\right)^2}\right)\right]\left[\because\ 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\Rightarrow \cos\left[\cos^{-1}\left(\frac{48\times49}{50\times49}\right)\right]=\sin\left[2\tan^{-1}\left(\frac{18}{24}\right)\right]$$

$$\Rightarrow \cos\left[\cos^{-1}\left(\frac{24}{25}\right)\right]=\sin\left(2\tan^{-1}\frac{3}{4}\right)$$

$$\Rightarrow \cos\left[\cos^{-1}\left(\frac{24}{25}\right)\right]=\sin\left(\sin^{-1}\frac{2\times\frac{3}{4}}{1+\frac{9}{16}}\right)\quad\left[\because\ 2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}\right]$$

$$\Rightarrow \frac{24}{25}=\sin\left(\sin^{-1}\frac{3/2}{25/16}\right)$$

$$\frac{25}{24}=\frac{48}{50}\Rightarrow\frac{24}{25}=\frac{24}{25}$$

$\because$ *LHS = RHS* (*Hence proved*)

11. Solve the following equation $\cos\left(\tan^{-1}x\right)=\sin\left(\cot^{-1}\frac{3}{4}\right)$.

Sol. We have, $\cos\left(\tan^{-1}x\right)=\sin\left(\cot^{-1}\frac{3}{4}\right)$.

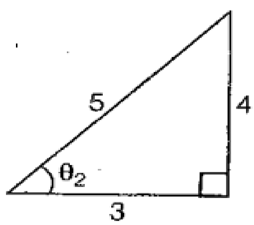
$\Rightarrow \cos\left(\cos^{-1}\frac{1}{\sqrt{x^2+1}}\right)=\sin\left(\sin^{-1}\frac{4}{5}\right)$

Let $\tan^{-1}x=\theta_1 \quad \Rightarrow \tan\theta_1=\frac{x}{1}$

$\Rightarrow \cos\theta_1=\frac{1}{\sqrt{x^2+1}} \Rightarrow \theta_1=\cos^{-1}\frac{1}{\sqrt{x^2+1}}$

And $\cot^{-1}\frac{3}{4}=\theta_2 \quad \Rightarrow \cot\theta_2=\frac{3}{4}$

$\Rightarrow \sin\theta_2=\frac{4}{5} \quad \Rightarrow \theta_2=\sin^{-1}\frac{4}{5}$



$\Rightarrow \quad \frac{1}{\sqrt{x^2+1}}=\frac{4}{5}$

$\left\{\because \cos\left(\cos^{-1}x\right)=x, x\in[-1,1]\, and \sin\left(\sin^{-1}x\right)=x, x\in[-1,1]\right\}$

On squaring both sides, we get

$\Rightarrow 16\left(x^2+1\right)=25$

$\Rightarrow 16x^2=9$

$\Rightarrow x^2=\left(\frac{3}{4}\right)^2$

$\therefore\ x=\pm\frac{3}{4}=\frac{-3}{4},\frac{3}{4}$

Inverse Trigonometric Functions

Long Answer Type Questions

12. Prove that $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2.$

Sol. We have,

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2.$$

$$\because \quad LHS=\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) \quad \text{................(i)}$$

$$\left[let\ x^2=\cos 2\theta=\left(\cos^2\theta-\sin^2\theta\right)=1-2\sin^2\theta=2\cos^2\theta-1\right]$$

$$\Rightarrow \cos^{-1}x^2=2\theta \Rightarrow \theta=\frac{1}{2}\cos^{-1}x^2$$

$$\therefore \quad \sqrt{1+x^2}=\sqrt{1+\cos 2\theta}$$

$$=\sqrt{1+2\cos^2\theta-1}=\sqrt{2}\cos\theta$$

And $\sqrt{1-x^2}=\sqrt{1-\cos 2\theta}$

$$=\sqrt{1-1+2\sin^2\theta}=\sqrt{2}\sin\theta$$

$$\therefore\ LHS=\tan^{-1}\left(\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)=\tan^{-1}\left(\frac{tab\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}.\tan\theta}\right)$$

$$=\tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right] \quad \left[\because \tan(x+y)=\frac{\tan x+\tan y}{1-\tan x.\tan y}\right]$$

$$=\frac{\pi}{4}+\theta=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$$

= RHS Hence proved.

13. Find the simplified form of

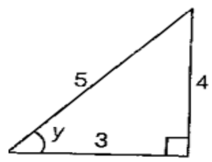
$\cos^{-1}\left(\frac{3}{5}\cos x+\frac{4}{5}\sin x\right)$, **where** $x\in\left[\frac{-3\pi}{4},\frac{\pi}{4}\right]$

Sol. $\cos^{-1}\left(\frac{3}{5}\cos x+\frac{4}{5}\sin x\right),\ x\in\left[\frac{-3\pi}{4},\frac{\pi}{4}\right]$

$Let\ \cos y = \frac{3}{5}$

$\Rightarrow\ \sin y = \frac{4}{5}$

$\Rightarrow\ y = \cos^{-1}\frac{3}{5} = \sin^{-1}\frac{4}{5} = \tan^{-1}\left(\frac{4}{3}\right)$



$\therefore\ \cos^{-1}[\cos y.\cos x + \sin y.\sin x]$

$= \cos^{-1}[\cos(y-x)] \quad [\therefore\ \cos(A-B) = \cos A.\cos B + \sin A.\text{Sin } B]$

$= y - x = \tan^{-1}\frac{4}{3} - x \quad \left[\because\ y = \tan^{-1}\frac{4}{3}\right]$

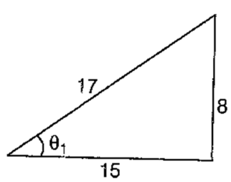
14. Prove that $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{77}{85}$.

Sol. We have, $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{77}{85}$.

$\therefore\quad LHS = \sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5}$

$= \tan^{-1}\frac{8}{15} + \tan^{-1}\frac{3}{4}$

Let $\sin^{-1}\frac{8}{17} = \theta_1 \Rightarrow \sin\theta_1 = \frac{8}{17}$



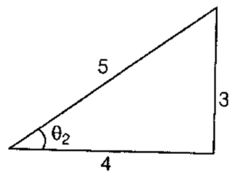
$\Rightarrow \tan\theta_1 = \frac{8}{15} \Rightarrow \theta_1 = \tan^{-1}\frac{8}{15}$

And $\sin^{-1}\frac{3}{5} = \theta_2 \Rightarrow \sin\theta_2 = \frac{3}{5}$

$\Rightarrow \tan\theta_2 = \frac{3}{4} \Rightarrow \theta_2 = \tan^{-1}\frac{3}{4}$

$= \tan^{-1}\left[\frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\times\frac{3}{4}}\right] \left[\because\ \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$$= \tan^{-1}\left[\frac{\frac{32+45}{60}}{\frac{60-24}{60}}\right] = \tan^{-1}\left(\frac{77}{36}\right)$$



Let $\theta_3 = \tan^{-1}\frac{77}{36} \Rightarrow \tan\theta_3 = \frac{77}{36}$

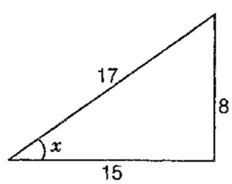
$$\Rightarrow \sin\theta_3 = \frac{77}{\sqrt{5929}+1296} = \frac{77}{85}$$

$$\therefore\ \theta_3 = \sin^{-1}\frac{77}{85}$$

$= \sin^{-1}\frac{77}{85} = RHS$ Hence proved.

Alternate Method

To Prove, $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{77}{85}$



Let $\sin^{-1}\frac{8}{17} = x$

$$\Rightarrow\ \sin x = \frac{8}{17}$$

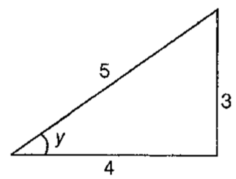
$$\Rightarrow\ \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\left(\frac{8}{17}\right)^2}$$

$$= \sqrt{\frac{289-64}{289}} = \sqrt{\frac{225}{289}} = \frac{15}{17}$$

Let $\sin^{-1}\frac{3}{5} = y$

$$\Rightarrow \sin y = \frac{3}{5} \Rightarrow \sin^2 y = \frac{9}{25}$$

$$\therefore\ \cos^2 y = 1-\frac{9}{25}$$



$\Rightarrow \cos^2 y = \left(\frac{4}{5}\right)^2 \Rightarrow \cos\ y = \frac{4}{5}$

Now, $\sin(x+y) = \sin x.\cos y + \cos x.\sin y$

$= \frac{8}{17}.\frac{4}{5} + \frac{15}{17}.\frac{3}{5}$

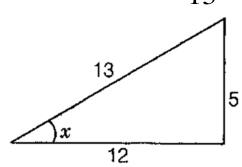
$= \frac{32}{85} + \frac{45}{85} = \frac{77}{85}$

$\Rightarrow (x+y) = \sin^{-1}\left(\frac{77}{85}\right)$

$\Rightarrow \sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{77}{85}$

15. Show that $\sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5} = \tan^{-1}\frac{63}{16}$.

Sol. We have, $\sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5} = \tan^{-1}\frac{63}{16} \quad (i)$



Let $\sin^{-1}\frac{5}{13} = x$

$\Rightarrow \sin x = \frac{5}{13}$

And $\cos^2 x = 1 - \sin^2 x$

$= 1 - \frac{25}{169} = \frac{144}{169}$

$\Rightarrow \cos x = \sqrt{\frac{144}{169}} = \frac{12}{13}$

$\therefore \tan x = \frac{\sin x}{\cos x} = \frac{5/13}{12/13} = \frac{5}{12} \quad (ii)$

$\Rightarrow \tan x = 5/12 \quad (iii)$

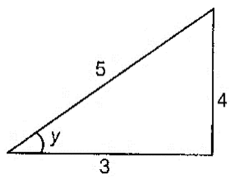
Again, let $\cos^{-1}\frac{3}{5} = y \Rightarrow \cos y = \frac{3}{5}$

$\therefore \sin y = \sqrt{1 - \cos^2 y}$

$$= \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}}$$

$$\sin y = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\Rightarrow \tan y = \frac{\sin y}{\cos y} = \frac{4/5}{3/5} = \frac{4}{3} (iii)$$



We know that,

$$\tan(x+y) = \frac{\tan x + \tan y}{1-\tan x.\tan y}$$

$$\Rightarrow \tan(x+y) = \frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}.\frac{4}{3}} \Rightarrow \tan(x+y) = \frac{\frac{15+48}{36}}{\frac{36-20}{36}}$$

$$\Rightarrow \tan(x+y) = \frac{63/36}{16/36}$$

$$\Rightarrow \tan(x+y) = \frac{63}{16}$$

$$\Rightarrow x+y = \tan^{-1}\frac{63}{16}$$

$$\Rightarrow \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{4}{3} = \tan^{-1}\frac{63}{16}$$ Hence proved.

16. Prove that $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \sin^{-1}\frac{1}{\sqrt{5}}$

Sol. We have, $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \sin^{-1}\frac{1}{\sqrt{5}}$

Let $\tan^{-1}\frac{1}{4} = x$

$$\Rightarrow \tan x = \frac{1}{4}$$

$$\Rightarrow \tan^2 x = \frac{1}{16}$$

$$\Rightarrow \sec^2 x - 1 = \frac{1}{16}$$

$$\Rightarrow \sec^2 x = 1 + \frac{1}{16} = \frac{17}{16}$$

$$\Rightarrow \frac{1}{\cos^2 x} = \frac{17}{16}$$

$$\Rightarrow \cos^2 x = \frac{16}{17}$$

$$\Rightarrow \cos x = \frac{4}{\sqrt{17}}$$

$$\Rightarrow \sin^2 x = 1 - \cos^2 x = 1 - \frac{16}{17} = \frac{1}{17}$$

$$\Rightarrow \sin x = \frac{1}{\sqrt{17}} .....(ii)$$

Again, let $\tan^{-1}\frac{2}{9} = y$

$$\Rightarrow \tan y = \frac{2}{9} \Rightarrow \tan^2 y = \frac{4}{81}$$

$$\Rightarrow \sec^2 y - 1 = \frac{4}{81}$$

$$\Rightarrow \sec^2 y = \frac{4}{81} + 1 = \frac{85}{81}$$

$$\Rightarrow \cos^2 y = \frac{81}{85} \Rightarrow \cos y = \frac{9}{\sqrt{85}}$$

$$\Rightarrow \sin^2 y = 1 - \cos^2 y = 1 - \frac{81}{85} = \frac{4}{85}$$

$$\Rightarrow \sin y = \frac{2}{\sqrt{85}} \quad ....(iii)$$

We know that, $\sin(x+y) = \sin x.\cos y + \cos x.\sin y$

$$= \frac{1}{\sqrt{17}}.\frac{9}{\sqrt{85}} + \frac{4}{\sqrt{17}}.\frac{2}{\sqrt{85}}$$

$$= \frac{17}{\sqrt{17}.\sqrt{85}} = \frac{\sqrt{17}}{\sqrt{17}.\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$\Rightarrow (x+y) = \sin^{-1}\frac{1}{\sqrt{5}}$$

$\Rightarrow \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \sin^{-1}\frac{1}{\sqrt{5}}$ Hence proved.

17. Find the value of $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$.

Sol. We have, $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$

$$= 2.2\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

$$= 2.\left[\tan^{-1}\frac{\frac{2}{5}}{1-\left(\frac{1}{5}\right)^2}\right] - \tan^{-1}\frac{1}{239} \quad \left[\because\ 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= 2.\left[\tan^{-1}\left(\frac{\frac{2}{5}}{1-\frac{1}{25}}\right)\right] - \tan^{-1}\frac{1}{239}$$

$$= 2.\left[\tan^{-1}\left(\frac{\frac{2}{5}}{24/25}\right)\right] - \tan^{-1}\frac{1}{239}$$

$$= 2\tan^{-1}\frac{5}{12} - \tan^{-1}\frac{1}{239}$$

$$= \tan^{-1}\frac{2.\frac{5}{12}}{1-\left(\frac{5}{12}\right)^2} - \tan^{-1}\frac{1}{239}\left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= \tan^{-1}\left(\frac{\frac{5}{6}}{1-\frac{25}{144}}\right) - \tan^{-1}\frac{1}{239}$$

$$= \tan^{-1}\left(\frac{144\times 5}{119\times 6}\right) - \tan^{-1}\frac{1}{239}$$

$$= \tan^{-1}\left(\frac{120}{199}\right) - \tan^{-1}\frac{1}{239}$$

$$= \tan^{-1}\left(\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}.\frac{1}{239}}\right)\left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= \tan^{-1}\left(\frac{120\times 239-119}{119\times 239+120}\right)$$

$$= \tan^{-1}\left[\frac{28680-119}{28441+120}\right] = \tan^{-1}\frac{28561}{28561}$$

$$= \tan^{-1}(1) = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$$

18. Show that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$ **and justify why the other value** $\frac{4+\sqrt{7}}{3}$ **is ignored?**

Sol. We have, $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3}$

$\therefore\ LHS=\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)\right]$

$Let\ \frac{1}{2}\sin^{-1}\frac{3}{4}=\theta \Rightarrow \sin^{-1}\frac{3}{4}=2\theta$

$\Rightarrow \sin 2\theta=\frac{3}{4}\Rightarrow \frac{2\tan\theta}{1+\tan^2\theta}=\frac{3}{4}$

$\Rightarrow 3+3\tan^2\theta=8\tan\theta$

$\Rightarrow 3\tan^2\theta-8\tan\theta+3=0$

Let $\tan\theta=y$

$\therefore 3y^2+8y+3=0$

$\Rightarrow\ y=\frac{+8\pm\sqrt{64-4\times3\times3}}{2\times3}=\frac{8\pm\sqrt{28}}{6}$

$=\frac{2\left[4\pm\sqrt{7}\right]}{2.3}$

$\Rightarrow \tan\theta=\frac{4\pm\sqrt{7}}{3}$

$\Rightarrow \theta=\tan^{-1}\left[\frac{4\pm\sqrt{7}}{3}\right]$

$\left\{but\ \frac{4+\sqrt{7}}{3}>\frac{1}{2}.\frac{\pi}{2}, \sin ce \max\left[\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)\right]=1\right\}$

$\therefore\ LHS=\tan\tan^{-1}\left(\frac{4-\sqrt{7}}{3}\right)=\frac{4-\sqrt{7}}{3}$ = RHS

Note *Since,* $-\frac{\pi}{2}\le\sin^{-1}\frac{3}{4}\le\frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{4}\le\frac{1}{2}\sin^{-1}\frac{3}{4}\le\frac{\pi}{4}$

$\therefore\ \tan\left(\frac{-\pi}{4}\right)\le\tan\frac{1}{2}\left(\sin^{-1}\frac{3}{4}\right)\le\tan\frac{\pi}{4}$

$\Rightarrow -1\le\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)\le 1$

19. If $a_1, a_2, a_3, \ldots\ldots\ldots\ldots\ldots\ldots a_n$ is an arithmetic progression with common difference d, then evaluate the following expression.

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\tan^{-1}\left(\frac{d}{1+a_3a_4}\right)+\ldots.+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right]$$

Sol. We have, $a_1 = a_1 a_2 = a + d, a_3 = a + 2d$

And $d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \ldots = a_n - a_{n-1}$

Given that,

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\tan^{-1}\left(\frac{d}{1+a_3a_4}\right)+\ldots.+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right]$$

$$= \tan\left[\tan^{-1}\frac{a_2-a_1}{1+a_2.a_1}+\tan^{-1}\frac{a_3-a_2}{1+a_3.a_2}+\ldots.+\tan^{-1}\frac{a_n-a_{n-1}}{1+a_n.a_{n-1}}\right]$$

$$= \tan\left[\left(\tan^{-1}a_2-\tan^{-1}a_1\right)+\left(\tan^{-1}a_3-\tan^{-1}a_2\right)+\ldots..+\left(\tan^{-1}a_n-\tan^{-1}a_{n-1}\right)\right]$$

$$= \tan\left[\tan^{-1}a_n-\tan^{-1}a_1\right]$$

$$= \tan\left[\tan^{-1}\frac{a_n-a_l}{1+a_n.a_1}\right] \quad \left[\because \tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= \frac{a_n-a_1}{1+a_n.a_1} \quad \left[\because \tan\left(\tan^{-1}x\right)=x\right]$$