Chapter 9

MOTION IN THREE DIMENSIONS

End of Art 129
EXAMPLES

1. In Art. 128, $V^2 = 2gl\cos\alpha$, and $\therefore\ T = 3mg\cos\theta$.

2. Here $\alpha = \beta$ and $\dot\theta = 0$ when $\theta = 90°$. Hence Art. 127 gives

$$0 = \frac{V^2\sin^2\beta}{l^2}\left(\frac{1}{\sin^2\beta} - 1\right) - \frac{2g}{l}\cos\beta, \text{ so that } V^2 = 2lg\sec\beta.$$

3. Here $l = \frac{a}{\sqrt2}$; $V^2 = \frac{7ag}{3}$; $n^2 = \frac{7\sqrt2}{12}$; and $\cos\alpha = \frac{2\sqrt2}{3}$.

Hence $\dot\theta$ vanishes again when

$$\frac{7\sqrt2}{6}\left(\frac{2\sqrt2}{3} + \cos\theta\right) = \sin^2\theta, \text{ i.e. when } \cos\theta = -\frac{\sqrt2}{3},$$

i.e. when height above centre $= \frac{a}{\sqrt2}\cdot\frac{\sqrt2}{3} = \frac{a}{3}$.

Also $\frac{R}{m} = \frac{7ag}{3} \div \frac{a}{\sqrt2} + g\left(3\cos\theta - \frac{4\sqrt2}{3}\right) = g\,[3\cos\theta + \sqrt2]$, and thus R is just zero at the highest point of the path.

4. Art. 125 gives $\ddot\theta - \sin\theta\cos\theta\dot\phi^2 = 0$, and $\frac{1}{\sin\theta}\frac{d}{dt}(\sin^2\theta\dot\phi) = 0$.

$$\therefore\ \dot\phi\sin^2\theta = C, \text{ and } \ddot\theta = C^2\frac{\cos\theta}{\sin^3\theta}. \text{ Hence } \dot\theta^2 = C^2\left(\frac{1}{\sin^2\beta} - \frac{1}{\sin^2\theta}\right).$$

$$\therefore\ \frac{d\phi}{d\theta} = \frac{\sin\beta}{\sin\theta\sqrt{\sin^2\theta - \sin^2\beta}} = \frac{\operatorname{cosec}^2\theta}{\sqrt{\cot^2\beta - \cot^2\theta}}.$$

$$\therefore\ \phi = \cos^{-1}\frac{\cot\theta}{\cot\beta}, \text{ if } \phi = 0 \text{ when } \theta = \beta, \text{ etc.}$$

5. Here $\dot\phi\sin^2\theta = \text{const.} = \frac{V\cos\beta}{a}$. Hence

$$\ddot\theta = \frac{V^2\cos^2\beta}{a^2}\frac{\cos\theta}{\sin^3\theta} - \frac{g\sin\theta}{a}, \text{ so that } \dot\theta^2 = -\frac{V^2\cos^2\beta}{a^2\sin^2\theta} + \frac{2g\cos\theta}{a} + \frac{V^2}{a^2}.$$

$$\therefore\ \frac{R}{m} = a\dot\theta^2 + a\sin^2\theta\dot\phi^2 + g\cos\theta = \frac{V^2}{a} + 3g\cos\theta.$$

$\dot\theta$ vanishes when $\cos^2\beta = \sin^2\theta\left[1 + \frac{2ga}{V^2}\cos\theta\right]$, and hence R just vanishes then, if this equation is satisfied by $\cos\theta = -\frac{V^2}{3ga}$, *i.e.* if

$$\cos^2\beta = \left(1 - \frac{V^4}{9g^2a^2}\right)\times\frac{1}{3}, \text{ i.e. if } \frac{V^4}{9g^2a^2} + 2 = 3\sin^2\beta.$$

6. $\alpha = 90°$, and $V = l\omega$. Hence $\dot\theta^2 = \omega^2\left(1 - \frac{1}{\sin^2\theta}\right) + \frac{2g}{a}\cos\theta$.

$\therefore\ \dot z^2 = a^2\sin^2\theta\dot\theta^2 = -a^2\omega^2\cos^2\theta + 2ga\cos\theta\sin^2\theta = 2gz\left[1 - \frac{\omega^2}{2g}z - \frac{z^2}{a^2}\right]$

$$= 2gz - \omega^2 z^2, \text{ since } g \div \omega^2 a \text{ is small.}$$

$$\therefore\ t = \int_0^z \frac{dz}{\sqrt{2gz - \omega^2 z^2}} = \left[\frac{1}{\omega}\cos^{-1}\frac{\omega^2 z - g}{g}\right]_0^z = \frac{1}{\omega}\cos^{-1}\frac{\omega^2 z - g}{g} - \frac{\pi}{\omega}.$$

$$\therefore\ \omega^2 z = g(1 - \cos\omega t) = 2g\sin^2\frac{\omega t}{2}.$$

7. As in Art. 125, putting $\theta = \alpha$ and $\dot\phi = \omega$, we have

$$\ddot r - r\sin^2\alpha\omega^2 = -g\cos\alpha,\quad -r\sin\alpha\cos\alpha\omega^2 = -\frac{R}{m} + g\sin\alpha,$$

and $\frac{1}{r\sin\alpha}\frac{d}{dt}(r^2\sin^2\alpha\omega) = S$, where R and S are the reactions of the tube in and perpendicular to the vertical plane through the tube.

Hence $\dot r^2 = r^2\omega^2\sin^2\alpha - 2g\cos\alpha r + \frac{g^2\cot^2\alpha}{\omega^2} = \omega^2\sin^2\alpha\left[\frac{g\cos\alpha}{\omega^2\sin^2\alpha} - r\right]^2$.

$$\therefore\ \omega t\sin\alpha = -\log\left(\frac{g\cos\alpha}{\omega^2\sin^2\alpha} - r\right) + \log\frac{g\cos\alpha}{\omega^2\sin^2\alpha}.$$

$$\therefore\ r = \frac{g\cos\alpha}{\omega^2\sin^2\alpha}\left[1 - e^{-\omega\sin\alpha t}\right].$$

Also $\frac{R}{m} = g\sin\alpha + r\sin\alpha\cos\alpha\omega^2 =$ etc., and so also S on substitution.

8. $\ddot r - r\sin^2\alpha\dot\phi^2 = -g\cos\alpha$, and $\frac{1}{r\sin\alpha}\frac{d}{dt}(r^2\sin^2\alpha\dot\phi) = 0$.

$$\therefore\ r^2\dot\phi = \text{const.} = \frac{h}{\sin\alpha\cos\alpha}\sqrt{\frac{2gh}{n^2+n}} = C.$$

$\therefore\ \ddot r = \frac{C^2\sin^2\alpha}{r^3} - g\cos\alpha$, and $\dot r^2 = -\frac{C^2\sin^2\alpha}{r^2} - 2gr\cos\alpha + \frac{C^2\sin^2\alpha}{h^2\sec^2\alpha} + 2gh$.

$\dot r$ is again zero when $0 = -\frac{2gh^3}{r^2\cos^2\alpha(n^2+n)} - 2gr\cos\alpha + \frac{2gh}{n^2+n} + 2gh$,

i.e. when $r\cos\alpha = \frac{h}{n}$.

9. We have $m(\ddot r - r\sin^2\alpha\dot\phi^2) = -T_1 - mg\cos\alpha$,

$$\frac{1}{r\sin\alpha}\frac{d}{dt}(r^2\sin^2\alpha\dot\phi) = 0, \text{ and } M\frac{d^2}{dt^2}(l - r) = Mg - T_1.$$

$$\therefore\ (M+m)\ddot r - mr\sin^2\alpha\dot\phi^2 = -mg\cos\alpha - Mg, \text{ and } r^2\dot\phi = C.$$

For the steady motion $\ddot r = 0$, $r = d$, $m\omega^2 d\sin^2\alpha = g(M + m\cos\alpha)$, $T = \frac{2\pi}{\omega}$, and $C = d^2\omega$.

Put $r = d + \xi$, where ξ is small, so that $\dot\phi = \omega\left(1 - \frac{2\xi}{d}\right)$, and we have

$$(M+m)\ddot\xi = -3m\omega^2\sin^2\alpha\xi.$$

$$\therefore\ \text{required time} = 2\pi\sqrt{\frac{M+m}{3m\omega^2\sin^2\alpha}} = T\operatorname{cosec}\alpha\sqrt{\frac{M+m}{3m}}.$$

10. $\ddot{r} - r\sin\alpha\cos\alpha\dot{\phi}^2 = -g\cos\alpha$, and $\frac{1}{r}\frac{d}{dt}(r^2\dot{\phi}\sin^2\alpha) = 0$.

$$\therefore\ r^2\dot{\phi} = \text{const.} = l^2\omega, \text{ and } \ddot{r} - \frac{l^4\omega^2\sin\alpha\cos\alpha}{r^3} = -g\cos\alpha.$$

For steady motion, $\ddot{r} = 0$, $r = l$, and $\therefore\ l\omega^2\sin\alpha\cos\alpha = g\cos\alpha$.

Put $r = l + \xi$, and we have $\ddot{\xi} = g\cos\alpha\left[\left(1+\frac{\xi}{l}\right)^{-3} - 1\right] = -\frac{3g\cos\alpha}{l}\xi$, etc.

11. For the steady motion $m_1\omega^2 l = (m_2 + m_3)g$, and $\cos\alpha = \frac{m_1}{m_2+m_3}$.

After m_3 has dropped off, we have

$$m_2(-\ddot{r}) = m_2 g - T, \text{ and } m_1[\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2] = m_1 g\cos\theta - T,$$

so that
$$T\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = g(1 + \cos\theta) + r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2.$$

Also
$$\frac{1}{r\sin\theta}(r^2\sin^2\theta\dot{\phi}) = 0, \text{ so that } r^2\dot{\phi}\sin^2\theta = l^2\omega\sin^2\alpha.$$

$$\therefore \text{ initially } T = \frac{m_1 m_2}{m_1 + m_2}[g(1 + \cos\alpha) + l\sin^2\alpha\omega^2], \text{ since } \dot{\theta} \text{ is small,}$$

$$= \frac{m_2(m_1 + m_2 + m_3)}{m_1 + m_2}g = (m_2 + m_3)g - \frac{m_2 m_3}{m_1 + m_2}g, \text{ etc.}$$

12. $\dot{\phi} = \lambda$; $\dot{\theta} = \dot{\phi}\cot\alpha\sin\theta = \lambda\cot\alpha\sin\theta$, and $\dot{r} = 0$.

$\therefore\ \beta$ = acceleration along the radius

$$= \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2 = -a\lambda^2\sin^2\theta\operatorname{cosec}^2\alpha,$$

γ = acceleration perpendicular to the radius

$$= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) - r\sin\theta\cos\theta\dot{\phi}^2 = a\lambda^2\sin\theta\cos\theta(\cot^2\alpha - 1),$$

and δ = acceleration perpendicular to the plane of β and γ

$$= \frac{1}{r\sin\theta}\frac{d}{dt}(r^2\sin^2\theta\dot{\phi}) = 2a\lambda^2\sin\theta\cos\theta\cot^2\alpha.$$

Hence $\sqrt{\gamma^2 + \delta^2} = a\lambda^2\sin\theta\cos\theta(\cot^2\alpha + 1) = a\lambda^2\operatorname{cosec}^2\alpha \,.\, \sin\theta\cos\theta$.

$\therefore\ \sqrt{\beta^2 + \gamma^2 + \delta^2} = a\lambda^2\operatorname{cosec}^2\alpha \,.\, \sin\theta$, which $\propto$ cos (latitude) and

$$\frac{\sqrt{\gamma^2 + \delta^2}}{\beta} = -\cot\theta = -\tan(\text{latitude}). \text{ Hence, etc.}$$

13. $\ddot{r} - r\sin^2\alpha\dot{\phi}^2 = -F\sin\alpha$, $-r\sin\alpha\cos\alpha\dot{\phi}^2 = \frac{R}{m} - F\cos\alpha$,

and $\frac{\sin\alpha}{r}\frac{d}{dt}(r^2\dot{\phi}) = 0$. Also we are given that $\frac{r\sin\alpha\dot{\phi}}{\dot{r}} = \tan\beta$.

Hence $r^2\dot{\phi} = \text{constant} = \frac{aV\sin\beta}{\sin\alpha}$, where $r = a$ initially.

$$\therefore\ \dot{r} = \frac{aV\cos\beta}{r}, \text{ and } \ddot{r} = -\frac{a^2V^2\cos^2\beta}{r^3}.$$

$$\therefore\ F\sin\alpha = \frac{a^2V^2\cos^2\beta}{r^3} + \frac{a^2V^2\sin^2\beta}{r^3} = \frac{a^2V^2}{r^3}.$$

Hence the force acts towards the axis and $= \frac{\mu}{r^3}$, where $V = \frac{\sqrt{\mu\sin\alpha}}{a}$.

Also
$$\frac{R}{m} = F\cos\alpha - r\sin\alpha\cos\alpha\dot{\phi}^2 = F\cos\alpha\cos^2\beta.$$

14. If V be the constant velocity and β the angle it makes with the plane, then

$$\dot{r}\cos\alpha = V\sin\beta, \text{ and } \sqrt{\dot{r}^2\sin^2\alpha + r^2\sin^2\alpha\dot{\phi}^2} = V\cos\beta.$$

$$\therefore\ r\sin\alpha\dot{\phi} = \frac{V}{\cos\alpha}\sqrt{\cos^2\alpha - \sin^2\beta} = VK, \text{ say.}$$

Then $\quad F_1 = \text{acceleration along } r = \ddot{r} - r\sin^2\alpha\dot{\phi}^2 = -\frac{V^2 . K^2}{r},$

$$F_2 = \text{acceleration perpendicular to } r = -r\sin\alpha\cos\alpha\dot{\phi}^2 = -\frac{V^2K^2}{r}\cot\alpha,$$

and $F_3 = \text{acceleration perpendicular to } F_1 \text{ and } F_2 = \frac{\sin\alpha}{r}\frac{d}{dt}(r^2\dot{\phi}) = \frac{VK\dot{r}}{r}$

$$= \frac{V^2K}{r} . \frac{\sin\beta}{\cos\alpha}.$$

Clearly $F_1\cos\alpha - F_2\sin\alpha = 0$, so that the acceleration is perpendicular to the axis, and $\sqrt{F_1^2 + F_2^2 + F_3^2} \propto \frac{1}{r}$, etc.

15. $\ddot{r} - r\sin^2\alpha\dot{\phi}^2 = \frac{\mu}{r^2}$, and $\frac{1}{r\sin\alpha}\frac{d}{dt}(r^2\sin^2\alpha\dot{\phi}) = 0.$

$$\therefore\ r^2\dot{\phi} = \text{const.} = \frac{1}{\sin^2\alpha} . c\sqrt{\frac{2\mu\sin^3\alpha}{c^3}} = \sqrt{\frac{2\mu}{c\sin\alpha}}.$$

$$\therefore\ \ddot{r} = \frac{2\mu\sin\alpha}{cr^3} + \frac{\mu}{r^2}. \quad \therefore\ \dot{r}^2 = -\frac{2\mu\sin\alpha}{cr^2} - \frac{2\mu}{3r^3} + \frac{8}{3}\frac{\mu\sin^2\alpha}{c^2},$$

since $\dot{r} = 0$ when $r = \frac{c}{\sin\alpha}$. Put $r = \frac{\xi}{\sin\alpha}$ and we have

$$\dot{\xi}^2 = \frac{2\mu\sin^5\alpha}{3c^2\xi^3}\left[4\xi^3 - 3c^2\xi - c^3\right] = \frac{2\mu\sin^5\alpha}{3c^2\xi^3}(\xi - c)(2\xi + c)^2.$$

Also $\quad \dot{\phi} = \sqrt{\frac{2\mu\sin^3\alpha}{c}} . \frac{1}{\xi^2}. \quad \therefore\ \frac{d\xi}{d\phi} = \frac{(2\xi + c)\sqrt{\xi(\xi - c)}}{c\sqrt{3}}\sin\alpha.$

Put $\quad \xi = \frac{c}{1 - 3y^2}. \quad \therefore\ \phi . \sin\alpha = \int\frac{2dy}{1 - y^2} = \log\frac{1 + y}{1 - y}.$

$$\therefore\ y = \tanh\frac{\phi\sin\alpha}{2} \text{ and } 3\tanh^2\left(\frac{\phi\sin\alpha}{2}\right) = 3y^2 = 1 - \frac{c}{\xi}.$$

16. Let $\xi = A\sin nt$, where A is small. Then

$$-l\dot{\theta}^2 - l\sin^2\theta\dot{\phi}^2 + \ddot{\xi}\cos\theta = g\cos\theta - \frac{T}{m},$$

$$l\ddot{\theta} - l\sin\theta\cos\theta\dot{\phi}^2 - \ddot{\xi}\sin\theta = -g\sin\theta,$$

and $\quad \frac{1}{\sin\theta}\frac{d}{dt}(\sin^2\theta\dot{\phi}) = 0$, so that $\dot{\phi}\sin^2\theta = C$,

and $\quad \therefore\ l\ddot{\theta} = \frac{lC^2\cos\theta}{\sin^3\theta} - An^2\sin nt\sin\theta - g\sin\theta.$

For steady motion, $A = 0$, $\ddot{\theta} = 0$ and $\theta = \alpha$. $\therefore\ C^2 = \frac{g}{l}\frac{\sin^4\alpha}{\cos\alpha}.$

For the ensuing motion put $\theta=\alpha+\psi$, where ψ is small. Then

$$l\ddot{\psi}=g\sin\alpha\,(1-\psi\tan\alpha)(1+\psi\cot\alpha)^{-3}-g(\sin\alpha+\psi\cos\alpha)-n^2A\sin\alpha\sin nt$$

$$=-g\psi\,\frac{1+3\cos^2\alpha}{\cos\alpha}-n^2A\sin\alpha\sin nt.$$

$$\therefore\ \psi=B\cos pt+D\sin pt-\frac{n^2A\sin\alpha}{l}\,\frac{\sin nt}{p^2-n^2},$$

$$=\frac{n^2A\sin\alpha}{l(p^2-n^2)}\left[\frac{n}{p}\sin pt-\sin nt\right],\ \text{where } p^2=\frac{g(1+3\cos^2\alpha)}{l\cos\alpha},$$

since ψ and $\dot{\psi}$ vanish when $t=0$.

Also $$\dot{\phi}=\frac{C}{\sin^2\theta}=\sqrt{\frac{g}{l\cos\alpha}}\,[1-2\psi\cot\alpha],\ \text{etc.}$$

The motion becomes unstable if $n^2=p^2=\dfrac{g(1+3\cos^2\alpha)}{l\cos\alpha}$, and then

$$\psi=B\cos pt+D\sin pt+\frac{pA\sin\alpha}{2l}\,t\cos pt=\frac{A\sin\alpha}{2l}[-\sin pt+pt\cos pt].$$

17. $$-a\dot{\theta}^2-a\sin^2\theta\dot{\phi}^2=-\frac{R}{m}+\frac{\mu\sin\theta}{\cos^4\theta}\,.\sin\theta,$$

$$a\ddot{\theta}-a\cos\theta\sin\theta\dot{\phi}^2=\frac{\mu\sin\theta}{\cos^4\theta}\,.\cos\theta,\ \text{and}\ \frac{1}{\sin\theta}(\dot{\phi}\sin^2\theta)=0.$$

$$\therefore\ \dot{\phi}\sin^2\theta=\text{const.}=\sin\gamma\sqrt{\frac{\mu}{a}}\,.\sec\gamma=\sqrt{\frac{\mu}{a}}\,.\tan\gamma.$$

$$\therefore\ a\ddot{\theta}=\frac{\mu\sin\theta}{\cos^3\theta}+\frac{\mu\tan^2\gamma\cos\theta}{\sin^3\theta}.$$

$$\therefore\ a\dot{\theta}^2=\mu\left[\frac{1}{\cos^2\theta}-\frac{1}{\cos^2\gamma}\right]-\mu\tan^2\gamma\left[\frac{1}{\sin^2\theta}-\frac{1}{\sin^2\gamma}\right]=\mu\,\frac{\sin^2\theta-\sin^2\gamma}{\sin^2\theta\cos^2\theta\cos^2\gamma}.$$

$$\therefore\ \left(\frac{d\theta}{d\phi}\right)^2=\frac{1}{\sin^2\gamma}\tan^2\theta\,(\sin^2\theta-\sin^2\gamma).$$

$$\therefore\ \phi=\int\frac{\sin\gamma\cos\theta\,d\theta}{\sin\theta\sqrt{\sin^2\theta-\sin^2\gamma}}=\cos^{-1}\left(\frac{\sin\gamma}{\sin\theta}\right),\ \text{if}\ \phi=0\ \text{when}\ \theta=\gamma.$$

$\therefore\ \cos\phi\sin\theta=\sin\gamma$, *i.e.* $x=a\sin\gamma$, which is a small circle.

Also $$\frac{R}{m}=\frac{\mu\sin^2\theta}{\cos^4\theta}+\frac{\mu(\sin^2\theta-\sin^2\gamma)}{\sin^2\theta\cos^2\theta\cos^2\gamma}+\frac{\mu\tan^2\gamma}{\sin^2\theta}=\frac{\mu}{\cos^4\theta}.$$

18. $$-a\dot{\theta}^2-a\sin^2\theta\dot{\phi}^2=-\frac{R}{m}+\frac{F}{m}\cos\theta,$$

$$a\ddot{\theta}-a\sin\theta\cos\theta\dot{\phi}^2=-\frac{F}{m}\sin\theta,\ \text{and}\ \frac{1}{\sin\theta}\frac{d}{dt}(\sin^2\theta\dot{\phi})=0.$$

Also $\dfrac{a\sin\theta\dot{\phi}}{a\dot{\theta}}=\tan\alpha$, since the path is a rhumb line.

Hence $\dot{\phi}\sin^2\theta=\text{const.}=A$, and then $\dot{\theta}=\dfrac{A\cot\alpha}{\sin\theta}$.

$$\therefore\ \frac{F}{m}\sin\theta=-a\ddot{\theta}+a\sin\theta\cos\theta\dot{\phi}^2=\frac{A^2a\cot^2\alpha\cos\theta}{\sin^3\theta}+\frac{A^2a\cos\theta}{\sin^3\theta}.$$

$$\therefore\ F=\frac{mA^2a}{\sin^2\alpha}\,\frac{\cos\theta}{\sin^4\theta},\ \text{etc.}$$

19. $a\dot{\theta}^2 + a\sin^2\theta\dot{\phi}^2 = -\frac{R}{m} + \frac{\mu}{a^3\sin^3\theta}\sin\theta,$

$$a\ddot{\theta} - a\cos\theta\sin\theta\dot{\phi}^2 = -\frac{\mu}{a^3\sin^3\theta}\cos\theta, \text{ and } \frac{1}{\sin\theta}\frac{d}{dt}(\dot{\phi}\sin^2\theta) = 0.$$

$$\therefore\ \dot{\phi}\sin^2\theta = C, \text{ and } a\ddot{\theta} = \frac{aC^2\cos\theta}{\sin^3\theta} - \frac{\mu\cos\theta}{a^3\sin^3\theta}.$$

$$\therefore\ \dot{\theta}^2 = -\left(C^2 - \frac{\mu}{a^4}\right)\cdot\frac{1}{\sin^2\theta} + D.$$

The path will cut the meridians at a constant angle β, if

$$\frac{\dot{\phi}\sin\theta}{\dot{\theta}} = \tan\beta, \text{ i.e. if } D + \left(\frac{\mu}{a^4} - C^2\right)\cdot\frac{1}{\sin^2\theta} = \frac{C^2\cot^2\beta}{\sin^2\theta},$$

i.e. if $D = 0$, and $C = \frac{\sqrt{\mu}\sin\beta}{a^2}$.

Also, if γ be the initial value of θ, then initially

$$\dot{\theta} = \frac{\sqrt{\mu}}{a^2}\frac{\cos\beta}{\sin\gamma} \text{ and } \dot{\phi}\sin\gamma = \frac{\sqrt{\mu}}{a^2}\frac{\sin\beta}{\sin\gamma}, \text{ so that } V = \frac{\sqrt{\mu}}{a\sin\gamma}.$$

20. $a\ddot{\theta} - a\sin\theta\cos\theta\dot{\phi}^2 = -P\cos\theta = -\mu a^n\sin^n\theta\cos\theta$, and

$$\frac{1}{\sin\theta}\frac{d}{dt}(\sin^2\theta\dot{\phi}) = 0, \text{ so that } \dot{\phi}\sin^2\theta = \frac{V}{a},$$

The first equation, as in previous questions, then gives

$$\dot{\theta}^2 = -\frac{V^2}{a^2}\cot^2\theta + \frac{2\mu a^{n-1}}{n+1}(1 - \sin^{n+1}\theta) + \omega^2,$$

where ω is the small value of $\dot{\theta}$ given at the disturbance.

Put $\theta = \frac{\pi}{2} + \psi$, and we have $\dot{\psi}^2 = \omega^2 - \psi^2 \cdot \frac{V^2 - \mu a^{n+1}}{a^2} = \omega^2 - B^2\psi^2$.

$$\therefore\ t = \frac{1}{B}\sin^{-1}\frac{B\psi}{\omega}, \text{ and } \psi = \frac{\omega}{B}\sin Bt.$$

Also $$\dot{\phi} = \frac{V}{a\sin^2\theta} = \frac{V}{a}(1 + \psi^2) = \frac{V}{a}\left[1 + \frac{\omega^2}{2B^2} - \frac{\omega^2}{2B^2}\cos 2Bt\right].$$

$$\therefore\ \phi = \frac{V}{a}\left[\left(1 + \frac{\omega^2}{2B^2}\right)t - \frac{\omega^2}{4B^3}\sin 2Bt\right].$$

The new path cuts the old one when $\theta = \frac{\pi}{2}$, *i.e.* when $\psi = 0$, *i.e.* when $Bt = 0, \pi, 2\pi$, etc., and then

$$\phi = \frac{2\pi}{m}, \text{ if } \frac{2\pi}{m} = \frac{V}{a}\left[\left(1 + \frac{\omega^2}{2B^2}\right)\frac{\pi}{B}\right] = \frac{V\pi}{aB} \text{ approx.,}$$

so that $$m^2 = 4\cdot\left[1 - \frac{\mu a^{n+1}}{V^2}\right].$$

21. $\ddot{r} - r\sin^2\alpha\dot{\phi}^2 = -\frac{\mu}{r^2}$, and $\frac{\sin\alpha}{r}\frac{d}{dt}(r^2\dot{\phi}) = 0$,

$$\therefore\ r^2\dot{\phi} = C, \text{ and } \ddot{r} = \frac{C^2\sin^2\alpha}{r^3} - \frac{\mu}{r^2}.$$

Hence $\dot{r}^2 = -\frac{C^2\sin^2\alpha}{r^2} + \frac{2\mu}{r} + B$, so that $\left(\frac{dr}{d\phi}\right)^2 = \frac{r^2}{C^2}[Br^2 + 2\mu r - C^2\sin^2\alpha]$.

If $d\psi$ is the angle between two consecutive generators, then

$$r\,d\psi = r \sin\alpha\, d\phi.$$

On putting $r = \frac{C \sin\alpha}{\xi}$, we hence have $\left(\frac{d\xi}{d\psi}\right)^2 = B + \frac{2\mu\xi}{C\sin\alpha} - \xi^2$,

so that
$$\psi = \sin^{-1}\frac{\xi - \frac{\mu}{C\sin\alpha}}{\sqrt{B + \frac{\mu^2}{C^2\sin^2\alpha}}} + \delta,$$

i.e.
$$C\sin\alpha = r\left[\frac{\mu}{C\sin\alpha} + \sqrt{B + \frac{\mu^2}{C^2\sin^2\alpha}}\,\sin(\psi - \delta)\right],$$

which is the polar equation of a conic section.

23. $\ddot{r} - r\sin^2\alpha\dot{\phi}^2 = \mu\left(\frac{a\cos^2\alpha}{r^3} - \frac{1}{2r^2}\right)$, and $\frac{\sin\alpha}{r}\frac{d}{dt}(r^2\dot{\phi}) = 0$.

$$\therefore\ r^2\dot{\phi} = \text{const.} = \frac{a}{\sin\alpha}\cdot\sqrt{\frac{\mu}{a}}\sin\alpha = \sqrt{\mu a}.$$

$$\therefore\ \ddot{r} = \frac{\mu a}{r^3} - \frac{\mu}{2r^2}, \text{ and } \dot{r}^2 = -\frac{\mu a}{r^2} + \frac{\mu}{r}.$$

Hence $\left(\frac{dr}{d\phi}\right)^2 = \frac{r^2(r-a)}{a}$. $\quad\therefore\ \phi = \cos^{-1}\frac{2a-r}{r}$, if $\phi = 0$ when $r = a$.

$$\therefore\ \frac{2a}{r} = 1 + \cos\phi, \text{ so that } x\cos\alpha + z\sin\alpha = 2a\cos\alpha\sin\alpha.$$

This is a plane section inclined at $\frac{\pi}{2} + \alpha$ to Ox, and it is therefore parallel to the other generator in the plane of xz. Hence, etc.

24. The equation to the plane of the curve is $lx + my + nz = na$, or for points on the cone we have

$$\frac{a}{r} = \frac{l}{n}\cos\phi\sin\alpha + \frac{m}{n}\sin\phi\sin\alpha + \cos\alpha = \cos\alpha + A\cos(\phi - \beta).$$

If the initial plane from which ϕ is measured is properly chosen, this becomes

$$\frac{a}{r} = \cos\alpha + A\cos\phi. \quad \text{.............................(1)}$$

The equations of motion are

$$\ddot{r} - r\sin^2\alpha\dot{\phi}^2 = -F, \text{ and } \frac{d}{dt}(r^2\dot{\phi}) = 0. \quad \therefore\ r^2\dot{\phi} = C.$$

Hence (1) gives

$$\dot{r} = \frac{AC}{a}\sin\phi, \text{ and } \ddot{r} = \frac{AC^2}{a}\cdot\frac{\cos\phi}{r^2}.$$

$$\therefore\ F = \frac{C^2\sin^2\alpha}{r^3} - \frac{AC^2}{ar^2}\cdot\frac{1}{A}\left(\frac{a}{r} - \cos\alpha\right) = \frac{C^2\cos\alpha}{a}\left(\frac{1}{r^2} - \frac{a\cos\alpha}{r^3}\right).$$

25. By Art. 126, we have, if ψ is the inclination to the transverse plane,

$$a\dot{\phi}^2=\frac{R}{m},\ a\ddot{\phi}=-\mu\frac{R}{m}\cos\psi,\text{ and }\ddot{z}=-\mu\frac{R}{m}\sin\psi.$$

$$\therefore\ \frac{a\ddot{\phi}}{\ddot{z}}=\cot\psi=\frac{a\dot{\phi}}{\dot{z}}.\quad \therefore\ \log\dot{z}=\log\dot{\phi}+\text{const.}\quad \therefore\ \dot{z}=A\dot{\phi}=a\tan\alpha\dot{\phi},$$

and so the path cuts the generators at the same angle $\frac{\pi}{2}-\alpha$.

Also $$\frac{\ddot{\phi}}{\dot{\phi}^2}=-\mu\cos\psi=-\mu\cos\alpha.$$

$$\therefore\ \frac{1}{\dot{\phi}}=\mu\cos\alpha t+\frac{a}{V\cos\alpha}.\quad \therefore\ \mu\cos\alpha\phi=\log\left[1+\frac{\mu V\cos^2\alpha}{a}t\right].$$

Now $$a\,d\phi=ds\cos\psi=ds\cos\alpha.\quad \therefore\ s=\frac{a}{\cos\alpha}\phi=\text{etc.}$$

End of Art 133
EXAMPLES

1. $x=a\cos\theta$, $y=a\sin\theta$, and $z=a\theta\tan\alpha$.

$$\therefore\ 2g(z-z_0)=\dot{x}^2+\dot{y}^2+\dot{z}^2=a^2\dot{\theta}^2\sec^2\alpha.$$

$$\therefore\ t\sqrt{\frac{2g\sin\alpha\cos\alpha}{a}}=\int_{\theta_0}^{\theta_0+2\pi}\frac{d\theta}{\sqrt{\theta-\theta_0}}=\left[2(\theta-\theta_0)^{\frac{1}{2}}\right]_{\theta_0}^{2\pi+\theta_0}=2\,.\sqrt{2\pi}.$$

2. $v^2=A-\int 2\mu(x\,dx+y\,dy+z\,dz)=A-\mu a^2(1+\theta^2\tan^2\alpha)=V_0^2-\mu a^2\theta^2\tan^2\alpha.$

We easily have $\frac{d^2x}{ds^2}=-\frac{\cos^2\alpha\cos\theta}{a}$, $\frac{d^2y}{ds^2}=-\frac{\cos^2\alpha\sin\theta}{a}$ and $\frac{d^2z}{ds^2}=0$, so that the direction cosines of the principal normal are $\cos\theta$, $\sin\theta$, and 0. Also $\rho=\frac{a}{\cos^2\alpha}$. Hence $\frac{v^2}{\rho}=-\frac{R}{m}+\mu x\cos\theta+\mu y\sin\theta=-\frac{R}{m}+\mu a.$

$\therefore\ \frac{R}{m}=\mu a-\frac{V_0^2\cos^2\alpha}{a}+\mu a\sin^2\alpha\,.\,\theta^2$. Now R can only vanish when it is least, *i.e.* when $\theta=0$, and hence V_0, the greatest velocity, $=a\sqrt{\mu}\sec\alpha$.

3. With the notation of Art. 133, we have $\rho^2=4az$. Then $\rho^2\dot{\phi}=\sqrt{4ah}\,V$. The equation of energy gives $\dot{s}^2+\rho^2\dot{\phi}^2=V^2+2g(h-z)$, and $\dot{s}$ is zero again when $\frac{h}{z}.V^2=V^2+2g(h-z)$, *i.e.* when $z=\frac{V^2}{2g}$.

Again $$\dot{z}^2+\dot{\rho}^2=\dot{s}^2=V^2+2gh-2gz-V^2.\frac{4ah}{\rho^2}.$$

$$\therefore\ \dot{z}^2=\frac{z}{z+a}\left[2g(h-z)+V^2\frac{z-h}{z}\right]=\frac{z-h}{z+a}(V^2-2gz)\ldots\ldots\ldots(1)$$

Also resolving parallel to the axis, we have $m\ddot{z}=R\frac{2a}{\sqrt{4a^2+4az}}-mg$.

$$\therefore\ \frac{R}{m}=\sqrt{\frac{a+z}{a}}\left[g+\frac{1}{2}(V^2-2gz)\frac{a+h}{(z+a)^2}-g\frac{z-h}{z+a}\right]$$

$$=\frac{(a+h)(V^2+2ga)}{2\sqrt{a}(z+a)^{\frac{3}{2}}}.\ \text{Hence, etc.}$$

4. Art. 133 gives $\rho^2\dot{\phi}=bV$, and

$$V^2+2g\left[\frac{b^2}{4a}-z\right]=\dot{\rho}^2+\dot{z}^2+\rho^2\dot{\phi}^2=\dot{\rho}^2\left(1+\frac{\rho^2}{4a^2}\right)+\frac{b^2V^2}{\rho^2}.$$

$$\therefore\ \dot{\rho}^2(\rho^2+4a^2)=2ga(b^2-\rho^2)-4a^2V^2\left(\frac{b^2}{\rho^2}-1\right).$$

$$\therefore\ \ddot{\rho}(\rho^2+4a^2)+\rho\dot{\rho}^2=-2ga\rho+\frac{4a^2V^2b^2}{\rho^3}.$$

Now the motion is steady, *i.e.* $\ddot{\rho}=0$, when $\rho=b$. $\therefore\ 2aV^2=gb^2$.
Put $\rho=b+\xi$, where ψ is small, and we have

$$\ddot{\rho}(4a^2+b^2)=-2ga(b+\xi)+2gab\left(1+\frac{\xi}{b}\right)^{-3}=-8ga\xi, \text{ etc.}$$

5. By Art. 133 $\rho^2\dot{\phi}=C$, and $2\int_{z_0}^{z}F\,dz=\dot{s}^2+\rho^2\dot{\phi}^2=\rho^2\dot{\phi}^2(1+\cot^2\beta)$, since the particle crosses the meridians at a constant angle β.

$$\therefore\ 2\int_{z_0}^{z}F\,dz=\frac{C^2}{\rho^2\sin^2\beta}=\frac{C^2}{4a\sin^2\beta z}.$$

$$\therefore\ F=-\frac{C^2}{8a\sin^2\beta z^2}=-\frac{2aC^2}{\sin^4\beta}\cdot\frac{1}{\rho^4}.$$

6. $\rho^2\dot{\phi}=C$, and the equation of Energy gives

$$\tfrac{1}{2}[\dot{\rho}^2+\dot{z}^2+\rho^2\dot{\phi}^2]=\tfrac{1}{2}V^2-\int\frac{\mu}{\rho^3}d\rho=\tfrac{1}{2}V^2+\frac{\mu}{2\rho^2}-\frac{\mu}{2\rho_0^2}.$$

Hence, since $\rho^2=4az$, $\dot{\rho}^2\left(1+\frac{\rho^2}{4a^2}\right)+\frac{C^2}{\rho^2}=V^2-\frac{\mu}{\rho_0^2}+\frac{\mu}{\rho^2}$.

If $V^2=\frac{\mu}{\rho_0^2}$, this gives $\rho\dot{\rho}\sqrt{\rho^2+4a^2}=2a\sqrt{\mu-C^2}$.

$$\therefore\ \frac{d\rho}{d\phi}\cdot\frac{\sqrt{\rho^2+4a^2}}{\rho}=\frac{2a\sqrt{\mu-C^2}}{C}=\text{const.}=B.$$

If we put $\rho=2a\tan\psi$, we have $\frac{B\phi}{2a}=\int\frac{d\psi}{\sin\psi\cos^2\psi}=d\psi\left[\frac{1}{\sin\psi}+\frac{\sin\psi}{\cos^2\psi}\right]$

$$=\tfrac{1}{2}\log\frac{1-\cos\psi}{1+\cos\psi}+\frac{1}{\cos\psi}=\frac{\sqrt{\rho^2+4a^2}}{2a}+\tfrac{1}{2}\log\frac{\sqrt{\rho^2+4a^2}-2a}{\sqrt{\rho^2+4a^2}+2a}, \text{ etc.}$$

7. $z=\rho\cot\alpha$ in this case, so that the last equation of Art. 133 gives

$$\frac{1}{\rho^4}\left(\frac{d\rho}{d\phi}\right)^2\frac{1}{\sin^2\alpha}=C-\frac{1}{\rho^2}=\frac{1}{a^2}-\frac{1}{\rho^2}.$$

Putting $\rho=\frac{1}{u}$, we have $\phi\sin\alpha=\int\frac{a\,du}{\sqrt{1-a^2u^2}}=\sin^{-1}au$, if $\phi\sin\alpha=\frac{\pi}{2}$ when $r=a$, *i.e.* $r\sin(n\phi)=a$, where $n=\sin\alpha$.

8. $x^2\dot{\phi}=h$, and $\dot{s}^2+x^2\dot{\phi}^2=A+2g(y-y_0)$.

The curve will cross the meridians at a constant angle β if $\dot{s}=x\dot{\phi}\cot\beta$, and then

$$\frac{1}{\sin^2\beta}\cdot\frac{h^2}{x^2}=A-2gy_0+2g\frac{a^3}{x^2}.$$

This is satisfied if $A=2gy_0$, and $h^2=2ga^3\sin^2\beta$.

9. If ψ is the inclination of the normal to the vertical, then

$$\cot\psi = \frac{dr}{dz} = \frac{4z^2}{9ar}.$$

Hence, when $\psi = \frac{\pi}{4}$, $z = \frac{3a}{2}$ and $r = a$; and when $\psi = \frac{\pi}{3}$, $z = \frac{a}{2}$ and $r = \frac{a}{3\sqrt{3}}$.

The equations of Art. 133 give

$$r^2\dot{\phi} = h = V.a, \text{ and } \dot{z}^2 + \dot{r}^2 + r^2\dot{\phi}^2 = V^2 - 2g\left(z - \frac{3a}{2}\right),$$

i.e.

$$\dot{z}^2\left[1 + \frac{2z}{3a}\right] = V^2 - \frac{27V^2a^3}{8z^3} - 2g\left(z - \frac{3a}{2}\right),$$

and

$$\therefore \ddot{z}\left[1 + \frac{2z}{3a}\right] + \frac{1}{3a}\dot{z}^2 = \frac{81}{16}\frac{V^2a^3}{z^4} - g.$$

Hence, when $z = \frac{a}{2}$, $\dot{z}^2 = \frac{3}{2}(ga - 13V^2)$, and $\ddot{z} = \frac{525V^2}{8a} - \frac{9g}{8}$.

If the particle leave the curve at this point, then $\ddot{z}$ must $= -g$, and hence $V^2 = \frac{ag}{525}$.

Chapter 9

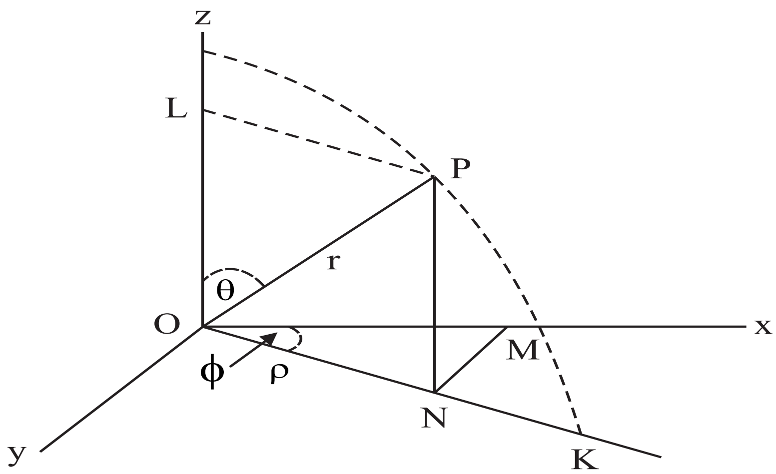
MOTION IN THREE DIMENSIONS

125. *To find the accelerations of a particle in terms of polar coordinates.*

Let the coordinates of any point P be r, θ, and ϕ, where r is the distance of P from a fixed origin O, θ is the angle that OP makes with a fixed axis Oz, and ϕ is the angle that the plane zOP makes with a fixed plane zOx.

Draw PN perpendicular to the plane xOy and let $ON = \rho$.

Then the accelerations of P are $\dfrac{d^2x}{dt^2}, \dfrac{d^2y}{dt^2}$ and $\dfrac{d^2z}{dt^2}$, where, x, y and z are the coordinates of P.



Since the polar coordinates of N, which is always in the plane xOy, are ρ and π, its accelerations are, as in Art. 49,

$\dfrac{d^2\rho}{dt^2} - \rho\left(\dfrac{d\phi}{dt}\right)^2$ along ON, and

$\dfrac{1}{\rho}\dfrac{d}{dt}\left(\rho^2\dfrac{d\phi}{dt}\right)^2$ perpendicuar to ON.

Also the acceleration of P relative to N is $\dfrac{d^2z}{dt^2}$ along NP.

Hence the accelerations of P are

$\dfrac{d^2\rho}{dt^2} - \rho\left(\dfrac{d\phi}{dt}\right)^2$ along LP,

$\dfrac{1}{\rho}\dfrac{d}{dt}\left(\rho^2\dfrac{d\phi}{dt}\right)$ perpendicuar to the plane zPK, and $\dfrac{d^2z}{dt^2}$ parallel to Oz.

Now, since $z = r\cos\theta$ and $\rho = r\sin\theta$, it follows, as in Art. 50, that accelerations $\dfrac{d^2z}{dt^2}$ and $\dfrac{d^2\rho}{dt^2}$, along and perpendicular to Oz in the plane zPK, are equivalent $\dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2$ along OP and $\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)$ perpendicular to OP in the plane zPK.

Also the acceleration $-\rho\left(r^2\dfrac{d\theta}{dt}\right)^2$ along LP is equivalent to $-\rho\sin\theta\left(\dfrac{d\phi}{dt}\right)^2$ along OP and $-\rho\cos\theta\left(\dfrac{d\phi}{dt}\right)^2$ perpendicular to OP.

Hence if α, β, γ be the accelerations of P respectively along OP, perpendicular to OP in the plane zPK in the direction of θ increasing, and perpendicular to the plane zPK in the direction of ϕ increasing, we have

$$\begin{aligned}\alpha &= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - \rho\sin\theta\left(\frac{d\phi}{dt}\right)^2 \\ &= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2 \qquad ...(1),\end{aligned}$$

$$\beta = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) - \rho\cos\theta\left(\frac{d\phi}{dt}\right)^2$$

$$= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) - r\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2 \qquad ...(2),$$

and $$\gamma = \frac{1}{\rho}\frac{d}{dt}\left(r^2\frac{d\phi}{dt}\right) = \frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\sin^2\theta\frac{d\phi}{dt}\right) \qquad ...(3).$$

126. Cylindrical coordinates.

It is sometimes convenient to refer the motion of P to the coordinates z, ρ, and ϕ, which are called cylindrical coordinates.

As in the previous article the accelerations are then $\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2$ along LP, $\frac{1}{\rho}\frac{d}{dt}\left(\rho^2\frac{d\phi}{dt}\right)$ perpendicular to the plane zPK, and $\frac{d^2z}{dt^2}$ parallel to Oz.

127. *A particle is attached to one end of a string, of length l, the other end of which is tied to a fixed point O. When the string is inclined at an acute angle α to the downward-drawn vertical the particle is projected horizontally and perpendicular to the string with a velocity V; to find the resulting motion.*

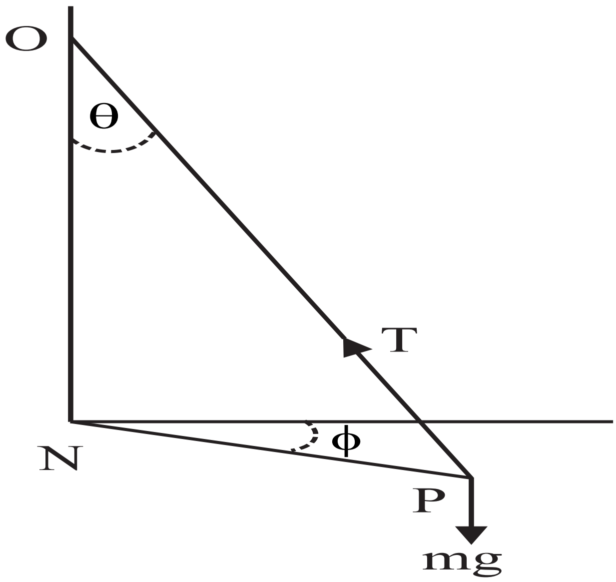
In the expressions (1), (2) and (3) of Art. 125 for the accelerations we here have $r = l$.

The equations of motion are thus

$$-l\ \dot{\theta}^2 - l\ \sin^2\theta\,\dot{\phi}^2 = -\frac{T}{m} + g\cos\theta \qquad ...(1),$$

$$l\ \ddot{\theta}^2 - l\ \cos\theta\sin\theta\,\dot{\phi}^2 = -g\sin\theta \qquad ...(2),$$

and $$\frac{1}{\sin\theta}\frac{d}{dt}(\sin^2\theta\,\dot{\phi}) = 0 \qquad ...(3).$$



The last equation gives

$$\sin^2\theta\,\dot{\phi} = \text{ constant } = \sin^2\alpha[\dot{\phi}]_0 = \frac{V\sin\alpha}{l} \qquad ...(4).$$

On substituting for ϕ in (2), we have

$$\ddot{\theta} - \frac{V^2\sin^2\alpha}{l^2}\frac{\cos\theta}{\sin^3\theta} = -\frac{g}{l}\sin\theta \qquad ...(5).$$

$$\therefore\quad \dot{\theta}^2 + \frac{V^2\sin^2\alpha}{l^2}.\frac{1}{\sin^2\theta} = \frac{2g}{l}\cos\theta + A,$$

$$\text{where}\quad 0 + \frac{V^2\sin^2\alpha}{l^2}.\frac{1}{\sin^2\alpha} = \frac{2g}{l}\cos\alpha + A$$

$$\therefore\quad \dot{\theta}^2 = \frac{V^2\sin^2\alpha}{l^2}.\left[\frac{1}{\sin^2\alpha} - \frac{1}{\sin^2\theta}\right] - \frac{2g}{l}(\cos\alpha - \cos\theta) \quad ...(6)$$

$$= \frac{2g}{l}(\cos\alpha - \cos\theta)\left(2n^2\frac{\cos\alpha + \cos\theta}{\sin^2\theta} - 1\right), \quad \text{where}\quad V^2 = 4lgn^2.$$

Hence $\dot{\theta}$ again zero when $2n^2(\cos\alpha + \cos\theta) = \sin^2\theta$,

i.e. when $\cos\theta = -n^2 \pm \sqrt{1 - 2n^2\cos\alpha + n^4}$.

The lower sign gives an inadmissible value for θ. The only inclination at which $\dot{\theta}$ again vanishes is when $\theta = \theta_1$,

where $\cos\theta_1 = -n^2 + \sqrt{1 - 2n^2\cos\alpha + n^4}$.

The motion is therefore confined between values α and θ_1, of θ.

The motion of the particle is always above or below the starting point, according as $\theta_1 \lesseqgtr \alpha$,

i.e. according as $\cos\theta_1 \lesseqgtr \alpha$,

i.e. " " $\sqrt{1 - 2n^2\cos\alpha + n^4} \lesseqgtr n^2 + \cos\alpha$,

i.e. " " $1 - 2n^2\cos\alpha \lesseqgtr \cos^2\alpha + 2n^2\cos\alpha$,

i.e. " " $n^2 \gtreqless \dfrac{\sin^2\alpha}{4\cos\alpha}$,

i.e. " " $V^2 \gtreqless lg\sin\alpha\tan\alpha$

The tension of the string at any instant is now given by equation (1). In the foregoing it is assumed that T does not vanish during the motion.

The square of the velocity at any instant

$$= (l\ \dot{\theta})^2 + (l\ \sin\theta\,\dot{\phi})^2 = l^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta).$$

Hence the Principle of Energy gives

$$\frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) = \frac{1}{2}mV^2 - mgl(\cos\alpha - \cos\theta).$$

[On substituting for $\dot{\phi}$ from (4) we have equation (6).]

(1) then gives

$$\begin{aligned}\frac{T}{m} &= g\cos\theta + \frac{(vel.)^2}{l} \\ &= g\cos\theta + \frac{V^2 - 2gl(\cos\alpha - \cos\theta)}{l} \\ &= \frac{V^2}{l} + g(3\cos\theta - 2\cos\alpha).\end{aligned}$$

128. In the previous example $\ddot{\theta}$ is zero when $\theta = a.$, *i.e.* the particle revolves at a constant depth below the centre O as in the ordinary conical pendulum, if $V^2 = gl\dfrac{\sin^2\alpha}{\cos\alpha}$.

Suppose the particle to have been projected with this velocity, and when it is revolving steadily let it receive a small displacement in the plane NOP, so that the value of $\dot{\phi}$ was not instantaneously altered. Putting $\theta = \alpha + \psi$, where Ψ is small, the equation (5) of the last article gives

$$\begin{aligned}\ddot{\psi} &= \frac{g\sin^4\alpha}{l\cos\alpha}\frac{\cos(\alpha+\psi)}{\sin^3(\alpha+\psi)} - \frac{g}{l}\sin(\alpha+\Psi) \\ &= \frac{g\sin\alpha}{l}\left[\frac{1-\Psi\tan\alpha}{(1+\Psi\cot\alpha)^3} - (1+\Psi\cot\alpha)\right],\end{aligned}$$

neglecting squares of Ψ,

$$\begin{aligned} &= -\frac{g\sin\alpha}{l}\psi(\tan\alpha + 4\cot\alpha) \\ &= -\frac{g}{l}\frac{1+3\cos^2\alpha}{\cos\alpha}\psi,\end{aligned}$$

so that the time of a small oscillation about the position of relative equilibrium is $2\pi\sqrt{\dfrac{l}{g}\dfrac{\cos\alpha}{1+3\cos^2\alpha}}$.

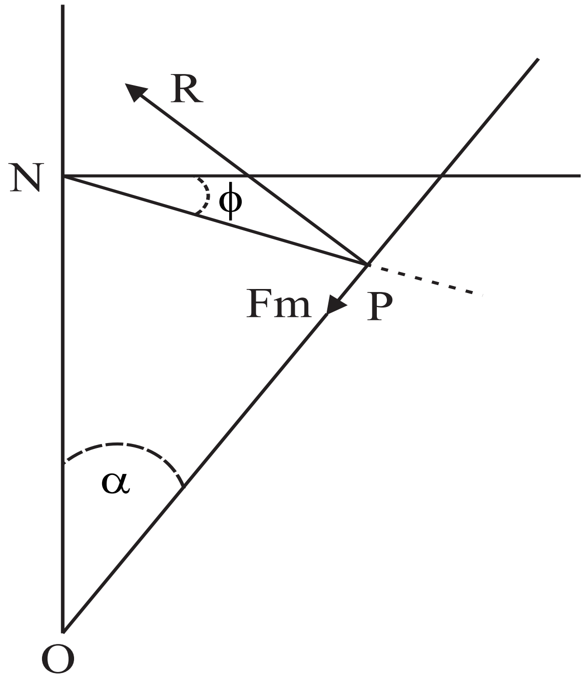
Again, from (4), on putting $\theta = \alpha + \psi$, we have

$$\dot{\phi} = \sqrt{\frac{g}{l\cos\alpha}\,\frac{1}{(1+\psi\cot\alpha)^2}} = \sqrt{\frac{g}{l\cos\alpha}}[1-2\psi\cot\alpha],$$

so that during the oscillation there is a small change in the value of ϕ whose period is the same as that of Ψ.

129. *A particle moves on the inner surface of a smooth cone, of vertical angle* 2α*, being acted on by a force towards the vertex of the cone, and its direction of motion always cuts the generators at a constant angle* β*; find the motion and the law of force.*

Let $F.m$ he the force, where m is the mass of the particle, and R the reaction of the cone. Then in the accelerations of Art. 125 we have $\theta = \alpha$ and therefore $\dot{\theta} = 0$.



Hence the equations of motion are

$$\frac{d^2r}{dt^2} - r\sin^2\alpha\left(\frac{d\phi}{dt}\right)^2 = -F \qquad ...(1),$$

$$-r\sin\alpha\cos\alpha\left(\frac{d\phi}{dt}\right)^2 = -\frac{R}{m} \qquad ...(2),$$

and $$\frac{\sin\alpha}{r}\frac{d}{dt}\left(r^2\frac{d\phi}{dt}\right) = 0 \qquad ...(3).$$

Also, since the direction of motion always cuts OP at an angle β,

$$\therefore \quad \frac{r\sin\alpha\,\dot{\phi}}{\dot{r}} = \tan\beta \qquad ...(4).$$

(3) gives $$r^2\frac{d\phi}{dt} = \text{ constant } = A \qquad ...(5),$$

and therefore, from (4), $$\frac{dr}{dt} = \sin\alpha\cot\beta.\frac{A}{r} \qquad ...(6).$$

Substituting in (1), we have

$$-F = -\sin^2\alpha\cot^2\beta.\frac{A^2}{r^3} - \sin^2\alpha.\frac{A^2}{r^3},$$

i.e. $$F = \frac{A^2\sin^2\alpha}{\sin^2\beta}.\frac{1}{r^3} = \frac{\mu}{r^3} \qquad ...(7).$$

Also $v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\sin^2\alpha\left(\frac{d\phi}{dt}\right)^2 = \frac{A^2\sin^2\alpha}{r^2\sin^2\beta}$, so that $v = \frac{\sqrt{\mu}}{r}$.

Again, (2) gives $\frac{R}{m} = \frac{A^2\sin\alpha\cos\alpha}{r^3} = F\frac{\sin^2\beta\cos\alpha}{\sin\alpha}$.

From (4), the path is given by $r = r_0.e^{\sin\alpha\cot\beta.\phi}$.

EXAMPLES

1. A heavy particle moves in a smooth sphere; show that, if the velocity be that due to the level of the centre, the reaction of the surface will vary as the depth below the centre.
2. A particle is projected horizontally along the interior surface of a smooth hemisphere whose axis is vertical and whose vertex is downwards; the point of projection being at an angular distance β

from the lowest point, show that the initial velocity so that the particle may just ascend to the rim of the hemisphere is $\sqrt{2ag\sec\beta}$.

3. A heavy particle is projected horizontally along the inner surface of a smooth spherical shell of radius $\frac{a}{\sqrt{2}}$ with velocity $\sqrt{\frac{7ag}{3}}$ at a depth $\frac{2a}{3}$ below the centre. Show that it will rise to a height $\frac{a}{3}$ above the centre, and that the pressure on the sphere just vanishes at the highest point of the path.
4. A particle moves on a smooth sphere under no forces except the pressure of the surface; show that its path is given by the equation $\cot\theta = \cot\beta\cos\phi$, where θ and ϕ are its angular coordinates.
5. A heavy particle is projected with velocity V from the end of a horizontal diameter of a sphere of radius a along the inner surface, the direction of projection making an angle β with the equator. If the particle never leaves the surface, prove that

$$3\sin^2\beta < 2 + \left(\frac{V^2}{3ga}\right)^2.$$

6. A particle constrained to move on a smooth spherical surface is projected horizontally from a point at the level of the centre so that its angular velocity relative to the centre is ω. If $\omega^2 a$ be very great compared with g, show that its depth z below the level of the centre at time t is $\frac{2g}{\omega^2}\sin^2\frac{\omega t}{2}$ approximately.
7. A thin straight hollow smooth tube is always inclined at an angle α to the upward drawn vertical, and revolves with uniform velocity ω about a vertical axis which intersects it. A heavy particle is projected from the stationary point of the tube with ve-

locity $\frac{g}{\omega}\cot\alpha$; show that in time t it has described a distance $\frac{g\cos\alpha}{\omega^2\sin^2\alpha}[1-e^{\omega\sin\alpha.t}]$. Find also the reaction of the tube.

8. A smooth hollow right circular cone is placed with its vertex downward and axis vertical, and at a point on its interior surface at a height h above the vertex a particle is projected horizontally along the surface with a velocity $\sqrt{\frac{2gh}{n^2+n}}$. Show that the lowest point of its path will be at a height $\frac{h}{n}$ above the vertex of the cone.

9. A smooth circular cone, of angle $2a$, has its axis vertical and its vertex, which is pierced with a small hole, downwards. A mass M hangs at rest by a string which passes through the vertex, and a mass m attached to the upper end describes a horizontal circle on the inner surface of the cone. Find the time T of a complete revolution, and show that small oscillations about the steady motion take place in the time T cosec $\alpha\sqrt{\frac{M+m}{3m}}$.

10. A smooth conical surface is fixed with its axis vertical and vertex downwards. A particle is in steady motion on its concave side in a horizontal circle and is slightly disturbed. Show that the time of a small oscillation about this state of steady motion is $2\pi\sqrt{\frac{1}{3g\cos\alpha}}$, where α is the semi-vertical angle of the cone and l is the length of the generator to the circle of steady motion.

11. Three masses m_1, m_2 and m_3 are fastened to a string which passes through a ring, and m_1 describes a horizontal circle as a conical pendulum while m_2 and m_3 hang vertically. If m_3 drop off, show that the instaneous change of tension of the string is $\frac{gm_1m_3}{m_1+m_2}$.

12. A particle describes a rhumb-line on a sphere in such a way that its longitude increases uniformly; show that the resultant acceleration varies as the cosine of the latitude and that its direction makes with the normal an angle equal to the latitude.

    [A Rhumb-hne is a curve on the sphere cutting all the meridians at a constant angle α; its equation is $\frac{\dot{\phi}\sin\theta}{\dot{\theta}} = \tan\alpha$.]

13. A particle moves on a smooth right circular cone under a force which is always in a direction perpendicular to the axis of the cone; if the particle describe on the cone a curve which cuts all the generators at a given constant angle, find the law of force and the initial velocity, and show that at any instant the reaction of the cone is proportional to the acting force.

14. A point moves with constant velocity on a cone so that its direction of motion makes a constant angle with a plane perpendicular to the axis of the cone. Show that the resultant acceleration is perpendicular to the axis of the cone and varies inversely as the distance of the point from the axis.

15. At the vertex of a smooth cone of vertical angle $2a$, fixed with its axis vertical and vertex downwards, is a centre of repulsive force $\frac{\mu}{(\text{distance})^4}$. A weightless particle is projected horizontally with velocity $\sqrt{\frac{2\mu\sin^3\alpha}{c^3}}$ from a point, distant c from the axis, along the inside of the surface. Show that it will describe a curve on the cone whose projection on a horizontal plane is

$$1 - \frac{c}{r} = 3\tanh\left(\frac{\theta}{2}\sin\alpha\right).$$

16. Investigate the motion of a conical pendulum when disturbed from its state of steady motion by a small vertical harmonic oscillation of the point of support. Can the steady motion be rendered unstable by such a disturbance?
17. A particle moves on the inside of a smooth sphere, of radius a, under a force perpendicular to and acting from a given diameter, which equals $\mu \dfrac{\sin\theta}{\cos^4\theta}$ when the particle is at an angular distance θ from that diameter; if when the angular distance of the particle is γ, it is projected with velocity $\sqrt{\mu a}\sec\gamma$ in a direction perpendicular to the plane through itself and the given diameter, show that its path is a small circle of the sphere, and find the reaction of the sphere.
18. A particle moves on the surface of a smooth sphere along a rhumb-line, being acted on by a force parallel to the axis of the rhumb-line. Show that the force varies inversely as the fourth power of the distance from the axis and directly as the distance from the medial plane perpendicular to the axis.
19. A particle moves on the surface of a smooth sphere and is acted on by a force in the direction of the perpendicular from the particle on a diameter and equal to $\dfrac{\mu}{(\text{distance})^3}$. Show that it can be projected so that its path will cut the meridians at a constant angle.
20. A particle moves on the interior of a smooth sphere, of radius a, under a force producing an acceleration $\mu\omega^n$ along the perpendicular or drawn to a fixed diameter. It is projected with velocity V along the great circle to which this diameter is perpendicular and is slightly disturbed from its path; show that the new path will cut the old one m times in a revolution, where $m^2 = 4\left[1 - \dfrac{\mu a^{n+1}}{V^2}\right]$.

21. A particle moves on a smooth cone under the action of a force to the vertex varying inversely as the square of the distance. If the cone be developed into a plane, show that the path becomes a conic section.

22. A particle, of mass m, moves on the inner surface of a cone of revolution, whose semi-vertical angle is α, under the action of a repulsive force $\frac{\mu}{(\text{distance})^3}$ from the axis; the moment of momentum of the particle about the axis being $m\sqrt{\mu}\tan\alpha$, show that its path is an arc of a hyperbola whose eccentricity is sec α.

[With the notation of Art. 129 we obtain $\dot{\phi}^2 = \frac{\mu}{\cos^2\alpha\sin^2\alpha}.\frac{1}{r^4}$ and $\ddot{r} = \frac{\mu}{\cos^2\alpha\sin^2\alpha}.\frac{1}{r^3}$, giving $\dot{r}^2 = \frac{\mu}{\cos^2\alpha\sin^2\alpha}\left(\frac{1}{d^3} - \frac{1}{r^2}\right)$, where d is a constant. Hence $\left(\frac{dr}{d\phi}\right)^2 = r^2.\frac{r^2-d^2}{d^2}$.

Hence $\phi = \gamma - \sin^{-1}\frac{d}{r}$.

$\therefore \frac{d}{r} = \sin(\gamma-\phi) = \cos\phi$, if the initial plane for ϕ be properly chosen. This is the plane $x = d\sin\alpha$, which is a plane parallel to the axis of the cone. The locus is thus a hyperbolic section of the cone, the parallel section of which through the vertex consists of two straight lines inclined at 2α. Hence, etc.]

23. If a particle move on the inner surface of a right circular cone under the action of a force from the vertex, the law of repulsion being

$$m\mu\left[\frac{a\cos^2\alpha}{r^3} - \frac{1}{2r^2}\right],$$

where $2a$ is the vertical angle of the cone, and if it be projected from an apse at distance a with velocity $\sqrt{\frac{\mu}{a}}\sin\alpha$, show that the path will be a parabola.

[Show that the plane of the motion is parallel to a generator of the cone.]

24. A particle is constrained to move on a smooth conical surface of vertical angle $2a$, and describes a plane curve under the action of an attraction to the vertex, the plane of the orbit cutting the axis of the cone at a distance a from the vertex. Show that the attractive force must vary as $\frac{1}{r^2} - \frac{a\cos\alpha}{r^3}$.

25. A particle moves on a rough circular cylinder under the action of no external forces. Initially the particle has a velocity V in a direction making an angle α with the transverse plane of the cylinder; show that the space described in time t is

$$\frac{a\sec^2\alpha}{\mu}\log\left[1+\frac{\mu V\cos^2\alpha}{a}t\right].$$

[Use the equations of Art. 126.]

130. *A point is moving along any curve in three dimensions; to find its accelerations along (1) the tangent to the curve, (2) the principal normal, and (3) the binormal.*

If (x,y,z) be the coordinates of the point at time t, the accelerations parallel to the axes of coordinates are $\ddot{x},\ddot{y}$ and $\ddot{z}$.

Now $\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}$. $\therefore$ $\frac{d^2x}{dt^2} = \frac{dx}{ds}\frac{d^2s}{dt^2} + \frac{d^2x}{ds^2}\left(\frac{ds}{dt}\right)^2$ ...(1).

So $$\frac{d^2y}{dt^2} = \frac{dy}{ds}\frac{d^2s}{dt^2} + \frac{d^2y}{ds^2}\left(\frac{ds}{dt}\right)^2 \qquad ...(2)$$

and $$\frac{d^2z}{dt^2} = \frac{dz}{ds}\frac{d^2s}{dt^2} + \frac{d^2z}{ds^2}\left(\frac{ds}{dt}\right)^2 \qquad ...(3).$$

The direction cosines of the tangent are $\frac{dx}{ds}, \frac{dy}{ds}$ and $\frac{dz}{ds}$.
Hence the acceleration along it

$$= \frac{dx}{ds}\frac{d^2x}{dt^2} + \frac{dy}{ds}\frac{d^2y}{dt^2} + \frac{dz}{ds}\frac{d^2z}{dt^2}$$

$$= \frac{d^2s}{dt^2}\left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2\right] + \left(\frac{dy}{dt}\right)^2\left[\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} + \frac{dz}{ds}\frac{d^2z}{ds^2}\right]$$

$$= \frac{d^2s}{dt^2} \qquad ...(4),$$

since $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1$, and

$$\therefore \frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} + \frac{dz}{ds}\frac{d^2z}{ds^2} = 0.$$

The direction cosines of the principal normal are $\rho\frac{d^2x}{ds^2}, \rho\frac{d^2y}{ds^2}$ and $\rho\frac{d^2z}{ds^2}$, where ρ is the radius of curvature.
Hence the acceleration along it

$$= \rho\frac{d^2x}{ds^2}\frac{d^2x}{dt^2} + \rho\frac{d^2y}{ds^2}\frac{d^2y}{dt^2} + \rho\frac{d^2z}{ds^2}\frac{d^2z}{dt^2}$$

$$= \rho\frac{d^2s}{dt^2}\left[\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} + \frac{dz}{ds}\frac{d^2z}{ds^2}\right] + \rho\left(\frac{ds}{dt}\right)^2\left[\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right]$$

$$= \rho \left(\frac{ds}{dt}\right)^2 \times \frac{1}{\rho^2} = \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 \qquad ...(5)$$

The direction cosines of the binormal are proportional to $\frac{dy}{ds}\frac{d^2z}{ds^2} - \frac{dz}{ds}\frac{d^2y}{ds^2}$, $\frac{dz}{ds}\frac{d^2x}{ds^2} - \frac{dx}{ds}\frac{d^2z}{ds^2}$ and $\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}$.

On multiplying (1), (2) and (3) in succession by these and adding, the result is zero, *i.e.* the acceleration in the direction of the binormal vanishes.

The foregoing results might have been seen at once from equations (1), (2), (3). For if (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) are the direction cosines of the tangent, the principal normal, and the binormal, these equations may be written

$$\frac{d^2x}{dt^2} = l_1\frac{d^2s}{dt^2} + l_2\left\{\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2\right\},$$

$$\frac{d^2y}{dt^2} = m_1\frac{d^2s}{dt^2} + m_2\left\{\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2\right\}, \text{ and}$$

$$\frac{d^2z}{dt^2} = n_1\frac{d^2s}{dt^2} + n_2\left\{\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2\right\}.$$

These equations show that the accelerations along the axes are the components of an acceleration $\frac{d^2s}{dt^2}$ along the tangent, an acceleration $\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2$ along the principal normal, and nothing in the direction of the binormal.

We therefore see that, as in the case of a particle describing a plane curve, the accelerations are $\frac{d^2s}{dt^2}$, or $v\frac{dv}{ds}$, along the tangent and $\frac{v^2}{\rho}$

along the principal normal, which lies in the osculating plane of the curve.

131. *A particle moves in a curve, there being no friction, under forces such as occur in nature. Show that the change in its kinetic energy as it passes from one position to the other is independent of the path pursued and depends only on its initial and final positions.*

Let X, Y, Z be the components of the forces. By the last article, resolving along the tangent to the path, we have

$$m\frac{d^2s}{dt^2} = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}. \quad \therefore \quad \frac{1}{2}.m\left(\frac{ds}{dt}\right)^2 = \int (Xdx + Ydy + Zdz).$$

Now, by Art. 95, since the forces are such as occur in nature, *i.e.* are one-valued functions of distances from fixed points, the quantity $Xdx + Ydy + Zdz$ is the differential of some function $\phi(x, y, z)$, so that

$$\frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{d^2s}{dt^2}\right)^2 = \phi(x, y, z) + C,$$

where $\frac{1}{2}mv_0^2 = \phi(x_0, y_0, z_0) + C$, (x_0, y_0, z_0) being the starting point and v_0 the initial velocity.

Hence $\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \phi(x, y, z) - \phi(x_0, y_0, z_0)$.

The right-hand member of this equation depends only on the position of the initial point and on that of the point of the path under consideration, and is quite independent of the path pursued.

The reaction R of the curve in the direction of the principal normal is given by the equation

$$\frac{v^2}{\rho} = R, \qquad \text{where } \rho \text{ is the radius of curvature of the curve.}$$

132. **Motion on any surface.** If the particle move on a surface whose equation is $f(x,y,z)=0$, let the direction cosines at any point (x,y,z) of its path be (l_1,m_1,n_1), so that

$$\frac{l_1}{\frac{df}{dx}}=\frac{m_1}{\frac{df}{dy}}=\frac{n_1}{\frac{df}{dz}}=\frac{1}{\sqrt{\left(\frac{df}{dx}\right)^2+\left(\frac{df}{dy}\right)^2+\left(\frac{df}{dz}\right)^2}}.$$

Then, if R be the normal reaction, we have

$$m\frac{d^2x}{dt^2}=X+Rl_1,\ \ m\frac{d^2y}{dt^2}=Y+Rm_1,\ \text{and}\ m\frac{d^2z}{dt^2}=Z+Rn_1,$$

where X,Y,Z are the components of the impressed forces.

Multiplying these equations by $\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}$ and adding, we have

$$\frac{1}{2}m\frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]$$

$$=X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt};\ \text{for the coefficient of } R$$

$$=l_1\frac{dx}{dt}+m_1\frac{dy}{dt}+n_1\frac{dz}{dt}=\left(l_1\frac{dx}{ds}+m_1\frac{dy}{ds}+n_1\frac{dz}{ds}\right)\frac{ds}{dt}$$

$$=\frac{ds}{dt}\times\left(\begin{array}{l}\text{the cosine of the angle between a tangent}\\ \text{line to the surface and the normal}\end{array}\right)$$

$$=0$$

Hence, on integration,

$$\frac{1}{2}mv^2=\int(Xdx+Ydy+Zdz),$$

as in the last article.

Also, on eliminating R, the path on the surface is given by

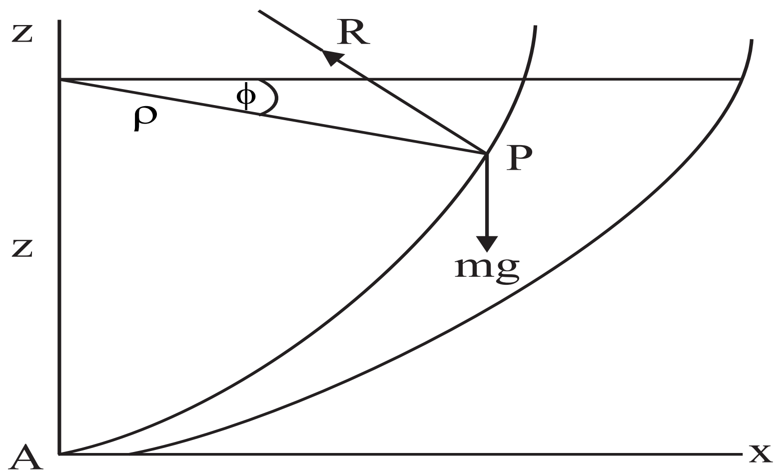
$$\frac{m\frac{d^2x}{dt^2}-X}{l_1}=\frac{m\frac{d^2y}{dt^2}-Y}{m_1}=\frac{m\frac{d^2z}{dt^2}-Z}{n_1},$$

giving two equations from which, by eliminating t, we should get a second surface cutting the first in the required path.

133. *Motion under gravity of a particle on a smooth surface of revolution whose axis is vertical.*

Use the coordinates z, ρ and ϕ of Art. 126, the z-axis being the axis of revolution of the surface. The second equation of that article gives

$$\frac{1}{\rho}\frac{d}{dt}\left(\rho^2\frac{d\phi}{dt}\right)=0. \qquad i.e.\ \ \rho^2\frac{d\phi}{dt}=\text{ constant }=h \qquad ...(1).$$



Also, if s be the arc AP measured from any fixed point A, the velocities of P are $\frac{ds}{dt}$ along the tangent at P to the generating curve, and $\rho\frac{d\phi}{dt}$ perpendicular to the plane zAP.

Hence the Principle of Energy gives

$$\frac{1}{2}\left\{\left(\frac{ds}{dt}\right)^2+\rho^2\left(\frac{d\phi}{dt}\right)^2\right\}=\text{ const. }-gz \qquad ...(2)$$

Equations (1) and (2) give the motion.

Equation (1) states that the moment of the momentum of the particle about the axis of z is constant.

By equating the forces parallel to Oz to $m\frac{d^2z}{dt^2}$, we easily have the value of the reaction R.

If the equation to the generating curve be $z = f(\rho)$ then, since

$$\dot{s}^2 = \dot{z}^2 + \dot{\rho}^2 = [1 + \{f'(\rho)\}^2]\left(\frac{d\rho}{d\phi}\right)^2 \dot{\phi}^2,$$

equation (2) easily given

$$\frac{1}{\rho^4}\left(\frac{d\rho}{d\phi}\right)^2 [1 + \{f'(\rho)\}^2] + \frac{1}{\rho^2} = \text{ constant } - \frac{2g}{h^2} f(\rho),$$

which gives the differential equation of the projection of the motion on a horizontal plane.

EXAMPLES

1. A smooth helix is placed with its axis vertical and a small bead slides down it under gravity; show that it makes its first revolution from rest in time $2\sqrt{\frac{\pi a}{g \sin\alpha \cos\alpha}}$, where α is the angle of the helix.
2. A particle, without weight, slides on a smooth helix of angle α and radius a under a force to a fixed point on the axis equal to $m\mu$ (distance). Show that the reaction of the curve cannot vanish unless the greatest velocity of the particle is $a\sqrt{\mu}$ sec α.
3. A smooth paraboloid is placed with its axis vertical and vertex downwards, the latus-rectum of the generating parabola being $4a$.

A heavy particle is projected horizontally with velocity V at a height h above the lowest point; show that the particle is again moving horizontally when its height is $\frac{V^2}{2g}$ Show also that the reaction of the paraboloid at any point is inversely proportional to the corresponding radius of curvature of the generating parabola.

4. A particle is describing steadily a circle, of radius b, on the inner surface of a smooth paraboloid of revolution whose axis is vertical and vertex downwards, and is slightly disturbed by an impulse in a plane through the axis; show that its period of oscillation about the steady motion is $\pi\sqrt{\frac{l^2+b^2}{gl}}$, where l is the semi-latus-rectum of the paraboloid.
5. A particle moving on a paraboloid of revolution under a force parallel to the axis crosses the meridians at a constant angle; show that the force varies inversely as the fourth power of the distance from the axis.
6. A particle moves on a smooth paraboloid of revolution under the action of a force directed to the axis which varies inversely as the cube of the distance from the axis; show that the equation of the projection of the path on the tangent plane at the vertex of the paraboloid may, under certain conditions of projection, be written $\sqrt{4a^2+r^2}+a\log\frac{\sqrt{4a^2+r^2}-2a}{\sqrt{4a^2+r^2}+2a}=k.\theta$. where $4a$ is the latus-rectum of the generating parabola.
7. A particle moves on a right circular cone under no forces; show that, whatever be the initial motion, the projection of the path on a plane perpendicular to the axis is one of the similar curves given by $r\sin n\theta=c$.

8. A smooth heavy particle moves on a surface of revolution formed by the revolution of the curve $x^2y = a^3$ about the axis of y, which is vertical with its positive direction downwards. Show that, if projected with a suitable speed from any point, the particle will cross all the meridians at the same angle.
9. A heavy particle is projected horizontally along a smooth surface of revolution whose equation in cylindrical coordinates is $8z^3 = 27ar^2$, the axis of z being vertical and upwards.
   Prove that, if the normal at the point of projection is inclined at 45^o to the vertical and the particle leaves the surface where the normal is inclined at 60^o to the vertical, the velocity of projection must be $\sqrt{\dfrac{ga}{525}}$.

ANSWERS WITH HINTS

Art. 129 EXAMPLES

7. Let R and S are reactions of the tube in and perpendicular to the vertical plane through tube. We get $R = m(g\sin\alpha + r\sin\alpha\cos\alpha\omega^2)$
$S = \dfrac{1}{r\sin\alpha}\dfrac{d}{dt}(r^2\sin^2\alpha\omega)$ where $r = \dfrac{g\cos\alpha}{\omega^2\sin^2\alpha}(1-e^{-\omega\sin\alpha t})$

9. $m(\ddot{r} - r\sin^2\alpha\dot{\phi}^2) = -T_1 - mg\cos\alpha$,
$\dfrac{1}{r\sin\alpha}\dfrac{d}{dt}(r^2\sin^2\alpha\,\dot{\phi}) = 0$, and $M\dfrac{d^2}{dt^2}(l-r) = Mg - T_1$

13. $\dfrac{\mu}{r^3}, \dfrac{\sqrt{\mu\sin\alpha}}{a}$

16. Unstable

17. $\dfrac{m\mu}{\cos^4\theta}$