Complex Numbers and Quadratic Equations

- A number of the form $a + ib$, where a and b are real numbers and $i = \sqrt{-1}$, is defined as a complex number.
- For the complex numbers $z = a + ib$, a is called the real part (denoted by Re z) and b is called the imaginary part (denoted by Im z) of the complex number z.

  Example: For the complex number $z = \frac{-5}{9} + i\frac{\sqrt{3}}{17}$, Re $z = \frac{-5}{9}$ and Im $z = \frac{\sqrt{3}}{17}$

- Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if $a = c$ and $b = d$.

- **Addition of complex numbers**

  Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ can be added as,
  $z_1 + z_2 = (a + ib) + (c + id) = (a + c) + \mathrm{i}(b + d)$

- **Properties of addition of complex numbers:**
  - **Closure Law:** Sum of two complex numbers is a complex number. In fact, for two complex numbers z_1 and z_2, such that $z_1 = a + ib$ and $z_2 = c + id$, we obtain $z_1 + z_2 = (a + c) + i(b + d)$.
  - **Commutative Law:** For two complex numbers z_1 and z_2,

    $z_1 + z_2 = z_2 + z_1$.

  - **Associative Law:** For any three complex numbers z_1, z_2 *and* z_3,

    $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.

  - **Existence of Additive Identity:** There exists a complex number $0 + i0$ (denoted by 0), called the additive identity or zero complex number, such that for every complex number z, $z + 0 = z$.
  - **Existence of Additive Inverse:** For every complex number $z = a+ ib$, there exists a complex number $-a + i(-b)$ [denoted by $-z$], called the additive inverse or negative of z, such that $z + (-z) = 0$.

- **Subtraction of complex numbers**

  Given any two complex numbers z_1, and z_2, the difference z_1 ⍰ z_2 is defined as
  $z_1 - z_2 = z_1 + (-z_2)$.

- **Multiplication of complex numbers:**

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ can be multiplied as,
$z_1 z_2 = (ac - bd) + i(ad + bc)$

- Properties of multiplication of complex numbers:
  - **Closure law:** The product of two complex numbers is a complex number.

    In fact, for two complex numbers z_1 and z_2, such that $z_1 = a + ib$ and $z_2 = c + id$, we obtain $z_1 z_2 = (ac - bd) + i(ad + bc)$.

  - **Commutative Law:** For any two complex numbers z_1 and z_2, $z_1z_2 = z_2z_1$.
  - **Associative Law:** For any three complex numbers z_1, z_2 and z_3,

    $(z_1z_2)\, z_3 = z_1\, (z_2z_3)$.

  - **Existence of Multiplicative Identity:** There exist a complex number $1 + i\,0$ (denoted as 1), called the multiplicative identity, such that for every complex numbers z, $z.1 = z$.
  - **Existence of Multiplicative Inverse:** For every non-zero complex number $z = a + ib$ $(a \neq 0, b \neq 0)$, we have the complex number

    $\frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}$ (denoted by $\frac{1}{z}$ or z^{-1}), called the multiplicative inverse of z, such that $z\frac{1}{z} = 1$.

    Example: The multiplicative inverse of the complex number $z = 2 – 3i$ can be found as,
    $z^{-1} = \frac{2}{(2)^2+(-3)^2} + i\frac{(-3)}{(2)^2+(-3)^2} = \frac{2}{13} - \frac{3}{13}i$

  - **Distributive Law:** For any three complex numbers z_1, z_2 and z_3,
    - $z_1\,(z_2 + z_3) = z_1z_2 + z_1z_3$
    - $(z_1 + z_2)\, z_3 = z_1z_3 + z_2z_3$

- **Division of Complex Numbers**

  Given any two complex number z_1 and z_2, where $z_2 \neq 0$, the quotient $\frac{z_1}{z_2}$ is defined as $\frac{z_1}{z_2} = z_1\frac{1}{z_2}$

  Example: For $z_1 = 1 + i$ and $z_2 = 2 – 3i$, the quotient $\frac{z_1}{z_2}$ can be found as,

$\frac{z_1}{z_2} = \frac{1+i}{2-3i}$

$= \left(1+i\right)\left(\frac{1}{2-3i}\right)$

$= \left(1+i\right)\left(\frac{2}{(2)^2+(-3)^2} + i\frac{(-3)}{(2)^2+(-3)^2}\right)$

$= \left(1+i\right)\left(\frac{2}{13} - \frac{3}{13}i\right)$

$= \left[1\times\frac{2}{13} - 1\times\left(-\frac{3}{13}\right)\right] + i\left[1\times\left(-\frac{3}{13}\right) + 1\times\frac{2}{13}\right]$

$= \frac{5}{13} - \frac{1}{13}i$

- **Property of Complex Numbers**
  - For any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$.
  - If a and b are negative real numbers, then

- **Modulus of a Complex Number**

  The modulus of a complex number $z = a + ib$, is denoted by $|z|$, and is defined as the non-negative real number $\sqrt{a^2 + b^2}$, i.e., $\left|z\right| = \sqrt{a^2 + b^2}$.

  Example 1: If $z = 2 - 3i$, then $\left|z\right| = \sqrt{(2)^2 + (-3)^2} = \sqrt{13}$

- **Conjugate of Complex Number**

  The conjugate of a complex number $z = a + ib$, is denoted by $\bar{z}$, and is defined as the complex number $a - ib$, i.e., $\bar{z} = a - ib$.

  Example 2: Find the conjugate of $\frac{2}{3+5i}$.

  Solution: We have

$$\frac{2}{3+5i}$$

$$= \frac{2}{3+5i} \times \frac{3-5i}{3-5i}$$

$$= \frac{2(3-5i)}{(3)^2 - (5i)^2}$$

$$= \frac{2(3-5i)}{9-25i^2}$$

$$= \frac{6-10i}{9+25} \qquad \left(\because i^2 = -1\right)$$

$$= \frac{6-10i}{34}$$

$$= \frac{6}{34} - \frac{10i}{34}$$

$$= \frac{3}{17} - \frac{5i}{17}$$

Thus, the conjugate of $\frac{2}{3+5i}$ is $\frac{3}{17} + \frac{5i}{17}$.

- Properties of modulus and conjugate of complex numbers:

  For any three complex numbers z, z_1, z_2,

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  - $z^{-1} = \frac{\overline{z}}{|z|^2}$ or $z.\overline{z} = |z|^2$
  - $|z_1 z_2| = |z_1||z_2|$
  - $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, provided $|z_2| \neq 0$
  - $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
  - $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$

- **Solutions of the quadratic equation when D < 0.**

  The solutions of the quadratic equation $ax^2 + bx + c = 0$, where a, b, c $\in R,\ a \neq 0$ are given by $x = \frac{-b \pm \sqrt{D}}{2a},\ where\ D = b^2 - 4ac < 0$.

  Example: Solve $2x^2 + 3ix + 2 = 0$

  Solution: Here $a = 2$, $b = 3i$ and $c = 2$

$D = b^2 - 4ac$

$= (3i)^2 - 4\times2\times2$

$= -9 - 16$

$= -25 < 0$

$\therefore x = \frac{-b \pm \sqrt{D}}{2a}$

$\Rightarrow x = \frac{-3i \pm \sqrt{-25}}{2\times2}$

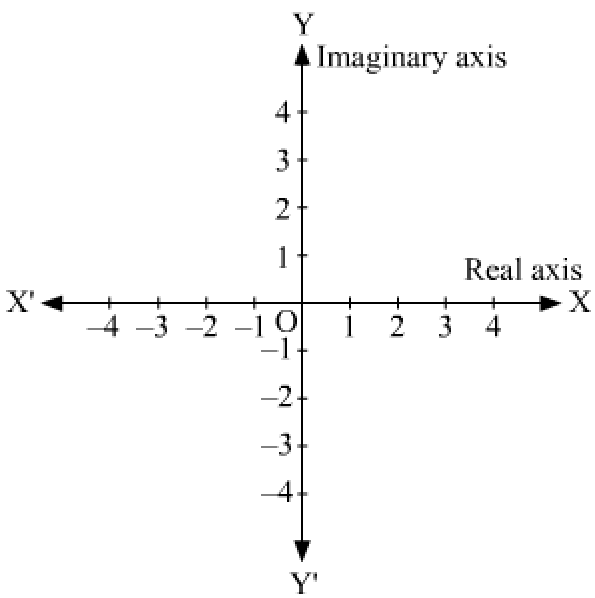
$\Rightarrow x = \frac{-3i \pm 5i}{4}$

$\Rightarrow x = \frac{-3i+5i}{4}$ or $x = \frac{-3i-5i}{4}$

$\Rightarrow x = \frac{2i}{4}$ or $x = -\frac{8i}{4}$

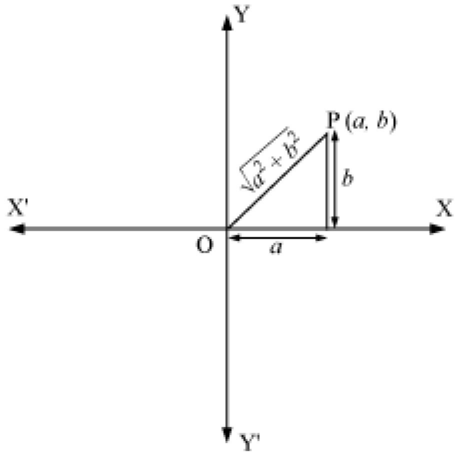
$\Rightarrow x = \frac{i}{2}$ or $x = -2i$

- **Argand plan:** Each complex number represents a unique point on Argand plane. An Argand plane is shown in the following figure.



Here, *x*-axis is known as the **real axis** and *y*-axis is known as the **imaginary axis**.

- **Complex Number on Argand plane:** The complex number $z = a + ib$ can be represented on an Argand plane as



In this figure, $OP = \sqrt{a^2 + b^2} = |z|$

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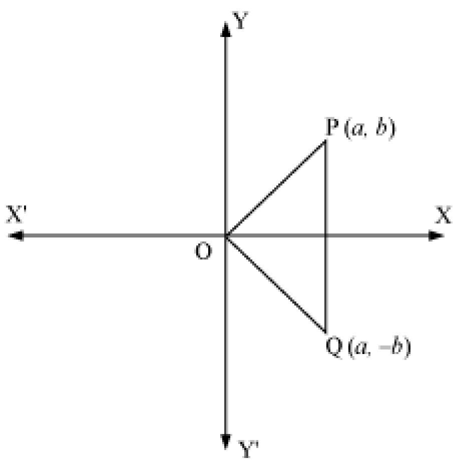
- .Thus, the modulus of a complex number z = $a + ib$ is the distance between the point P(x, y) and the origin O.

- **Conjugate of Complex Number on Argand plane:**
- The conjugate of a complex number $z = a + ib$ is $\bar{z} = a - ib$, z and $\bar{z}$ can be represented by the points P(a, b) and Q(a, −b) on the Argand plane as



Thus, on the Argand plane, the conjugate of a complex number is the mirror image of the complex number with respect to the real axis.

- **Polar representation of Complex Numbers**

The polar form of the complex number $z = x + iy$,
is $r(\cos\theta + \sin\theta)$ where $r = \sqrt{x^2 + y^2}$ (modulus of z)
and $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$ (θ is known as the argument of z).
The value of θ is such that $-\pi < \theta \le \pi$, which is called the principle argument of z.

Example 2: Represent the complex number $z = \sqrt{2} - i\sqrt{2}$ in polar form.

Solution: $z = \sqrt{2} - i\sqrt{2}$

Let $\sqrt{2} = r\cos\theta$ and $-\sqrt{2} = r\sin\theta$

By squaring and adding them, we have

$$2 + 2 = r^2\left(\cos^2\theta + \sin^2\theta\right)$$

$$\Rightarrow r^2 = 4$$

$$\Rightarrow r = \sqrt{4} = 2$$

Thus,

$$\cos\theta = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\sin\theta = \frac{\sqrt{2}}{2} = \frac{-1}{\sqrt{2}} = \sin\left(2\pi - \frac{\pi}{4}\right)$$

$$\Rightarrow \theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

Thus, the required polar form is $2\left(\cos\frac{7\pi}{4} + \sin\frac{7\pi}{4}\right)$.