

## 5. Permutations and Combinations

- **Factorial notation:** The notation  $n!$  represents the product of the first  $n$  natural numbers, i.e.,

$$n! = n \times (n-1) \times (n-2) \times \dots \times 5 \times 4 \times 3 \times 2 \times 1$$

$$0! = 1$$

**Example:**

$$\frac{12!}{8!} = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$= 12 \times 11 \times 10 \times 9$$

$$= 11880$$

- **Fundamental Principle of Counting:** If an event occurs in  $m$  different ways, following which another event occurs in  $n$  different ways, then the total number of occurrence of the events in the given order is  $m \times n$ . This is called the fundamental principle of counting.

**Example:**

Find the number of 5-letter words, with or without meaning, which can be formed out of the letters of the word MATHS, where the repetition of digits is not allowed.

**Solution:**

There are as many words as there are ways of filling 5 vacant places  $\square\square\square\square\square$  by the 5 letters.

The first place can be filled with any of the 5 letters in 5 different ways, following which the second place can be filled with any of the remaining 4 letters in 4 different ways, following which the third place can be filled in 3 different ways, following which the fourth place can be in 2 different ways, following which the fifth place can be filled in 1 way.

Thus, the number of ways in which the 5 places can be filled, by the multiplication principle, is  $5 \times 4 \times 3 \times 2 \times 1 = 120$ .

**Note:** If repetition of letters had been allowed, then the required number of words would be  $5 \times 5 \times 5 \times 5 \times 5 = 3125$ .

- **Permutation when all objects are distinct:** A permutation is an arrangement in a definite order of a number of objects taken some or all at a time.

The number of permutations of  $n$  different things taken  $r$  at a time, when

- repetition is not allowed, is  ${}^n P_r = \frac{n!}{(n-r)!}$ , where  $0 \leq r \leq n$ .
- repetition is allowed, is  $n^r$ , where  $0 \leq r \leq n$ .

**Example 1:** Twenty five students are participating in a competition. In how many ways, can the first three prizes be won in such a way that a prize cannot be shared by more than one student?

**Solution:** The total number of ways in which first three prizes can be won is the number of arrangements of 25 different things taken 3 at a time.

$$\begin{aligned}
 \text{So, required number of ways} &= {}^{25}P_3 \\
 &= \frac{25!}{(25-3)!} \\
 &= \frac{25!}{22!} \\
 &= \frac{25 \times 24 \times 23 \times 22!}{22!} \\
 &= 25 \times 24 \times 23 = 13800
 \end{aligned}$$

**Example 2:** Find the total number of four digit numbers that can be formed by using the digits 0, 2, 5, and 6?

**Solution:** A four digit number has four places i.e., units, tens, hundreds and thousands. Units, tens and hundreds place can be filled with either 0, 2, 5, or 6 where as thousands place can be filled with 2, 5 or 6 only.

Number of ways to fill the units place = 4

Number of ways to fill the tens place = 4

Number of ways to fill the hundreds place = 4

Number of ways to fill the thousands place = 3 ways.

∴ Total number of four digit numbers =  $4 \times 4 \times 4 \times 3 = 192$

- **Concept of permutations when all objects are not distinct**

- The number of permutations of  $n$  objects, when  $p$  objects are of the same kind and the rest are all different, is  $\frac{n!}{p!}$ .

- In general, the number of permutations of  $n$  objects, when  $p_1$  objects are of one kind,  $p_2$  are of the second kind, ...,  $p_k$  are of the  $k^{\text{th}}$  kind and the rest, if any, are of different kinds, is  $\frac{n!}{p_1! p_2! \dots p_k!}$ .

**Example:** Find the number of permutations of the letters of the word ARRANGEMENT.

**Solution:** Here, there are 11 objects (letters) of which there are 2A's, 2R's, 2N's, 2E's and the rest are all different.

∴ Required number of arrangements

$$\begin{aligned}
 &= \frac{11!}{2!2!2!2!} \\
 &= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 3 \times 2}{2 \times 2 \times 2 \times 2} \\
 &= 2494800
 \end{aligned}$$

## Circular Permutation

When the arrangements of  $n$  different things are around a circle, the arrangements are circular permutations. Circular arrangements are considered different when the relative order of the things is changed.

- The number of circular permutations of  $n$  different things is  $(n - 1)!$ .
- The number of circular permutations of  $n$  different things taken all at a time is  $\frac{n-1!}{2}$  if clockwise and anti-clockwise orders are not different.
- The number of circular permutations of  $n$  different things taken  $r$  at a time is  $\frac{P_{n-r}}{r}$  or  $\frac{C_{n-r}}{r} \cdot (r-1)!$ .
- The number of circular permutations of  $n$  different things taken  $r$  at a time is  $\frac{P_{n-2r}}{2r}$  or  $\frac{C_{n-2r}}{2r} \cdot (2r-1)!$  if clockwise and anti-clockwise orders are not different.

- The number of circular permutations of  $n$  things, of which  $r$  are alike, is  $\frac{n-1}{r}!$ .
- **Combinations:** The number of combinations of  $n$  different things taken  $r$  at a time is denoted by  ${}^nC_r$ , which is given by

$${}^nC_r = \frac{n!}{r!(n-r)!}, \text{ where } 0 \leq r \leq n.$$

In particular,  ${}^nC_0 = {}^nC_n = 1$

**Example 1:** A box contains 8 red bulbs and 5 blue bulbs. Determine the number of ways in which 4 red and 2 blue bulbs can be selected.

**Solution:**

It is given that a box contains 8 red bulbs and 5 blue bulbs.

Now, 4 red bulbs can be selected from 8 red bulbs in  ${}^8C_4$  number of ways, and 2 blue bulbs can be selected from 5 blue bulbs in  ${}^5C_2$  number of ways.

Hence, 4 red bulbs and 2 blue bulbs can be selected from a box containing 8 red bulbs and 5 blue bulbs in  ${}^8C_4 \times {}^5C_2$  number of ways.

Now,

$$\begin{aligned} & {}^8C_4 \times {}^5C_2 \\ &= \frac{8!}{4! \times (8-4)!} \times \frac{5!}{2! \times (5-2)!} \\ &= \frac{8!}{4! \times 4!} \times \frac{5!}{2! \times 3!} \\ &= \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} \times \frac{5 \times 4 \times 3}{2 \times 1} \\ &= 70 \times 10 \\ &= 700 \end{aligned}$$

Thus, the number of ways of selecting the bulbs is 700.

$$\bullet \quad {}^nC_{n-r} = \frac{n!}{(n-r)! [n - (n-r)]!} = \frac{n!}{(n-r)! r!} = {}^nC_r$$

In other words, selecting  $r$  objects out of  $n$  objects is the same as rejecting  $(n-r)$  objects.

$$\bullet \quad {}^nC_a = {}^nC_b \Rightarrow a = b \text{ or } a = n, \text{ i.e., } n = a + b$$

**Example 2:** If  ${}^{19}C_{3r} = {}^{19}C_{2r+4}$ , then find the value of  $r$ .

**Solution:**

$$\begin{aligned} & {}^{19}C_{3r} = {}^{19}C_{2r+4} \\ & \Rightarrow 3r + (2r + 4) = 19 \quad \text{or} \quad 3r = 2r + 4 \\ & \Rightarrow 3r + (2r + 4) = 19 \quad \Rightarrow r = 4 \\ & \Rightarrow 5r + 4 = 19 \\ & \Rightarrow 5r = 19 - 4 = 15 \\ & \Rightarrow r = 3 \end{aligned}$$

$\therefore$  The value of  $r$  is either 3 or 4.

- ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

**Example 3:** If  ${}^nC_{r-1} + {}^nC_r + {}^{n+1}C_{r+1} + {}^{n+2}C_{r+2} = {}^{n+a}C_{r+(a-1)}$ , then find the value of  $a$ .

**Solution:**

$${}^nC_{r-1} + {}^nC_r + {}^{n+1}C_{r+1} + {}^{n+2}C_{r+2}$$

$$= {}^{n+1}C_r + {}^{n+1}C_{r+1} + {}^{n+2}C_{r+2}$$

$$= {}^{n+1}C_{r+1} + {}^{n+2}C_{r+2}$$

$$= {}^{n+3}C_{r+2}$$

$$\therefore {}^{n+3}C_{r+2} = {}^{n+a}C_{r+(a-1)}$$

$$\Rightarrow a = 3$$

$$(\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r)$$