DAY SEVEN

# Binomial Theorem and Mathematical Induction

*Learning & Revision for the Day*

- Binomial Theorem
- Binomial Theorem for Positive Index
- Properties of Binomial Coefficient
- Applications of Binomial Theorem
- Binomial Theorem for Negative/Rational Index
- Principle of Mathematical Induction

## Binomial Theorem

Binomial theorem describes the algebraic expansion of powers of a binomial. According to this theorem, it is possible to expand $(x + y)^n$ into a sum involving terms of the form $ax^b y^c$, where the exponents $b$ and $c$ are non-negative integers with $b + c = n$. The coefficient $a$ of each term is a specific positive integer depending on $n$ and $b$, is known as the binomial coefficient $\binom{n}{b}$.

## Binomial Theorem for Positive Index

An algebraic expression consisting of two terms with (+) ve or (–)ve sign between them, is called binomial expression.

If $n$ is any positive integer,

then $(x + a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} a + \cdots + {}^nC_n a^n$

$$= \sum_{r=0}^{n} {}^nC_r \cdot x^{n-r} a^r,$$ where $x$ and $a$ are **real** (complex) **numbers**.

(i) The coefficient of terms equidistant from the beginning and the end, are equal.

(ii) $(x - a)^n = {}^nC_0 x^n - {}^nC_1 x^{n-1} a + \cdots + (-1)^n \, {}^nC_n a^n$

(iii) $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n$

(iv) Total number of terms in the expansion $(x + a)^n$ is $(n + 1)$.

(v) If $n$ is a positive integer, then the number of terms in $(x + y + z)^n$ is $\frac{(n+1)(n+2)}{2}$.

(vi) The number of terms in the expansion of

$$(x+a)^n+(x-a)^n=\begin{cases}\dfrac{n+2}{2}, & \text{if } n \text{ is even}\\ \dfrac{n+1}{2}, & \text{if } n \text{ is odd}\end{cases}$$

(vii) The number of terms in the expansion of

$$(x+a)^n-(x-a)^n=\begin{cases}\dfrac{n}{2}, & \text{if } n \text{ is even}\\ \dfrac{n+1}{2}, & \text{if } n \text{ is odd}\end{cases}$$

## General Term and Middle Term

(i) Let $(r+1)$th term be the **general term** in the expansion of $(x+a)^n$.

$$T_{r+1}={}^nC_r x^{n-r} a^r$$

(ii) If expansion is $(x-a)^n$, then the **general term** is

$$(-1)^r \cdot {}^nC_r \; x^{n-r} a^r.$$

(iii) The **middle term** in the expansion of $(a+x)^n$.

(a) ***Case* I** If $n$ is even, then $\left(\dfrac{n}{2}+1\right)$th term is middle term.

(b) ***Case* II** If $n$ is odd, then $\dfrac{(n+1)}{2}$th term and $\dfrac{(n+3)}{2}$th terms are middle terms.

(iv) $(p+1)^{\text{th}}$ term from end $=(n-p+1)^{\text{th}}$ term from beginning.

(v) For making a term independent of $x$ we put $r=n$ in general term of $(x+a)^n$, so we get ${}^nC_n a^n$, that is independent of $x$.

NOTE If the coefficients of $r$th, $(r+1)$th, $(r+2)$th term of $(1+x)^n$ are in AP, then $n^2-(4r+1)n+4r^2=2$

## Greatest Term

If $T_r$ and $T_{r+1}$ be the $r$th and $(r+1)$th terms in the expansion of $(1+x)^n$, then

$$\frac{T_{r+1}}{T_r}=\frac{{}^nC_r\cdot x^r}{{}^nC_{r-1}\cdot x^{r-1}}=\frac{n-r+1}{r}\cdot x$$

Let numerically, $T_{r+1}$ be the greatest term in the above expansion. Then, $T_{r+1}\ge T_r$ or $\dfrac{T_{r+1}}{T_r}\ge 1$.

$$\therefore \quad \frac{n-r+1}{r}|x|\ge 1 \text{ or } r\le\frac{(n+1)}{(1+|x|)}|x| \qquad \ldots\text{(i)}$$

(i) Now, substituting values of $n$ and $x$ in Eq. (i), we get $r\le m+f$ or $r\le m$, where $m$ is a positive integer and $f$ is a fraction such that $0<f<1$.

(ii) When $r\le m+f$, $T_{m+1}$ is the greatest term, when $r\le m$, $T_m$ and $T_{m+1}$ are the greatest terms and both are equal.

(iii) The coefficients of the middle terms in the expansion of $(a+x)^n$ are called **greatest coefficients.**

# Properties of Binomial Coefficients

In the binomial expansion of $(1+x)^n$,

$(1+x)^n={}^nC_0+{}^nC_1\cdot x+{}^nC_2\cdot x^2+\ldots+{}^nC_r\cdot x^r+\ldots {}^nC_n\cdot x^n$,

where, ${}^nC_0, {}^nC_1,\ldots,{}^nC_n$ are the coefficients of various powers of $x$ are called **binomial coefficients** and it is also written as $C_0, C_1,\ldots,C_n$ or $\binom{n}{0},\binom{n}{1},\ldots,\binom{n}{n}$

- ${}^nC_r={}^nC_{n-r}$
- ${}^nC_{r_1}={}^nC_{r_2}\Rightarrow r_1=r_2$ or $r_1+r_2=n$
- ${}^nC_r+{}^nC_{r-1}={}^{n+1}C_r$
- $\dfrac{{}^nC_r}{{}^nC_{r-1}}=\dfrac{n-r+1}{r}$
- $r\cdot {}^nC_r=n\cdot {}^{n-1}C_{r-1}$
- $\dfrac{{}^nC_r}{r+1}=\dfrac{{}^{n+1}C_{r+1}}{n+1}$
- $C_0+C_1+C_2+\ldots+C_n=2^n$
- $C_0+C_2+C_4+\ldots=C_1+C_3+C_5+\ldots=2^{n-1}$
- $C_0-C_1+C_2-C_3+\ldots+(-1)^n\cdot C_n=0$
- $C_0^2+C_1^2+C_2^2+\ldots+C_n^2={}^{2n}C_n=\dfrac{(2n)!}{(n!)^2}$
- $C_0^2-C_1^2+C_2^2-C_3^2+\ldots=\begin{cases}(-1)^{n/2}\cdot {}^nC_{n/2}, & \text{if } n \text{ is even}\\ 0, & \text{if } n \text{ is odd}\end{cases}$
- $C_0\cdot C_r+C_1\cdot C_{r+1}+\ldots+C_{n-r}\cdot C_n={}^{2n}C_{n-r}=\dfrac{(2n)!}{(n-r)!(n+r)!}$
- $C_1-2C_2+3C_3-\ldots=0$
- $C_0+2C_1+3C_2+\ldots+(n+1)\cdot C_n=(n+2)2^{n-1}$
- $C_0-C_2+C_4-C_6+\ldots=\sqrt{2^n}\cdot\cos\dfrac{n\pi}{4}$
- $C_1-C_3+C_5-C_7+\ldots=\sqrt{2^n}\cdot\sin\dfrac{n\pi}{4}$

# Applications of Binomial Theorem

### 1. $R$-$f$ Factor Relation

Here, we are going to discuss problems involving $(\sqrt{A}+B)^n=I+f$, where $I$ and $n$ are positive integers $0\le f\le 1$, $|A-B^2|=k$ and $|\sqrt{A}-B|<1$.

### 2. Divisibility Problem

In the expansion, $(1+\alpha)^n$. We can conclude that, $(1+\alpha)^n-1$ is divisible by $\alpha$, i.e. it is a multiple of $\alpha$.

### 3. Differentiability Problem

Sometimes to generalise the result we use the differentiation.

$$(1+x)^n={}^nC_0+{}^nC_1x+{}^nC_2x^2+\ldots+{}^nC_nx^n$$

On differentiating w.r.t. $x$, we get

$n(1+x)^{n-1} = 0 + {}^nC_1 + 2 \cdot x \cdot {}^nC_2 + \ldots + n \cdot {}^nC_n \cdot x^{n-1}$

Put $x = 1$, we get, $n 2^{n-1} = {}^nC_1 + 2\,{}^nC_2 + \ldots + n\,{}^nC_n$

## Binomial Theorem for Negative/Rational Index

Let $n$ be a rational number and $x$ be a real number such that $|x| < 1$, then $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$

- If $n$ is a positive integer, then $(1+x)^n$ contains $(n+1)$ terms i.e. a finite number of terms. When $n$ is any negative integer or rational number, then expansion of $(1+x)^n$ contains infinitely many terms.
- When $n$ is a positive integer, then expansion of $(1+x)^n$ is valid for all values of $x$. If $n$ is any negative integer or rational number, then expansion of $(1+x)^n$ is valid for the values of $x$ satisfying the condition $|x| < 1$.
  (i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$
  (ii) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$
  (iii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$
  (iv) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$

## Principle of Mathematical Induction

In algebra, there are certain results that are formulated in terms of $n$, where $n$ is a positive integer. Such results can be proved by a specific technique, which is known as the principle of mathematical induction.

### First Principle of Mathematical Induction

*It consists of the following three steps*

***Step* I** Actual verification of the proposition for the starting value of $i$.

***Step* II** Assuming the proposition to be true for $k$, $k \ge i$ and proving that it is true for the value $(k+1)$ which is next higher integer.

***Step* III** To combine the above two steps. Let $p(n)$ be a statement involving the natural number $n$ such that

(i) $p(1)$ is true i.e. $p(n)$ is true for $n = 1$.

(ii) $p(m+1)$ is true, whenever $p(m)$ is true

i.e. $p(m)$ is true $\Rightarrow p(m+1)$ is true. Then, $p(n)$ is true for all natural numbers $n$.

Product of $r$ consecutive integers is divisible by $r!$

DAY PRACTICE SESSION 1

# FOUNDATION QUESTIONS EXERCISE

**1** If $(1+ax)^n = 1 + 8x + 24x^3 + \ldots$, then the values of $a$ and $n$ are

(a) 2, 4 (b) 2, 3 (c) 3, 6 (d) 1, 2

**2** The coefficient of $x^n$ in the expansion of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ are in the ratio → **NCERT Exemplar**

(a) 1 : 2 (b) 1 : 3
(c) 3 : 1 (d) 2 : 1

**3** The value of $(1.002)^{12}$ upto fourth place of decimal is

(a) 1.0242 (b) 1.0245
(c) 1.0004 (d) 1.0254

**4** The coefficient of $x^4$ in the expansion of $(1+x+x^2+x^3)^n$ is

(a) ${}^nC_4$ (b) ${}^nC_4 + {}^nC_2$
(c) ${}^nC_4 + {}^nC_2 + {}^nC_2$ (d) ${}^nC_4 + {}^nC_2 + {}^nC_1 \cdot {}^nC_2$

**5** If the middle term of $\left(\frac{1}{x} + x\sin x\right)^{10}$ is equal to $7\frac{7}{8}$, then the value of $x$ is → **NCERT Exemplar**

(a) $2n\pi + \frac{\pi}{6}$ (b) $n\pi + \frac{\pi}{6}$
(c) $n\pi + (-1)^n\frac{\pi}{6}$ (d) $n\pi + (-1)\frac{\pi}{3}$

**6** If the 7th term in the binomial expansion of $\left(\frac{3}{\sqrt[3]{84}} + \sqrt{3}\ln x\right)^9$, $x > 0$ is equal to 729, then $x$ can be → **JEE Mains 2013**

(a) $e^2$ (b) $e$ (c) $e/2$ (d) $2e$

**7** If the number of terms in the expansion of $\left(1 - \frac{2}{x} + \frac{4}{x^2}\right)^n$, $x \ne 0$ is 28, then the sum of the coefficients of all the terms in this expansion, is → **JEE Mains 2016**

(a) 64 (b) 2187 (c) 243 (d) 729

**8** In the binomial expansion of $(a-b)^n$, $n \ge 5$, the sum of 5th and 6th terms is zero, then $\frac{a}{b}$ is equal to

(a) $\frac{5}{n-4}$ (b) $\frac{6}{n-5}$
(c) $\frac{n-5}{6}$ (d) $\frac{n-4}{5}$

**9** In the expansion of the following expression $1 + (1+x) + (1+x)^2 + \ldots + (1+x)^n$, the coefficient of $x^4 (0 \le k \le n)$ is

(a) ${}^{n+1}C_{k+1}$ (b) ${}^nC_k$
(c) ${}^nC_{n-k-1}$ (d) None of these

**10** The coefficient of $t^{24}$ in the expansion of $(1+t^2)^{12}(1+t^{12})(1+t^{24})$ is

(a) ${}^{12}C_6 + 2$ (b) ${}^{12}C_5$ (c) ${}^{12}C_6$ (d) ${}^{12}C_7$

**11** The coefficient of $x^{53}$ in the following expansion $\sum_{m=0}^{100} {}^{100}C_m (x-3)^{100-m} \cdot 2^m$ is

(a) ${}^{100}C_{47}$ (b) ${}^{100}C_{53}$ (c) $-{}^{100}C_{53}$ (d) $-{}^{100}C_{100}$

**12** If $p$ is a real number and if the middle term in the expansion of $\left(\frac{p}{2}+2\right)^8$ is 1120, then the value of $p$ is → **NCERT Exemplar**

(a) $\pm 3$ (b) $\pm 1$
(c) $\pm 2$ (d) None of these

**13** The constant term in the expansion of $\left(1+x+\frac{2}{x}\right)^6$, is

(a) 479 (b) 517 (c) 569 (d) 581

**14** If in the expansion of $(1+x)^m(1-x)^n$, the coefficient of $x$ and $x^2$ are 3 and –6 respectively, them $m$ is

(a) 6 (b) 9 (c) 12 (d) 24

**15** If $n$ is a positive integer, then $(\sqrt{3}+1)^{2n} - (\sqrt{3}-1)^{2n}$ is

(a) an irrational number → **AIEEE 2012**
(b) an odd positive integer
(c) an even positive integer
(d) a rational number other than positive integers

**16** If the $(r+1)$ th term in the expansion of $\left(\sqrt[3]{\frac{a}{\sqrt{b}}} + \sqrt{\frac{b}{\sqrt[3]{a}}}\right)^{21}$ has the same power of $a$ and $b$, then the value of $r$ is

(a) 9 (b) 10 (c) 8 (d) 6

**17** If $x^{2k}$ occurs in the expansion of $\left(x+\frac{1}{x^2}\right)^{n-3}$, then

(a) $n-2k$ is a multiple of 2 (b) $n-2k$ is a multiple of 3
(c) $k=0$ (d) None of these

**18** The ratio of the coefficient of $x^{15}$ to the term independent of $x$ in the expansion of $\left(x^2+\frac{2}{x}\right)^{15}$, is → **JEE Mains 2013**

(a) 7 : 16 (b) 7 : 64 (c) 1 : 4 (d) 1 : 32

**19** The greatest term in the expansion of $\sqrt{3}\left(1+\frac{1}{\sqrt{3}}\right)^{20}$ is

(a) $\binom{20}{7}\frac{1}{27}$ (b) $\binom{20}{6}\frac{1}{81}$
(c) $\frac{1}{9}\binom{20}{9}$ (d) None of these

**20** The largest term in the expansion of $(3+2x)^{50}$, where $x=\frac{1}{5}$ is

(a) 5th (b) 3th (c) 7th (d) 6th

**21** If the sum of the coefficients in the expansion of $(x-2y+3z)^n$ is 128, then the greatest coefficient in the expansion of $(1+x)^n$ is

(a) 35 (b) 20 (c) 10 (d) None of these

**22.** If for positive integers $r>1$, $n>2$, the coefficient of the $(3r)$th and $(r+2)$th powers of $x$ in the expansion of $(1+x)^{2n}$ are equal, then

(a) $n=2r$ (b) $n=3r$
(c) $n=2r+1$ (d) None of these

**23** If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n} \frac{r}{{}^nC_r}$ is equal to

(a) $(n-1)a_n$ (b) $na_n$
(c) $\frac{1}{2}na_n$ (d) None of these

**24** $\sum_{r=0}^{n} (-1)^r ({}^nC_r) \frac{1+rx}{1+nx}$ is equal to

(a) 1 (b) –1 (c) $n$ (d) 0

**25** $\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \ldots + \binom{30}{20}\binom{30}{30}$ is equal to

(a) ${}^{30}C_{11}$ (b) ${}^{60}C_{10}$ (c) ${}^{30}C_{10}$ (d) ${}^{65}C_{55}$

**26** The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \ldots + ({}^{21}C_{10} - {}^{10}C_{10})$ is → **JEE Mains 2017**

(a) $2^{21}-2^{11}$ (b) $2^{21}-2^{10}$ (c) $2^{20}-2^9$ (d) $2^{20}-2^{10}$

**27** The sum of the series ${}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \ldots + {}^{20}C_{10}$ is → **AIEEE 2007**

(a) $-{}^{20}C_{10}$ (b) $\frac{1}{2}{}^{20}C_{10}$ (c) 0 (d) ${}^{20}C_{10}$

**28** If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nX^n$, then the value of $C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n$ will be

(a) $(n+2)2^{n-1}$ (b) $(n+1)2^n$
(c) $(n+1)2^{n-1}$ (d) $(n+2)2^n$

**29** If $n > (8+3\sqrt{7})^{10}, n \in N$, then the least value of $n$ is

(a) $(8+3\sqrt{7})^{10} - (8-3\sqrt{7})^{10}$
(b) $(8+3\sqrt{7})^{10} + (8-3\sqrt{7})^{10}$
(c) $(8+3\sqrt{7})^{10} - (8-3\sqrt{7})^{10} + 1$
(d) $(8+3\sqrt{7})^{10} - (8-3\sqrt{7})^{10} - 1$

**30** $49^n + 16n - 1$ is divisible by

(a) 3 (b) 19 (c) 64 (d) 29

**31** If $A = 1000^{1000}$ and $B = (1001)^{999}$, then

(a) $A > B$ (b) $A = B$
(c) $A < B$ (d) None of these

**32** If ${}^{n-1}C_r = (k^2-3) \cdot {}^nC_{r+1}$, then $k$ belongs to

(a) $(-\infty, -2]$ (b) $[2, \infty)$ (c) $[-\sqrt{3}, \sqrt{3}]$ (d) $(\sqrt{3}, 2]$

**33** The remainder left out when $8^{2n} - (62)^{2n+1}$ is divided by 9, is

(a) 0 (b) 2 (c) 7 (d) 8

**34** If $x$ is positive, the first negative term in the expansion of $(1+x)^{27/5}$ is → **AIEEE 2003**

(a) 7th term (b) 5th term (c) 8th term (d) 6th term

**35** Let $P(n): n^2+n+1\ (n \in N)$ is an even integer. Therefore, $P(n)$ is true

(a) for $n>1$ (b) for all $n$ (c) for $n>2$ (d) None of these

**36** For all $n \in N, 1\times 1! + 2\times 2! + 3\times 3! + \ldots + n\times n!$ is equal to → **NCERT Exemplar**

(a) $(n+1)!-2$ (b) $(n+1)!$
(c) $(n+1)!-1$ (d) $(n+1)!-3$

**37** For each $n \in N$, $2^{3n}-1$ is divisible by

(a) 8 (b) 16
(c) 32 (d) None of these

**38** Let $S(k) = 1+3+5+\ldots+(2k-1) = 3+k^2$.
Then, which of the following is true? → **AIEEE 2004**

(a) $S(1)$ is correct
(b) $S(k) \Rightarrow S(k+1)$
(c) $S(k) \not\Rightarrow S(k+1)$
(d) Principle of mathematical induction can be used to prove the formula

## DAY PRACTICE SESSION 2

# PROGRESSIVE QUESTIONS EXERCISE

**1** The coefficient of $x^{2m+1}$ in the expansion of
$E = \dfrac{1}{(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots(1+x^{2m})}, |x|<1$ is

(a) 3 (b) 2 (c) 1 (d) 0

**2** $C_1 - \dfrac{C_2}{2} + \dfrac{C_3}{3} - \ldots + (-1)^{n-1}\dfrac{C_n}{n}$ is equal to

(a) $1-\dfrac{1}{2}+\dfrac{1}{3}-\ldots+\dfrac{(-1)^{n-1}}{n}$ (b) $1+\dfrac{1}{2}+\dfrac{1}{3}+\ldots+\dfrac{1}{n}$
(c) $1+\dfrac{1}{2}+\dfrac{1}{3}+\ldots+\dfrac{1}{n-1}$ (d) None of these

**3** If the coefficient of $x^5$ in $\left[ax^2+\dfrac{1}{bx}\right]^{10}$ is $a$ times and equal to the coefficient of $x^{-5}$ in $\left[ax-\dfrac{1}{b^2x^2}\right]^{10}$, then the value of $ab$ is

(a) $(b)^{-3}$ (b) $-(b)^6$ (c) $(b)^{-1}$ (d) None of these

**4** The sum of coefficients of integral powers of $x$ in the binomial expansion of $(1-2\sqrt{x})^{50}$, is → **JEE Mains 2015**

(a) $\dfrac{1}{2}(3^{50}+1)$ (b) $\dfrac{1}{2}(3^{50})$
(c) $\dfrac{1}{2}(3^{50}-1)$ (d) $\dfrac{1}{2}(2^{50}+1)$

**5** The term independent of $x$ in expansion of $\left(\dfrac{x+1}{x^{2/3}-x^{1/3}+1}-\dfrac{x-1}{x-x^{1/2}}\right)$ is → **JEE Mains 2013**

(a) 4 (b) 120 (c) 210 (d) 310

**6** If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, then $C_0^2 + C_1^2 + C_2^2 + C_3^2 + \ldots + C_n^2$ is equal to

(a) $\dfrac{n!}{n!n!}$ (b) $\dfrac{(2n)!}{n!n!}$
(c) $\dfrac{(2n)!}{n!}$ (d) None of these

**7** If $a$ and $d$ are two complex numbers, then the sum to $(n+1)$ terms of the following series
$aC_0 - (a+d)C_1 + (a+2d)C_2 - \ldots + \ldots$ is

(a) $\dfrac{a}{2^n}$ (b) $na$
(c) 0 (d) None of these

**8** $\displaystyle\sum_{p=1}^{n}\sum_{m=p}^{n}\binom{n}{m}\binom{m}{p}$ is equal to

(a) $3^n$ (b) $2^n$
(c) $3^n+2^n$ (d) $3^n-2^n$

**9** The sum of the series
$\displaystyle\sum_{r=0}^{n}(-1)^r\, {}^nC_r\left(\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\ldots+m\text{ terms}\right)$ is

(a) $\dfrac{2^{mn}-1}{2^{mn}(2^n-1)}$ (b) $\dfrac{2^{mn}-1}{2^n-1}$
(c) $\dfrac{2^{mn}+1}{2^n+1}$ (d) None of these

**10** The value of $x$, for which the 6th term in the expansion of $\left\{2^{\log_2\sqrt{9^{x-1}+7}}+\dfrac{1}{2^{(1/5)\log_2(3^{x-1}+1)}}\right\}^7$ is 84, is equal to

(a) 4 (b) 3 (c) 2 (d) 5

**11** If the last term in the binomial expansion of $\left(2^{1/3}-\dfrac{1}{\sqrt{2}}\right)^n$ is $\left(\dfrac{1}{3^{5/3}}\right)^{\log_3 8}$, then the 5th term from the beginning is

(a) 210 (b) 420
(c) 105 (d) None of these

**12** The sum of the coefficients of all odd degree terms in the expansion of
$(x+\sqrt{x^3-1})^5 + (x-\sqrt{x^3-1})^5, (x>1)$ is → **JEE Mains 2018**

(a) $-1$ (b) 0 (c) 1 (d) 2

**13** The greatest value of the term independent of $x$, as $\alpha$ varies over $R$, in the expansion of $\left(x\cos\alpha + \frac{\sin\alpha}{x}\right)^{20}$ is

(a) ${}^{20}C_{10}$ (b) ${}^{20}C_{15}$ (c) ${}^{20}C_{19}$ (d) None of these

**14** **Statement I** For each natural number $n, (n+1)^7 - n^7 - 1$ is divisible by 7.

**Statement II** For each natural number $n, n^7 - n$ is divisible by 7. **→ AIEEE 2011**

(a) Statement I is false, Statement II is true
(b) Statement I is true, Statement II is true, Statement II is correct explanation of Statement I.
(c) Statement I is true, Statement II is true; Statement II is not a correct explanation of Statement I
(d) Statement I is true, Statement II is false

**15** If the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6}:1$, then

**Statement I** The value of $n$ is 10.

**Statement II** $\dfrac{2^{\frac{n-4}{4}} \cdot 3^{-1}}{2 \cdot 3^{\frac{4+n}{4}}} = \sqrt{6}$ **→ NCERT Exemplar**

(a) Statement I is true; Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true; Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

# ANSWERS

**SESSION 1**

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|---|---|---|---|---|---|---|---|---|---|
| **1** (a) | **2** (d) | **3** (a) | **4** (d) | **5** (c) | **6** (b) | **7** (d) | **8** (d) | **9** (a) | **10** (a) |
| **11** (c) | **12** (c) | **13** (d) | **14** (c) | **15** (a) | **16** (a) | **17** (b) | **18** (d) | **19** (a) | **20** (c) |
| **21** (a) | **22** (c) | **23** (c) | **24** (d) | **25** (c) | **26** (d) | **27** (b) | **28** (a) | **29** (b) | **30** (c) |
| **31** (a) | **32** (d) | **33** (b) | **34** (c) | **35** (d) | **36** (c) | **37** (d) | **38** (b) | | |

**SESSION 2**

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|---|---|---|---|---|---|---|---|---|---|
| **1** (c) | **2** (b) | **3** (b) | **4** (a) | **5** (c) | **6** (b) | **7** (c) | **8** (d) | **9** (a) | **10** (c) |
| **11** (a) | **12** (d) | **13** (d) | **14** (b) | **15** (c) | | | | | |

# Hints and Explanations

**1** Given that, $(1 + ax)^n = 1 + 8x + 24x^2 + \ldots$

$\Rightarrow \quad 1 + \frac{n}{1}ax + \frac{n(n-1)}{1\cdot 2}a^2x^2 + \ldots$

$= 1 + 8x + 24x^2 + \ldots$

On comparing the coefficients of $x, x^2$, we get

$na = 8, \frac{n(n-1)}{1\cdot 2}a^2 = 24$

$\Rightarrow \quad na(n-1)a = 48$

$\Rightarrow \quad 8(8-a) = 48 \Rightarrow 8 - a = 6$

$\Rightarrow \quad a = 2 \Rightarrow n = 4$

**2** Coefficient of $x^n$ in $(1+x)^{2n} = {}^{2n}C_n$

and coefficient of $x^n$ in $(1+x)^{2n-1} = {}^{2n-1}C_n$

$\therefore$ Required ratio

$$= \frac{{}^{2n}C_n}{{}^{2n-1}C_n} = \frac{\frac{(2n)!}{n!n!}}{\frac{(2n-1)!}{n!(n-1)!}} = 2:1$$

**3** We have, $(1.002)^{12}$ or it can be rewritten as $(1 + 0.002)^{12}$

$\Rightarrow \quad (1.002)^{12} = 1 + {}^{12}C_1(0.002) + {}^{12}C_2(0.002)^2 + {}^{12}C_3(0.002)^3 + \ldots$

We want the answer upto 4 decimal places and as such we have left further expansion.

$\therefore (1.002)^{12} = 1 + 12(0.002) + \frac{12\cdot 11}{1\cdot 2}(0.002)^2 + \frac{12\cdot 11\cdot 10}{1\cdot 2\cdot 3}(0.002)^3 + \ldots$

$= 1 + 0.024 + 2.64\times 10^{-4} + 1.76\times 10^{-6} + \ldots$

$= 1.0242$

**4** $(1 + x + x^2 + x^3)^n = \{(1+x)^n (1+x^2)^n\}$

$= (1 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + {}^nC_4x^4 + \ldots {}^nC_nx^n)$

$(1 + {}^nC_1x^2 + {}^nC_2x^4 + \ldots + {}^nC_nx^{2n})$

Therefore, the coefficient of $x^4$

$= {}^nC_2 + {}^nC_2\,{}^nC_1 + {}^nC_4$

$= {}^nC_4 + {}^nC_2 + {}^nC_1\,{}^nC_2$

**5** $\left(\frac{1}{x} + x\sin x\right)^{10}$

Here, $n = 10$ [even]

$\Rightarrow$ Middle term $= \left(\frac{10}{2} + 1\right)$th $=$ 6th

$T_6 = {}^{10}C_5\left(\frac{1}{x}\right)^{10-5}(x\sin x)^5$

$\Rightarrow \quad 252(\sin x)^5 = 7\frac{7}{8} = \frac{63}{8}$

$\Rightarrow \quad (\sin x)^5 = \frac{1}{32} \Rightarrow \sin x = \frac{1}{2}$

$\Rightarrow \quad \sin x = \sin \pi/6$

$\therefore \quad x = n\pi + (-1)^n\frac{\pi}{6}$

**6** $T_7 = {}^9C_6\left(\frac{3}{\sqrt[3]{84}}\right)^3(\sqrt{3}\ln x)^6 = 729$

$\Rightarrow \quad \frac{84\times 3^3}{84}\times 3^3\times(\ln x)^6 = 729$

$= (\ln x)^6 = 1$

$\Rightarrow \quad x = e$

**7** Clearly number of terms in the expansion of

$\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n$ is $\frac{(n+2)(n+1)}{2}$ or ${}^{n+2}C_2$.

[assuming $\frac{1}{x}$ and $\frac{1}{x^2}$ distinct]

$\therefore \quad \frac{(n+2)(n+1)}{2}=28$

$\Rightarrow \quad (n+2)(n+1)=56=(6+1)(6+2)$

$\Rightarrow \quad n=6$

Hence, sum of coefficients

$=(1-2+4)^6=3^6=729$

**8** Since, in a binomial expansion of $(a-b)^n$, $n \geq 5$, the sum of 5th and 6th terms is equal to zero.

$\therefore \quad {}^nC_4 a^{n-4}(-b)^4+{}^nC_5 a^{n-5}(-b)^5=0$

$\Rightarrow \quad \frac{n!}{(n-4)!4!}a^{n-4}\cdot b^4 - \frac{n!}{(n-5)!5!}a^{n-5}b^5=0$

$\Rightarrow \frac{n!}{(n-5)!4!}a^{n-5}\cdot b^4\left(\frac{a}{n-4}-\frac{b}{5}\right)=0$

$\Rightarrow \quad \frac{a}{b}=\frac{n-4}{5}$

**9** The given expression is $1+(1+x)+(1+x)^2+\ldots+(1+x)^n$ being in GP.

Let, $S=1+(1+x)+(1+x)^2+\ldots+(1+x)^n$

$=\frac{(1+x)^{n+1}-1}{(1+x)-1}=x^{-1}[(1+x)^{n+1}-1]$

$\therefore$ The coefficient of $x^k$ in $S$.

= The coefficient of $x^{k+1}$ in $[(1+x)^{n+1}-1]$

$={}^{n+1}C_{k+1}$

**10** We have, $(1+t^2)^{12}(1+t^{12})(1+t^{24})$

$=(1+{}^{12}C_1t^2+{}^{12}C_2t+\ldots+{}^{12}C_6t^{12}+\ldots+{}^{12}C_{12}t^{24}+\ldots)(1+t^{12}+t^{24}+t^{36})$

$\therefore$ Coefficient of $t^{24}$ in

$(1+t^2)^{12}(1+t^{12})(1+t^{24})$

$={}^{12}C_6+{}^{12}C_{12}+1={}^{12}C_6+2$

**11** The given sigma expansion

$\sum_{m=0}^{100}{}^{100}C_m(x-3)^{100-m}\cdot 2^m$ can be written

as $[(x-3)+2]^{100}=(x-1)^{100}=(1-x)^{100}$

$\therefore$ Coefficient of $x^{53}$ in

$(1-x)^{100}=(-1)^{53}\,{}^{100}C_{53}=-{}^{100}C_{53}$

**12** Given expression is $\left(\frac{p}{2}+2\right)^8$

Here, $\quad n=8$ [even]

$\Rightarrow$ Middle term $=\left(\frac{8}{2}+1\right)$th term

$=5$ th term

$T_5={}^8C_4(p/2)^{8-4}(2^4)$

$\Rightarrow \quad \frac{8\times7\times6\times5}{4\times3\times2\times1}\times\frac{p^4}{2^4}\times2^4=1120$

$\Rightarrow \quad p^4=16$

$\Rightarrow \quad p=\pm 2$

**13** $\left(1+x+\frac{2}{x}\right)^6=1+\binom{6}{1}\left(x+\frac{2}{x}\right)+\binom{6}{2}\left(x+\frac{2}{x}\right)^2+\ldots+\binom{6}{6}\left(x+\frac{2}{x}\right)^6$

$\therefore$ Constant term

$=1+\binom{6}{2}\binom{2}{1}2^1+\binom{6}{4}\binom{4}{2}2^2+\binom{6}{6}\binom{6}{3}2^3$

$=1+60+360+160=581$

**14** $(1+x)^m(1-x)^n$

$=\left\{1+mx+\frac{m(m-1)x^2}{2!}+\ldots\right\}\left[1-nx+\frac{n(n-1)}{2!}x^2-\ldots\right]$

$=1+(m-n)x+\left[\frac{n^2-n}{2}-mn+\frac{(m^2-m)}{2}\right]x^2+\ldots$

Given, $\quad m-n=3 \Rightarrow n=m-3$

and $\quad \frac{n^2-n}{2}-mn+\frac{m^2-m}{2}=-6$

$\Rightarrow \quad \frac{(m-3)(m-4)}{2}-m(m-3)+\frac{m^2-m}{2}=-6$

$\Rightarrow \quad m^2-7m+12-2m^2+6m+m^2-m+12=0$

$\Rightarrow \quad -2m+24=0 \Rightarrow m=12$

**15** $(\sqrt{3}+1)^{2n}={}^{2n}C_0(\sqrt{3})^{2n}+{}^{2n}C_1(\sqrt{3})^{2n-1}+{}^{2n}C_2(\sqrt{3})^{2n-2}+\ldots+{}^{2n}C_{2n}(\sqrt{3})^{2n-2n}$

$(\sqrt{3}-1)^{2n}={}^{2n}C_0(\sqrt{3})^{2n}(-1)^0+{}^{2n}C_1(\sqrt{3})^{2n-1}(-1)^1+{}^{2n}C_2(\sqrt{3})^{2n-2}(-1)^2+\ldots+{}^{2n}C_{2n}(\sqrt{3})^{2n-2n}(-1)^{2n}$

Adding both the binomial expansions above, we get

$(\sqrt{3}+1)^{2n}-(\sqrt{3}-1)^{2n}=2[{}^{2n}C_1(\sqrt{3})^{2n-1}+{}^{2n}C_3(\sqrt{3})^{2n-3}+{}^{2n}C_5(\sqrt{3})^{2n-5}+\ldots+{}^{2n}C_{2n-1}(\sqrt{3})^{2n-(2n-1)}]$

which is most certainly an irrational number because of odd powers of $\sqrt{3}$ in each of the terms.

**16** $\therefore$ General term is

$T_{r+1}={}^{21}C_r\left(\sqrt[3]{\frac{a}{\sqrt{b}}}\right)^{21-r}\left(\sqrt{\frac{b}{\sqrt[3]{a}}}\right)^r$

$={}^{21}C_r\, a^{7-\frac{r}{2}}\cdot b^{\frac{2r}{3}-\frac{7}{2}}$

$\because$ Power of $a$ = Power of $b$ [given]

$\Rightarrow \quad 7-\frac{r}{2}=\frac{2}{3}r-\frac{7}{2}$

$\therefore \quad r=9$

**17** The general term in the expansion of $\left(x+\frac{1}{x^2}\right)^{n-3}$ is given by

$T_{r+1}={}^{n-3}C_r(x)^{n-3-r}\left(\frac{1}{x^2}\right)^r$

$={}^{n-3}C_r x^{n-3-3r}$

As $x^{2k}$ occurs in the expansion of $\left(x+\frac{1}{x^2}\right)^{n-3}$, we must have

$n-3-3r=2k$ for some non-negative integer $r$.

$\Rightarrow \quad 3(1+r)=n-2k$

$\Rightarrow \quad n-2k$ is a multiple of 3.

**18** $T_{r+1}={}^{15}C_r(x^2)^{15-r}\cdot\left(\frac{2}{x}\right)^r$

$={}^{15}C_r\, x^{30-2r}\cdot 2^r\cdot x^{-r}$

$={}^{15}C_r\cdot x^{30-3r}\cdot 2^r \quad \ldots(i)$

For coefficient of $x^{15}$, put $30-3r=15$

$\Rightarrow 3r=15 \Rightarrow r=5$

$\therefore$ Coefficient of $x^{15}={}^{15}C_5\cdot 2^5$

For coefficient of independent of $x$ i.e. $x^0$ put $30-3r=0$

$\Rightarrow \quad r=10$

$\therefore$ Coefficient of $x^0={}^{15}C_{10}\cdot 2^{10}$

By condition $\Rightarrow \frac{\text{Coefficient of } x^{15}}{\text{Coefficient of } x^0}$

$=\frac{{}^{15}C_5\cdot 2^5}{{}^{15}C_{10}\cdot 2^{10}}=\frac{{}^{15}C_{10}\cdot 2^5}{{}^{15}C_{10}\cdot 2^{10}}=1:32$

**19** Greatest term in the expansion of $(1+x)^n$ is $T_{r+1}$

where, $r=\left[\frac{(n+1)x}{1+x}\right]$

Here, $n=20$, $x=\frac{1}{\sqrt{3}}$

$\therefore \quad r=\left[\frac{21}{\sqrt{3}+1}\right]$

$=[10.5(\sqrt{3}-1)]=(7.69)\approx 7$

Hence, greatest term is

$\sqrt{3}\binom{20}{7}\left(\frac{1}{\sqrt{3}}\right)^7=\binom{20}{7}\frac{1}{27}$.

**20** $\because \quad (3+2x)^{50}=3^{50}\left(1+\frac{2x}{3}\right)^{50}$

Here, $\quad T_{r+1}=3^{50}\,{}^{50}C_r\left(\frac{2x}{3}\right)^r$

and $\quad T_r=3^{50}\,{}^{50}C_{r-1}\left(\frac{2x}{3}\right)^{r-1}$

But $\quad x=\frac{1}{5}$ (given)

$\therefore \quad \frac{T_{r+1}}{T_r}\geq 1 \Rightarrow \frac{{}^{50}C_r}{{}^{50}C_{r-1}}\cdot\frac{2}{3}\cdot\frac{1}{5}\geq 1$

$\Rightarrow \quad 102-2r\geq 15r \Rightarrow r\leq 6$

**21** Sum of the coefficients in the expansion of
$(x-2y+3z)^n$ is $(1-2+3)^n = 2^n$
(put $x = y = z = 1$)
$\therefore \quad 2^n = 128 \Rightarrow n = 7$
Therefore, the greatest coefficient in the expansion of $(1+x)^7$ is ${}^7C_3$ or ${}^7C_4$ because both are equal to 35.

**22** In the expansion of $(1+x)^{2n}$, the general term $= {}^{2n}C_k x^k, 0 \le k \le 2n$
As given for $r > 1, n > 2$,
$${}^{2n}C_{3r} = {}^{2n}C_{r+2}$$
$\Rightarrow$ Either $3r = r+2$
or $3r = 2n-(r+2)$
$(\because {}^nC_x = {}^nC_y \Rightarrow x + y = n$ or $x = y)$
$\Rightarrow r = 1$ or $n = 2r+1$
We take the relation only
$$n = 2r+1 \quad (\because r > 1)$$

**23** Let $b = \sum_{r=0}^{n} \frac{r}{{}^nC_r} = \sum_{r=0}^{n} \frac{n-(n-r)}{{}^nC_r}$
$= n\sum_{r=0}^{n} \frac{1}{{}^nC_r} - \sum_{r=0}^{n} \frac{n-r}{{}^nC_r}$
$= na_n - \sum_{r=0}^{n} \frac{n-r}{{}^nC_{n-r}} \quad (\because {}^nC_r = {}^nC_{n-r})$
$= na_n - b \Rightarrow 2b = na_n \Rightarrow b = \frac{n}{2}a_n$

**24** Let $E = \sum_{r=0}^{n}(-1)^r\, {}^nC_r\left(\frac{1+rx}{1+nx}\right)$
$= \left(\frac{1}{1+nx}\right)\sum_{r=0}^{n}(-1)^r\, {}^nC_r(1+rx)$
$= \left(\frac{1}{1+nx}\right)\left\{\sum_{r=0}^{n}(-1)^r \cdot {}^nC_r + x\sum_{r=0}^{n} r(-1)^r\, {}^nC_r\right\}$
$= \left(\frac{1}{1+nx}\right)(0+0) = 0$
$[\because {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots (-1)^n\, {}^nC_n = 0]$

**25** Let
$A = \binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} - \ldots + \binom{30}{20}\binom{30}{30}$
or $A = {}^{30}C_0 . {}^{30}C_{10} - {}^{30}C_1 . {}^{30}C_{11} + {}^{30}C_2 . {}^{30}C_{12} - \ldots + {}^{30}C_{20} . {}^{30}C_{30}$
= Coefficient of $x^{20}$ in $(1+x)^{30}(1-x)^{30}$
= Coefficient of $x^{20}$ in $(1-x^2)^{30}$
= Coefficient of $x^{20}$ in $\sum_{r=0}^{30}(-1)^r\, {}^{30}C_r (x^2)^r$
$= (-1)^{10}\, {}^{30}C_{10}$
(for coefficient of $x^{20}$, let $r = 10$)
$= {}^{30}C_{10}$

**26** $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + \ldots + ({}^{21}C_{10} - {}^{10}C_{10})$
$= ({}^{21}C_1 + {}^{21}C_2 + \ldots + {}^{21}C_{10}) - ({}^{10}C_1 + {}^{10}C_2 + \ldots + {}^{10}C_{10})$
$= \frac{1}{2}({}^{21}C_1 + {}^{21}C_2 + \ldots + {}^{21}C_{20}) - (2^{10}-1)$
$= \frac{1}{2}({}^{21}C_1 + {}^{21}C_2 + \ldots + {}^{21}C_{21} - 1) - (2^{10}-1)$
$= \frac{1}{2}(2^{21}-2) - (2^{10}-1) = 2^{20} - 1 - 2^{10} + 1$
$= 2^{20} - 2^{10}$

**27** We know that,
$(1+x)^{20} = {}^{20}C_0 + {}^{20}C_1 x + \ldots + {}^{20}C_{10}x^{10} + \ldots + {}^{20}C_{20}x^{20}$
On putting $x = -1$ in the above expansion, we get
$0 = {}^{20}C_0 - {}^{20}C_1 + \ldots - {}^{20}C_a + {}^{20}C_{10} - {}^{20}C_{11} + \ldots + {}^{20}C_{20}$
$\Rightarrow 0 = {}^{20}C_0 - {}^{20}C_1 + \ldots - {}^{20}C_9 + {}^{20}C_{10} - {}^{20}C_9 + \ldots + {}^{20}C_0$
$\Rightarrow 0 = 2({}^{20}C_0 - {}^{20}C_1 + \ldots - {}^{20}C_9) + {}^{20}C_{10}$
$\Rightarrow {}^{20}C_{10} = 2({}^{20}C_0 - {}^{20}C_1 + \ldots + {}^{20}C_{10})$
$\Rightarrow {}^{20}C_0 - {}^{20}C_1 + \ldots + {}^{20}C_{10} = \frac{1}{2}\, {}^{20}C_{10}$

**28** Since, $x(1+x)^n = xC_0 + C_1x^2 + C_2x^3 + \ldots + C_nx^{n+1}$
On differentiating w.r.t. $x$, we get
$(1+x)^n + nx(1+x)^{n-1} = C_0 + 2C_1x + 3C_2x^2 + \ldots + (n+1)C_nx^n$
Put $x = 1$, we get
$C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n = 2^n + n2^{n-1} = 2^{n-1}(n+2)$

**29** Let $f = (8-3\sqrt{7})^{10}$, here $0 < f < 1$
$\therefore \quad (8+3\sqrt{7})^{10} + (8-3\sqrt{7})^{10}$ is an integer, hence this is the value of $n$.

**30** We have,
$49^n + 16n - 1 = (1+48)^n + 16n - 1$
$= 1 + {}^nC_1(48) + {}^nC_2(48)^2 + \ldots + {}^nC_n(48)^n + 16n - 1$
$= (48n + 16n) + {}^nC_2(48)^2 + {}^nC_3(48)^3 + \ldots + {}^nC_n(48)^n$
$= 64n + 8^2[{}^nC_2 \cdot 6^2 + {}^nC_3 \cdot 6^3 \cdot 8 + {}^nC_4 \cdot 6^4 8^2 + \ldots + {}^nC_n \cdot 6^n \cdot 8^{n-2}]$
Hence, $49^n + 16n - 1$ is divisible by 64.

**31** Since, $\left(1+\frac{1}{n}\right)^n < 3$ for $\forall n \in N$
Now, $\frac{(1001)^{999}}{(1000)^{1000}} = \frac{1}{1001}\left(\frac{1001}{1000}\right)^{1000}$
$= \frac{1}{1001}\left(1+\frac{1}{1000}\right)^{1000} < \frac{1}{1001} \cdot 3 < 1$
$(1001)^{999} < (1000)^{1000}$
$\therefore \quad B < A$

**32** Since, ${}^{n-1}C_r = (k^2-3)\frac{n}{r+1}\, {}^{n-1}C_r$
$\Rightarrow k^2 - 3 = \frac{r+1}{n}$
$\Rightarrow 0 < k^2 - 3 \le 1$
$\left[\because n \ge r \Rightarrow \frac{r+1}{n} \le 1 \text{ and } n, r > 0\right]$
$\Rightarrow 3 < k^2 \le 4$
Hence, $k \in [-2, -\sqrt{3}) \cup (\sqrt{3}, 2]$

**33** $8^{2n} - (62)^{2n+1} = (1+63)^n - (63-1)^{2n+1}$
$= (1+63)^n + (1-63)^{2n+1}$
$= [1 + {}^nC_1 \cdot 63 + {}^nC_2 \cdot (63)^2 + \ldots + (63)^n] + [1 - {}^{2n-1}C_1 \cdot 63 + {}^{(2n+1)}C_2 \cdot (63)^2 - \ldots + (-1)(63)^{(2n+1)}]$
$= 2 + 63[{}^nC_1 + {}^nC_2(63) + \ldots + (63)^{n-1} - {}^{(2n+1)}C_1 + {}^{(2n+1)}C_2(63) - \ldots + (-1)(63)^{(2n)}]$
Hence, remainder is 2.

**34** Since, $(r+1)$th term in the expansion of $(1+x)^{27/5}$
$= \frac{\frac{27}{5}\left(\frac{27}{5}-1\right)\ldots\left(\frac{27}{5}-r+1\right)}{r!}x^r$
Now, this term will be negative, if the last factor in numerator is the only one negative factor.
$\Rightarrow \frac{27}{5} - r + 1 < 0 \Rightarrow \frac{32}{5} < r$
$\Rightarrow 6.4 < r \Rightarrow$ least value of $r$ is 7.
Thus, first negative term will be 8th.

**35** Given, $P(n): n^2 + n + 1$
At $n = 1, P(1): 3$, which is not an even integer.
Thus, $P(1)$ is not true.
Also, $n(n+1)+1$ is always an odd integer.

**36** Let the statement $P(n)$ be defined as
$P(n): 1 \times 1! + 2 \times 2! + 3 \times 3! \ldots + n \times n! = (n+1)! - 1$
for all natural numbers $n$.
Note that $P(1)$ is true, since
$P(1): 1 \times 1! = 1 = 2 - 1 = 2! - 1$
Assume that $P(n)$ is true for some natural number $k$, i.e.
$P(k): 1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + k \times k! = (k+1)! - 1 \quad \ldots(i)$
To prove $P(k+1)$ is true, we have
$P(k+1): 1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + k \times k! + (k+1) \times (k+1)!$
$= (k+1)! - 1 + (k+1)! \times (k+1)$ [by Eq. (i)]
$= (k+1+1)(k+1)! - 1$
$= (k+2)(k+1)! - 1 = (k+2)! - 1$

Thus, $P(k+1)$ is true, whenever $P(k)$ is true. Therefore, by the principle of mathematical induction, $P(n)$ is true for all natural numbers $n$.

**37** Now, $2^{3n}-1=(2^3)^n-1=(1+7)^n-1$

$$=1+{}^nC_1\cdot 7+{}^nC_2\cdot 7^2+\ldots+{}^nC_n\cdot 7^n-1$$

$$=7[{}^nC_1+{}^nC_2 7+\ldots+{}^nC_n\cdot 7^{n-1}]$$

Hence, 7 divides $2^{3n}-1$ for all $n\in N$.

**38** $S(k)=1+3+5+\ldots+(2k-1)=3+k^2$

Put $k=1$ in both sides, we get

LHS = 1 and RHS = 3 + 1 = 4

$\Rightarrow$ LHS $\neq$ RHS

Put $(k+1)$ in both sides in the place of $k$, we get

LHS $=1+3+5+\ldots+(2k-1)+(2k+1)$

RHS $=3+(k+1)^2=3+k^2+2k+1$

Let LHS = RHS

$$1+3+5+\ldots+(2k-1)+(2k+1)=3+k^2+2k+1$$

$\Rightarrow\ 1+3+5+\ldots+(2k-1)=3+k^2$

If $S(k)$ is true, then $S(k+1)$ is also true.

Hence, $S(k)\Rightarrow S(k+1)$

## SESSION 2

**1** Multiplying the numerator and denominator by $1-x$, we have

$$E=\frac{1-x}{(1-x)(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^m})}$$

$$=\frac{1-x}{(1-x^2)(1+x^2)(1+x^4)\ldots(1+x^{2^m})}$$

$$=\frac{1-x}{(1-x^4)(1+x^4)\ldots(1+x^{2^m})}$$

$$=\frac{1-x}{(1-x^{2^{m+1}})}=(1-x)(1-x^{2^{m+1}})^{-1}$$

$$=(1-x)(1+x^{2^{m+1}}+x^{2^{m+2}}+\ldots)$$

$\therefore$ Coefficient of $x^{2^{m+1}}$ is 1.

**2** Since, $(1-x)^n=C_0-C_1\cdot x+C_2\cdot x^2-C_3\cdot x^3+\ldots$

$\Rightarrow 1-(1-x)^n=C_1\cdot x-C_2\cdot x^2+C_3\cdot x^3-\ldots$

$\Rightarrow \dfrac{1-(1-x)^n}{x}=C_1-C_2\cdot x+C_3\cdot x^2-\ldots$

$\Rightarrow \displaystyle\int_0^1(C_1-C_2\cdot x+C_3\cdot x^2-\ldots)\,dx=\int_0^1\frac{1-(1-x)^n}{1-(1-x)}\,dx$

$\Rightarrow \dfrac{C_1}{1}-\dfrac{C_2}{2}+\dfrac{C_3}{3}-\ldots=\displaystyle\int_0^1\frac{1-x^n}{1-x}\,dx$

$$\left[\because \int_0^1 f(x)\,dx=\int_0^1 f(1-x)\,dx\right]$$

$$=\int_0^1(1+x+x^2+\ldots+x^{n-1})\,dx$$

$$=\left[x+\frac{x^2}{2}+\ldots+\frac{x^n}{n}\right]_0^1=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$$

**3** General term is

$$T_{r+1}={}^{10}C_r\cdot(a\cdot x^2)^{10-r}\cdot\left(\frac{1}{bx}\right)^r$$

$$={}^{10}C_r\cdot(a)^{10-r}\left(\frac{1}{b}\right)^r(x)^{20-3r}$$

Since, $x^5$ occurs in $T_{r+1}$.

$\therefore\quad 20-3r=5$

$\Rightarrow\ 3r=15\Rightarrow r=5$

So, the coefficient of $x^5$ is ${}^{10}C_5(a)^5(b)^{-5}$.

Again, let $x^{-5}$ occurs in $T_{r+1}$ of

$$\left[a\cdot x-\frac{1}{b^2\cdot x^2}\right]^{10}\text{ is }{}^{10}C_r(ax)^{10-r}\left(-\frac{1}{b^2x^2}\right)^r$$

$$={}^{10}C_r(a)^{10-r}\left(-\frac{1}{b^2}\right)^r(x)^{10-3r}$$

$$10-3r=-5\Rightarrow 15=3r\Rightarrow r=5$$

So, the coefficient of $x^{-5}$ is $-{}^{10}C_5\dfrac{a^5}{b^{10}}$.

According to the given condition,

$${}^{10}C_5\frac{a^5}{b^5}=-a\,{}^{10}C_5\frac{a^5}{b^{10}}$$

$\Rightarrow\quad -b^5=a\Rightarrow -b^6=ab$

**4** Let $T_{r+1}$ be the general term in the expansion of $(1-2\sqrt{x})^{50}$.

$$\therefore\quad T_{r+1}={}^{50}C_r\,(1)^{50-r}(-2x^{1/2})^r={}^{50}C_r\cdot 2^r\cdot x^{r/2}(-1)^r$$

For the integral power of $x$ and $r$ should be even integer.

$$\therefore \text{Sum of coefficients}=\sum_{r=0}^{25}{}^{50}C_{2r}(2)^{2r}$$

$$=\frac{1}{2}[(1+2)^{50}+(1-2)^{50}]=\frac{1}{2}[3^{50}+1]$$

**Alternate Method**

We have,

$$(1-2\sqrt{x})^{50}=C_0-C_1\cdot 2\sqrt{x}+C_2(2\sqrt{x})^2+\ldots+C_{50}(2\sqrt{x})^{50}\quad\ldots\text{(i)}$$

$$(1+2\sqrt{x})^{50}=C_0+C_1\cdot 2\sqrt{x}+C_2(2\sqrt{x})^2+\ldots+C_{50}(2\sqrt{x})^{50}\quad\ldots\text{(ii)}$$

On adding Eqs. (i) and (ii), we get

$$(1-2\sqrt{x})^{50}+(1+2\sqrt{x})^{50}=2[C_0+C_2(2\sqrt{x})^2+\ldots+C_{50}(2\sqrt{x})^{50}]$$

$$\Rightarrow\ \frac{(1-2\sqrt{x})^{50}+(1+2\sqrt{x})^{50}}{2}=C_0+C_2(2\sqrt{x})^2+\ldots+C_{50}(2\sqrt{x})^{50}$$

On putting $x=1$, we get

$$\frac{(1-2\sqrt{1})^{50}+(1+2\sqrt{1})^{50}}{2}=C_0+C_2(2)^2+\ldots+C_{50}(2)^{50}$$

$$\Rightarrow\frac{(-1)^{50}+(3)^{50}}{2}=C_0+C_2(2)^2+\ldots+C_{50}(2)^{50}$$

$$\Rightarrow\frac{1+3^{50}}{2}=C_0+C_2(2)^2+\ldots+C_{50}(2)^{50}$$

**5** $$\left[\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{(x-1)}{x-x^{1/2}}\right]^{10}$$

$$=\left[\frac{(x^{1/3})^3+1^3}{x^{2/3}-x^{1/3}+1}-\frac{\{(\sqrt{x})^2-1\}}{\sqrt{x}(\sqrt{x}-1)}\right]^{10}$$

$$=\left[\frac{(x^{1/3}+1)(x^{2/3}+1-x^{1/3})}{x^{2/3}-x^{1/3}+1}-\frac{\{(\sqrt{x})^2-1\}}{\sqrt{x}(\sqrt{x}-1)}\right]^{10}$$

$$=\left[(x^{1/3}+1)-\frac{(\sqrt{x}+1)}{\sqrt{x}}\right]^{10}=(x^{1/3}-x^{-1/2})^{10}$$

$\therefore$ The genreal term is

$$T_{r+1}={}^{10}C_r(x^{1/3})^{10-r}(-x^{-1/2})^r={}^{10}C_r(-1)^r x^{\frac{10-r}{3}-\frac{r}{2}}$$

For independent for $x$, put

$$\frac{10-r}{3}-\frac{r}{2}=0\Rightarrow 20-2r-3r=0$$

$\Rightarrow\quad 20=5r\Rightarrow r=4$

$$\therefore\quad T_5={}^{10}C_4=\frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1}=210$$

**6** We have,

$$(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n\quad\ldots\text{(i)}$$

$$\text{and}\left(1+\frac{1}{x}\right)^n=C_0+C_1\frac{1}{x}+C_2\left(\frac{1}{x}\right)^2+\ldots+C_n\left(\frac{1}{x}\right)^n\quad\ldots\text{(ii)}$$

On multiplying Eqs. (i) and (ii) and taking coefficient of constant terms in right hand side $=C_0^2+C_1^2+C_2^2+\ldots+C_n^2$

In right hand side $(1+x)^n\left(1+\dfrac{1}{x}\right)^n$ or in $\dfrac{1}{x^n}(1+x)^{2n}$ or term containing $x^n$ in $(1+x)^{2n}$. Clearly, the coefficient of $x^n$ in $(1+x)^{2n}$ is equal to ${}^{2n}C_n=\dfrac{(2n)!}{n!n!}$.

**7** We can write,

$aC_0-(a+d)C_1+(a+2d)C_2-\ldots$ upto $(n+1)$ terms

$$=a(C_0-C_1+C_2-\ldots)+d(-C_1+2C_2-3C_3+\ldots)\quad\ldots\text{(i)}$$

We know,

$$(1-x)^n=C_0-C_1x+C_2x^2-\ldots+(-1)^nC_nx^n\quad\ldots\text{(ii)}$$

On differentiating Eq. (ii) w.r.t. $x$, we get

$$-n(1-x)^{n-1}=-C_1+2C_2x-\ldots+(-1)^nC_nnx^{n-1}\quad\ldots\text{(iii)}$$

On putting $x=1$ in Eqs. (ii) and (iii), we get

$$C_0-C_1+C_2-\ldots+(-1)^nC_n=0\quad\ldots\text{(iv)}$$

$$\text{and }-C_1+2C_2-\ldots+(-1)^nC_n=0\quad\ldots\text{(v)}$$

From Eq. (i),

$aC_0-(a+d)C_1+(a+2d)C_2-\ldots+$ upto $(n+1)$ terms

$$= a\cdot 0 + d\cdot 0 = 0$$

[from Eqs. (iv) and (v)]

**8** Since, $\binom{n}{m}\binom{m}{p}=\dfrac{n!}{(n-m)!\,p!\,(m-p)!}$

$$=\binom{n}{p}\binom{n-p}{m-p}$$

$\therefore$ Given series can be rewritten as

$$\sum_{p=1}^{n}\sum_{m=p}^{n}\binom{n}{p}\binom{n-p}{m-p}$$

$$=\sum_{p=1}^{n}\binom{n}{p}\sum_{m=p}^{n}\binom{n-p}{m-p}$$

$$=\sum_{p=1}^{n}\binom{n}{p}\sum_{t=0}^{n-p}\binom{n-p}{t}$$

$$=\sum_{p=1}^{n}\binom{n}{p}2^{n-p} \qquad [\text{put } m-p=t]$$

$$=2^n\sum_{p=1}^{n}\binom{n}{p}\cdot\frac{1}{2^p}=2^n\left[\left(1+\frac{1}{2}\right)^n-1\right]$$

$$=3^n-2^n$$

**9** $\displaystyle\sum_{r=0}^{n}(-1)^r$

$$^nC_r\left\{\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\ldots \text{upto } m \text{ terms}\right\}$$

$$=\sum_{r=0}^{n}(-1)^r\,{}^nC_r\cdot\frac{1}{2^r}+\sum_{r=0}^{n}(-1)^r\cdot{}^nC_r\frac{3^r}{2^{2r}}+\sum_{r=0}^{n}(-1)^r\,{}^nC_r\frac{7^r}{2^{3r}}+\ldots$$

$$=\left(1-\frac{1}{2}\right)^n+\left(1-\frac{3}{4}\right)^n+\left(1-\frac{7}{8}\right)^n+\ldots$$

upto $m$ terms

$$=\frac{1}{2^n}+\frac{1}{2^{2n}}+\frac{1}{2^{3n}}\ldots \text{ upto } m \text{ terms}$$

$$=\frac{\frac{1}{2^n}\left\{1-\left(\frac{1}{2^n}\right)^m\right\}}{\left(1-\frac{1}{2^n}\right)}=\frac{2^{mn}-1}{2^{mn}(2^n-1)}$$

**10** We have,

$$\left[2^{\log_2\sqrt{9^{x-1}+7}}+\frac{1}{2^{(1/5)\log_2(3^{x-1}+1)}}\right]^7$$

$$=\left[\sqrt{9^{x-1}+7}+\frac{1}{(3^{x-1}+1)^{1/5}}\right]^7$$

$$\therefore T_6={}^7C_5(\sqrt{9^{x-1}+7})^{7-5}\left[\frac{1}{(3^{x-1}+1)^{1/5}}\right]^5$$

$$={}^7C_5(9^{x-1}+7)\frac{1}{(3^{x-1}+1)}$$

$$\Rightarrow\ 84={}^7C_5\frac{(9^{x-1}+7)}{(3^{x-1}+1)}$$

$$\Rightarrow\ 9^{x-1}+7=4(3^{x-1}+1)$$

$$\Rightarrow\ \frac{3^{2x}}{9}+7=4\left(\frac{3^x}{3}+1\right)$$

$$\Rightarrow\ 3^{2x}-12(3^x)+27=0$$

$$\Rightarrow\ y^2-12y+27=0 \quad (\text{put } y=3^x)$$

$$\Rightarrow\ (y-3)(y-9)=0$$

$$\Rightarrow\ y=3,9$$

$$\Rightarrow\ 3^x=3,9$$

$$\Rightarrow\ x=1,2$$

**11** Last term of $\left(2^{1/3}-\dfrac{1}{\sqrt{2}}\right)^n$ is

$$T_{n+1}={}^nC_n(2^{1/3})^{n-n}\left(-\frac{1}{\sqrt{2}}\right)^n$$

$$={}^nC_n(-1)^n\frac{1}{2^{n/2}}=\frac{(-1)^n}{2^{n/2}}$$

Also, we have

$$\left(\frac{1}{3^{5/3}}\right)^{\log_3 8}=3^{-(5/3)\log_3 2^3}=2^{-5}$$

Thus, $\dfrac{(-1)^n}{2^{n/2}}=2^{-5}\Rightarrow\dfrac{(-1)^n}{2^{n/2}}=\dfrac{(-1)^{10}}{2^5}$

$$\Rightarrow\ \frac{n}{2}=5\Rightarrow n=10$$

Now, $T_5=T_{4+1}={}^{10}C_4(2^{1/3})^{10-4}\left(-\dfrac{1}{\sqrt{2}}\right)^4$

$$=\frac{10!}{4!6!}(2^{1/3})^6(-1)^4(2^{-1/2})^4$$

$$=210(2)^2(1)(2^{-2})=210$$

**12** **Key idea** $=(a+b)^n+(a-b)^n$

$$=2({}^nC_0a^n+{}^nC_2a^{n-2}b^2+{}^nC_4a^{n-4}b^4\ldots)$$

We have

$$(x+\sqrt{x^3-1})^5+(x-\sqrt{x^3-1})^5,\ x>1$$

$$=2({}^5C_0x^5+{}^5C_2x^3(\sqrt{x^3-1})^2+{}^5C_4x(\sqrt{x^3-1})^4)$$

$$=2(x^5+10x^3(x^3-1)+5x(x^3-1)^2)$$

$$=2(x^5+10x^6-10x^3+5x^7-10x^4+5x)$$

Sum of coefficients of all odd degree terms is

$$2(1-10+5+5)=2$$

**13** The general term in the expansion of $\left(x\cos\alpha+\dfrac{\sin\alpha}{x}\right)^{20}$ is

$$^{20}C_r(x\cos\alpha)^{20-r}\left(\frac{\sin\alpha}{x}\right)^r$$

$$={}^{20}C_rx^{20-2r}(\cos\alpha)^{20-r}(\sin\alpha)^r$$

For this term to be independent of $x$, we get

$$20-2r=0$$

$$\Rightarrow\ r=10$$

Let $\beta$ = Term indeoendent of $x$

$$={}^{20}C_{10}(\cos\alpha)^{10}(\sin\alpha)^{10}$$

$$={}^{20}C_{10}(\cos\alpha\sin\alpha)^{10}$$

$$={}^{20}C_{10}\left(\frac{\sin 2\alpha}{2}\right)^{10}$$

Thus, the greatest possible value of $\beta$ is $^{20}C_{10}\left(\dfrac{1}{2}\right)^{10}$.

**14** Let $P(n)=(n)^7-n$

By mathematical induction

For $n=1$,

$P(1)=0$, which is divisible by 7.

For $n=k$

$$P(k)=k^7-k$$

Let $P(k)$ be divisible by 7.

$\therefore\quad k^7-k=7\lambda$, for some $\lambda\in N$ ... (i)

For $n=k+1$,

$$P(k+1)=(k+1)^7-(k+1)$$

$$=({}^7C_0k^7+{}^7C_1k^6+{}^7C_2k^5+\ldots+{}^7C_6\cdot k+{}^7C_7)-(k+1)$$

$$=(k^7-k)+7\{k^6+3k^5+\ldots+k\}$$

$$=7\lambda+7\{k^6+3k^5+\ldots+k\} \text{[Using Eq. (i)]}$$

$\Rightarrow$ Divisible by 7.

So, both statements are true and Statement II is correct explanation of Statement I.

**15** We know that, in the expansion of $(a+b)^n$, $p$th term from the end is $(n-p+2)$th term from the beginning.

So, 5th term from the end is

$=(n-5+2)$ th term from the beginning

$=(n-3)$ th term from the beginning

$=(n-4+1)$ th term from the beginning ...(i)

$\therefore$ We have,

$$\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n=\left(2^{1/4}+\frac{1}{3^{1/4}}\right)^n$$

Now, 5th term from the beginning is

$$T_{4+1}={}^nC_4\,(2^{1/4})^{n-4}\,(3^{-1/4})^4$$

$$={}^nC_4\,2^{\frac{n-4}{4}}\,3^{-1} \qquad \ldots\text{(ii)}$$

And 5th term from the end is

$$T_{(n-4)+1}={}^nC_{n-4}(2^{1/4})^{n-n+4}(3^{1/4})^{n-4}$$

$$={}^nC_4\ 2\cdot 3^{-\frac{n+4}{4}}$$

$$[\because {}^nC_r={}^nC_{n-r}] \quad \ldots\text{(iii)}$$

So, from the given condition, we have

$$\frac{\text{Fifth term from the beginning}}{\text{Fifth term from the end}}=\frac{\sqrt{6}}{1}$$

$$\Rightarrow\ \frac{{}^nC_4\cdot 2^{\frac{n-4}{4}}\cdot 3^{-1}}{{}^nC_4\cdot 2\cdot 3^{\frac{-n+4}{4}}}=\frac{\sqrt{6}}{1}$$

$$\Rightarrow\ 2^{\frac{n-4}{4}-1}\cdot 3^{-1-\left(\frac{4-n}{4}\right)}=\sqrt{6}$$

$$\Rightarrow\ 2^{\frac{n-8}{4}}\,3^{\frac{n-8}{4}}=6^{1/2}$$

$$\Rightarrow\ (2\times 3)^{\frac{n-8}{4}}=(2\cdot 3)^{1/2}$$

$$\Rightarrow\ \frac{n-8}{4}=1/2$$

$$\Rightarrow\ n=2+8 \quad \therefore n=10$$