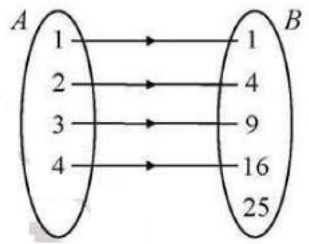
FUNCTIONS (XII, R. S. AGGARWAL)

EXERCISE 2A (Pg. No.: 34)

1. Define a function. What do you mean by the domain and range of a function? Give examples.

Sol. Let A and B be two non empty sets. Then, a rule f which associates to each element $x \in A$, a unique element, denoted by $f(x)$ of B, is called a function from A to B and we write $f: A \to B$



Domain, co-domain and range of function let $f: A \to B$, then, A is called the domain of f and B is called the co-domain of f. And $f(A) = \{f(x) : x \in A\}$ is called the range of f.

Example : let $A = \{1, 2, 3, 4\}$ and $B = \{1, 4, 9, 16, 25\}$

Consider the rule $f: A \to B : f(x) = x^2 \; \forall x \in A$.

Then, each element in A has its unique image in B. So, f is a function from A to B.

$f(1) = (1)^2 = 1, f(x) = (2)^2 = 4, f(3) = (3)^2 = 9, f(4) = (4)^2 = 16$

Domain $(f) = \{1, 2, 3, 4\} = A$, Co-domain $(f) = \{1, 4, 9, 16, 25\} = B$ and range $(f) = \{1, 4, 9, 16\}$. Clearly, $25 \in B$ does not have its pre-image in A.

2. Define each of the following:

(i) injective function (ii) surjective function (iii) bijective function (iv) many-one function (v) into function

Sol. **(i) Injective function :** A function $f: A \to B$ is said to be one-one if distinct element in A have distinct image in B.

Example: let N be the set of all natural number let $f: N \to N : f(x) = 2x \forall x \in N$

Then $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2 \therefore f$ is one- one

(ii) Surjective function : A function $f: A \to B$ is said to be onto if every element in B has at least one pre-image in A. Thus, if f is onto, then for each $y \in B$ at least one element $x \in A$ such that $y = f(x)$. Also, f is onto $\Leftrightarrow$ range $(f) = B$.

Example : let N be the set of all natural number at least E be the set of all even natural numbers.

Let $f: N \to E : f(x) = 2x \forall x \in N$. Then $y = 2x \Rightarrow x = \frac{1}{2}y$.

Thus, for each $y \in E$, there exists $\frac{1}{2}y \in N$ such that $f\left(\frac{1}{2}y\right) = \left(2 \times \frac{1}{2}y\right) = y \therefore$ its onto.

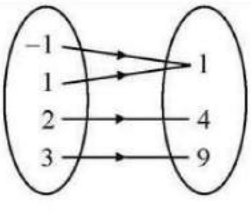
(iii) Bijective function: A one-one onto function is said to be bijective. If there is a one-to-one correspondence between A and B.

(iv) Many one function : A function $f: A \to B$ is said to be many one if two or more then two element in A have the same image in B.

Example. Let $A = \{-1, 1, 2, 3\}$ and $B = \{1, 4, 9\}$

Let $f: A \to B : f(x) = x^2 \; \forall x \in A$.

Then each element in a has a unique image under f in B.



$\therefore f$ is a function from A to B such that

$f(-1)=(-1)^2=1, f(1)=(1)^2=1$

$f(2)=(2)^2=4$, and $f(3)=(3)^2=9$

Clearly, two different elements, namely -1 and 1 have the same image $1 \in B$.

Hence f is many one.

(v) Into function : A function $f : A \to B$ is said to be into if there exists even a single element in B having no pre-image in A. Clearly, f is into $\Leftrightarrow$ range $(f) \subset B$.

Example : Let A = {2, 3, 5, 7} and B = {0, 1, 3, 5, 7}. Let $f : A \to B : f(x) = (x-2)$.

Then, $f(2)=(2-2)=0$, $f(3)=(3-2)=1$, $f(5)=5-2=3$ and $f(7)=7-2=5$.

Thus, every element in A has a unique image in B.

Now, $\exists\ 7 \in B$ having no pre-image in A. $\therefore f$ is into. **Note :** Range (f) = {0, 1, 3, 5} $\subset B$

3. Give an example of a function which is

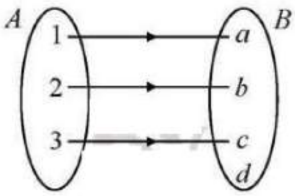
(i) one-one but not onto (ii) one-one and onto

(iii) neither one-one nor onto (iv) onto but not one-one.

Sol. (i) $A=\{1,2,3\}$

$\& B=\{a,b,c,d\}$

$f=\{(1,a)(2,b)(3,c)\}$



(ii) We observe the following properties of $f(x)=2x$.

Infectivity: Let $x_1, x_2 \in R$ such that $f(x_1)=f(x_2)$. then, $f(x_1)=f(x_2) \Rightarrow 2x_1=2x_2 \Rightarrow x_1=x_2$

So, $f : R \to R$ is one-one

Subjectivity: Let y be any real number in R (co-domain)

Then, $f(x)=y \Rightarrow 2x=y \Rightarrow x=\frac{y}{2}$ clearly, $\frac{y}{2} \in R$ for $y \in R$ such that $f\left(\frac{y}{2}\right)=2\left(\frac{y}{2}\right)=y$.

Thus, for each $y \in R$ (co-domain) there exists $x=\frac{y}{2} \in R$ (domain) such that $f(x)=y$.

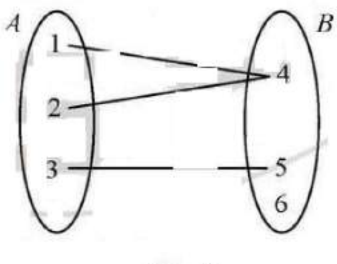
This means that each element in co-domain has its pre-image in domain.

Hence $f : R \to R$ is a bijective.

(iii) $A=\{1,2,3\}$

$B=\{4,5,6\}$

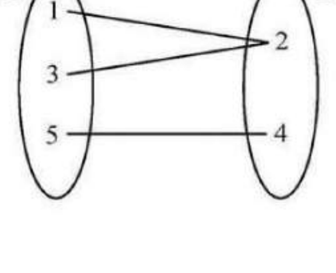
$\Rightarrow f=\{(1,4),(2,4),(3,5)\}$



(iv) $A=\{1,3,5\}$

$B=\{2,4\}$

$\Rightarrow f=\{(1,2)(3,2)(5,4)\}$



4. Let $f: R \to R$ be defined by $f(x)=\begin{cases} 2x+3, \text{ when } x<-2 \\ x^2-2, \text{ when } -2 \le x \le 3 \\ 3x-1, \text{ when } x>3 \end{cases}$

Find (i) $f(2)$ (ii) $f(4)$ (iii) $f(-1)$ (iv) $f(-3)$

Sol. **(i)** $f(x)=x^2-2 \Rightarrow f(2)=(2)^2-2=4-2=2 \therefore f(2)=2$

(ii) $f(x)=3x-1 \Rightarrow f(4)=3\times 4-1=12-1=11 \therefore f(4)=11$

(iii) $f(x)=x^2-2 \Rightarrow f(-1)=(-1)^2-2=1-2=-1 \therefore f(-1)=-1$

(iv) $f(x)=2x+3 \Rightarrow f(-3)=2(-3)+3=-6+3=-3 \quad \therefore f(-3)=-3$

5. Show that the function $f: R \to R: f(x)=1+x^2$ is many-one into

Sol. Clearly for every negative and positive value of x will get same value of $f(x)$ hence $f(x)$ is not one-one i.e., many one.

Clearly $x^2+1 \ge 1$ for all x. Hence, Range $=[1, \infty) \subset R \quad \therefore$ function is into function.

6. Show that the function $f: R \to R: f(x)=x^4$ is many-one and into.

Sol. Clearly for every negative and positive value of x we will get same value of $f(x)$.

Hence, $f(x)$ is not one-one i.e., many one. Clearly $x^4 \ge 0$ for all x

Hence range $=[0, \infty) \subset R \quad \therefore$ function is into function.

7. Show that the function $f: R \to R: f(x)=x^5$ is one-one and onto.

Sol. **Injectivity:** Let $x, y \in R$ such that $f(x)=f(y)$

$\Rightarrow x^5=y^5 \Rightarrow x^5-y^5=0 \Rightarrow (x^{5/2})^2-(y^{5/2})^2=0 \Rightarrow (x^{5/2}-y^{5/2})(x^{5/2}+y^{5/2})=0$

$\Rightarrow x^{5/2}-y^{5/2}=0 \Rightarrow x=y$. Thus $f(x)=f(y) \Rightarrow x=y$ for all $x, y \in R$. So, f is an injective map.

Surjectivity: Let y be arbitrary element of R. Then $f(x)=y \Rightarrow x^5=y \Rightarrow x^5-y=0$

We know that an odd degree equation has at least one real root. Therefore, for every real value of y, the equation $x^5-y=0$ has a real root α such that

$\alpha^5-y=0 \Rightarrow \alpha^5=y \Rightarrow f(\alpha)=y$

Thus for every $y \in R$ these exists $\alpha \in R$ such that $f(\alpha)=y$.

So, f is a surjective map. Hence $f: R \to R$ is a bijective.

8. Let $f:\left[0, \frac{\pi}{2}\right] \to R: f(x)=\sin x$ and $g:\left[0, \frac{\pi}{2}\right] \to R: g(x)=\cos x$. Show that each one of f and g is one-one but $(f+g)$ is not one-one.

Sol. We observe that for any tow district elements x_1 and x_2 in $\left[0, \frac{\pi}{2}\right]$

$\sin x_1 \ne \sin x_2$ and $\cos x_1 \ne \cos x_2$

$\Rightarrow f(x_1) \ne f(x_2)$ and $g(x_1) \ne g(x_2) \Rightarrow f$ and g are one-one

We have, $(f+g)(x)=f(x)+g(x)=\sin x+\cos x$

$\Rightarrow (f+g)(0) = \sin 0 + \cos 0 = 1$ and $(f+g)\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1$

$\therefore 0 \neq \frac{\pi}{2}$, but $(f+g)(0) = (f+g)\left(\frac{\pi}{2}\right)$. So, $f+g$ is not one-one.

9. Show that the function

(i) $f : N \to N : f(x) = x^2$ is one-one into (ii) $f : Z \to Z : f(x) = x^2$ is many-one into.

Sol. **(i)** Let N be the set of all natural numbers. Let $f : N \to N : f(x) = x^2 \;\forall x \in N$

Then $f(x_1) \Rightarrow f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1^2 - x_2^2 = 0 \Rightarrow (x_1 - x_2)(x_1 + x_2) = 0$

$\Rightarrow (x_1 - x_2) = 0 \Rightarrow x_1 = x_2$. $\therefore f$ is one- one.

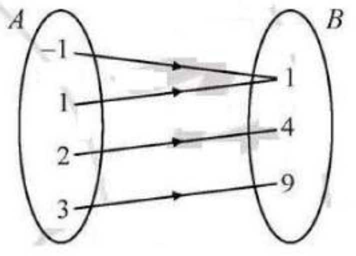
Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 4, 9, 16, 25\}$. Let $f : A \to B$, $f(x) = x^2$

Then, $f(1) = (1)^2 = 1, f(2) = (2)^2 = 4, f(3) = (3)^2 = 9, f(4) = (4)^2 = 16$.

Thus, every moment in a has a unique image in B. Now, $\exists 25 \in B$ having no pre-image in A $\therefore f$ is into.

(ii) Let $A = \{-1, 1, 2, 3\}$ and $B = \{1, 4, 9\}$.

Let $f : A \to B : f(x) = x^2 \;\forall\, x \in A$



Then, each element in A has a unique image under f in B.

$\therefore f$ is a function from A to B such that

$f(-1) = (-1)^2 = 1, f(1) = (1)^2 = 1$

$f(2) = (2)^2 = 4,\; f(3) = (3)^2 = 9$

Clearly, two different elements, namely -1 and 1 have the same image $1 \in B$.
Hence f is many-one.

10. Show that the function

(i) $f : N \to N : f(x) = x^3$ is one-one into (ii) $f : Z \to Z : f(x) = x^3$ is one-one into

Sol. **(i)** Let N be the let of all natural number, let if: $N \to N, f(x) = x^3 \;\forall x \in N$

Then $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3$ $\therefore f$ is one-one.

Here $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3$

$\Rightarrow x_1 = x_2$ $\therefore$ function is one-one and there is no pre-image for the numbers which are not prefect cubes in N e.g., 7 in co-domain has no pre-image in domain.
not onto. Hence, function is not onto.

(ii) Let Z be the set of all integers. Let $f : z \to z, f(x) = x^3 \;\forall x \in R$

Then $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3$

$\Rightarrow x_1 = x_2$. $\therefore f$ is one-one. The explanation is same as in per-image question

11. Show that the function $f : R \to R : f(x) = \sin x$ is neither one-one nor onto.

Sol. As range is $[-1, 1]$ and Co-domain is R

$\therefore$ Range C co-domain $\therefore$ $f(x)$ is onto, also $f(0) = f(\pi) = 0$

Hence, $f(x)$ is many-one function.

12. Prove that the function $f: N \to N : f(n) = (n^2 + n + 1)$ is one-one but not onto.

Sol. $f: N \to N,\ f(x) = n^2 + n + 1$

Let $f(x_1) = f(x_2) \Rightarrow n_1^2 + n_1 + 1 = n_2^2 + n_2 + 1 \quad \Rightarrow (n_1^2 - n_2^2) + (n_1 - n_2) = 0$

$\Rightarrow (n_1 - n_2)(n_1 + n_2) + (n_1 - n_2) = 0 \quad \Rightarrow (n_1 - n_2)\{n_1 + n_2 + 1\} = 0$

$\Rightarrow (n_1 - n_2) = 0,\ (n_1 + n_2 + 1) \neq 0 \quad \Rightarrow n_1 = n_2$

13. Show that the function $f: N \to Z$, defined by $f(n) = \begin{cases} \frac{1}{2}(n-1), & \text{when } n \text{ is odd} \\ -\frac{1}{2}n, & \text{when } n \text{ is even} \end{cases}$

is both one-one and onto.

Sol. f is an injection: let $N, m \in N$ if n and m are even than $f(n) = f(m)$

$\Rightarrow \frac{-n}{2} = \frac{-m}{2} \Rightarrow -n = -m \Rightarrow n = m$

If n and m are odd then $f(n) = f(m)$

$\Rightarrow \frac{n-1}{2} = \frac{m-1}{2} \Rightarrow n - 1 = m - 1 \Rightarrow n = m$

Thus in both case we have $f(n) = f(m) \Rightarrow n = m$

And if n is even and m is odd then the $f(n) = f(m)$

$\Rightarrow -\frac{n}{2} = \frac{m-1}{2} \quad \Rightarrow n + m = 1$

Which is not possible because m, n are natural number. Now for every $+ve$ integer n terms is a number $2n+1$ in domain show that $f(2n+1) = \frac{1}{2}(2n+1-1) = n$ and for every negative number n in Z there is a number $-2n$ in N such that $f(-2n) = -\frac{1}{2}2n = n$ and for 0 in Z the number in N is 1 such that $f(1) = 0$

Hence, the function is one-one and onto.

14. Find the domain and range of the function $f: R \to R : f(x) = x^2 + 1$.

Sol. Domain $(f) = R$, Also $y = x^2 + 1 \Rightarrow x = \pm\sqrt{y-1}$

x is defined when $y - 1 \geq 0$ i.e. $y \geq 1$. $\therefore$ Range$(f) = \{y \in R : y \geq 1\}$

15. Which of the following relations are functions? Give reasons. In case of a function, find its domain and range.

(i) $f = \{(-1,2),(1,8),(2,11),(3,14)\}$ (ii) $g = \{(1,1),(1,-1),(4,2),(9,3),(16,4)\}$

(iii) $h = \{(a,b),(b,c),(c,b),(d,c)\}$

Sol. **(i)** $f = \{(-1,2),(1,8),(2,11),(3,14)\}$

f is a function, domain $(f) = \{-1,1,2,3\}$ and range $(f) = \{2,8,11,14\}$

(ii) $g = \{(1,1),(1,-1),(4,2),(9,3),(16,4)\}$. g is not a function.

(iii) h is a function, domain $(h) = \{a,b,c,d\}$ and range $(h) = \{b,c\}$.

16. Find the domain and range of the real function, defined by $f(x)=\frac{x^2}{(1+x^2)}$. Show that f is many-one.

Sol. When x is real, $1+x^2 \neq 0$, so, domain $(f)=R$

$$y=\frac{x^2}{1+x^2} \Rightarrow y+x^2y=x^2 \Rightarrow y=x^2-x^2y \Rightarrow y=x^2(1-y) \Rightarrow x^2=\frac{y}{1-y} \Rightarrow x=\sqrt{\frac{y}{1-y}}$$

For x to be real, $\frac{y}{1-y} \geq 0$ and $(1-y) \neq 0$. $\therefore$ $y(1-y) \geq 0 \Rightarrow 0 \leq y \leq 1$

$\therefore$ Range $(f)=\{y \in R: 0 \leq y \leq 1\}$

Also 1 and -1 have the same image $\left(\frac{1}{2}\right)$. Hence f is many-one

17. Show that the function $f: R \to R: f(x)=\begin{cases} 1, \text{if } x \text{ is rational} \\ -1, \text{if } x \text{ is irrational} \end{cases}$

is many-one into. Find (i) $f\left(\frac{1}{2}\right)$ (ii) $f(\sqrt{2})$ (iii) $f(\pi)$ (iv) $f(2+\sqrt{3})$.

Sol. Since all rational have the same image, namely 1, so f is many one.

Range $(f)=\{-1,1\} \subset R$, so f is into

(i) $f\left(\frac{1}{2}\right)=1$ (ii) $f(\sqrt{2})=-1$ (iii) $f(\pi)=-1$ (iv) $f(2+\sqrt{3})=-1$

EXERCISE 2B (Pg. No.: 42)

1. Let $A=\{1,2,3,4\}$. Let $f: A \to A$ and $g: A \to A$, defined by $f=\{(1,4),(2,1),(3,3),(4,2)\}$ and $g=\{(1,3),(2,1),(3,2),(4,4)\}$. Find (i) gof (ii) fog (iii) fof.

Sol. **(i)** Here range $(f)=\{1,2,3,4\}$ and domain (g) $=\{1,2,3,4\}$. Clearly, range $(f) \subseteq$ domain (g).

(ii) $\therefore gof$ is defined and domain (gof) = domain(f) $=\{1,2,3,4\}$

Now, $(gof)(1)=g\{f(1)\}=g(4)=4$; $(gof)(2)=g\{f(2)\}=g(1)=3$

$(gof)(3)=g\{f(3)\}=g(3)=2$; $(gof)(4)=g\{f(4)\}=g(2)=1$

Hence $gof=\{(1,4)(2,3)(3,2)(4,1)\}$

(iii) $\therefore$ fog is defined and domain (fog)= domain $(g)=\{1,2,3,4\}$

Now, $(fog)(1)=f\{g(1)\}=f(3)=3$; $(fog)(2)=f\{g(2)\}=f(1)=4$

$(fog)(3)=f(g(3))=f(3)=2$; $(fog)(4)=f\{g(4)\}=f\{4\}=2$

Hence fog $=\{(1,3),(2,4),(3,2),(4,2)\}$

(iv) $\therefore$ fof is defined and domain (fof) = domain (f) = $\{1,2,3,4\}$

Now, $(fof)(1)=f\{f(1)\}=f(4)=2$; $(fof)(2)=f\{f(2)\}=f(1)=4$

$(fof)(3)=f\{f(3)\}=f\{3\}=3$; $(fof)(4)=f\{f(4)\}=f\{2\}=1$

$\therefore fof=\{(1,2),(2,4),(3,3),(4,1)\}$

2. Let $f:\{3,9,12\}\to\{1,3,4\}$ and $g:\{1,3,4,5\}\to\{3,9\}$ be defined as $f=\{(3,1),(9,3),(12,4)\}$ and $g=\{(1,3),(3,3),(4,9),(5,9)\}$. Find (i) (*gof*) (ii) (*fog*).

Sol. **(i)** $\therefore gof$ is defined and domain $(gof)=$ domain $(f)=\{3,9,12\}$

$(gof)(3)=g\{f(3)\}=g\{1\}=3$; $(gof)(9)=g\{f(9)\}=g\{3\}=3$

$(gof)(12)=g\{f(12)\}=g\{4\}=9$; $\therefore gof=\{(3,3),(9,3),(12,9)\}$

(ii) $\therefore fog$ is defined and domain $(fog)=$ domain $(g)=\{1,3,4,5\}$

$(fog)(1)=f\{g(1)\}=f\{3\}=1$; $(fog)(3)=f\{g(3)\}=f\{3\}=1$

$(fog)(4)=f\{g(4)\}=f\{9\}=3$; $(fog)(5)=f\{g(5)\}=f\{9\}=3$

$\therefore fog=\{(1,1),(3,1),(4,3)(5,3)\}$

3. Let $f:R\to R:f(x)=x^2$ and $g:R\to R:g(x)=(x+1)$. Show that $(gof)\neq(fog)$.

Sol. Let $f(x)=x^2$, and $g(x)=x+1$

L.H.S., $gof=gof(x)=g\{f(x)\}=g\{x^2\}=(x^2)+1=x^2+1$

R.H.S., $fog=fog(x)=f\{g(x)\}=f\{x+1\}=(x+1)^2=x^2+2x+1$

Hence $gof\neq fog$

4. Let $f:R\to R:f(x)=(2x+1)$ and $g:R\to R:g(x)=(x^2-2)$. Write down the formulae for

(i) (*gof*) (ii) (fog) (iii) (fof) (iv) (gog).

Sol. Let $f(x)=2x+1$ and $g(x)=x^2-2$

(i) $gof=gof(x)=g\{f(x)\}=g\{2x+1\}=(2x+1)^2-2=4x^2+4x+1-2=4x^2+4x-1$

(ii) $fog=fog(x)=f\{g(x)\}=f\{x^2-2\}=2\{x^2-2\}+1=2x^2-4+1=(2x^2-3)$

(iii) $fof=fof(x)=f\{f(x)\}=f(2x+1)=2(2x+1)+1=4x+2+1=4x+3$

(iv) $gog(x)=g\{g(x)\}=g\{x^2-2\}=(x^2-2)^2-2=x^4-4x^2+4-2=x^4-4x^2+2$

5. Let $f:R\to R:f(x)=x^2+3x+1$ and $g:R\to R:g(x)=2x-3$ write down the formula for

(i) *gof* (ii) *fog* (iii) *gog*

Sol. Let $f(x)=x^2+3x+1$, and $g(x)=2x-3$

(i) $gof=gof(x)=g\{f(x)\}=g(x^2+3x+1)=2(x^2+3x+1)-3$

$=2x^2+6x+2-3=2x^2+6x-1$

(ii) $fog=fog(x)=f\{g(x)\}=f\{2x-3\}=(2x-3)^2+3(2x-3)+1$

$=4x^2-12x+9+6x-9+1=4x^2-6x+1$

(iii) $gog(x)=g\{g(x)\}=g\{2x-3\}=2(2x-3)-3=4x-6-3=4x-9$

6. Let $f: R \to R: f(x) = |x|$, prove that $fof = f$

Sol. $f(x) = |x|, fof = f\{f(x)\} = f(|x|) = |x|$ ... (i)

$f = |x|$ ... (ii)

Hence from (i) and (ii) $fof = f$

7. Let $f: R \to R: f(x) = x^2, g: R \to R: g(x) = \tan x$ and $h: R \to R: h(x) = \log x$.

Find a formula for $ho(gof)$. Show that $[ho(gof)]\sqrt{\frac{\pi}{4}} = 0$.

Sol. Let $f(x) = x^2$, $g(x) = \tan x$, $h(x) = \log x$

$$ho(gof) = ho\{gof(x)\} = ho\{g(f(x))\}$$

$$= ho\{g(x^2)\} = ho\{\tan(x^2)\} = h\{\tan(x^2)\} = \log\{\tan(x^2)\}$$

Now, $[ho(gof)]\sqrt{\frac{\pi}{4}} = \log\left\{\tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right\} = \log\left(\tan\frac{\pi}{4}\right) = \log(1) = 0$

8. Let $f: R \to R: f(x) = (2x-3)$ and $g: R \to R: g(x) = \frac{1}{2}(x+3)$. Show that $(fog) = I_R = (gof)$.

Sol. Let $f(x) = 2x - 3$, and $g(x) = \frac{x+3}{2}$

$$fog = fog(x) = f\{g(x)\} = f\left\{\frac{x+3}{2}\right\} = 2\left(\frac{x+3}{2}\right) - 3 = x + 3 - 3 = x$$

$$gof = gof(x) = g\{f(x)\} = g\{2x-3\} = \frac{2x-3+3}{2} = \frac{2x}{2} = x. \quad \therefore gof = fog = I_R$$

9. Let $f: Z \to Z: f(x) = 2x$. Find $g: Z \to Z: gof = I_z$.

Sol. Let $f(x) = 2x$ $gof = gof(x) = g\{f(x)\} = g\{2x\}$.

Let $x \in Z$ then x, $gof = Iz \Rightarrow (gof)(x) = Iz(x) \Rightarrow g[f(x)] = x$

$\Rightarrow g(2x) = x \Rightarrow g(y) = \frac{1}{2}y$, where $2x = y$. Thus $g: z \to z: g(y) = \frac{1}{2}y$.

10. Let $f: N \to N: f(x) = 2x, g: N \to N: g(y) = 3y + 4$ and $h: N \to N: h(z) = \sin z$.

Show that $ho(gof) = (hog)of$.

Sol. Let $f(x) = 2x$, $g(y) = 3y + 4, h(z) = \sin z$

L.H.S. $ho(gof) = ho\{gof(x)\} = ho\{g(f(x))\} = ho\{g(2x)\} = ho\{3(2x)+4\}$

$$= ho(6x+12) = h(6x+12) = \sin(6x+12)$$

R.H.S $(hog)\ of = (hog)f(x) = (hog)(2x)$

$$= ho\{g(2x)\} = h\{g(2x)\} = h\{3(2x+4)\} = h(6x+12) = \sin(6x+12)$$

Hence $ho(gof) = (hog)of$

11. If f be a greatest integer function and g be an absolute value function, find the value of $(fog)\left(\frac{-3}{2}\right)+(gof)\left(\frac{4}{3}\right)$.

Sol. $(fog)\left(\frac{-3}{2}\right)=f\left\{g\left(\frac{-3}{2}\right)\right\}=f\left\{\left|\frac{-3}{2}\right|\right\}=f\left(\frac{3}{2}\right)=\left[\frac{3}{2}\right]=1$

$(gof)\left(\frac{4}{3}\right)=g\left\{f\left(\frac{4}{3}\right)\right\}=g\left[\frac{4}{3}\right]=g(1)=|1|=1$. Required sum $=(1+1)=2$

12. Let $f:R\to R:f(x)=x^2+2$ and $g:R\to R:g(x)=\frac{x}{x-1},x\neq 1$. Find the fog and gof and hence find $(fog)(2)$ and $(gof)(-3)$

Sol. (fog)x = f {g(x)}

$=f\left\{\frac{x}{x-1}\right\}=\left\{\frac{x}{x-1}\right\}^2+2=\frac{x^2+2(x-1)^2}{(x-1)^2}=\frac{x^2+2(x^2-2x+1)}{(x-1)^2}=\frac{x^2+2x^2-4x+2}{(x-1)^2}=\frac{3x^2-4x+2}{(x-1)^2}$

(gof)x = g {f(x)}

$\Rightarrow$ (gof)x = g $\{x^2+2\}\Rightarrow (gof)x=\frac{x^2+2}{x^2+2-1}=\frac{x^2+2}{x^2+1}$

Now, (fog) (2)

$=\left(\frac{2}{2-1}\right)^2+2$

$=6$

and, (gof) (2)

$=\frac{2^2+2}{2^2+1}=\frac{6}{5}$

EXERCISE 2C (Pg.No.: 53)

Very-short-Answer Questions

1. Prove that the function $f:R\to R:f(x)=2x$ is one-one and onto.

Sol. $f(x_1)=f(x_2)\Rightarrow 2x_1=2x_2\Rightarrow x_1=x_2$. So, f is one-one

Let $y=2x$ Then $x=\frac{1}{2}y$.

Thus for each y in co domain R, there exists $\frac{1}{2}y$ such that $f\left(\frac{1}{2}y\right)=\left(2\times\frac{1}{2}y\right)=y$

$\therefore f$ is onto.

2. Prove that the function $f:N\to N:f(x)=3x$ is one-one and into.

Sol. $f(x_1)=f(x_2)\Rightarrow 3x_1=3x_2\Rightarrow x_1=x_2$, so f is one-one.

If we consider 2 in co domain N, there is no natural whose image is 2. So, f is into.

3. Show that the function $f:R\to R:f(x)=x^2$ is neither one-one nor onto.

Sol. Clearly $f(1)=(1)^2=1$ and $f(-1)=(-1)^2=1$. So, f is many one.

If we consider -1 in the co domain R, then there is no elements in R, whose square is -1.

$\therefore -1\in R$ has no pre-image in R, so f is many one into.

4. Show that the function $f: N \to N: f(x) = x^2$ is one-one and into.

Sol. $f(x_1) = f(x_2) \quad \Rightarrow x_1^2 = x_2^2 \quad \Rightarrow (x_1 - x_2)(x_1 + x_2) = 0$

$\Rightarrow x_1 - x_2 = 0 \quad \Rightarrow x_1 = x_2 \quad [\because x_1, x_2 \in N] \quad \therefore f$ is one-one.

If we consider 2 is the co domain N, then $\sqrt{2} \notin N$ and $f(\sqrt{2}) = (\sqrt{2})^2 = 2$. So, f is into.

5. Show that the function $f: R \to R: f(x) = x^4$ is neither one-one nor onto.

Sol. $f(1) = (1)^4 = 1$ and $f(-1) = (-1)^4 = 1$. So f is many-one

If we consider -1 in the co-domain in the co-domain R,

then there exists no $x \in R$ such that $f(x) = x^4 = -1$ so, f is into.

6. Show that the function $f: Z \to Z: f(x) = x^3$ is one-one and into.

Sol. Let $x_1, x_2 \in z$ and $f(x_1) = f(x_2)$

$\Rightarrow x_1^3 = x_2^3 \quad \Rightarrow x_1 = x_2 \quad \Rightarrow f$ is one-one. Let $2 \in Z$. Then, these exists no $x \in Z$ such that $x^3 = 2$.

Thus $2 \in Z$ has no pre-image in Z. so f is into.

7. Let R_0 be the set of all nonzero real numbers. Then, show that the function $f: R_0 \to R_0: f(x) = \frac{1}{x}$ is one-one and onto.

Sol. $f(x_1) = f(x_2) \quad \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2$. So, f is one-one.

$y = \frac{1}{x} \quad \Rightarrow x = \frac{1}{y}$

Thus, for each y in co domain R_0, these exists $\frac{1}{y}$ in domain R_0 such that $f\left(\frac{1}{y}\right) = \frac{1}{1/y} = y$

So, f is onto.

8. Show that the function $f: R \to R: f(x) = 1 + x^2$ is many-one into.

Sol. $f(-1) = (-1)^2 + 1 = 2$ and $f(1) = 1 + (1)^2 = 2$, $\therefore$ so, f is many one.

$y = (1 + x^2) \quad \Rightarrow x = \sqrt{y - 1}$

So, when $y < 1$, then $\sqrt{y-1}$ is imaginary in particular, $O \in R$ has no pre image in R. $\therefore f$ is into.

9. Let $f: R \to R: f(x) = \frac{2x - 7}{4}$ be an invertible function. Find f^{-1}

Sol. $y = \frac{2x - 7}{4} \quad \Rightarrow 4y = 2x - 7 \quad \Rightarrow 4y + 7 = 2x \quad \Rightarrow x = \frac{4y + 7}{2}$

$f^{-1}(y) = \frac{4y + 7}{2}$ for all $y \in R$.

10. Let $f: R \to R: f(x) = 10x + 3$. Find f^{-1}.

Sol. $f(x) = 10x + 3 \quad \Rightarrow y = 10x + 3 \quad \Rightarrow y - 3 = 10x \quad \Rightarrow x = \frac{y - 3}{10} \quad \Rightarrow f^{-1}(y) = \frac{y - 3}{10}$

11. $f: R \to R: f(x) = \begin{cases} 1, \text{if } x \text{ is rational} \\ -1, \text{if } x \text{ is irational} \end{cases}$. Show that f is many-one and into.

Sol. Since all rationals have the same image namely 1, so f is many one.

Range $(f) = \{-1,1\} \subset R$ so f is onto.

12. Let $f(x) = x+7$ and $g(x) = x-7$, $x \in R$. Find $(fog)(7)$.

Sol. Let $f(x) = x+7$ and $g(x) = x-7$

$fog = fog\,(x) = f\{g(x)\} = f(x-7) = x-7+7 \Rightarrow fog = x \Rightarrow fog\,(7) = 7$

13. Let $f: R \to R$ and $g: R \to R$ defined by $f(x) = x^2$ and $g(x) = (x+1)$. Show that $g\,of \neq f\,og$.

Sol. $f(x) = x^2$, and $g(x) = x+1$

$gof = gof\,(x) = g\{f(x)\} = g\{(x^2)\} = x^2+1; \quad fog = fog\,(x) = f\{g(x)\} = f(x+1) = (x+1)^2$

Hence $gof \neq fog$.

14. Let $f: R \to R: f(x) = (3-x^3)^{\frac{1}{3}}$. Find fof.

Sol. $f(x) = (3-x^3)^{\frac{1}{3}}$

$\Rightarrow fof\,(x) = f\{f(x)\} = f\{3-x^3\}^{\frac{1}{3}} = \left[3-\left\{(3-x^3)^{\frac{1}{3}}\right\}^3\right]^{\frac{1}{3}} = \left[3-(3-x^3)\right]^{\frac{1}{3}} = \left[x^3\right]^{\frac{1}{3}} = x$

15. Let $f: R \to R: f(x) = 3x+2$, find $f\{f(x)\}$

Sol. f(x) = 3x + 2

$\Rightarrow$ f{f(x)} = f(3x +2)

= 3(3x + 2) + 2= 9x + 6 + 2= 9x + 8

16. Let $A = \{1,3,4\}, B = \{1,2,5\}$ and $C = \{1,3\}$. Let $f: A \to B$ and $g: B \to C$, defined as

$f = \{(1,2),(3,5),(4,1)\}$ and $g = \{(1,3),(2,3),(5,1)\}$. Write down $(g\,of)$.

Sol. Dom $(gof) = \text{Dom}(f) = \{1,3,4\}$

$\Rightarrow gof\,(1) = g\{f(1)\} = g(1) = 3 \quad \Rightarrow gof\,(3) = g\{f(3)\} = g(5) = 1$

$\Rightarrow gof\,(4) = g\{f(4)\} = g(1) = 3. \quad \therefore (gof) = \{(1,3)(3,1)(4,3)\}$.

17. Let $A = \{1,2,3,4\}$ and $f = \{(1,4),(2,1)(3,3),(4,2)\}$. Write down $(f\,of)$.

Sol. Domain $(fof) = \text{Dom}(f) = \{1,2,3,4\}$

$fof\,(1) = f\{f(1)\} = f(4) = 2 \quad \Rightarrow fof\,(2) = f\{f(2)\} = f(1) = 4$

$fof\,(3) = f\{f(3)\} = f(3) = 3 \quad \Rightarrow fof\,(4) = f\{f(4)\} = f(2) = 1$

$\therefore fof = \{(1,2)(2,4)(3,3)(4,1)\}$

18. Let $f(x)=8x^3$ and $g(x)=x^{\frac{1}{3}}$. Find gof and fog.

Sol. $f(x)=8x^3$ and $g(x)=x^{\frac{1}{3}}$

$$gof=gof(x)=g\{f(x)\}=g(8x^3)=(8x^3)^{\frac{1}{3}}=8^{\frac{1}{3}}.x=2x$$

19. Let $f:R\to R:f(x)=10x+7$. Find the function $g:R\to R:g\ of=f\ og=I_R$.

Sol. Let $f(x)=10x+7$

$$gof=I_R\Rightarrow gof(x)=Ig(x)=x\Rightarrow g\{f(x)\}=x\Rightarrow g\{10x+7\}=x$$

Put $10x+7=y$ then $x=\frac{y-7}{10}$. $\therefore g(y)=\frac{y-7}{10}$. Hence $g:R\to R:g(x)=\frac{x-7}{10}$ for all $x\in R$.

20. Let $A=\{1,2,3\},B=\{4,5,6,7\}$ and let $f=\{(1,4),(2,5),(3,6)\}$ be a function from A to B. State whether f is one-one

Sol. Here, f(1) = 4,

f(2) = 5

and f(3) =6

$\because$ defferent elements in A have different images in B,

Hence f is one – one

EXERCISE 2D (Pg.No.: 56)

1. Let $A=\{2,3,4,5\}$ and $B=\{7,9,11,13\}$, and let $f=\{(2,7),(3,9),(4,11),(5,13)\}$.

Show that f is invertible and find f^{-1}.

Sol. We have $f(2)=7,\ f(3)=9,\ f(4)=11,\ f(5)=13$.

Dom $(f)=\{2,3,4,5\}=A$ and range $(f)=\{7,9,11,13\}=B$.

Clearly, different element in A have different image. $\therefore f$ is one-one.

Rang $(f)=B\Rightarrow f$ is onto. Thus f is one - one onto and there fore invertible.

Now $f(2)=7, f(3)=9, f(4)=11, f(5)=13$

$\Rightarrow f^{-1}(7)=2,\ f^{-1}(9)=3,\ f^{-1}(11)=4,\ f^{-1}(13)=5$. Hence $f^{-1}\{(7,2),(9,3),(11,4),(3,5)\}$.

2. Show that the function $f:R\to R:f(x)=2x+3$ is invertible and find f^{-1}.

Sol. We have $f(x_1)=f(x_2)\Rightarrow 2x_1+3=2x_2+3\Rightarrow 2x_1=2x_2\Rightarrow x_1=x_2$. $\therefore f$ is one-one.

Again $y=2x+3\Rightarrow y-3=2x\Rightarrow x=\frac{y-3}{2}$

Now, if $y\in R$ (co-domain of f), then these exists $x=\frac{y-3}{2}\in R$.

Such that $f(x)=f\left(\frac{y-3}{2}\right)=2\left(\frac{y-3}{2}\right)+3=y$. $\therefore f$ is onto.

Thus f is one-one onto and there fore invertible.

Now $y=f(x)\Rightarrow y=2x+3\Rightarrow x=\frac{y-3}{2}\Rightarrow f^{-1}(y)=\frac{y-3}{2}$ $\left[\because f(x)=y\Rightarrow x=f^{-1}(y)\right]$

Thus, we define : $f^{-1}: R \to R: f^{-1}(y)=\frac{y-3}{2}$ for all $y \in R$

3. Let $f: Q \to Q: f(x)=3x-4$. Show that f is invertible and find f^{-1}.

Sol. We have $f(x_1)=f(x_2) \Rightarrow 3x_1-4=3x_2+4 \Rightarrow 3x_1=3x_2 \Rightarrow x_1=x_2$. $\therefore f$ is one-one.

Again $y=3x-4 \Rightarrow y+4=3x \Rightarrow \frac{y+4}{3}=x$.

Now, if $y \in R$ (co-domain of f), then these exists $x=\frac{y+4}{3} \in R$.

Such that $f(x)=f\left(\frac{y+4}{3}\right)=3\left(\frac{y+4}{3}\right)-4=y$. $\therefore f$ is onto.

Thus, f is one one onto and there fore invertible.

Now $y=f(x) \Rightarrow y=4x-3 \Rightarrow x=\frac{y+4}{3} \Rightarrow f^{-1}(y)=\frac{y+4}{3}$ $[\because f(x)=y \Rightarrow x=f^{-1}(y)]$

Thus we define : $f^{-1}: R \to R: f^{-1}(y)=\frac{y+4}{3}$ for all $y \in R$.

4. Let $f: R \to R: f(x)=\frac{1}{2}(3x+1)$. Show that f is invertible and find f^{-1}.

Sol. We have $f(x_1)=f(x_2) \Rightarrow \frac{3x_1+1}{2}=\frac{3x_2+1}{2} \Rightarrow 3x_1+1=3x_2+1 \Rightarrow 3x_1=3x_2 \Rightarrow x_1=x_2$

$\therefore$ f is one-one. Again $y=\frac{3x+1}{2} \Rightarrow 2y=3x+1 \Rightarrow 2y-1=3x \Rightarrow x=\frac{2y-1}{3}$

Now, if $y \in R$ (co-domain of f), then there exists, $x=\frac{2y-1}{3} \in R$.

Such that $f(x)=f\left(\frac{2y-1}{3}\right)=\frac{3\left(\frac{2y-1}{3}\right)+1}{2}=y$. $\therefore f$ is onto.

Thus, f is one - one and therefore invertible.

Now, $y=f(x) \Rightarrow y=\frac{3x+1}{2} \Rightarrow x=\frac{2y-1}{3}$ $[\because f(x)=y \Rightarrow x=f^{-1}(y)]$

Thus we define, $f^{-1}: R \to R: f^{-1}(y)=\frac{2y-1}{3}$ for all $y \in R$.

5. If $f(x)=\frac{(4x+3)}{(6x-4)}, x \neq \frac{2}{3}$, show that $(fof)(x)=x$ for all $x \neq \frac{2}{3}$. Hence, find f^{-1}.

Hint : $fof=I \Rightarrow f^{-1}=f$.

Sol. $f(x)=\frac{4x+3}{6x-4}$; $fof=fof(x)=fo\{f(x)\}=f\left\{\frac{4x+3}{6x-4}\right\}$

$$=\frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}=\frac{\frac{16x+12+18x-12}{6x-4}}{\frac{24x+18-24x+16}{6x-4}}=\frac{24x}{24}=x$$

$\therefore$ Hence $(fof)(x)=x$; $fof=I \Rightarrow f^{-1}=I \Rightarrow f^{-1}=\frac{4y+3}{6y-4}$

6. Show that the function f on $A = R - \left\{\frac{2}{3}\right\}$, defined as $f(x) = \frac{4x+3}{6x-4}$ is one-one and onto. Hence find f^{-1}

Sol. one – one f :→

Let, x, and $x_2 \in$ domain (f)

Now, $f(x_1) = f(x_2)$

$\Rightarrow \frac{4x_1+3}{6x_1-4} = \frac{4x_2+3}{6x_2-4} \Rightarrow (4x_1+3)(6x_2-4) = (4x_2+3)(6x_1-4)$

$\Rightarrow 24x_1x_2 - 16x_1 + 18x_2 - 12 = 24x_1x_2 - 16x_2 + 18x_1 - 12 \Rightarrow 34x_2 = 34x_1 \Rightarrow x_1 = x_2$

Hence, f is one – one.

Onto:→

Let, $y \in$ co-domain (f)

Now, y = f(x)

$\Rightarrow y = \frac{4x+3}{6x-4} \Rightarrow 6xy - 4y = 4x + 3 \Rightarrow 6xy - 4x = 3 + 4y \Rightarrow x(6y-4) = 3 + 4y$

$\Rightarrow x = \frac{3+4y}{6y-4} \in$ domain $\forall\ y \in$ co-domain

Hence, f is an on to function

Since, $x = \frac{3+4y}{6y-4}, y \neq \frac{2}{3}$ $\quad \therefore f^{-1}(y) = \frac{3+4y}{6y-4}, y \neq \frac{2}{3}$ Ans.

7. Show that the function $f : N \to N : f(x) = x^2$ is one-one and into.

Sol. $f(x_1) = f(x_2) \quad \Rightarrow x_1^2 = x_2^2 \quad \Rightarrow (x_1 - x_2)(x_1 + x_2) = 0$

$\Rightarrow x_1 - x_2 = 0 \quad \Rightarrow x_1 = x_2 \quad [\because x_1, x_2 \in N] \quad \therefore f$ is one-one.

If we consider 2 is the co domain N, then $\sqrt{2} \notin N$ and $f(\sqrt{2}) = (\sqrt{2})^2 = 2$. So, f is into.

8. Let R_0 be the set of all nonzero real numbers. Then, show that the function $f : R_0 \to R_0 : f(x) = \frac{1}{x}$ is one-one and onto.

Sol. $f(x_1) = f(x_2) \quad \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2$. So, f is one-one.

$y = \frac{1}{x} \quad \Rightarrow x = \frac{1}{y}$

Thus, for each y in co domain R_0, these exists $\frac{1}{y}$ in domain R_0 such that $f\left(\frac{1}{y}\right) = \frac{1}{1/y} = y$

So, f is onto.

9. Let $f : N \to R : f(x) = 4x^2 + 12x + 15$. show that $f : N \to$ range (f) is invertible. Find f^{-1}

Sol. Let, x_1 and x_2 ε N (domain f)

Now, $f(x_1) = f(x_2)$

$\Rightarrow 4x_1^2 + 12x_1 + 15 = 4x_2^2 + 12x_2 + 15 \Rightarrow 4x_1^2 - 4x_2^2 + 12x_1 - 12x_2 = 0$

$\Rightarrow 4(x_1^2 - x_2^2) + 12(x_1 - x_2) = 0 \Rightarrow 4(x_1 - x_2)(x_1 + x_2) + 12(x_1 - x_2) = 0$

$\Rightarrow (x_1 - x_2)\{4(x_1 + x_2) + 12\} = 0$

$\because x_1 \ \&\ x_2 \in N \quad \therefore 4(x_1 + x_2) + 12 \neq 0 \quad \therefore x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$

Hence, f is one- one

$\because$ co-domain = range (given) $\therefore$ f is an onto function.

$\because$ f is one – one onto function $\therefore$ f is an invertible function.

Let, y = f(x)

$\Rightarrow y = 4x^2 + 12x + 15 \quad \Rightarrow 4x^2 + 12x + (15 - y) = 0$

$$\Rightarrow x = \frac{-12 + \sqrt{12^2 - 4 \times 4 \times (15 - y)}}{2 \times 4} \Rightarrow x = \frac{-12 + \sqrt{144 - 240 + 16y}}{8}$$

$$\Rightarrow x = \frac{-12 + \sqrt{16y - 96}}{8} \Rightarrow x = \frac{-12 + 4\sqrt{y - 6}}{8} \Rightarrow x = \frac{\sqrt{y - 6} - 3}{2} \Rightarrow \overset{-1}{f}(y) = \frac{\sqrt{y - 6} - 3}{2} \text{ Ans.}$$

10. Let $f : N - \{2\}$ and $B = R - \{1\}$. If $f : A \to B : f(x) = \frac{x - 1}{x - 2}$, show that f is one-one and onto. Hence, find f^{-1}

Sol. Let, x_1 & x_2 ε domain(f)

Now, $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1 - 1}{x_1 - 2} = \frac{x_2 - 1}{x_2 - 2} \Rightarrow (x_1 - 1)(x_2 - 2) = (x_1 - 2)(x_2 - 1)$$

$$\Rightarrow x_1 \cdot x_2 - 2x_1 - x_2 + 2 = x_1x_2 - x_1 - 2x_2 + 2 \Rightarrow -2x_1 + x_1 = -2x_2 + x_2 \Rightarrow -x_1 = -x_2 \Rightarrow x_1 = x_2$$

Hence, f is on – one fanc

Let, y $\in$ co-domain f.

Now, y = f(x)

$$\Rightarrow y = \frac{x - 1}{x - 2} \Rightarrow xy - 2y = x - 1 \Rightarrow xy - x = 2y - 1 \Rightarrow x(y - 1) = 2y - 1$$

$$\Rightarrow x = \frac{2y - 1}{y - 1} \in \text{ domain, } \forall\, y \neq 1$$

Hence, f is an onto function.

Here, $x = \frac{2y - 1}{y - 1} \qquad \Rightarrow f^{-1}(y) = \frac{2y - 1}{y - 1} \; \forall\, y \neq 1$

11. Let f and g be two functions from R into R, defined by $f(x) = |x| + x$ and $g(x) = |x| - x$ fro all $x \in R$. Find $f o g$ and $g o f$

Sol. Given, f(x) = |x| + x & g(x) = |x| - x

Now, (fog)x = f{g(x)}

=f { |x| - x } = | |x| - x| + |x| - x

If, x < 0. Then. |x| = -x, we have(fog) x = | -x – x| - x – x = |-2x| - 2x= - 2x – 2x = -4x

If, x = 0

(fog)x = 0

If x > 0, then, |x| = x, we have (fog)x = |x – x| + x – x = 0

$$\text{Hence, } (fog)x = \begin{cases} -4x, & \text{if } x < 0 \\ 0, & \text{if } x \geq 0 \end{cases}$$

Now, (got)x = g { f(x)} = g { |x| + x}= | | x | + x| - { | x | + x}

If, x < 0, we have (gof) x = | - x + x| - {-x + x} = 0

If, x ≠ 0, we have (gof)x = 0 If, x > 0, we have. (gof) x = |x + x| - {x + x}

= |2x| - 2x= 2x – 2x= 0

Hence, (gof)x = 0