Exercise 10.R

Q1CC

(a).

Suppose that x and y are both given as functions of a third variable t (called a parameter) by the equations x = f(t), y = g(t) (called parametric equations). Each value of t determines a point (x,y) which can plot in a coordinate plane. As t varies the point (x,y)=(f(t),g(t)) varies and traces out a curve C which we called a parametric curve.

(b).

Each value of t determines a point (f(t),g(t)) which we can plot in a coordinate plane.

Q1E

Parametric equations are $x = t^2 + 4t$ ---(1)

And $y = 2 - t$ ....(2)

Where $-4 \le t \le 1$

We sketch the curve with taking different values of t in the interval $[-4, 1]$

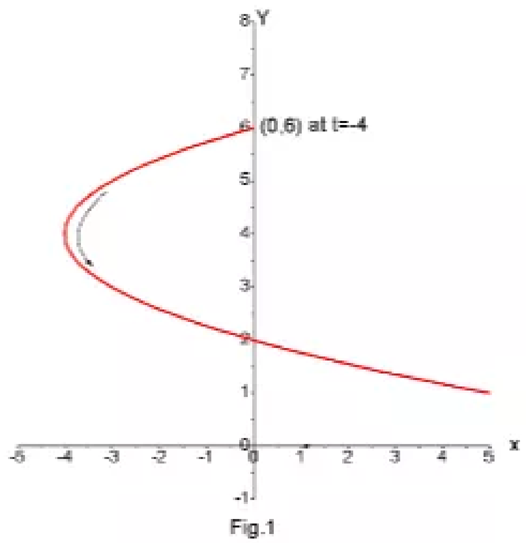


Fig.1

From equation (2) we have $y = 2 - t$

$$\Rightarrow t = 2 - y$$

Putting this value of t in equation (1)

$$\Rightarrow x = (2-y)^2 + 4(2-y)$$

$$= 4 + y^2 - 4y + 8 - 4y$$

$$\Rightarrow x = y^2 - 8y + 16 - 4$$

$$\Rightarrow (x+4) = (y-4)^2$$

Which represents parabola with vertex (-4, 4)

$$\boxed{x = y^2 - 8y + 12}$$

Q1P

Given parametric equations are

$$x = \int_1^t \frac{\cos u}{u} du, \qquad y = \int_1^t \frac{\sin u}{u} du$$

By fundamental theorem, we differentiate x and y with respect t

$$\frac{dx}{dt} = \frac{\cos t}{t} \qquad and \qquad \frac{dy}{dt} = \frac{\sin t}{t}$$

For vertical tangents we must have $\frac{dx}{dt} = 0$

$$\Rightarrow \frac{\cos t}{t} = 0$$

$$\Rightarrow \cos t = 0$$

$$\Rightarrow t = \pi/2$$

and for $t = 1$, $(x, y) = (0, 0)$

So nearest tangents occurs when $\boxed{t = \pi/2}$

So length between these points.

$$L = \int_1^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 \left(\frac{dy}{dt}\right)^2} \; dt$$

$$= \int_1^{\pi/2} \sqrt{\frac{\cos^2 t}{t^2} + \frac{\sin^2 t}{t^2}} \; dt$$

$$= \int_0^{\pi/2} \sqrt{\frac{1}{t^2}} \; dt \qquad \left[\sin^2\theta + \cos^2\theta = 1\right]$$

$$= \int_1^{\pi/2} \frac{1}{t} dt \qquad = [\ln t]_1^{\pi/2}$$

$$\Rightarrow \boxed{L = \ln(\pi/2)}$$

Q1TFQ

Given $x = f(t)$ and $y = g(t)$, $g'(1) = 0$

Hence the curve is horizontal tangent when t = 1.

Since $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$,

if $\frac{dy}{dt} = 0$, $\frac{dx}{dt} \neq 0$ then the curve is horizontal tangent.

$\boxed{\text{Hence the given statement is true.}}$

Q2CC

(a).

The slope of a tangent to parametric curve is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, if $\frac{dx}{dt} \neq 0$

(b).

If the curve is traced out once the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \le t \le \beta$

then the area under the parametric curve is $A = \int_{\alpha}^{\beta} g(t) f'(t)\, dt$

Q2E

Given parametric equations are $x = 1 + e^{2t}$ ....(1)

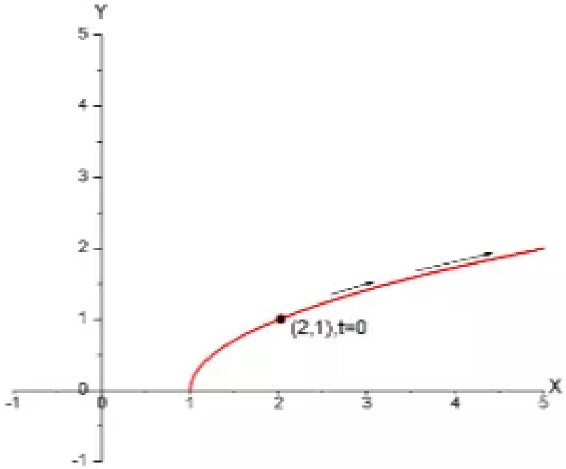
$y = e^{t}$ ....(2)

Since $y > 0$ $for\ all\ t$

Now $x = 1 + \left(e^{t}\right)^{2}$

$\Rightarrow \boxed{x = 1 + y^{2}}$ $, y > 0$

Now we sketch the curve



Q2P

(a)

How do you find the slope of a tangent to a parametric curve?

Suppose f and g are differentiable functions and

Parametric curve is defined by equations $x = f(t)$ and $y = g(t)$.

Where y is also a differentiable function of x

Then using Chain Rule

Write $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

If $\frac{dx}{dt} \ne 0$ then solve for $\frac{dy}{dx}$:

Rewrite the slope

$$\frac{dy}{dx} = \frac{dy / d\theta}{dx / d\theta}$$

This equation allows us to find the slope $\frac{dy}{dx}$ of tangent line without having to eliminate parameter t.

To see from equation that the curve has

Horizontal tangent when $\frac{dy}{dt}=0$ ($\frac{dx}{dt}\neq 0$)

Vertical tangent when $\frac{dx}{dt}=0$ ($\frac{dy}{dt}\neq 0$)

For example:

Consider the parametric equations,

$x=a(\theta-\sin\theta)$ And $y=a(1-\cos\theta), a>0$

To find $\frac{dy}{dx}$:

Recollect the formula $\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}$

So first find $\frac{dy}{d\theta}$

$y=a(1-\cos\theta)$

$\frac{dy}{d\theta}=a(0+\sin\theta)$ Since $\frac{d}{d\theta}(\cos\theta)=-\sin\theta$

$\frac{dy}{d\theta}=a\sin\theta$

For $\frac{dx}{d\theta}$ take $x=a(\theta-\sin\theta)$

$\frac{dx}{d\theta}=a(1-\cos\theta)$ Since $\frac{d}{dx}(\sin\theta)=\cos\theta$

Thus, $\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}$

$\frac{dy}{dx}=\frac{a\sin\theta}{a(1-\cos\theta)}$ Substitute $\frac{dy}{d\theta}=a\sin\theta, \frac{dx}{d\theta}=a(1-\cos\theta)$

Therefore, $\frac{dy}{dx}=\boxed{\frac{\sin\theta}{(1-\cos\theta)}}$.

To find $\frac{d^2y}{dx^2}$:

Recollect the formula

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}}$$

$$= \frac{\frac{d}{d\theta}\left(\frac{\sin\theta}{(1-\cos\theta)}\right)}{a(1-\cos\theta)}$$ Substitute $\frac{dy}{dx} = \frac{\sin\theta}{(1-\cos\theta)}, \frac{dx}{d\theta} = a(1-\cos\theta)$

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{(1-\cos\theta)\frac{d}{dx}(\sin\theta)-(\sin\theta)\frac{d}{dx}(1-\cos\theta)}{(1-\cos\theta)^2}\right)}{a(1-\cos\theta)}$$

Since $\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{v(x)\frac{d}{dx}(u(x))-u(x)\frac{d}{dx}(v(x))}{[v(x)]^2}$

Simplify the above steps,

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{(1-\cos\theta)\cos\theta-(\sin\theta)\sin\theta}{(1-\cos\theta)^2}\right)}{a(1-\cos\theta)}$$

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{\cos\theta-\cos^2\theta-\sin^2\theta}{(1-\cos\theta)^2}\right)}{a(1-\cos\theta)}$$

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{\cos\theta-(\cos^2\theta+\sin^2\theta)}{(1-\cos\theta)^2}\right)}{a(1-\cos\theta)}$$

To find the equation of the tangent line at the point $\theta = \frac{\pi}{6}$:

At $\theta = \frac{\pi}{6}$, then $\frac{dy}{dx} = \frac{\sin\theta}{(1-\cos\theta)}$

$$\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{6}} = \frac{\sin\left(\frac{\pi}{6}\right)}{\left(1-\cos\frac{\pi}{6}\right)}$$

$$\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{6}} = \frac{\frac{1}{2}}{\left(1-\frac{\sqrt{3}}{2}\right)}$$

$$\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{6}} = \frac{1}{\left(2-\sqrt{3}\right)}$$

$\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{6}} = 2+\sqrt{3}$ Rationalization

Hence, the equation of the tangent line at the point $\theta = \frac{\pi}{6}$ is $y - y_1 = m(x - x_1)$

$$\left[y - a\left(1-\cos\frac{\pi}{6}\right)\right] = \left(2+\sqrt{3}\right)(x - a\left(\frac{\pi}{6}-\sin\frac{\pi}{6}\right))$$

$$y - a\left(1-\frac{\sqrt{3}}{2}\right) = \left(2+\sqrt{3}\right)(x - a\left(\frac{\pi}{6}-\frac{1}{2}\right))$$

$y = \left(2+\sqrt{3}\right)(x - a\left(\frac{\pi}{6}-\frac{1}{2}\right)) + a\left(1-\frac{\sqrt{3}}{2}\right)$ Add on both sides with $a\left(1-\frac{\sqrt{3}}{2}\right)$

Therefore,

The equation of the tangent line at the point $\theta = \frac{\pi}{6}$ is

$$\boxed{y = \left(2+\sqrt{3}\right)(x - a\left(\frac{\pi}{6}-\frac{1}{2}\right)) + a\left(1-\frac{\sqrt{3}}{2}\right)}.$$

To find all points of horizontal tangency

Recollect the horizontal tangency slope equal to zero.

$$\frac{dy}{dx} = \frac{\sin\theta}{(1-\cos\theta)}$$

$$\frac{\sin\theta}{(1-\cos\theta)} = 0$$

$$\sin\theta = 0, 1-\cos\theta \neq 0$$

Therefore, points of horizontal tangency (x, y) is $\boxed{(a(2n+1)\pi, 2a)}$.

Determine where the curve is concave upward or concave downward.

To find the second derivative at $\theta = \frac{\pi}{6}$

$$\left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{6}} = \frac{-1}{a\left(\cos\frac{\pi}{6}-1\right)^2}$$

$$= \frac{-1}{a\left(\frac{\sqrt{3}}{2}-1\right)^2}$$

$$= \frac{-1}{a\left(\frac{\sqrt{3}}{2}-1\right)^2}$$

$$< 0$$

So, the second derivative at $\theta = \frac{\pi}{6}$ is negative, the concavity is downwards.

Concave is downward on all open θ intervals $\ldots,(-2\pi,0),(0,2\pi),(2\pi,4\pi),\ldots$

(b)

To find a formula for determining the area under a parametric curve

Given by the parametric equations $x = f(t)$ and $y = g(t)$.

Also need to further add in the assumption that the curve is traced out exactly once as *t* increases from α to β.

First recall how to find the area under $y = F(x)$ on $a \leq x \leq b$

Calculate an area formula by using the substitution Rule for definite integrals as follows:

$$A = \int_a^b F(x)dx$$

Q2TFQ

Given $x = f(t)$ and $y = g(t)$

Then

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Hence the statement is false.

Q3CC

(a).

The length of the parametric curve is $L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

(b).

The area of the surface obtained by rotating a parametric curve about the $x-$axis is $S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ where $x = f(t)$ and $y = g(t)$, $\alpha \le t \le \beta$ be parametric equations.

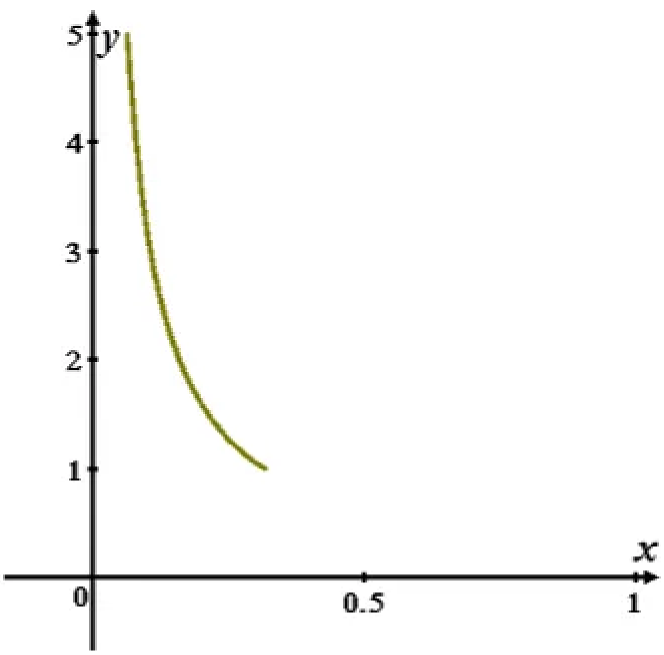
Q3E

Consider the parametric equations

$x = \cos\theta$, $y = \sec\theta$, where $0 \le \theta < \frac{\pi}{2}$

Need to sketch the parametric equations.

The sketch of the parametric equations is as shown below.



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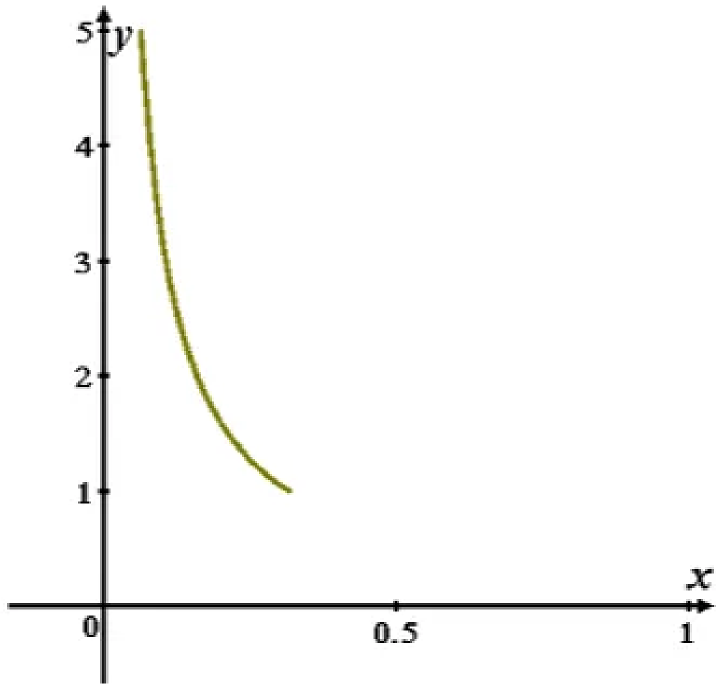


Now need to eliminate the parameter to find the Cartesian equation.

Consider

$y = \sec\theta$

$= \frac{1}{\cos\theta}$ Since $\sec x = \frac{1}{\cos x}$

$= \frac{1}{x}$ Substituting $\cos\theta = x$

Hence the Cartesian equation of the parametric equations is $\boxed{y = \frac{1}{x}}$.

Q3P

Consider the polar curves,

$r=1+c\sin\theta$ Where $0\le c\le 1$

To find the smallest viewing rectangle that contains every member of the family of polar curves

$r=1+c\sin\theta$ where $0\le c\le 1$

In terms of x and y , we have $x=r\cos\theta$

$x=(1+c\sin\theta)\cos\theta$ Substitute $r=1+c\sin\theta$

$x=\cos\theta+c\cdot\cos\theta\sin\theta$ Multiply

$x=\cos\theta+\frac{1}{2}c\sin 2\theta$ Since $\sin 2\theta=2\sin\theta\cos\theta$

And

$y=r\sin\theta$

$y=(1+c\sin\theta)\sin\theta$ Since $r=1+c\sin\theta$

$y=\sin\theta+c\sin^2\theta$ Multiply

Now $-1\le\sin\theta\le 1$

$-1+c\sin^2\theta\le\sin\theta+c\sin^2\theta\le 1+c\sin^2\theta$ Add on both sides with $c\sin^2\theta$

$-1+c\sin^2\theta\le y\le 1+c\sin^2\theta$ Since $y=\sin\theta+c\sin^2\theta$

$-1\le y\ \le 1+c\le 2$

If $y=2$ when $c=1$ and $\theta\ =\frac{\pi}{2}$

While $y=-1$ when $c=0$ and $\theta\ =\frac{3\pi}{2}$

Therefore, a viewing rectangle with $-1\le y\ \le 2$

To find the $x-$ values:

The equation $x=\cos\theta+\frac{1}{2}c\sin 2\theta$ and use the fact that $\sin 2\theta\ge 0$ for $0\le\theta\ \le\frac{\pi}{2}$

And $\sin 2\theta\le 0$ for $-\frac{\pi}{2}\le\theta\le 0$

Since $r=1+c\sin\theta$ is symmetric about the $y-$axis

So, only need to consider $-\frac{\pi}{2}\le\theta\ \le\frac{\pi}{2}$

Thus, $-\frac{\pi}{2}\le\theta\le 0,$ x has a maximum value when $c=0$ and then $x=r\cos\theta$ has a maximum value of 1at $\theta=0.$

Thus, the maximum value of x must occur on $\left[0,\frac{\pi}{2}\right]$ with $c=1$

Then $x = \cos\theta + \frac{1}{2}\sin 2\theta$

$$\frac{dx}{d\theta} = -\sin\theta + \frac{1}{2}\cos 2\theta \cdot 2$$

$$\frac{dx}{d\theta} = -\sin\theta + \cos 2\theta$$

$\frac{dx}{d\theta} = -\sin\theta + 1 - 2\sin^2\theta$ Since $\cos 2\theta = 1 - 2\sin^2\theta$

$$\frac{dx}{d\theta} = -2\sin^2\theta - \sin\theta + 1$$

Recollect $\frac{dx}{d\theta} = 0$

$-2\sin^2\theta - \sin\theta + 1 = 0$

$2\sin^2\theta + \sin\theta - 1 = 0$

$2\sin^2\theta + 2\sin\theta - \sin\theta - 1 = 0$ Factors

$2\sin\theta(\sin\theta + 1) - 1(\sin\theta + 1) = 0$

$(\sin\theta + 1)(2\sin\theta - 1) = 0$ Common like terms

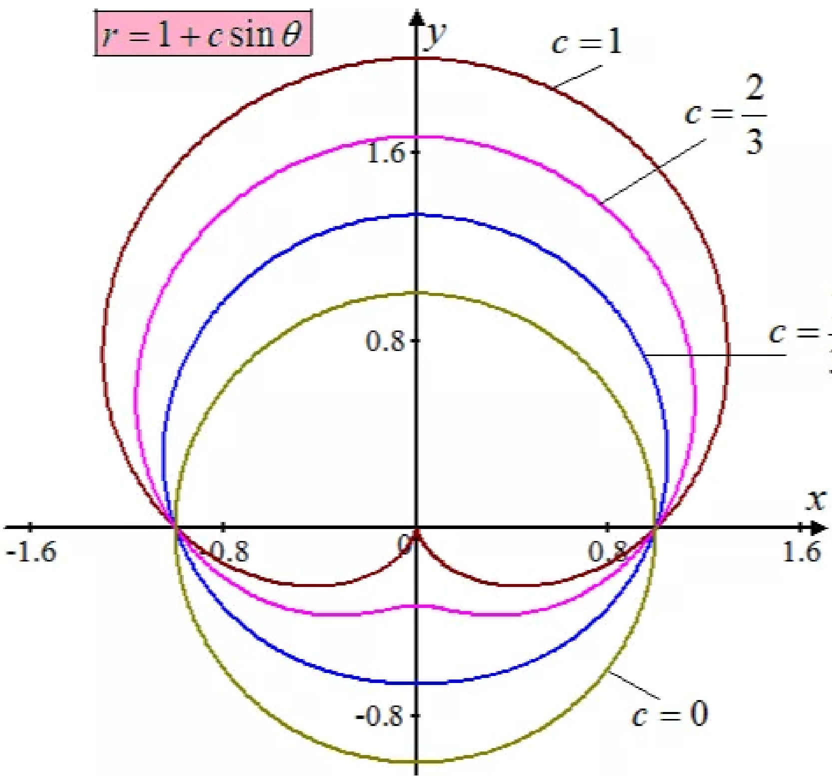
$(\sin\theta + 1) = 0$ or $(2\sin\theta - 1) = 0$ Use zero product property

$\sin\theta = -1$ Or $\sin\theta = \frac{1}{2}$

But $\sin\theta \neq -1$ for $0 \leq \theta \leq \frac{\pi}{2}$

If $\sin\theta = \frac{1}{2}$ then $\theta = \frac{\pi}{6}$ and $x = \cos\theta + \frac{1}{2}\sin 2\theta$

Observe the below graph of several members of the family in this viewing rectangle:



Q3TFQ

Given $x = f(t), y = g(t)$ and $a \le t \le b$

The length of the curve is

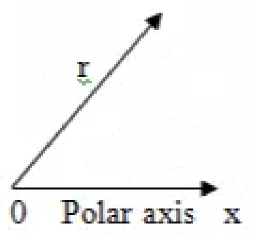
$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_a^b \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2}\, dt$$

Hence the given statement is true.

Q4CC

(a)



(b).

Equations to express the Cartesian coordinates (x, y) of a point in terms of the polar coordinates are $x = r\cos\theta, y = r\sin\theta$

(c).

Equations to find polar coordinates of a point if we know the Cartesian coordinates are $r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\frac{y}{x}$

Q4E

Given parametric equations are $x = 2\cos\theta$ ---(1)

And $y = 1 + \sin\theta$ ---(2)

We sketch the curve for different values of θ where $0 \le \theta \le 2\pi$

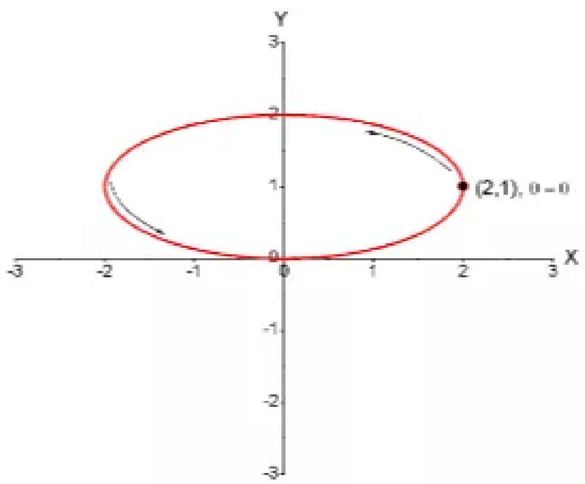


Fig.1

From (1) we have $\cos\theta = \frac{x}{2}$ ....(3)

And from (2) we have $\sin\theta = y-1$ ....(4)

Adding the squares of equation (3) and (4)

$$\cos^2 + \sin^2\theta = \left(\frac{x}{2}\right)^2 + (y-1)^2$$

$$\Rightarrow \boxed{1 = \frac{x^2}{4} + \frac{(y-1)^2}{1}}$$

Which represents and ellipse where a = 2 and b = 1

So major axis = 2a =4 and minor axis = 2b =2

And center of the ellipse is at the point (0, 1)

Q4P

Let's call the bug starts in the first quadrant the first bug, second quadrant the second bug, third quadrant the third bug, and fourth quadrant the fourth bug.

Let $r = f(\theta)$ be the path of the first bug. The first bug's path can be calculated as the slope of the tangent line to the curve.

The slope of the tangent line at any point $(x, y) = (r\cos\theta, r\sin\theta)$, and it is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

On applying the product rule,

$$\frac{dy}{dx} = \frac{r\cos\theta + \sin\theta\frac{dr}{d\theta}}{-r\sin\theta + \cos\theta\frac{dr}{d\theta}}$$

Rewrite the above equation as,

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} \quad \text{...... (1)}$$

Observe that, the first bug is at the point (x, y), then the second bug is at the point $(-y, x)$.

The line connection of first and second bug has the slope $\frac{y-x}{x+y}$

That is,

$$\frac{dy}{dx} = \frac{y-x}{x+y}$$

To convert it into polar coordinates, substitute $x = r\cos\theta$ and $y = r\sin\theta$.

$$\frac{dy}{dx} = \frac{r\sin\theta - r\cos\theta}{r\cos\theta + r\sin\theta}$$

$$\frac{dy}{dx} = \frac{\sin\theta - \cos\theta}{\cos\theta + \sin\theta} \quad \text{...... (2)}$$

The equations (1) and (2) describes the slope of the tangent line equal to each other,

That is,

$$\frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{\sin\theta - \cos\theta}{\cos\theta + \sin\theta}$$

Using Pythagorean identity,

$$f'(\theta) = -f(\theta)$$

Thus,

$f(\theta) = ke^{-\theta}$,for some constant k.

The bugs starts a distance of $\frac{\sqrt{2}}{2}a$ units from the origin.

So,

$$\frac{\sqrt{2}}{2}a = ke^{-\frac{\pi}{4}}$$

Therefore,

$$\boxed{k = \frac{\sqrt{2}e^{\frac{\pi}{4}}}{2}a}$$

Q4TFQ

The point (r, θ) in terms of Cartesian coordinates is $r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\frac{y}{x}$

Hence the given statement is true.

Q5CC

(a). Slope of a tangent line to the polar curve is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

(b).

The area of a region bounded by a polar curve is $A = \int_a^b \frac{1}{2}r^2\, d\theta$

(c).

The length of a polar curve is $L = \int_a^b \sqrt{(r)^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

Q5E

Equations of the curve is $y = \sqrt{x}$

Let $x = t$ then $y = \sqrt{t}$ where $t \geq 0$

So first set of the parametric equations $\Rightarrow \boxed{\begin{array}{l} x = t \\ y = \sqrt{t} \end{array}}$ $t \geq o$

Let $x = t^2$ *then* $y = t$

So second set of the parametric equations $\Rightarrow \boxed{\begin{array}{l} x = t^2 \\ y = t \end{array}}$

Let $x = \tan^2\theta$

Then $y = \tan\theta$ *when* $0 \leq \theta < \pi/2$

So

Third set of the parametric equations $\Rightarrow \boxed{\begin{array}{l} x = \tan^2\theta \\ y = \tan\theta \end{array}}$, $0 \leq \theta \leq \pi/2$

Q5TFQ

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Given

$r = 1 - \sin 2\theta, r = \sin 2\theta - 1$

Given statement is true.

Since graphs of the above two curves are same and the graph is

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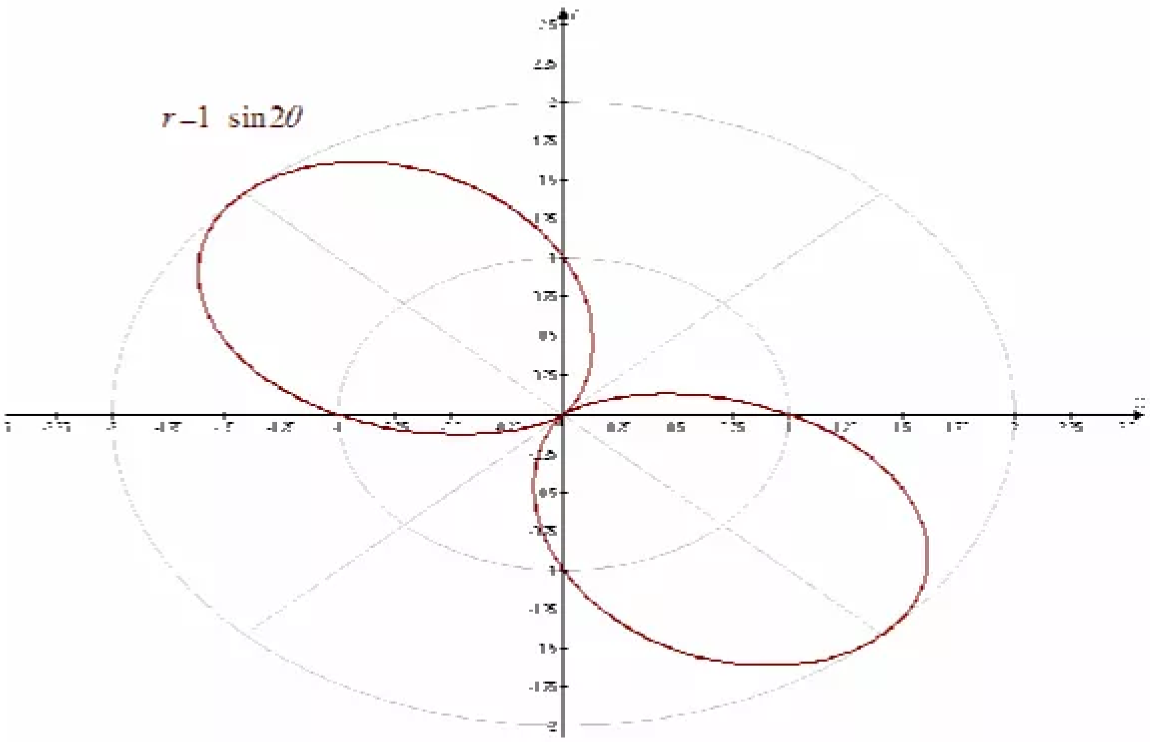
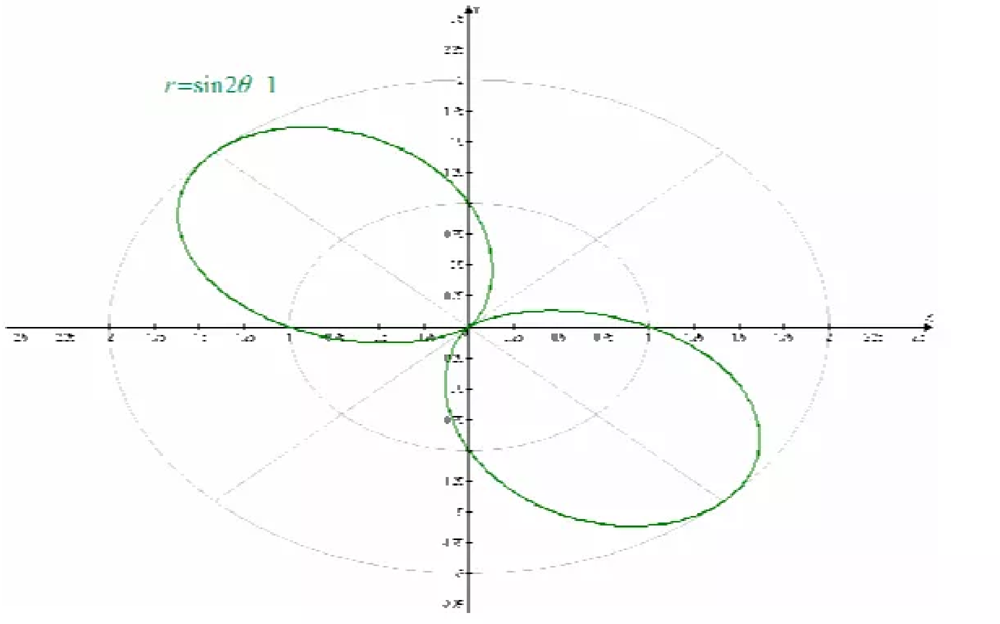




Q6CC

(a).
A parabola is the set of points in the plane that are equidistant from a fixed point F (called the focus) and a fixed line (called the directrix).

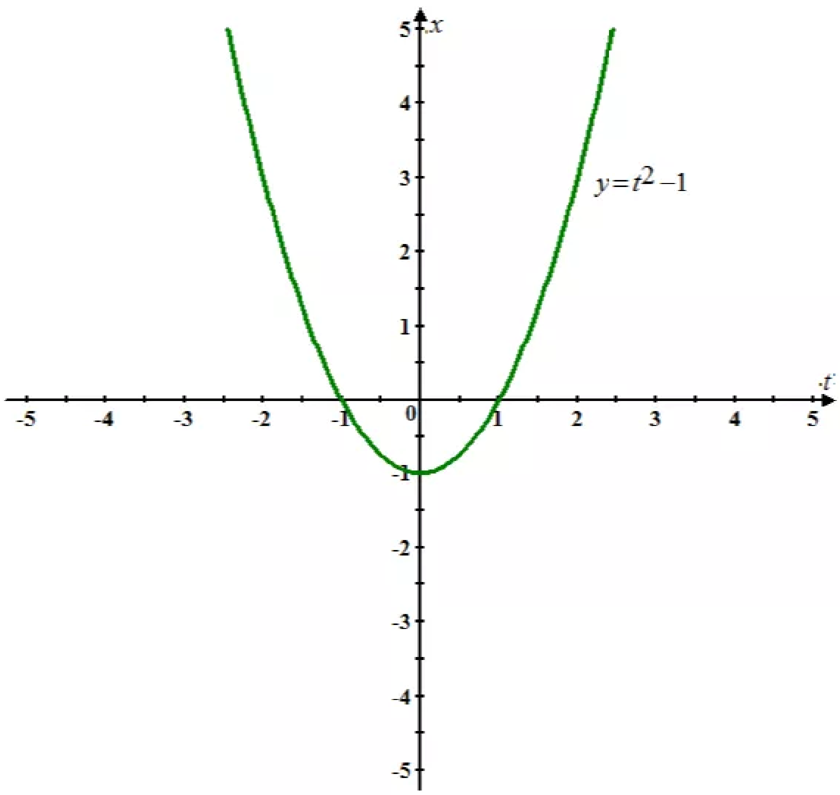
(b).
An equation of the parabola with focus $(0, p)$ and directrix $y=-p$ is $x^2=4py$

An equation of the parabola with focus $(p,0)$ and directrix $x=-p$ is $y^2=4px$

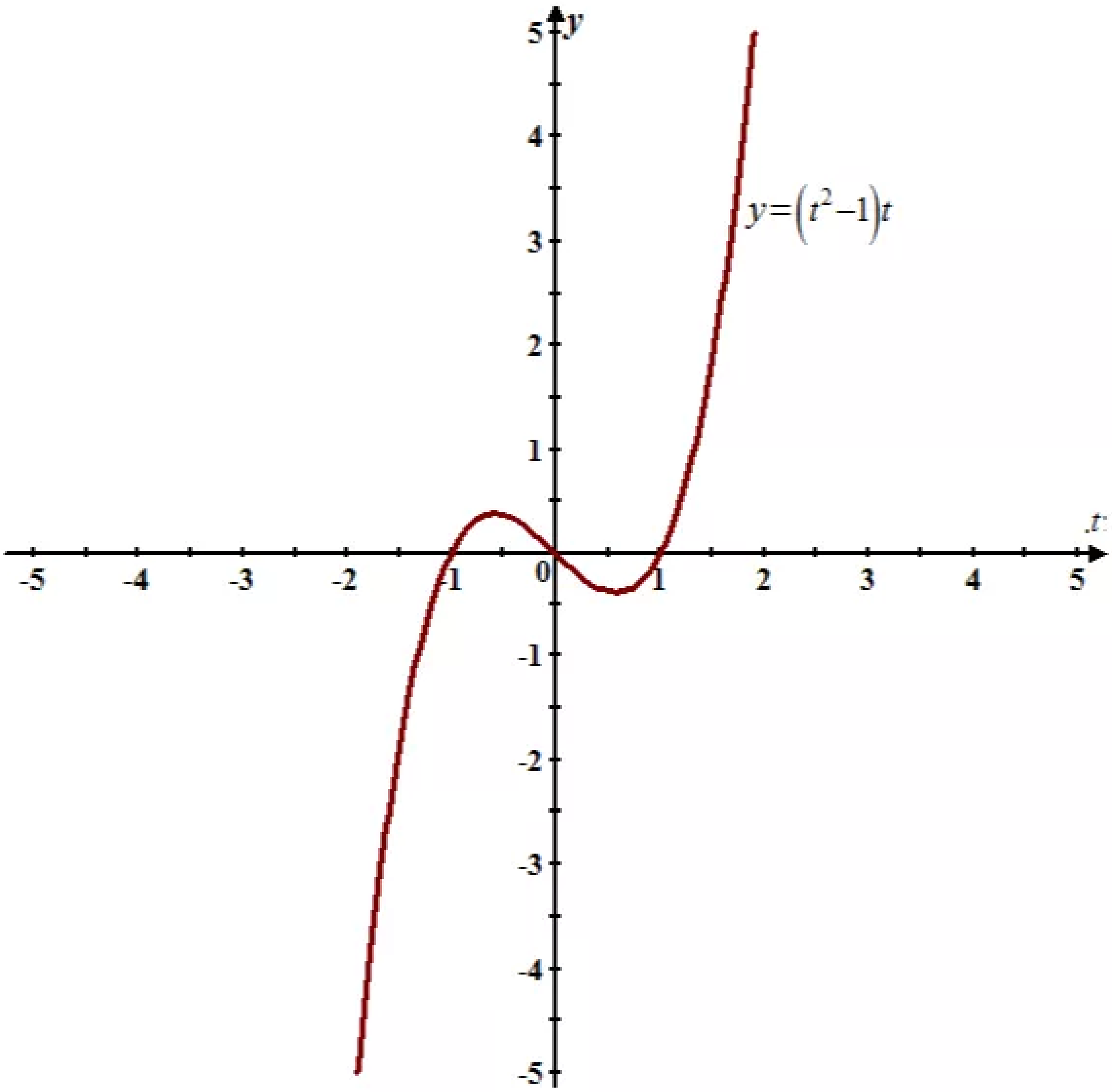
Q6E

Consider the graphs $x=f(t)$ and $y=g(t)$.

First graph:

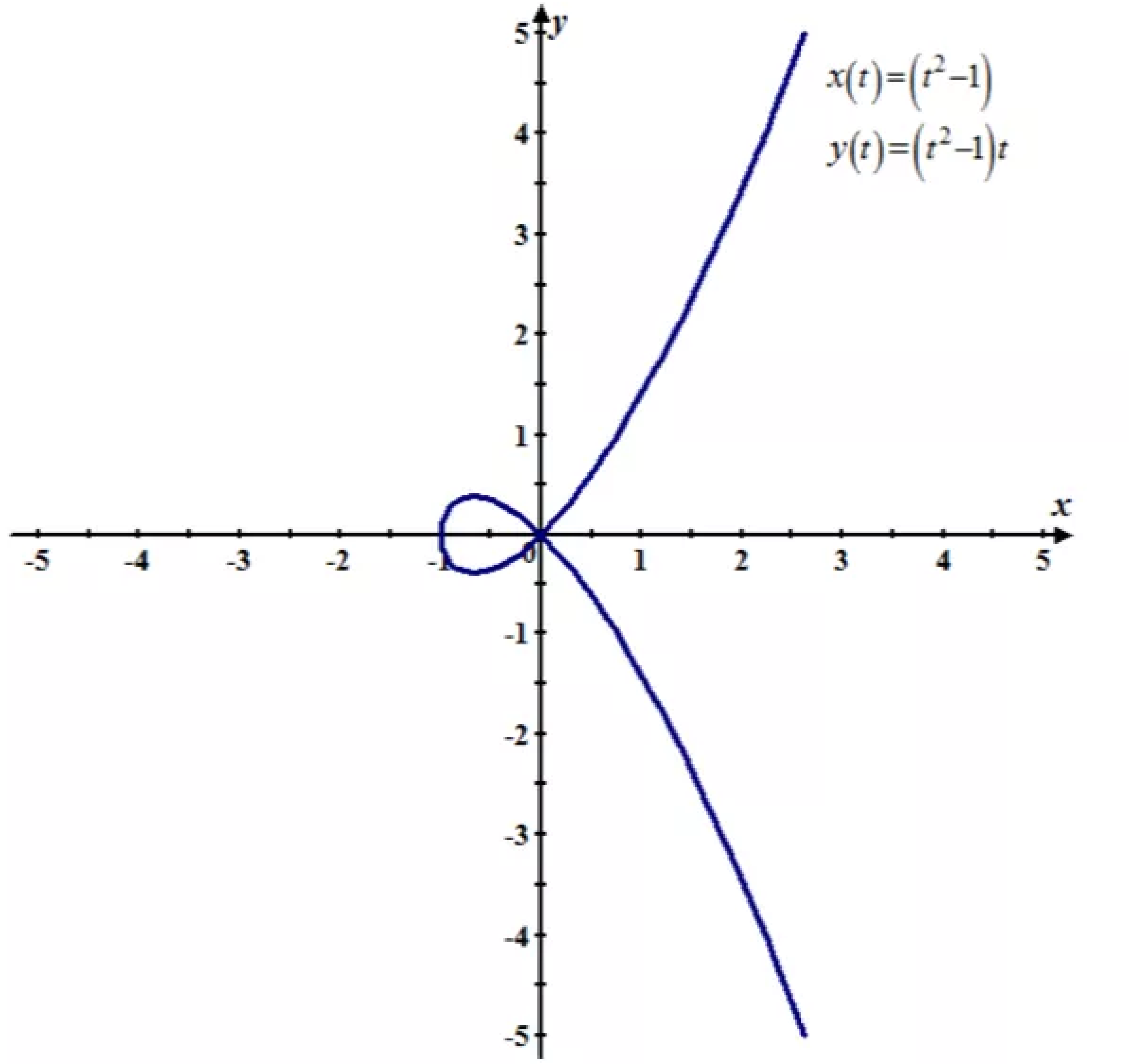


Second Graph:



The two graphs are $x = f(t)$ and $y = g(t)$

To sketch by matching up the parameter values and obtaining the corresponding (x, y) coordinates.



Q6TFQ

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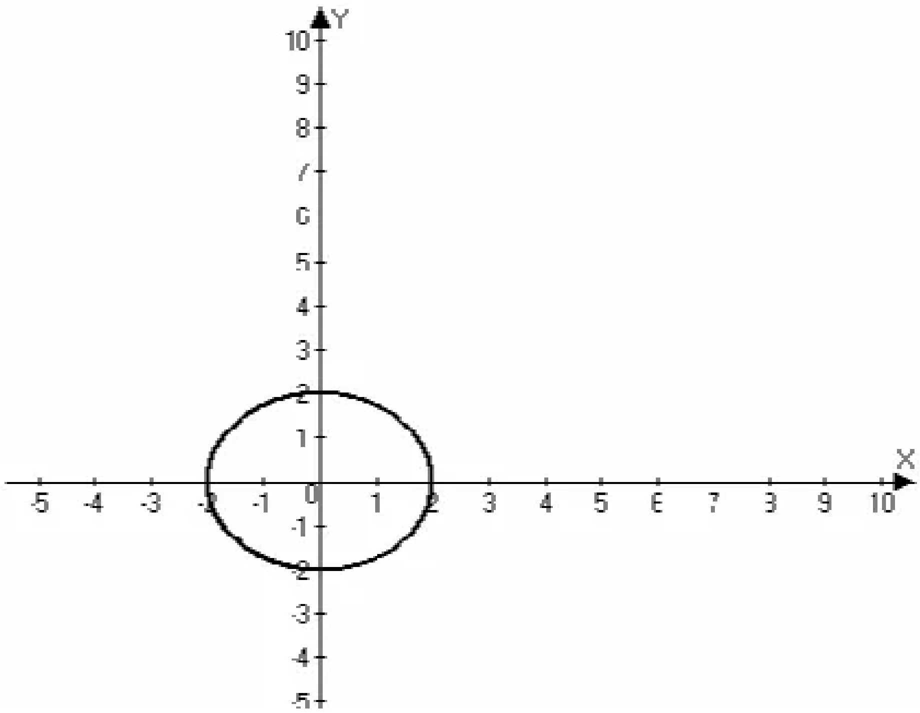
RID: 378

Given

$r = 2,\ x^2 + y^2 = 4$ and $x = 2\sin 3t,\ y = 2\cos 2t$

Given statement is true.

Since graphs of the above curves are same and the graph is



Q7CC

(a). An ellipse is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2 is a constant. These two fixed points are called the foci.

(b).

An equation of the ellipse with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a \ge b \ge 0$$

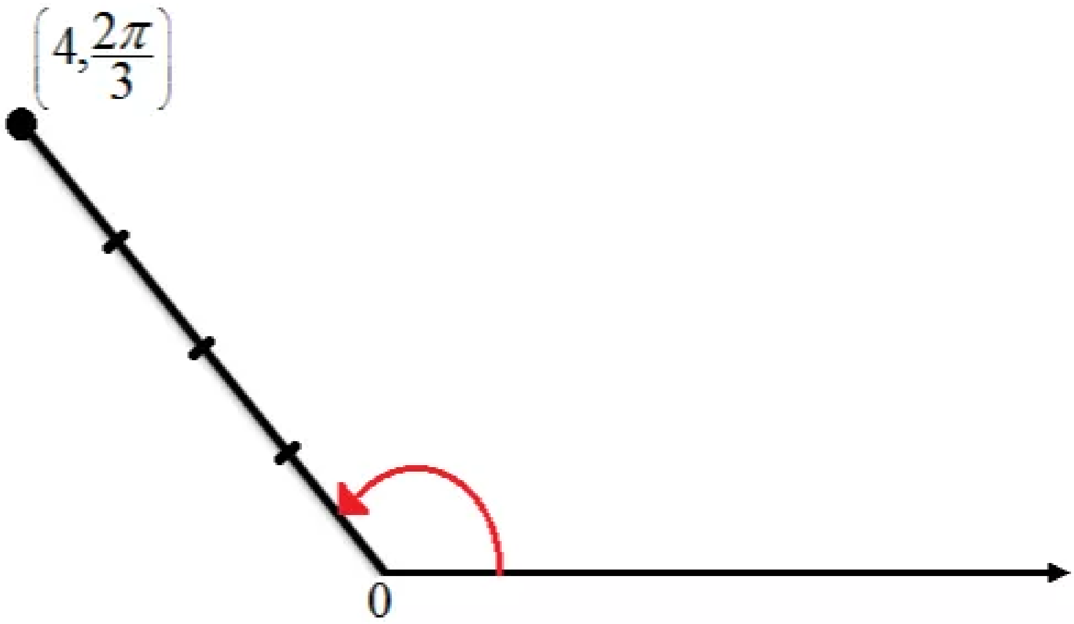
Q7E

a) Consider the polar coordinates of the point

$\left(4, \frac{2\pi}{3}\right)$

Need to plot the point and find its Cartesian coordinates of the point.

The plot of the polar coordinate is as shown below.



From point $\left(4, \frac{2\pi}{3}\right)$,

$r = 4$ and $\theta = \frac{2\pi}{3}$

To change the polar coordinates to Cartesian coordinates, use the equations

$x = r\cos\theta$ and $y = r\sin\theta$

Now

$x = r\cos\theta$

$= 4\cos\left(\frac{2\pi}{3}\right)$ Substituting $r = 4$ and $\theta = \frac{2\pi}{3}$

$= 4\left(-\frac{1}{2}\right)$

$= -2$

Now

$y = r\sin\theta$

$= 4\sin\left(\frac{2\pi}{3}\right)$ Substituting $r = 4$ and $\theta = \frac{2\pi}{3}$

$= 4\left(\frac{\sqrt{3}}{2}\right)$

$= 2\sqrt{3}$

Hence the Cartesian coordinates of the given polar coordinates are $\boxed{\left(-2, 2\sqrt{3}\right)}$.

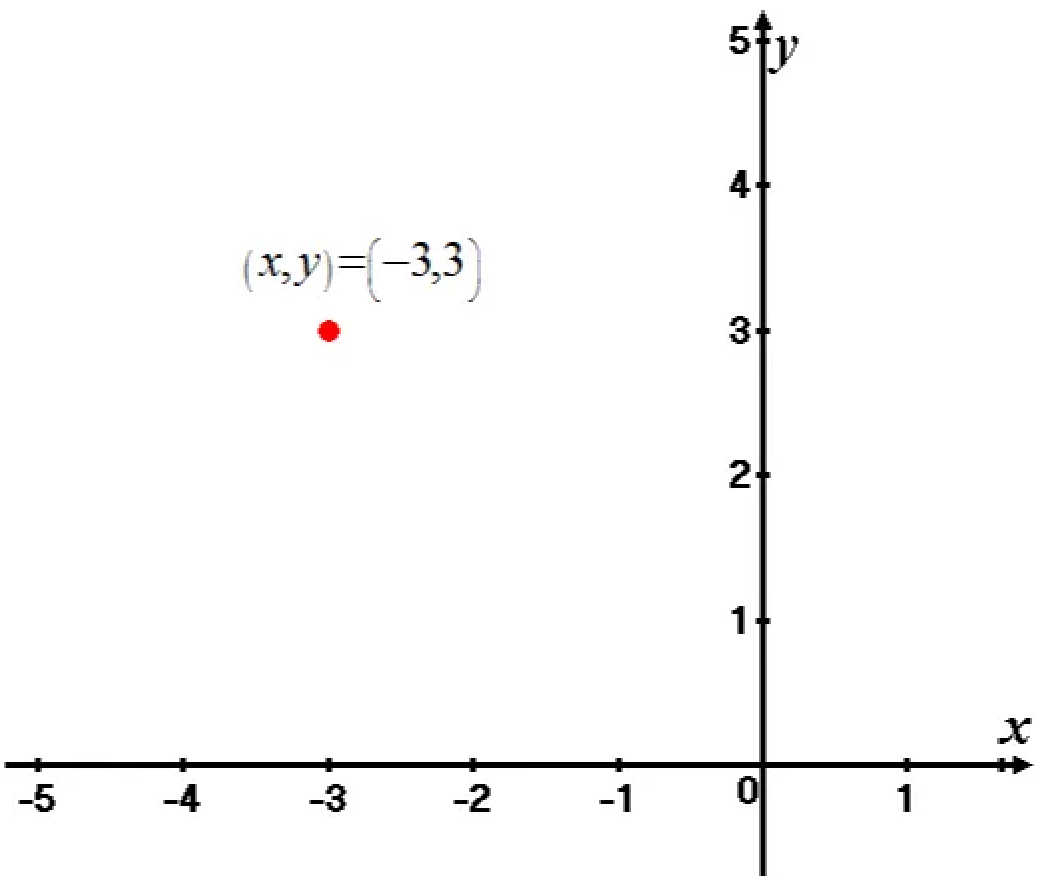
b) Consider the rectangular coordinates of the point

$(-3,3)$

Need to plot the point and to find two sets of polar coordinates of the point.

Let $(x, y) = (-3,3)$

The sketch of the point is as shown below:



To change the rectangular coordinates to polar coordinates, use the following formulas:

$\tan\theta = \frac{y}{x}$ and $r^2 = x^2 + y^2$

Now

$\tan\theta = \frac{y}{x}$

$= \frac{3}{-3}$ Substituting $x = 4, y = -4$

$= -1$

$\theta = \tan^{-1}(-1)$

$= \frac{3\pi}{4}, \frac{7\pi}{4}$ Since $0 \le \theta \le 2\pi$

And

$r^2 = x^2 + y^2$

$= (-3)^2 + (3)^2$

$= 9 + 9$

$= 18$

$r = \pm\sqrt{18}$

$= \pm\sqrt{9(2)}$

$= \pm 3\sqrt{2}$

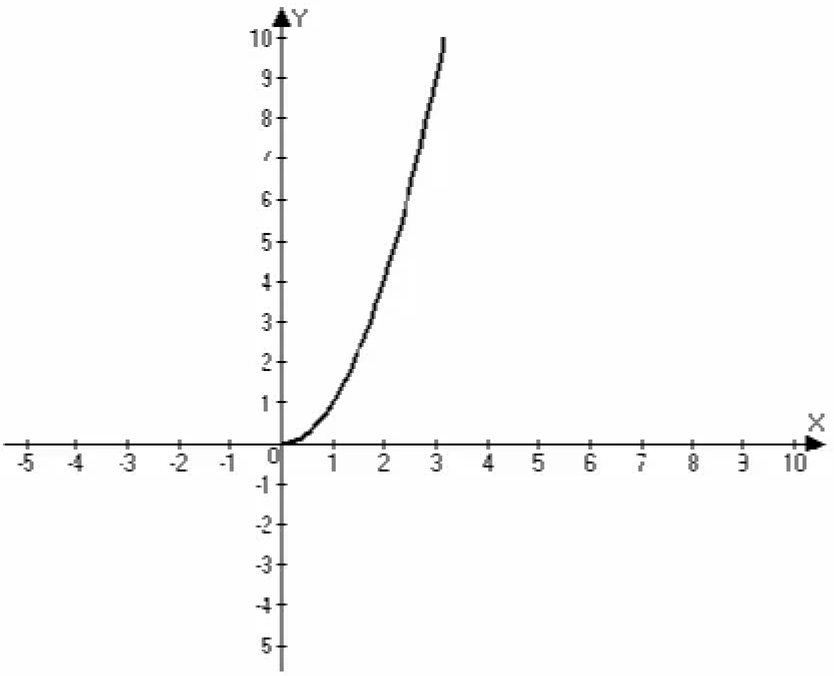
Hence the two set of polar coordinates are $\boxed{\left(-3\sqrt{2}, \frac{7\pi}{4}\right)}$ and $\boxed{\left(3\sqrt{2}, \frac{3\pi}{4}\right)}$.

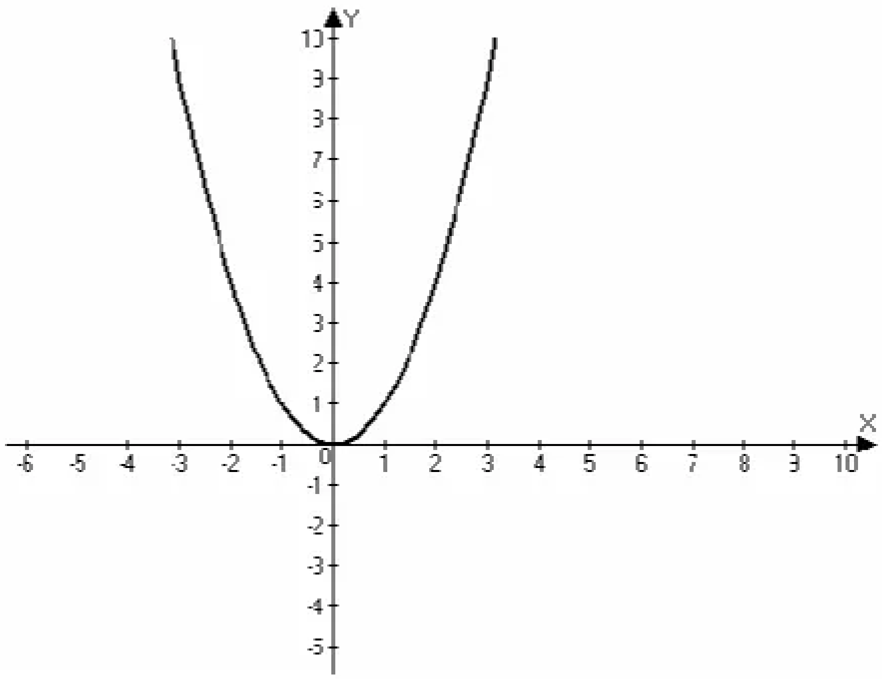
Q7TFQ

2657-10R-7TFQ

AID: 9514
RID: 378

Given
$x = t^2, y = t^4$ and $x = t^3, y = t^6$
Given statement is false.
Since graphs of the above curves are not same and the graphs are





Q8CC

(a). A Hyperbola is the set of points in a plane the difference of whose distances from two fixed points F_1 and F_2 is a constant.

(b). An equation for the hyperbola with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$ is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(c). the equations for the asymptotes of the hyperbola in (b) are $y = \pm \frac{b}{a} x$

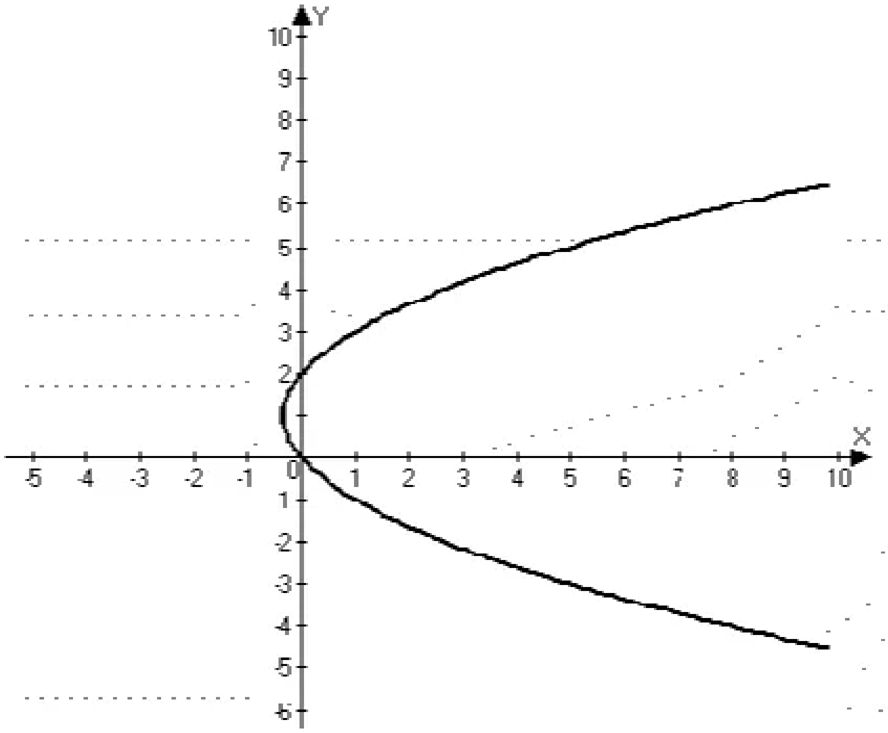
Q8TFQ

Given

$2y + 3x = y^2$

Given statement is true.

Since graph of the above curve is a parabola and the graph is



Q9CC

(a).

Let F be a fixed point (called the focus) and l be a fixed line (called the directrix) in the plane. Let e be a fixed positive number (called the eccentricity). The set of all points P in the plane $\ni \frac{|PF|}{|Pl|} = e$

(b). The conic is an ellipse if e < 1, a parabola if e = 1 and a hyperbola if e > 1 (c).

If $x = d$ is the directrix then the polar equation for the conic section is $r = \dfrac{ed}{1+e\cos\theta}$

If $x = -d$ is the directrix then the polar equation for the conic section is $r = \dfrac{ed}{1-e\cos\theta}$

If $y = d$ is the directrix then the polar equation for the conic section is $r = \dfrac{ed}{1+e\sin\theta}$

If $y = -d$ is the directrix then the polar equation for the conic section is $r = \dfrac{ed}{1-e\sin\theta}$

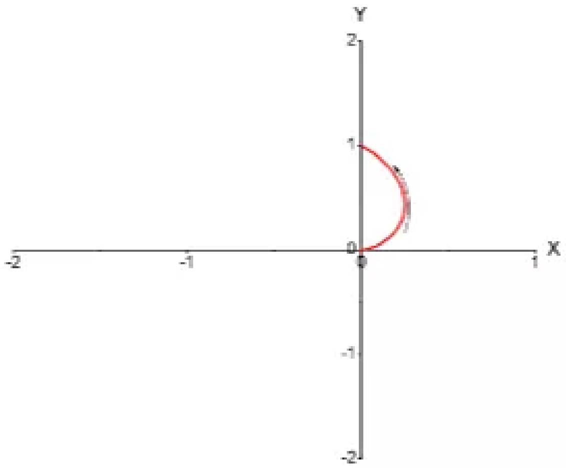
Q9E

We have the equation $r = 1-\cos\theta$
We calculate the value of r for differential values of θ
In the interval $0 \le \theta \le 2\pi$

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	0	1	2	1	0

Step 2 of 5

We see that as θ changes from 0 to $\pi/2$, r increases from 0 to 1
We sketch the part of curve



As θ increases from $\pi/2$ to π, r increases from 1 to 2
We sketch the second part of curve

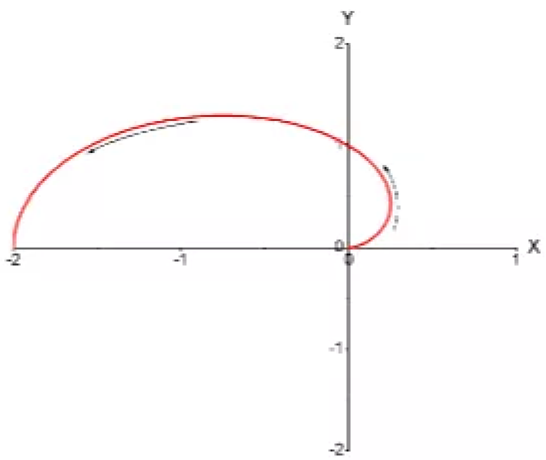


Fig.2

As θ increases from π to $3\pi/2$, r decreases from 2 to 1

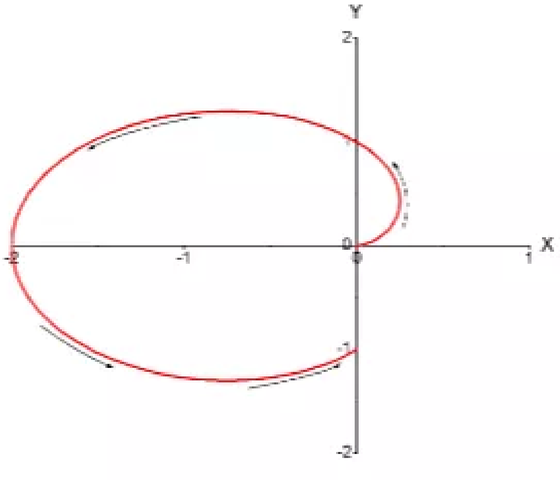


Fig.3

Q9TFQ

A tangent line to the parabola intersects the parabola only once
From the shape of the parabola the tangent line to the parabola intersects the parabola only once.
Hence the statement is true.

Q10E.

First we sketch the Cartesian curve of the equation $r = \sin 4\theta, \quad 0 \le \theta \le 2\pi$

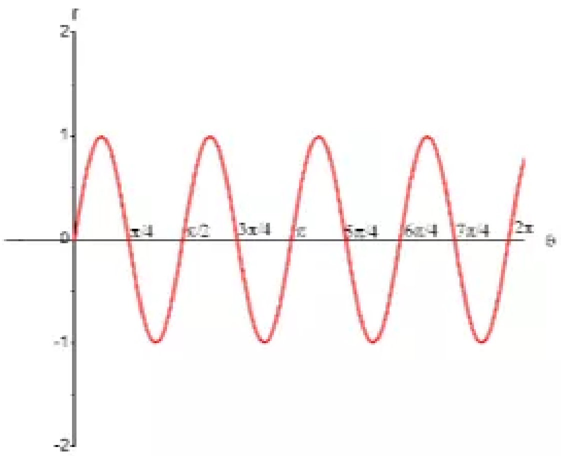


Fig.1

Step 2 of 3

We see that $r = 1$ at $\theta = \pi/8,\ 5\pi/8,\ 9\pi/8,\ 13\pi/8$
And $r = -1$ at $\theta = 3\pi/8,\ 7\pi/8,\ 11\pi/8,\ 15\pi/8$
And $r = 0$ at $\theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{6\pi}{4}, \frac{7\pi}{4}$ and 2π
So with the help of above information and graph, we sketch the polar curve of given equation

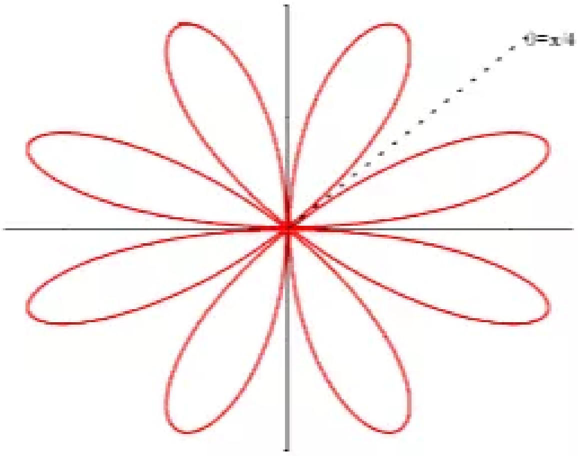


Fig.2

Q10TFQ

A hyperbola never intersects its directrix.
The statement is true.
Since the directrix is a fixed line that never intersects hyperbola.

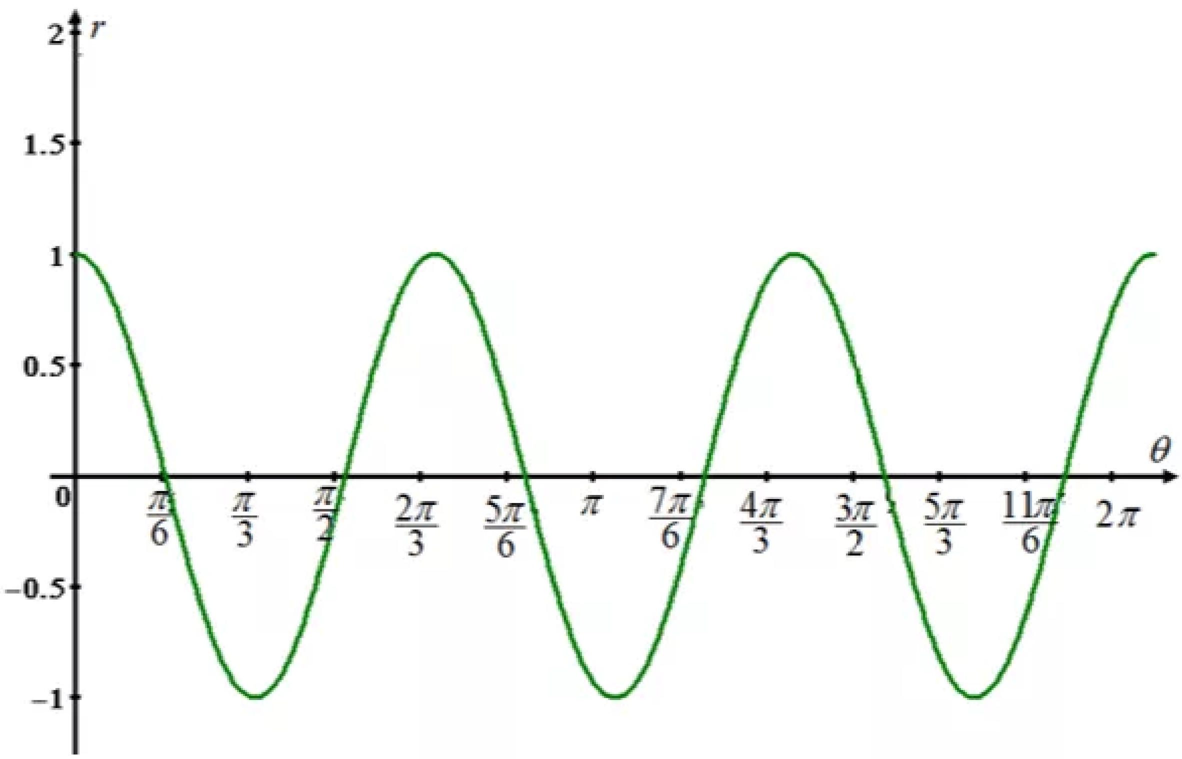
Q11E

Consider the equation of the polar curve

$r = \cos 3\theta$

Need to sketch the polar curve.

First sketch the curve $r = \cos 3\theta$, $0 \le \theta \le 2\pi$, in Cartesian coordinates. The sketch is as shown below.



From the figure, observe that r decreases from 1 to 0 as θ increases from 0 to $\frac{\pi}{6}$.

So draw the corresponding portion of the polar curve as indicated by the number 1 in the following figure.

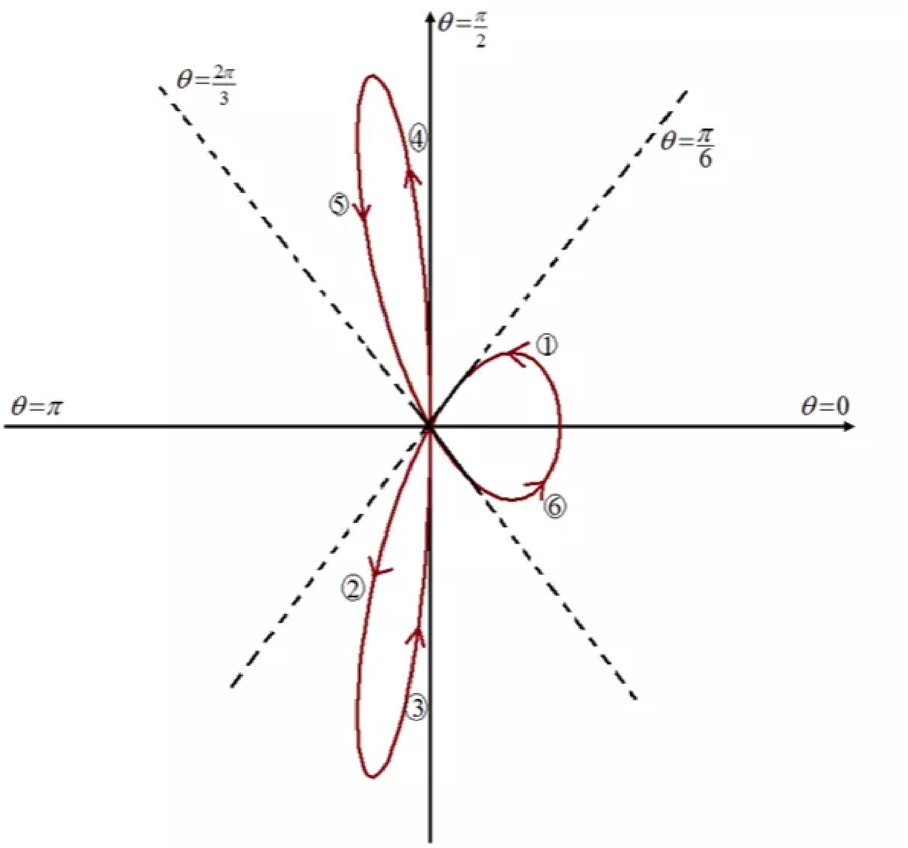
Now r decreases from 0 to -1 as θ increases from $\theta = \frac{\pi}{6}$ to $\frac{\pi}{3}$.

That is the distance from the origin O increases from 0 to 1.

But this portion of the polar curve lies on the opposite side of the pole in quadrant III instead of in quadrant I. This is indicated by 2.

Continuing in this way, the following figure with three petals is obtained.

Thus, the polar curve is as shown below:



Q12E

Given parametric equation is $r = 3 + \cos 3\theta$
We calculate the values of r for different values of θ

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
r	4	3	2	3	4	3	2	3	4	3	2	3	4

We see that curve starts from $(0, 4)$
As θ increases from 0 to $\pi/3$, r decreases from 4 to 2
As θ increases from $\pi/3$ to $2\pi/3$, r increases from 2 to 4
As θ increases from $2\pi/3$ to π, r decreases from 4 to 2
As θ increases from π to $4\pi/3$, r increases from 2 to 4
As θ increases from $4\pi/3$ to $5\pi/3$, r decreases from 4 to 2
And as θ increases from $5\pi/3$ to 2π, r increases from 2 to 4
Thus we get the complete graph

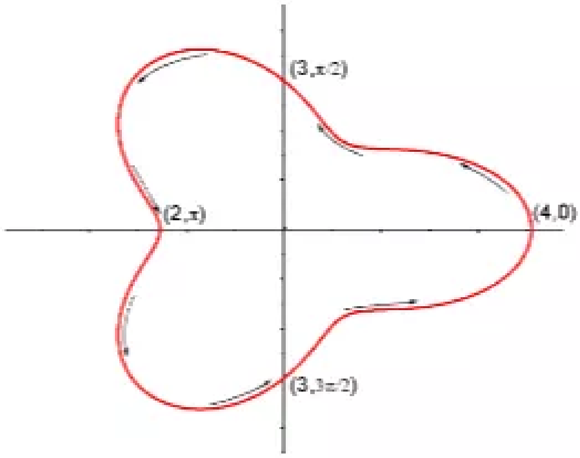


Fig.1

Q13E

Given polar equation is $r = 1 + \cos 2\theta$

We calculate the values of r for different values of θ

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	2	0	2	0	2

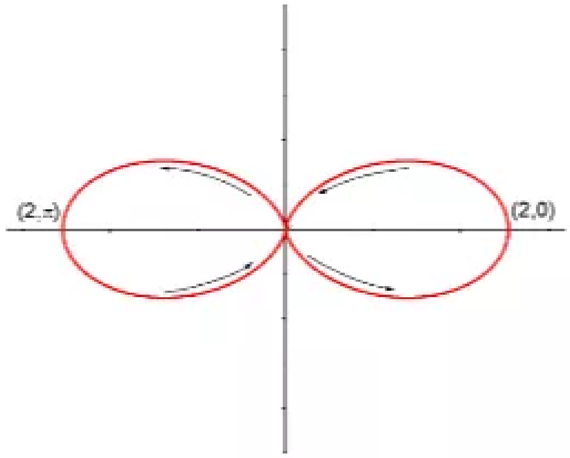


Fig.1

Q14E

Given equation of the curve is $r = 2\cos(\theta/2)$

We calculate the value of r for different values of θ in the interval $0 \le \theta \le 4\pi$

θ	0	π	2π	3π	4π
r	2	0	-2	0	2

We see that as θ increases from 0 to π, r decreases from 2 to 0

As θ increases from π to 2π, r decreases from 0 to -2

As θ increases from 2π to 3π, r increases from -2 to 0

As θ increases from 3π to 4π, r increases from 0 to 2

Thus we get the graph

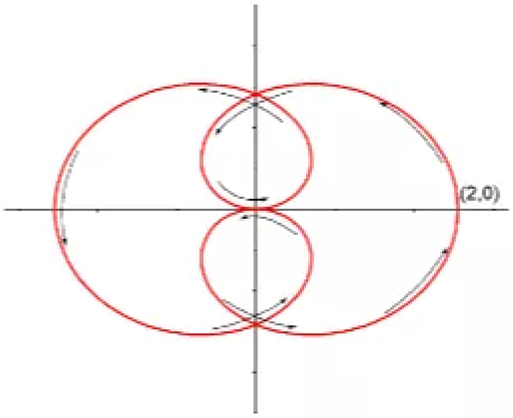


Fig.1

Q15E

Consider the equation of the polar curve

$$r = \frac{3}{1+2\sin\theta}$$

Need to sketch the polar curve.

Compare the equation $r = \frac{3}{1+2\sin\theta}$ with the general form $r = \frac{ed}{1+e\sin\theta}$ gives

$e = 2$

And $ed = 3$

$2d = 3$ Substituting $e = 2$

$d = \frac{3}{2}$ Dividing both sides by 2

Since $e = 2 > 1$

So the equation represents a **hyperbola**.

And the equation of the directrix is $y = \frac{3}{2}$.

The vertices occur when $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$.

Suppose $\theta = \frac{\pi}{2}$

Then

$$r = \frac{3}{1+2\sin\left(\frac{\pi}{2}\right)}$$

$$= \frac{3}{1+2(1)}$$

$$= \frac{3}{3}$$

$$= 1$$

Thus $r = 1$

Suppose $\theta = \frac{3\pi}{2}$

Then

$$r = \frac{3}{1 + 2\sin\left(\frac{3\pi}{2}\right)}$$

$$= \frac{3}{1 + 2(-1)}$$

$$= \frac{3}{1 - 2}$$

$$= \frac{3}{-1}$$

$$= -3$$

Thus $r = -3$

Also $\left(-3, \frac{3\pi}{2}\right) = \left(3, \frac{\pi}{2}\right)$

Hence the vertices of the hyperbola are $\left(1, \frac{\pi}{2}\right), \left(3, \frac{\pi}{2}\right)$.

The *x*-intercepts occur when $\theta = 0, \pi$

Suppose $\theta = 0$

Then

$$r = \frac{3}{1 + 2\sin(0)}$$

$$= \frac{3}{1 + 2(0)}$$

$$= \frac{3}{1 + 0}$$

$$= \frac{3}{1}$$

$$= 3$$

Thus $r = 3$

Suppose $\theta = \pi$

Then

$$r = \frac{3}{1+2\sin(\pi)}$$

$$= \frac{3}{1+2(0)}$$

$$= \frac{3}{1+0}$$

$$= \frac{3}{1}$$

$$= 3$$

Thus $r = 3$

In both the cases, $r = 3$

Therefore, the *x*-intercepts are $(3,0)$ and $(3,\pi)$.

Now need to find the asymptotes for the hyperbola.

Suppose $(1+2\sin\theta) \to 0^+$ and $(1+2\sin\theta) \to 0^-$

Then $r \to \pm\infty$

And $1+2\sin\theta = 0$

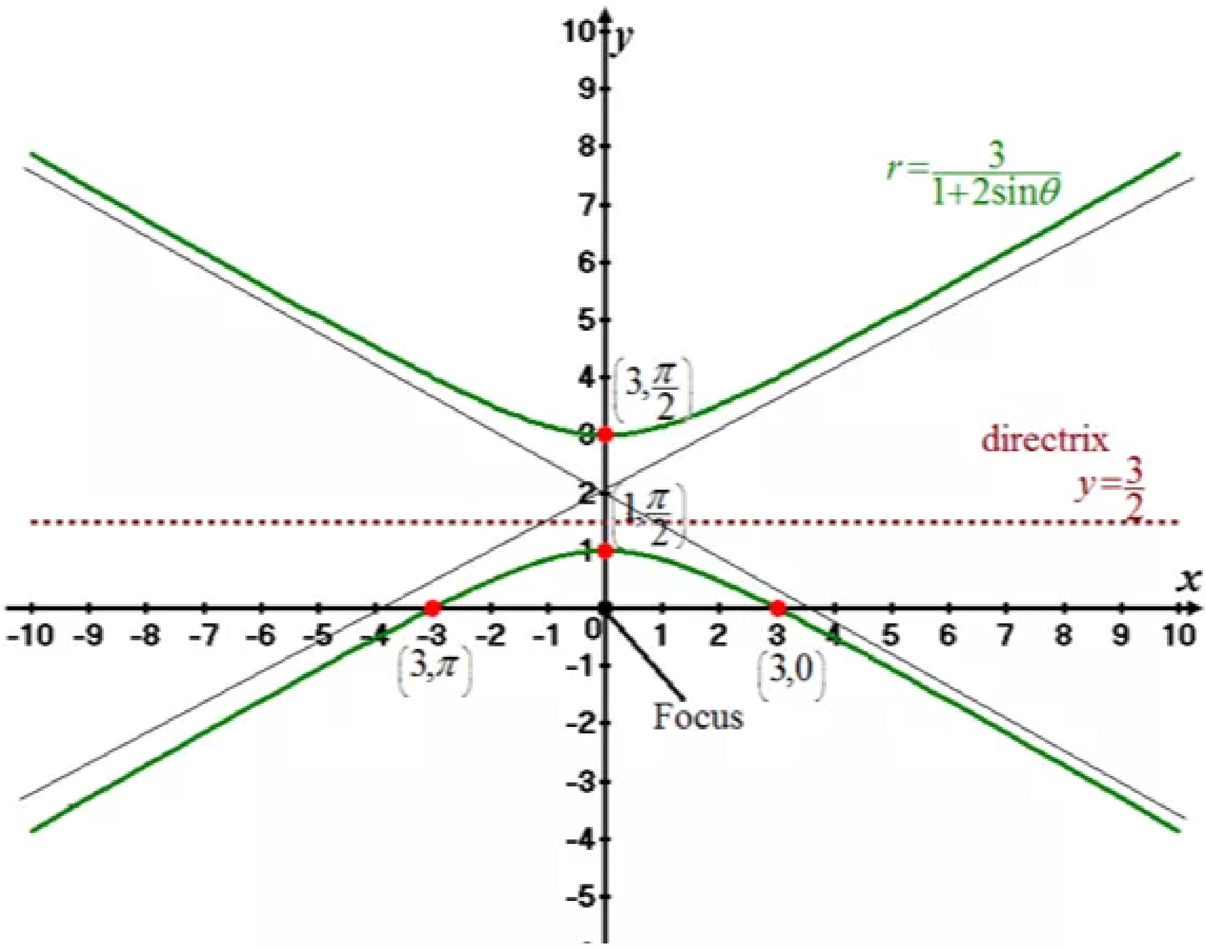
$2\sin\theta = -1$ Subtracting 1 from both sides

$\sin\theta = -\frac{1}{2}$ Dividing both sides by 2

$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$ Solving for θ

Thus, the asymptotes are parallel to the rays $\theta = \frac{7\pi}{6}$ and $\theta = \frac{11\pi}{6}$.

Thus, the hyperbola is sketched as follows:



Q16E

Consider the equation of the polar curve

$$r = \frac{3}{2-2\cos\theta}$$

Rewrite the equation as $r = \frac{3/2}{1-\cos\theta}$

Need to sketch the polar curve.

Compare the equation $r = \frac{3/2}{1-\cos\theta}$ with the general form $r = \frac{ed}{1-e\cos\theta}$ gives

$e = 1$

And $ed = 3/2$

$d = 3/2$ Substituting $e = 1$

Since $e = 1$

So the equation represents a **parabola**.

If the parabola opens towards right then the Cartesian equation of the directrix is $x = -d$.

Thus, this parabola has the directrix with Cartesian equation $x = -\frac{3}{2}$.

Here the negative sign is considered for the directrix equation because of the negative cosine term.

The vertices of the parabola occur when $\theta = 0$ and $\theta = \pi$.

Suppose $\theta = 0$

Then

$$r = \frac{3}{2 - 2\cos 0}$$
$$= \frac{3}{2 - 2(1)}$$
$$= \frac{3}{0}$$
$$= \infty$$

Thus, $r = \infty$

Suppose $\theta = \pi$

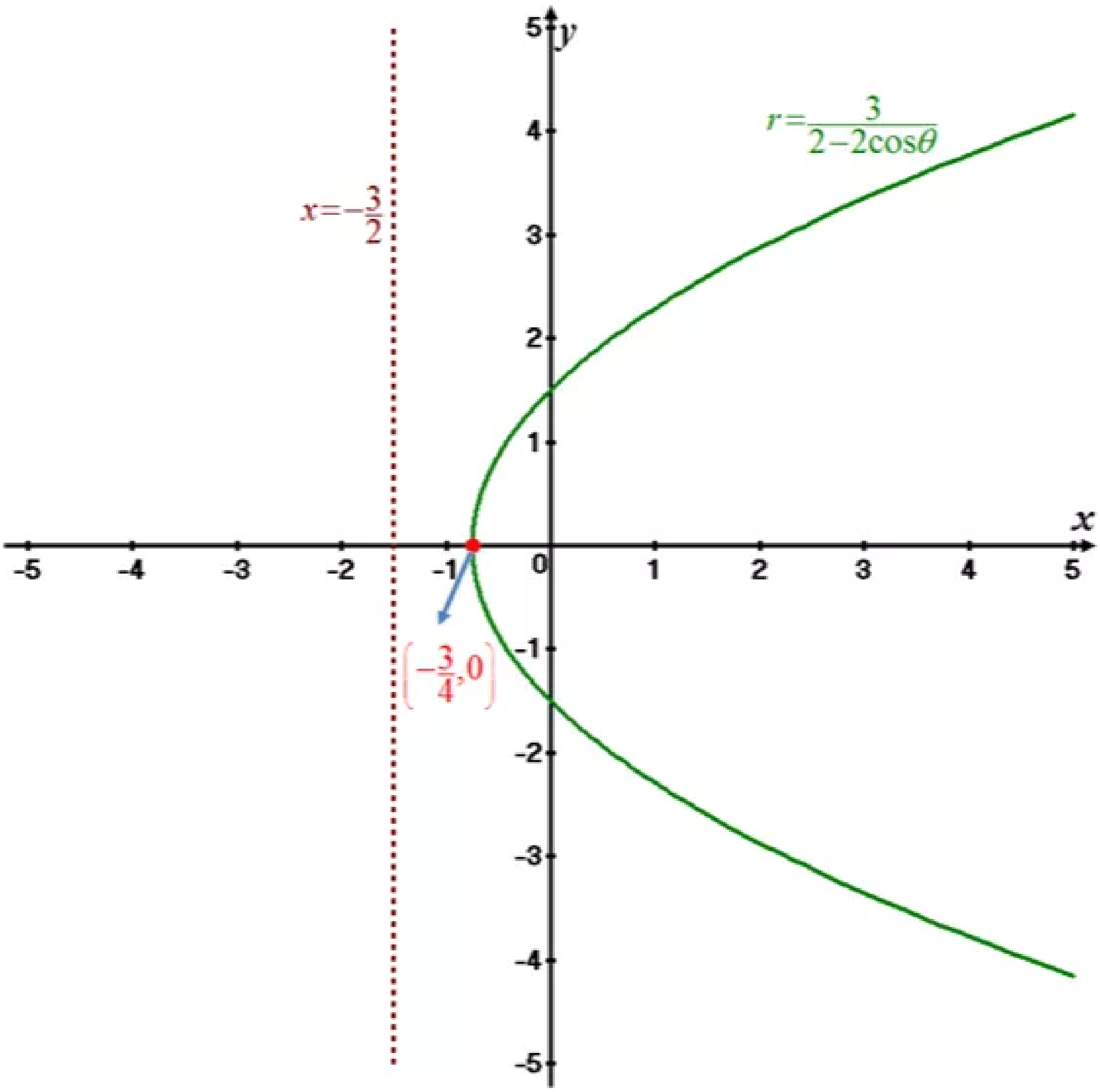
Then

$$r = \frac{3}{2 - 2\cos \pi}$$
$$= \frac{3}{2 - 2(-1)}$$
$$= \frac{3}{2 + 2}$$
$$= \frac{3}{4}$$

Thus, $r = \frac{3}{4}$

Thus, the vertex of the parabola is $\left(-\frac{3}{4}, 0\right)$.

The sketch of the parabola is as shown below:



Q17E

Since $\quad x = r\cos\theta$

$\qquad\quad y = r\sin\theta$

Therefore $\quad x + y = r\cos\theta + r\sin\theta$

$$= r\left(\cos\theta + \sin\theta\right)$$

$$\Rightarrow r\left(\cos\theta + \sin\theta\right) = 2$$

$$\Rightarrow \boxed{r = \frac{2}{\cos\theta + \sin\theta}}$$

Q18E

Since $x = r\cos\theta, \quad y = r\sin\theta$

Therefore

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$
$$= r^2\left(\cos^2\theta + \sin^2\theta\right)$$
$$= r^2$$
$$\Rightarrow \boxed{r^2 = 2} \text{ is a polar equation.}$$

Q19E

Given equation is $r = \sin\theta/\theta$

We sketch the graph of r as a function of θ in Cartesian Coordinates.

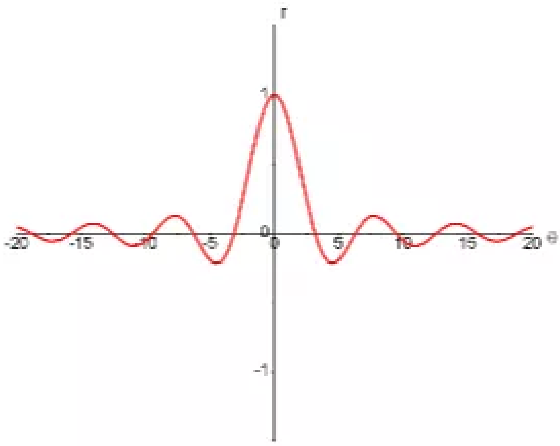


Fig.1

Step 2 of 3

We see that $as\,\theta \to \pm\infty, \quad r \to 0$

and $as\,\theta \to 0, \quad r \to 1$

Now we can sketch the curve

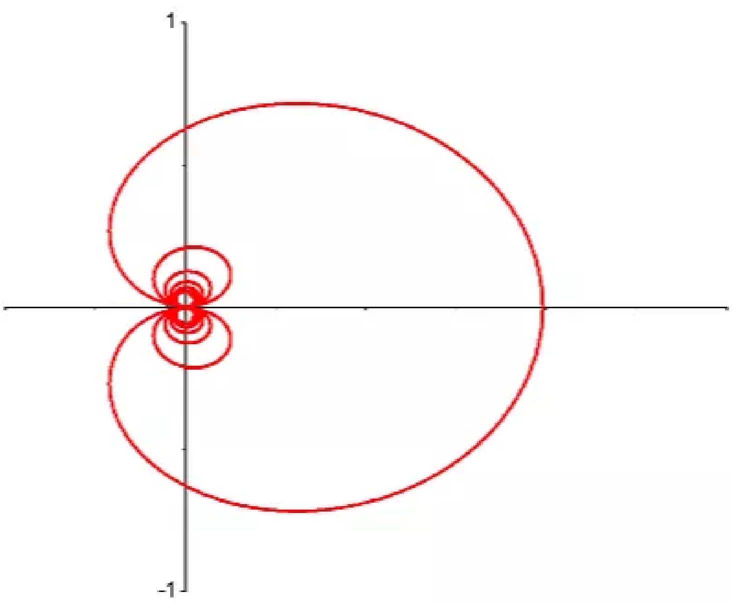


Fig.2

Q20E

Given equation is $r = 2/(4-3\cos\theta)$

Dividing the numerator and denominator by 4

We have $$r = \frac{1/2}{1-3/4\cos\theta}$$

Comparing with $$r = \frac{ed}{1-e\cos\theta}$$

We have $e = 3/4 \quad <1$ so this equation is an equation of ellipse

Since $$ed = 1/2$$
$$\Rightarrow d = 1/2e$$
$$= 2/3$$

And since in the given equation, denominator has negative sign

So directrix is $\boxed{x = -2/3}$

Now we sketch the curve and directrix

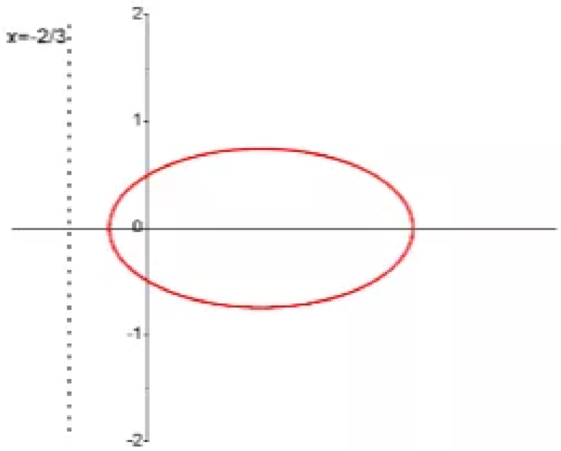


Fig.1

If graph of ellipse is rotated counterclockwise about the origin through an angle $2\pi/3$, then we have to replace θ by $(\theta - 2\pi/3)$ in the given equation and so equation of the curve becomes.

$$r = \frac{2}{4-3\cos(\theta - 2\pi/3)}$$

If graph is rotated clockwise about the origin through an angle $2\pi/3$, we have to replace θ by $(\theta + 2\pi/3)$ in the given equation

$$r = \frac{2}{4-3\cos(\theta + 2\pi/3)}$$

Now we sketch the curves for both the cases.

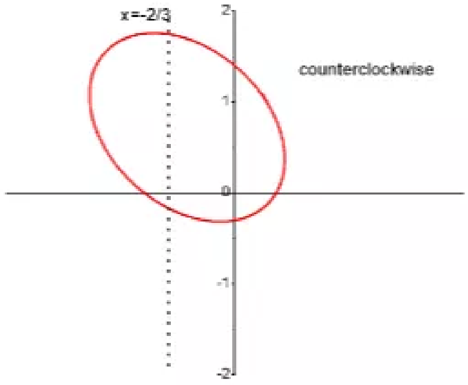


Fig.2

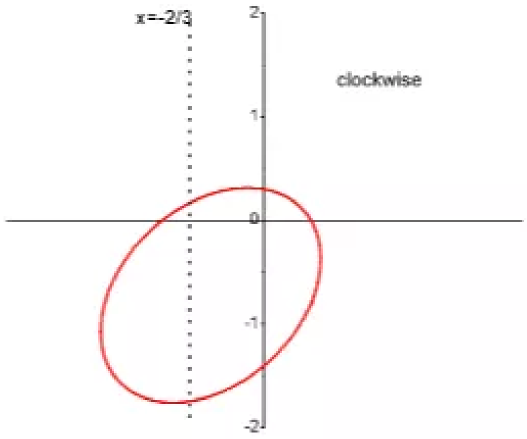


Fig.3

Q21E

We have $x = \ln t \qquad y = 1 + t^2$

Then $\dfrac{dx}{dt} = \dfrac{1}{t} \qquad \dfrac{dy}{dt} = 2t$

Therefore $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t}{1/t} = 2t^2$

The slope of the tangent line at a point where t = 1, is

$$\left.\frac{dy}{dx}\right]_{t=1} = 2(1)^2 = \boxed{2}$$

Q22E

We have $x = t^3 + 6t + 1 \qquad y = 2t - t^2$

Then $\dfrac{dx}{dt} = 3t^2 + 6 \qquad \dfrac{dy}{dt} = 2 - 2t$

Therefore $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2-2t}{3t^2+6}$

The slope of the tangent line at the point, where t = -1, is

$$\left.\frac{dy}{dx}\right]_{t=-1} = \frac{2-2(-1)}{3(-1)^2+6}$$

$$= \frac{2+2}{3+6} = \boxed{\frac{4}{9}}$$

Q23E

We have $r = e^{-\theta}$

Since $\dfrac{dy}{dx} = \dfrac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$

Then $\dfrac{dy}{dx} = \dfrac{-e^{-\theta}\sin\theta + e^{-\theta}\cos\theta}{-e^{-\theta}\cos\theta - e^{-\theta}\sin\theta}$

$= \dfrac{e^{-\theta}(\cos\theta - \sin\theta)}{-e^{-\theta}(\cos\theta + \sin\theta)}$

$= -\dfrac{(\cos\theta - \sin\theta)}{\cos\theta + \sin\theta}$

$= \dfrac{\sin\theta - \cos\theta}{\cos\theta + \sin\theta}$

The slope of the tangent line at a point where $\theta = \pi$, is

$$\left.\frac{dy}{dx}\right]_{\theta=\pi} = \frac{\sin\pi - \cos\pi}{\cos\pi + \sin\pi}$$

$$= \frac{0-(-1)}{-1+0} = \boxed{-1}$$

Q24E

We have $r = 3 + \cos 3\theta$

Then $\dfrac{dr}{d\theta} = -3\sin 3\theta$

Now $\dfrac{dy}{dx} = \dfrac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$

$= \dfrac{-3\sin 3\theta\sin\theta + (3+\cos 3\theta)\cos\theta}{-3\sin 3\theta\cos\theta - (3+\cos 3\theta)\sin\theta}$

The slope of the tangent line at a point where $\theta = \dfrac{\pi}{2} is$

$$\left.\frac{dy}{dx}\right]_{\theta=\pi/2} = \frac{-3\sin(3\pi/2)\sin(\pi/2) + (3+\cos(3\pi/2))\cos(\pi/2)}{-3\sin(3\pi/2)\cos(\pi/2) - (3+\cos(3\pi/2))\sin(\pi/2)}$$

$$= \frac{-3\cdot(-1)\cdot(1) + (3+0)\cdot 0}{-3\cdot(-1)\cdot 0 - (3+0)\cdot 1}$$

$$= \frac{3+0}{0-3} = \boxed{-1}$$

Q25E

Consider the parametric equations,

$x = t + \sin t, \quad y = t - \cos t$

First, to find $\frac{dx}{dt}$:

$$\frac{dx}{dt} = \frac{d}{dt}(t + \sin t)$$
$$= \frac{d}{dt}(t) + \frac{d}{dt}(\sin t)$$ Use the Sum Rule.

$= 1 + \cos t$ Use the Product Rule and $\frac{d}{dt}(\sin t) = \cos t$.

$\neq 0$

Next, to find $\frac{dy}{dt}$:

$$\frac{dy}{dt} = \frac{d}{dt}(t - \cos t)$$
$$= \frac{d}{dt}(t) - \frac{d}{dt}(\cos t)$$ Use the Difference Rule.

$= 1 - (-\sin t)$ Use the Product Rule and $\frac{d}{dt}(\cos t) = -\sin t$.

$= 1 + \sin t$

Finally, to find $\frac{dy}{dx}$:

Use the formula $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ if $\frac{dx}{dt} \neq 0$, to get

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$= \frac{1 + \sin t}{1 + \cos t}$$ Substitute.

$$= \boxed{\frac{1 + \sin t}{1 + \cos t}}.$$

First, to find $\frac{d}{dt}(dy/dx)$:

$$\frac{d}{dt}(dy/dx) = \frac{d}{dt}\left(\frac{1+\sin t}{1+\cos t}\right)$$

$$= \frac{(1+\cos t)\frac{d}{dt}(1+\sin t) - (1+\sin t)\frac{d}{dt}(1+\cos t)}{(1+\cos t)^2}$$ Use the Quotient Rule.

$$= \frac{(1+\cos t)\cdot(\cos t) - (1+\sin t)\cdot(-\sin t)}{(1+\cos t)^2}$$

$$= \frac{\cos t + \cos^2 t + \sin t + \sin^2 t}{(1+\cos t)^2}$$

$$= \frac{1+\cos t+\sin t}{(1+\cos t)^2}$$ Use the identity $\sin^2 t + \cos^2 t = 1$

Finally, to find $\frac{d^2y}{dx^2}$:

Use the formula $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{dx/dt}$ if $\frac{dx}{dt} \neq 0$, to get

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{dx/dt}$$ Substitute.

$$= \frac{\frac{1+\cos t+\sin t}{(1+\cos t)^2}}{1+\cos t}$$

$$= \boxed{\frac{1+\cos t+\sin t}{(1+\cos t)^3}}.$$

Q26E

We have $x = 1+t^2$, $y = t-t^3$

Then $\frac{dx}{dt} = 2t$ $\frac{dy}{dt} = 1-3t^2$

Therefore $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{1-3t^2}{2t}$$

Thus $$\boxed{\frac{dy}{dx} = \frac{1-3t^2}{2t}}$$

Since $\frac{dy}{dx} = \frac{1-3t^2}{2t} = \frac{1}{2t} - \frac{3}{2}t$

Then $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{1}{2}(-1)t^{-2} - \frac{3}{2}}{2t}$$

$$= \frac{1}{2}\frac{\left(-\frac{1}{t^2} - 3\right)}{2t}$$

$$= \frac{1}{4t}\left(\frac{-1-3t^2}{t^2}\right)$$

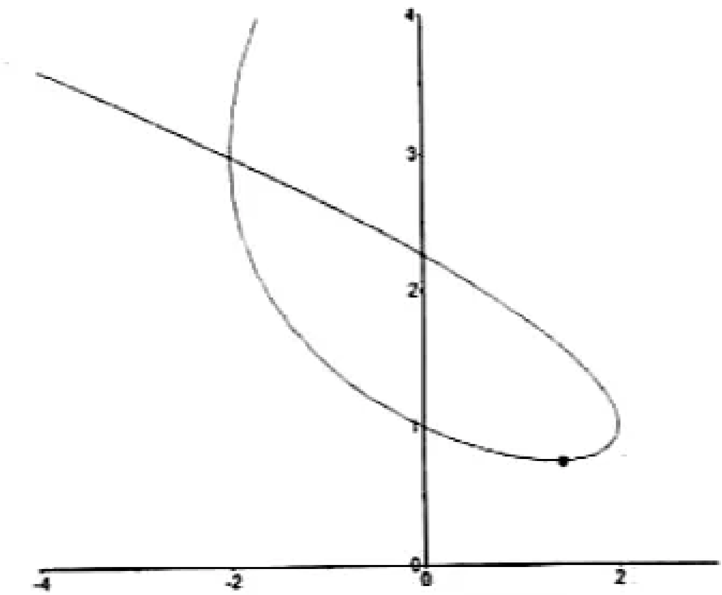
Thus $\boxed{\frac{d^2y}{dx^2} = \frac{-(1+3t^2)}{4t^3}}$

Q27E

We have the questions $x = t^3 - 3t$

and $y = t^2 + t + 1$

Now we sketch the curve



From the graph we see that lowest point is about (1.4, 0.75)

Slope of the curve $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$= \frac{2t+1}{3t^2-3}$$

Curve must have a horizontal tangent at its lowest point and for this

We must have $\frac{dy}{dx} = 0$

$$\Rightarrow 2t+1=0 \quad \Rightarrow \boxed{t=-1/2}$$

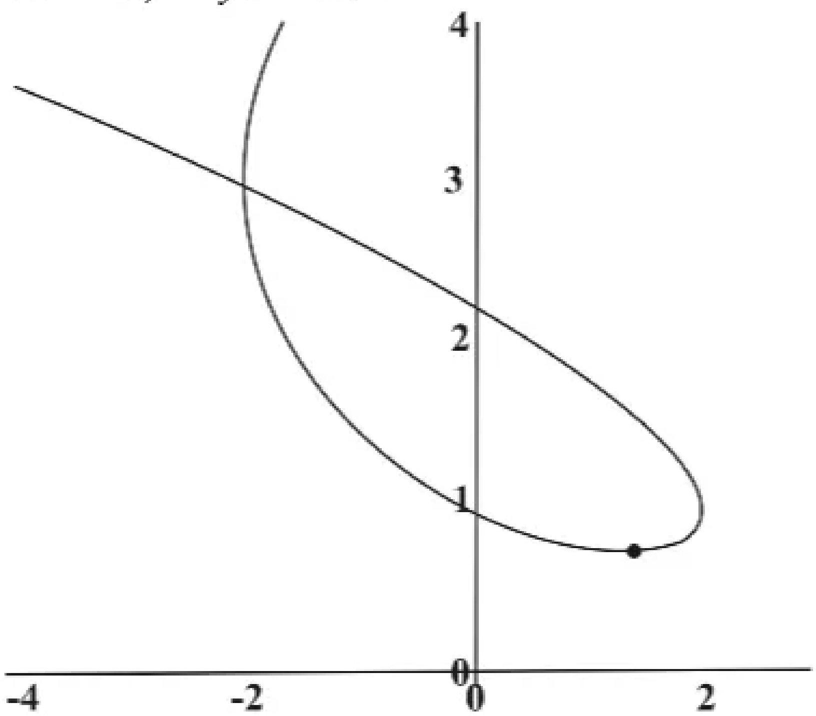
Calculating the value of x and y for $t=-1/2$, we get $x=\frac{11}{8}$ and $y=\frac{3}{4}$

So lowest point on the curve is $\boxed{\left(\frac{11}{8},\frac{3}{4}\right)}$

Q28E

First we sketch the curve with parametric equations

$$x=t^3-3t, \qquad y=t^2+t+1$$



For getting the right most point we have to find t at which curve has vertical tangents and for this we must have $\frac{dx}{dt}=0$

$$\Rightarrow 3t^2-3=0$$
$$\Rightarrow t^2-1=0$$
$$\Rightarrow t=\pm 1$$

For getting the right most point we have to find t at which curve has vertical tangents and for this we must have $\frac{dx}{dt}=0$

$$\Rightarrow 3t^2-3=0$$
$$\Rightarrow t^2-1=0$$
$$\Rightarrow t=\pm 1$$

For getting the right most point we have to find t at which curve has vertical tangents and for this we must have $\frac{dx}{dt}=0$

$$\Rightarrow 3t^2-3=0$$
$$\Rightarrow t^2-1=0$$
$$\Rightarrow t=\pm 1$$

Therefore the area of the loop $=3\left[\frac{t^5}{5}+\frac{t^4}{4}-\frac{t^2}{2}-t\right]_{-2}^{1}$

$$=3\left[\left(\frac{1}{5}+\frac{1}{4}-\frac{1}{2}-1\right)-\left(\frac{-32}{5}+\frac{16}{4}-\frac{4}{2}+2\right)\right]$$
$$=3\left[\left(\frac{-21}{20}\right)-\left(\frac{-48}{20}\right)\right]$$
$$=3\times\frac{27}{20}=\frac{81}{20}$$
$$\Rightarrow \boxed{A=4.05}$$

Q29E

Parametric equations are $x=2a\cos t-a\cos 2t$

And $y=2a\sin t-a\sin 2t$

Then $\frac{dx}{dt}=-2a\sin t+2a\sin 2t$

$$=2a\sin t+4a\sin t\cos t \qquad [2\sin t\cos t=\sin 2t$$

Or $\frac{dx}{dt}=2a\sin t\left(2\cos t-1\right)$

And $\frac{dy}{dt}=2a\cos t-2a\cos 2t$

$$=2a\cos t-2a\left(2\cos^2 t-1\right)$$
$$\Rightarrow \frac{dy}{dt}=2a\left(1+\cos t-2\cos^2 t\right) \qquad \left(\cos 2t=2\cos^2 t-1\right)$$

For vertical tangents we must have $\frac{dx}{dt} = 0$

$$\Rightarrow 2a \sin t(2\cos t - 1) = 0$$
$$\Rightarrow \sin t = 0 \quad or \quad 2\cos t - 1 = 0$$
$$\Rightarrow \sin t = 0 \quad or \quad \cos t = 1/2$$
$$\Rightarrow \boxed{t = \pi/3,\ \pi,\ \text{and}\ 5\pi/3} \qquad \text{For } 0 \le t \le 2\pi$$

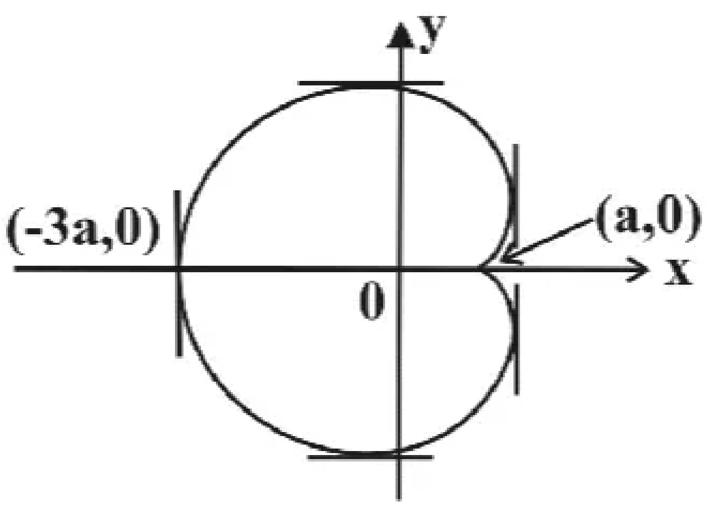
For horizontal tangents we must have $\frac{dy}{dt} = 0$

$$\Rightarrow 2a\left(1 + \cos t - 2\cos^2 t\right) = 0$$
$$\Rightarrow 2\cos^2 t - \cos t - 1 = 0$$
$$\Rightarrow 2\cos^2 t - 2\cos t + \cos t - 1 = 0$$
$$\Rightarrow 2\cos t(\cos t - 1) + 1(\cos t - 1) = 0$$
$$\Rightarrow (\cos t - 1)(2\cos t + 1) = 0$$
$$\Rightarrow \cos t = 1 \quad or \quad \cos t = -1/2$$
$$\Rightarrow \boxed{t = 2\pi/3,\ \text{and}\ 4\pi/3}$$

Now we calculate corresponding points for different values of t at which curve has horizontal or vertical tangent.

t	x	y
0	a	0
$\pi/3$	$3a/2$	$\sqrt{3}a/2$
$2\pi/3$	$-a/2$	$3\sqrt{3}a/2$
π	$-3a$	0
$4\pi/3$	$-a/2$	$-3\sqrt{3}a/2$
$5\pi/3$	$3a/2$	$-\sqrt{3}a/2$

Using these points we sketch the graph



Q30E

We have $\quad x = 2a\cos t - a\cos 2t,\ \ y = 2a\sin t - a\sin 2t$

Then $\quad \dfrac{dx}{dt} = -2a\sin t + 2a\sin 2t$

Then Area $\quad A = 2\int_{\pi}^{0} y\dfrac{dx}{dt}\,dt$

$$= 2\int_{\pi}^{0}(2a\sin t - a\sin 2t)(-2a\sin t + 2a\sin 2t)\,dt$$

$$= 2\int_{\pi}^{0}\left(-4a^2\sin^2 t + 2a^2\sin t\sin 2t + 4a^2\sin t\sin 2t - 2a^2\sin^2 2t\right)dt$$

$$= 2\int_{\pi}^{0}\left(-4a^2\sin^2 t - 2a^2\sin^2 2t + 6a^2\sin t\sin 2t\right)dt$$

$$= -2\left(2a^2\right)\int_{\pi}^{0}\left(2\sin^2 t + \sin^2 2t - 3\sin t\sin 2t\right)dt$$

$$= 4a^2\int_{0}^{\pi}\left[(1-\cos 2t) + \frac{1}{2}(1-\cos 4t) - 6\sin^2\cos t\right]dt$$

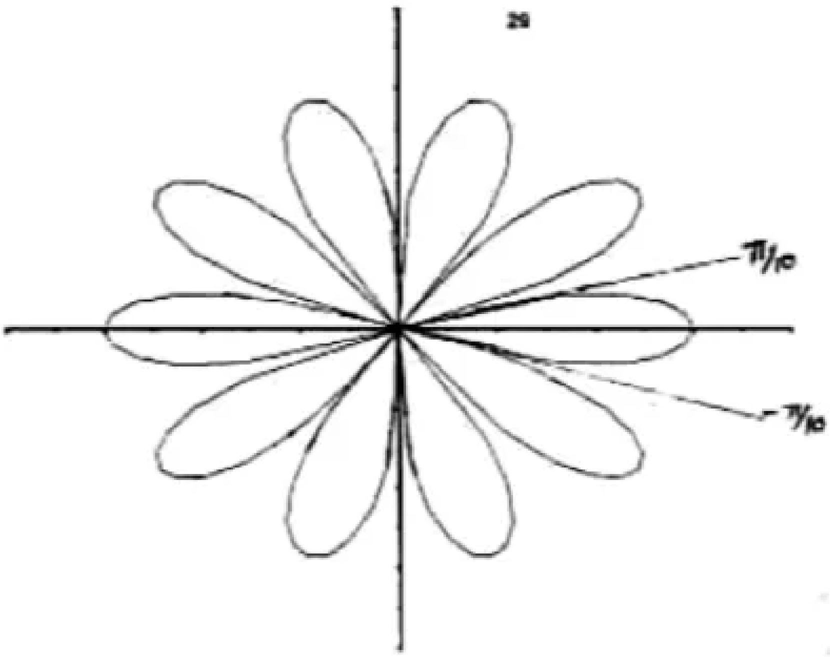
Therefore the area $= 4a^2\left[t - \dfrac{1}{2}\sin 2t + \dfrac{1}{2}\left(t - \dfrac{1}{4}\sin 4t\right) - 2\sin^3 t\right]_0^{\pi}$

$$= 4a^2\left[\pi - 0 + \frac{1}{2}(\pi - 0) - 0\right]$$

$$= 4a^2\left(3\pi/2\right) = \boxed{6\pi a^2}$$

Q31E

First we sketch the curve $r^2 = 9\cos 5\theta$



We see that this curve has 10 leaves and one leave is in the interval $-\frac{\pi}{10} \le \theta \le \frac{\pi}{10}$

So Area enclosed by the curve

$$A = 10\int_{-\pi/10}^{\pi/10} \frac{1}{2} r^2 d\theta$$

$$= 5\int_{-\pi/10}^{\pi/10} 9\cos 5\theta d\theta$$

$$= 90\int_{0}^{\pi/10} \cos 5\theta d\theta \qquad [(\cos 5\theta) \text{is an even fountion}]$$

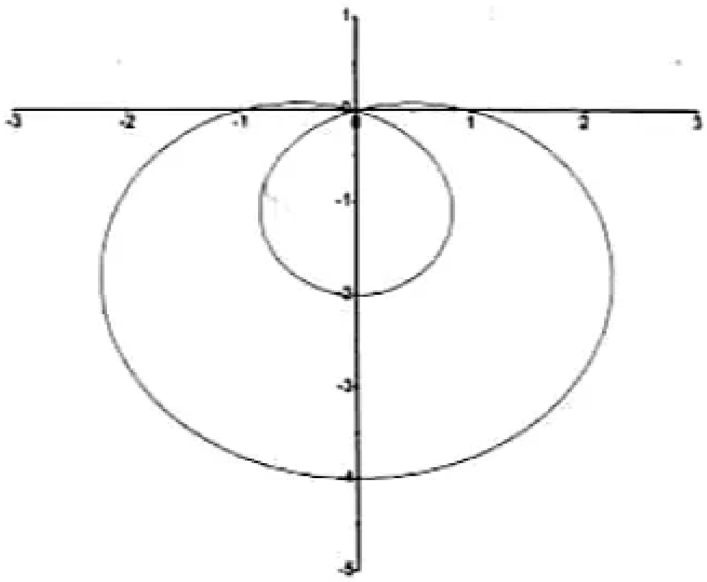
$$= \frac{90}{5}[\sin 5\theta]_0^{\pi/10}$$

$$= 18\left[\sin\frac{\pi}{2} - 0\right] = 18(1-0)$$

$$\Rightarrow \boxed{A = 18}$$

Q32E

First we sketch the curve



For inner loop we find the interval.

We set $r = 0$

$$\Rightarrow 1 - 3\sin\theta = 0$$

$$\Rightarrow \sin\theta = (1/3)$$

$$\Rightarrow \theta = \sin^{-1}(1/3)$$

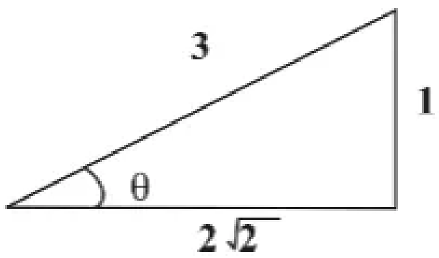
So loop is in between $\sin^{-1}\frac{1}{3} \le \theta \le \pi/2$ $\qquad \left[\begin{array}{l}\text{Since} \\ \text{at}(\theta = \pi/2), r = -2\end{array}\right.$

Then area encloses by inner loop is

$$A = 2\int_{\sin^{-1}(1/3)}^{\pi/2} \frac{1}{2} r^2 d\theta$$
$$= \int_{\sin^{-1}(1/3)}^{\pi/2} (1-3\sin\theta)^2 d\theta$$
$$= \int_{\sin^{-1}(1/3)}^{\pi/2} (1+9\sin^2\theta - 6\sin\theta) d\theta$$
$$= \int_{\sin^{-1}(1/3)}^{\pi/2} (1-6\sin\theta) d\theta + 9\int_{\sin^{-1}(1/3)}^{\pi/2} \sin^2\theta\, d\theta$$

Using trigonometric relation $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ and integrating, we get

$$A = \left[\theta + 6\cos\theta + \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta\right]_{\theta=\sin^{-1}\frac{1}{3}}^{\theta=\pi/2}$$
$$= \left[\frac{11}{2}\theta + 6\cos\theta - \frac{9}{2}\sin\theta\cos\theta\right]_{\theta=\sin^{-1}\frac{1}{3}}^{\theta=\pi/2}$$
$$= \frac{11}{2}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right) - 6\cos\left(\sin^{-1}\frac{1}{3}\right) + \frac{9}{2}\sin\left(\sin^{-1}\frac{1}{3}\right)\cos\left(\sin^{-1}\frac{1}{3}\right)$$
$$= \frac{11}{2}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right) - 6\times\frac{2\sqrt{2}}{3} + \frac{9}{2}\times\frac{1}{3}\times\frac{2\sqrt{2}}{3}$$
$$= \boxed{\frac{11}{2}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right) - 3\sqrt{2}}$$



$$\sin\theta = \frac{1}{3}$$
$$\cos\theta = \frac{2\sqrt{2}}{3}$$

Q33E

Since $r = 2$ *and* $r = 4\cos\theta$ are two given curves

For the points of intersection we must have

$$4\cos\theta = 2$$
$$\Rightarrow \cos\theta = \frac{1}{2}$$
$$\Rightarrow \theta = +\frac{\pi}{3}, -\frac{\pi}{3} \quad \text{for } -\pi \le \theta \le \pi$$

When $\theta = \pm\pi/3$, $\quad r = 4\cos(\pm\pi/3)$

$$= 4\left(\frac{1}{2}\right) = 2$$

The points of intersection of two given curves are $\boxed{(2, \pi/3) \text{ and } (2, -\pi/3)}$

Q34E

$$r = \cot\theta = \frac{\cos\theta}{\sin\theta} \quad \textit{and} \quad r = 2\cos\theta \;\; \textit{are two curves}$$

So

$$\frac{\cos\theta}{\sin\theta} = 2\cos\theta$$

By cross multiplication

$$\Rightarrow 2\sin\theta\cos\theta = \cos\theta$$

$$\Rightarrow 2\frac{\sin\theta\cos\theta}{\cos\theta} = 1$$

$$\Rightarrow 2\sin\theta = 1$$

$$\Rightarrow \sin\theta = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

Then $\quad r = \dfrac{\cos\pi/6}{\sin\pi/6} = \dfrac{\sqrt{3}/2}{1/2} = \sqrt{3}$

And

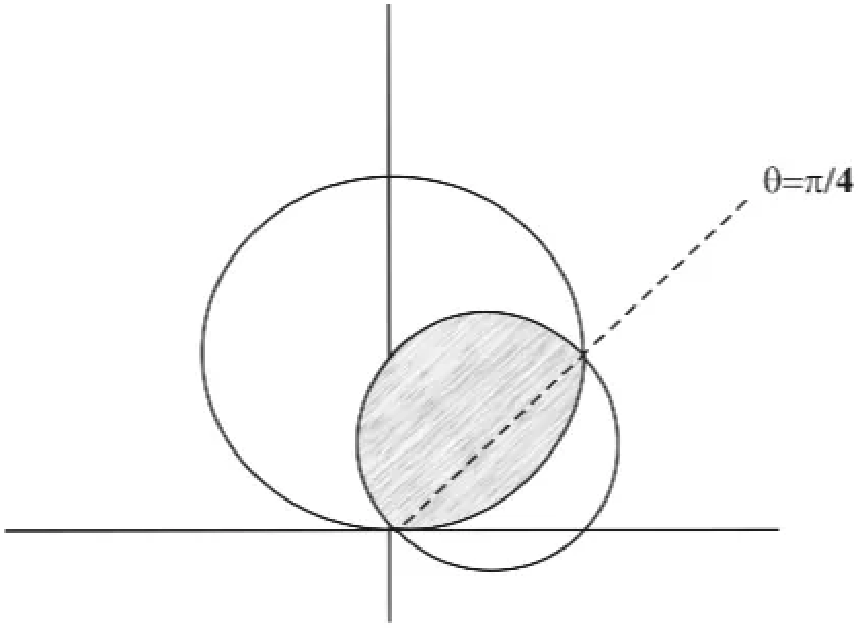
$$r = \frac{\cos 5\pi/6}{\sin 5\pi/6} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$$

Therefore, the points of intersection of two given curve are

$$\left(\sqrt{3}, \frac{\pi}{6}\right), \left(-\sqrt{3}, 5\pi/6\right)$$

Q35E

First we sketch the curve $r = 2\sin\theta$ and $r = \sin\theta + \cos\theta$



For getting points of intersection

We take $2\sin\theta = \sin\theta + \cos\theta$

$$\Rightarrow \sin\theta = \cos\theta \Rightarrow \theta = \frac{\pi}{4} \quad \text{And at the pole at which}$$

$$\theta = \frac{3\pi}{4} \text{ on the second curve}$$

So Area of shaded region

$$A = \int_0^{\pi/4} \frac{1}{2}(2\sin\theta)^2 d\theta + \int_{\pi/4}^{3\pi/4} \frac{1}{2}(\sin\theta + \cos\theta)^2 d\theta$$

$$= \int_0^{\pi/4} 2\sin^2\theta d\theta + \int_{\pi/4}^{3\pi/4} \frac{1}{2}\left(\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta\right)d\theta$$

$$= \int_0^{\pi/4} (1 - \cos 2\theta)d\theta + \frac{1}{2}\int_{\pi/4}^{3\pi/4} (1 + \sin 2\theta)d\theta$$

[here we used $2\sin\theta\cos\theta = \sin 2\theta, \sin^2\theta + \cos^2\theta = 1 \qquad \cos^2\theta = 1 - 2\sin^2\theta$]

Therefore $\Rightarrow A = \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} + \left[\frac{1}{2}\theta - \frac{1}{4}\cos 2\theta\right]_{\pi/4}^{3\pi/4}$

$$= \left[\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right] + \left[\frac{3\pi}{8} - \frac{1}{4}\cos\frac{3\pi}{2} - \frac{\pi}{8} + \frac{1}{4}\cos\frac{\pi}{2}\right]$$

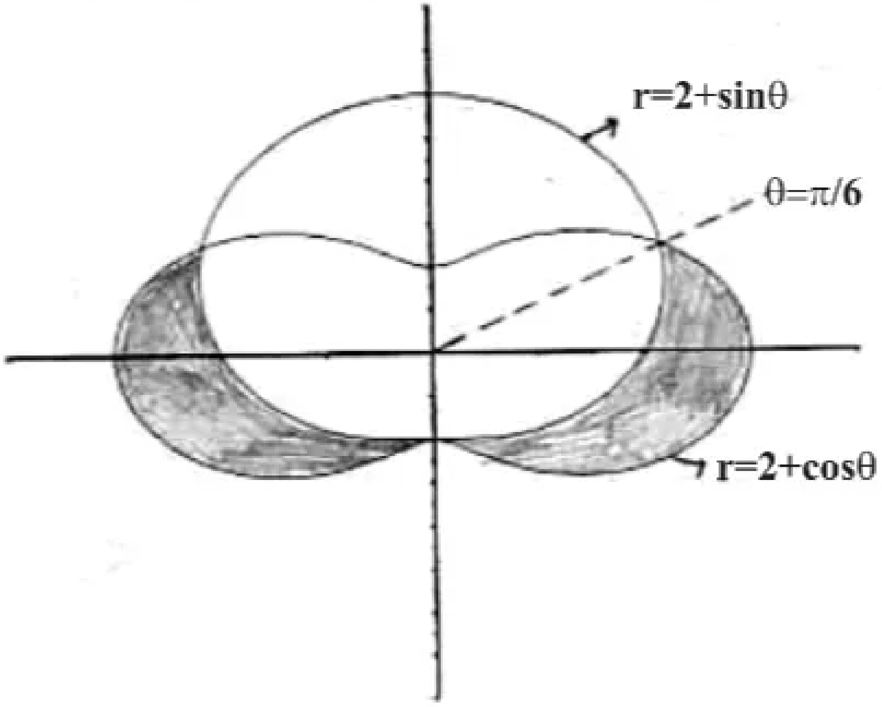
$$= \frac{\pi}{4} + \frac{\pi}{4} - \frac{1}{2}$$

$$= \pi/2 - \frac{1}{2}$$

$$\Rightarrow \boxed{A = \frac{1}{2}(\pi - 1)}$$

Q36E

First we sketch the curves $r = 2+\cos 2\theta$ *and* $r = 2+\sin\theta$



We have to find the area of shaded region which is symmetric about the axis.
First we find the points of intersection
For this $2+\cos 2\theta = 2+\sin\theta$

$$\Rightarrow \cos 2\theta = \sin\theta$$
$$\Rightarrow 1-2\sin^2\theta = \sin\theta$$
$$\Rightarrow 2\sin^2\theta+\sin\theta-1=0$$
$$\Rightarrow (\sin\theta+1)(2\sin\theta-1)=0$$
$$\Rightarrow \sin\theta=-1 \quad or \quad \sin\theta = 1/2$$
$$\Rightarrow \theta = -\pi/2 \quad or \quad \theta = \pi/6, 5\pi/6$$

Then Area of shaded region is (one side of the axis)

$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/6}(2+\cos 2\theta)^2\,d\theta - \frac{1}{2}\int_{-\pi/2}^{\pi/6}(2+\sin\theta)^2\,d\theta$$
$$\Rightarrow A = \frac{1}{2}\int_{-\pi/2}^{\pi/6}\left[(2+\cos 2\theta)^2-(2+\sin\theta)^2\right]d\theta$$
$$= \frac{1}{2}\int_{-\pi/2}^{\pi/6}\left[\left(4+\cos^2 2\theta+4\cos 2\theta\right)-\left(4+\sin^2\theta+4\sin\theta\right)\right]d\theta$$
$$= \frac{1}{2}\int_{-\pi/2}^{\pi/6}\left[\cos^2 2\theta+4\cos 2\theta-\sin^2\theta-4\sin\theta\right]d\theta$$

Since $\cos^2 2\theta = \frac{1+\cos 4\theta}{2}$ *and* $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

So $A = \frac{1}{2}\int_{-\pi/2}^{\pi/6}\left[\frac{1+\cos 4\theta}{2} + 4\cos 2\theta - \frac{(1-\cos 2\theta)}{2} - 4\sin\theta\right]d\theta$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/6}\left[\frac{\cos 4\theta}{2} + 4\cos 2\theta + \frac{\cos 2\theta}{2} - 4\sin\theta\right]d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/6}\left[\frac{\cos 4\theta}{2} + \frac{9}{2}\cos 2\theta - 4\sin\theta\right]d\theta$$

$$= \frac{1}{2}\left[\frac{\sin 4\theta}{8} + \frac{9}{4}\sin 2\theta + 4\cos\theta\right]_{-\pi/2}^{\pi/6}$$

$$= \frac{1}{2}\left[\frac{1}{8}\sin\left(\frac{2\pi}{3}\right) + \frac{9}{4}\sin\left(\frac{\pi}{3}\right) + 4\cos\left(\pi/6\right) - 0 - 0 - 0\right]$$

$$= \frac{1}{2}\left[\frac{\sqrt{3}}{16} + \frac{9\sqrt{3}}{8} + \frac{4\sqrt{3}}{2}\right] = \frac{1}{2}\left[\frac{\sqrt{3}+10\sqrt{3}+32\sqrt{3}}{16}\right] = \frac{51\sqrt{3}}{32}$$

Then total Area of shaded region $= 2A$

$$= 2\times 51\sqrt{3}/32$$

$$\Rightarrow \boxed{Area = 51\sqrt{3}/16}$$

Q37E

We have $x = 3t^2$, $y = 2t^3$ $0 \le t \le 2$

Then $\frac{dx}{dt} = 6t$, $\frac{dy}{dt} = 6t^2$

The length of the curve is

$$L = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^2 \sqrt{36t^2 + 36t^4}\,dt$$

$$= \int_0^2 6t\sqrt{1+t^2}\,dt$$

Now substituted $1+t^2 = u$ $\Rightarrow 2t\,dt = du$

And when $\begin{cases} t = 0, & u = 1 \\ t = 2, & u = 5 \end{cases}$

Therefore, $\mathrm{L} = \int_0^5 3\sqrt{u}\, du$

$$= 3\left[\frac{u^{3/2}}{3/2}\right]_1^5$$

$$= 2\left[5^{3/2} - 1\right]$$

$$= \boxed{2\left[5\sqrt{5} - 1\right]}$$

Q38E

We have $x = 2 + 3t, \qquad y = \cosh 3t$

Then $\dfrac{dx}{dt} = 3, \qquad \dfrac{dy}{dt} = 3\sinh 3t$

Q39E

We have the equation $r = \dfrac{1}{\theta} \qquad \pi \leq \theta \leq 2\pi$

$$\Rightarrow \frac{dr}{d\theta} = \frac{-1}{\theta^2}$$

$$\Rightarrow r^2 + \left(\frac{dr}{d\theta}\right)^2 = \left(\frac{1}{\theta}\right)^2 + \left(-\frac{1}{\theta^2}\right)^2$$

$$= \frac{1}{\theta^2} + \frac{1}{\theta^4} = \frac{\theta^2 + 1}{\theta^4}$$

Length of the curve is

$$L = \int_\pi^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_\pi^{2\pi} \sqrt{\frac{\theta^2 + 1}{\theta^4}}\, d\theta$$

$$= \int_\pi^{2\pi} \frac{\sqrt{\theta^2 + 1}}{\theta^2}\, d\theta$$

Using $\int \frac{\sqrt{u^2+a^2}}{u^2}du = \frac{-\sqrt{a^2+u^2}}{u} + \ln\left(u+\sqrt{a^2+u^2}\right) + C$

We have $L = \left[-\frac{\sqrt{1+\theta^2}}{\theta} + \ln\left|\theta + \sqrt{\theta^2+1}\right| \right]_{\pi}^{2\pi}$

$$= \left[-\frac{\sqrt{1+4\pi^2}}{2\pi} + \ln\left|2\pi+\sqrt{1+4\pi^2}\right| + \frac{\sqrt{1+\pi^2}}{\pi} - \ln\left|\pi+\sqrt{1+\pi^2}\right| \right]$$

Or $\boxed{L = \frac{\sqrt{1+\pi^2}}{\pi} - \frac{\sqrt{1+4\pi^2}}{2\pi} + \ln\left|\frac{2\pi+\sqrt{1+4\pi^2}}{\pi+\sqrt{1+\pi^2}}\right|}$

Or $\boxed{L = \frac{2\sqrt{1+\pi^2} - \sqrt{1+4\pi^2}}{2\pi} + \ln\left(\frac{2\pi+\sqrt{1+4\pi^2}}{\pi+\sqrt{1+\pi^2}}\right)}$

Q40E

Given equation is $r = \sin^3(\theta/3)$, $\quad 0 \le \theta \le \pi$

$$\Rightarrow \frac{dr}{d\theta} = 3\sin^2(\theta/3)\cos(\theta/3)\frac{1}{3} \qquad [\text{by chain rule}]$$

$$\Rightarrow \frac{dr}{d\theta} = \sin^2(\theta/3)\cos(\theta/3)$$

Then $r^2 + \left(\frac{dr}{d\theta}\right)^2 = \sin^6(\theta/3) + \sin^4(\theta/3)\cos^2(\theta/3)$

$$= \sin^4(\theta/3)\left[\sin^2(\theta/3) + \cos^2(\theta/3)\right]$$

$$= \sin^4(\theta/3) \qquad [\sin^2 A + \cos^2 A = 1]$$

Length of the curve

$$L = \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\Rightarrow L = \int_0^{\pi} \sqrt{\sin^4(\theta/3)}\, d\theta$$

$$\Rightarrow L = \int_0^{\pi} \sin^2(\theta/3)\, d\theta$$

Let $\dfrac{\theta}{3} = t$ then $d\theta = 3dt$ and $\theta = 0,\ t = 0$

When $\theta = \pi,\ t = \pi/3$

Then $L = 3\int_0^{\pi/3} \sin^2 t\ dt$

Since $\sin^2 t = \dfrac{1-\cos 2t}{2}$

Then $L = 3\int_0^{\pi/3} \left(\dfrac{1-\cos 2t}{2}\right)dt$

$$= \frac{3}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{\pi/3}$$

$$= \frac{3}{2}\left[\frac{\pi}{3} - \frac{\sin(2\pi/3)}{2} - 0 + 0\right]$$

$$= \frac{3}{2}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right]$$

$$\Rightarrow L = \frac{3}{2}\times\frac{1}{12}\left[4\pi - 3\sqrt{3}\right]$$

$$\Rightarrow \boxed{L = \frac{1}{8}\left(4\pi - 3\sqrt{3}\right)}$$

Q41E

We have $x = 4\sqrt{t}$, $y = \dfrac{t^3}{3} + \dfrac{1}{2t^2}$

Then $\dfrac{dx}{dt} = 4\dfrac{1}{2\sqrt{t}}$, $\dfrac{dy}{dt} = \dfrac{3t^2}{3} + \dfrac{1}{2}(-2)t^{-3}$

$= \dfrac{2}{\sqrt{t}}$ $\qquad = t^2 - \dfrac{1}{t^3}$

The surface are of the curve about the x- axis is

$$S=\int_1^4 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

$$=\int_1^4 2\pi\left(\frac{t^3}{3}+\frac{1}{2t^2}\right)\sqrt{\frac{4}{t}+\left(t^2-\frac{1}{t^3}\right)^2}\ dt$$

$$=\int_1^4 2\pi\left(\frac{t^3}{3}+\frac{1}{2t^2}\right)\sqrt{\frac{4}{t}+t^4+\frac{1}{t^6}-\frac{2}{t}}\ dt$$

$$=\int_1^4 2\pi\left(\frac{t^3}{3}+\frac{1}{2t^2}\right)\sqrt{t^4+\frac{1}{t^6}+\frac{2}{t}}\ dt$$

$$=\int_1^4 2\pi\left(\frac{t^3}{3}+\frac{1}{2t^2}\right)\left(t^2+\frac{1}{t^3}\right)dt$$

$$=\int_1^4 2\pi\left(\frac{1}{3}t^5+\frac{1}{3}+\frac{1}{2}+\frac{1}{2t^5}\right)dt$$

$$=2\pi\int_1^4\left(\frac{1}{3}t^5+\frac{5}{6}+\frac{1}{2}t^{-5}\right)dt$$

Therefore, $S=2\pi\left[\frac{1}{3}\left(\frac{t^6}{6}\right)+\frac{5}{6}t+\frac{1}{2}\left(\frac{t^{-4}}{-4}\right)\right]_1^4$

$$=2\pi\left[\frac{1}{18}\left(4^6-1^6\right)+\frac{5}{6}(4-1)-\frac{1}{8}\left(\frac{1}{4^4}-1\right)\right]$$

$$=2\pi\left[\frac{4095}{18}+\frac{5}{2}+\frac{1}{8}\left(\frac{255}{256}\right)\right]$$

$$=2\pi\frac{(465920+5120+255)}{2048}=\boxed{\frac{471295\pi}{1024}}$$

Q42E

We have $x=2+3t$ $\qquad y=\cosh 3t$

Then $\frac{dx}{dt}=3$, $\qquad \frac{dy}{dt}=3\sinh 3t$

The surfaces area of the curve about the x-axis is

$$S=\int_0^1 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

$$=\int_0^1 2\pi\ \cosh 3t\sqrt{9+9\sinh^2 3t}\ dt$$

$$=\int_0^1 6\pi\cosh 3t\sqrt{1+\sinh^2 3t}\ dt$$

$$=6\pi\int_0^1 \cosh^2 3t\ dt$$

$$=6\pi\int_0^1\frac{1}{2}(1+\cosh 6t)dt$$

$$\text{Therefore, } S = 3\pi\left[t + \frac{\sinh 6t}{6}\right]_0^1$$

$$= 3\pi\left[(1-0) + \frac{1}{6}(\sinh 6 - \sinh 0)\right]$$

$$= 3\pi\left[1 + \frac{1}{6}\frac{e^6 - e^{-6}}{2}\right]$$

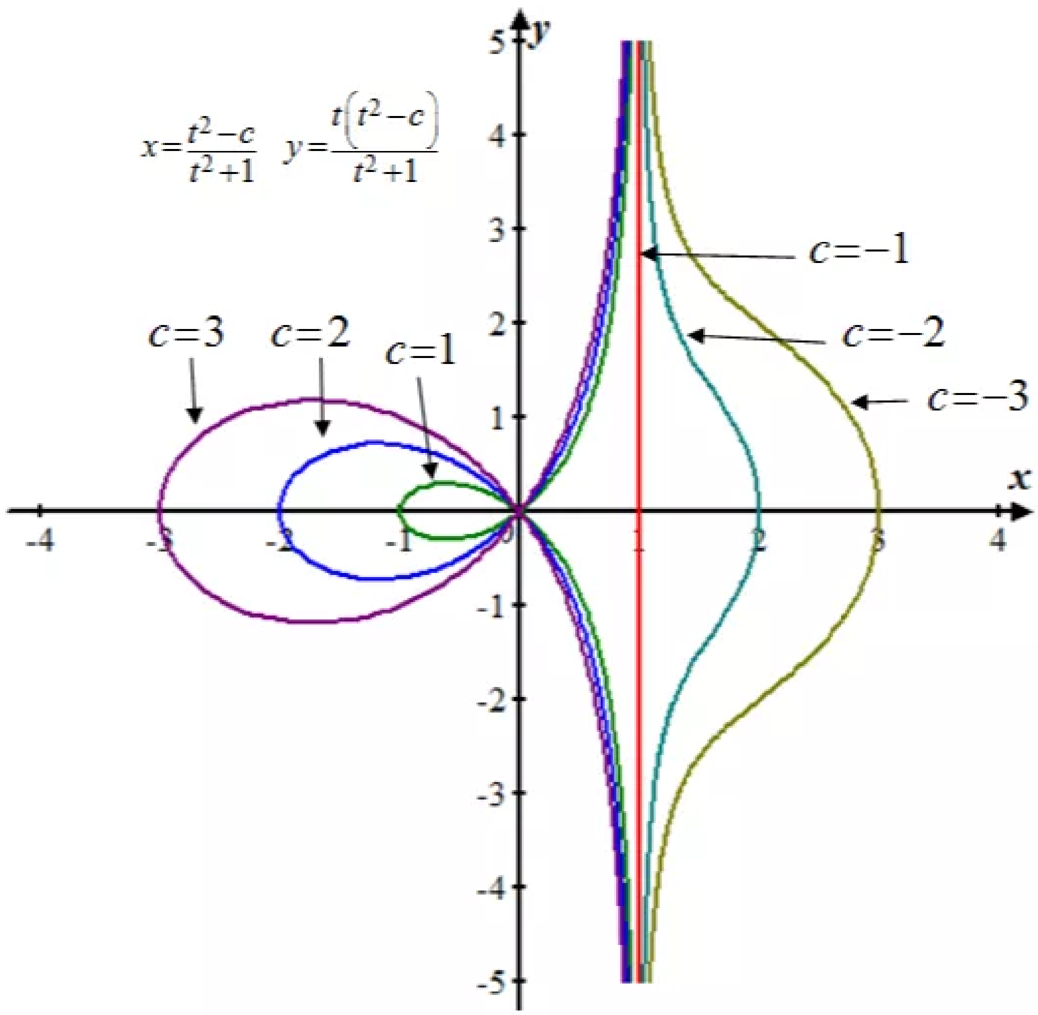
$$= \boxed{3\pi + \frac{\pi}{4}\left(e^6 - e^{-6}\right)}$$

Q43E

Consider the parametric equations,

$$x = \frac{t^2 - c}{t^2 + 1} \quad y = \frac{t\left(t^2 - c\right)}{t^2 + 1}$$

Sketch the graph of the function $x = \frac{t^2 - c}{t^2 + 1}$ $y = \frac{t\left(t^2 - c\right)}{t^2 + 1}$ for various values for c.



From the graph, we observe that all curves have the vertical asymptotes $x = 1$.

For $c < -1$, the curves bulges to the right. At $c = -1$, the curve is the line at $x = 1$

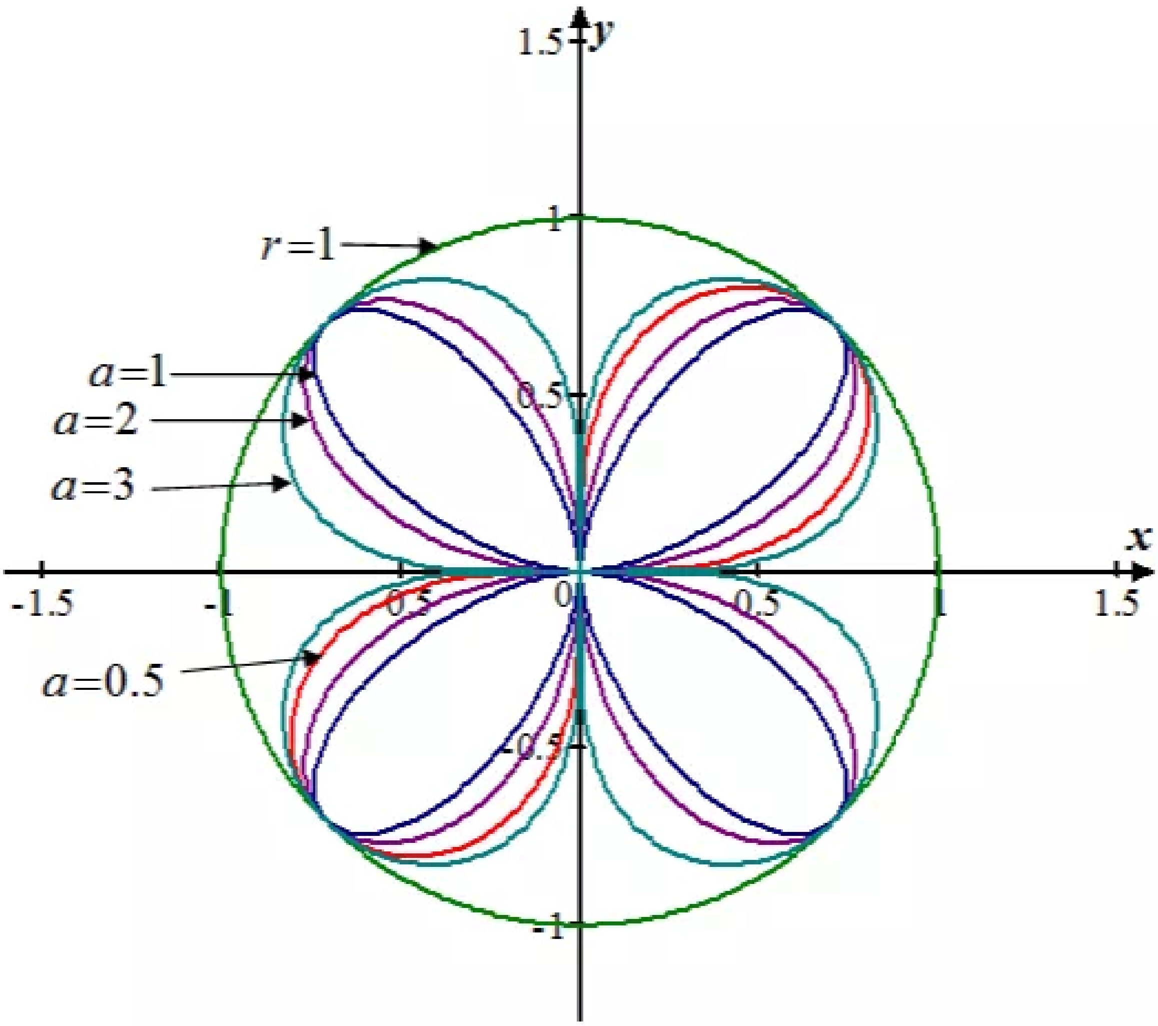
For $-1 < c < 0$, it bulges to the left. At $c = 0$ there is a cusp at $(0,0)$.

For $c > 0$, there is a loop.

Q44E

Consider the polar equation $r^a = |\sin 2\theta|$

Sketch the graph of the function $r^a = |\sin 2\theta|$ for various values of a.



As the value of a increases we see that the graph begins to look like a circle with a radius of 1 with its center at the origin. Initially at $a = 1$ we have the light blue curve we have four lobes. When we have $0 < a < 1$ we see the four lobes get narrower and narrower.

Q45E

Given equation is $\frac{x^2}{9} + \frac{y^2}{8} = 1$

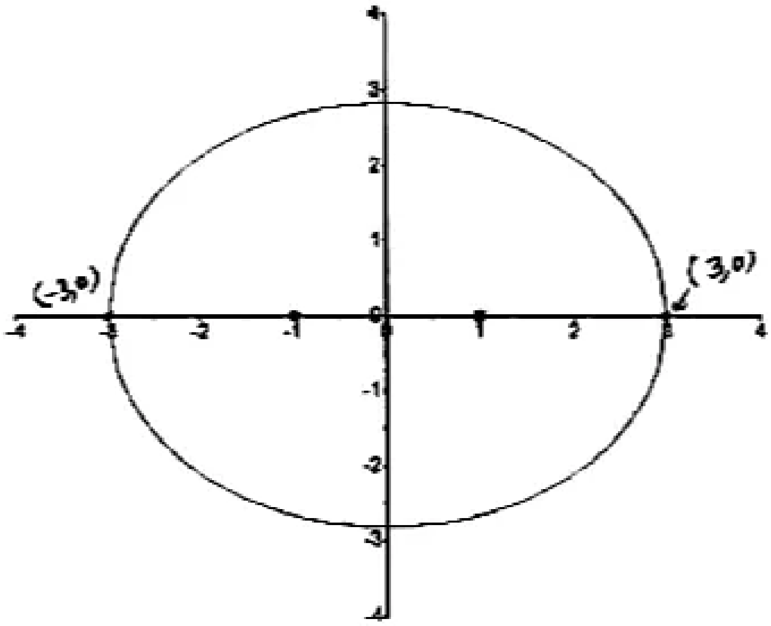
Comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

We have

$a = 3$ and $b = 2\sqrt{2}$ then $c^2 = a^2 - b^2 = 1$

Thus foci are $\boxed{(\pm 1, 0)}$ and vertices are $\boxed{(\pm 3, 0)}$

Now we sketch the curve which is an ellipse



Q46E

Given equation is $4x^2 - y^2 = 16$

Dividing by 16

$$\Rightarrow \frac{x^2}{4} - \frac{y^2}{16} = 1$$

$$\Rightarrow \frac{x^2}{2^2} - \frac{y^2}{4^2} = 1$$

Comparing with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

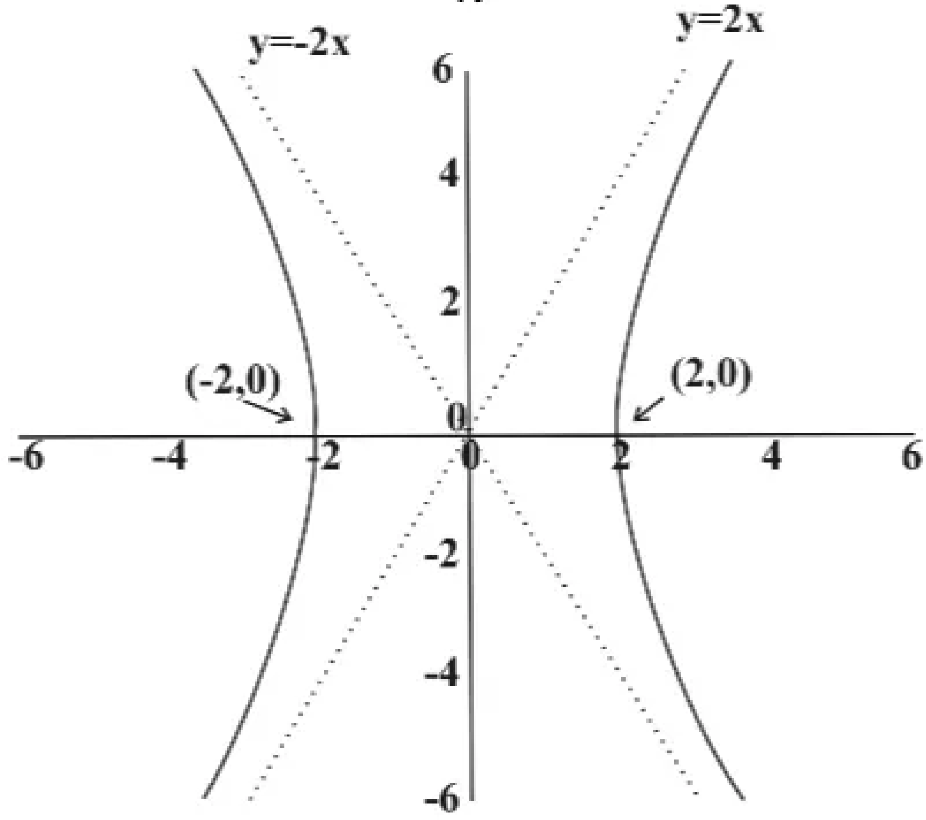
We have $a^2 = 4,$ $b^2 = 16$ then $c^2 = a^2 + b^2 = 20$

$$\Rightarrow c = \pm 2\sqrt{5}$$

Thus foci are $\boxed{\left(\pm 2\sqrt{5}, 0\right)}$ and vertices are $(\pm 2, 0)$,

Asymptotes are $y = \pm(b/a)x \Rightarrow \boxed{y = \pm 2x}$

Now we sketch the curve which is a hyperbola



Q47E

Given equation is $6y^2 + x - 36y + 55 = 0$

$\Rightarrow \quad 6\left(y^2 - 6y\right) + x + 55 = 0$

Making perfect square

$\Rightarrow 6\left(y^2 - 6y + 9 - 9\right) + x + 55 = 0$

$\Rightarrow 6\left(y^2 - 6y + 9\right) + x + 55 - 54 = 0$

$\Rightarrow 6(y-3)^2 + (x+1) = 0$

$\Rightarrow (x+1) = -6(y-3)^2$

$\Rightarrow (y-3)^2 = -\frac{1}{6}(x+1)$

Comparing with $(y-k)^2 = 4p(x-h)$, this is a parabola

We have $\quad h = -1, \quad k = 3 \quad p = -\frac{1}{24}$

Vertex of the parabola $\quad (h,k) = (-1,3)$

Focus of the parabola $\quad = (h+p, k)$

$= \left(-1 - \frac{1}{24}, 3\right) = \boxed{\left(-\frac{25}{24}, 3\right)}$

Since $p < 0$, so parabola opens to the left.

Directrix of parabola $x = -p + h$

$$\Rightarrow x = \frac{1}{24} - 1$$

$$\Rightarrow \boxed{x = \frac{-23}{24}}$$

Q48E

Given equation is $25x^2 + 4y^2 + 50x - 16y = 59$

$$\Rightarrow 25\left(x^2 + 2x\right) + 4\left(y^2 - 4y\right) = 59$$

Making perfect squares

$$\Rightarrow 25\left(x^2 + 2x + 1 - 1\right) + 4\left(y^2 - 4x + 4 - 4\right) = 59$$

$$\Rightarrow 25(x+1)^2 + 4(y-2)^2 = 100$$

Dividing by 100, $\Rightarrow \dfrac{(x+1)^2}{4} + \dfrac{(y-2)^2}{25} = 1$

This is an equation of $\boxed{ellipse}$

Comparing with $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$

We have h = -1, k =2

Center of the ellipse $= (h,k) = (-1,2)$

Since $a^2 = 25$ and $b^2 = 4$

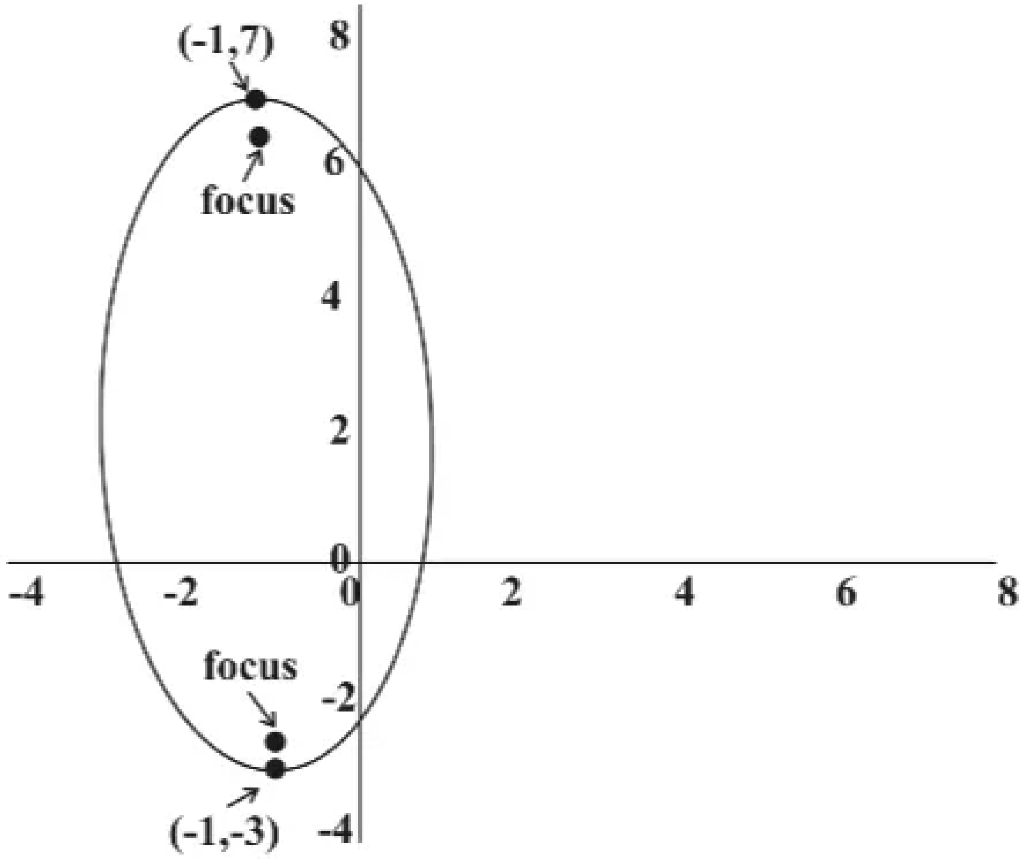
Then $c^2 = a^2 - b^2 = 25 - 4 = 21$

Then foci are $= (h, k \pm c) = \boxed{\left(-1, 2 \pm \sqrt{21}\right)}$

Vertices are $= (h, k \pm a) = (-1, 2 \pm 5)$

$$\boxed{= (-1,-3) \; and \; (-1,7)}$$

Now we sketch the curve



Q49E

Recollect the standard form of the equation of the ellipse which has foci $(\pm c, 0)$,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad a \geq b > 0$$

where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$

Find the equation of the ellipse with foci $(\pm 4, 0)$ and vertices $(\pm 5, 0)$.

Compare the given equation with the standard form.

$c = 4, a = 5$.

Substitute 4 for *c* and 5 for *a* in $c^2 = a^2 - b^2$.

$$\begin{aligned} c^2 &= a^2 - b^2 \\ 4^2 &= 5^2 - b^2 \\ 16 &= 25 - b^2 \\ b^2 &= 25 - 16 \\ b^2 &= 9 \\ b &= \sqrt{9} \\ &= 3 \end{aligned}$$

Therefore the equation of the ellipse with foci $(\pm 4, 0)$ and vertices $(\pm 5, 0)$ is given by the following the equation.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$$

$$\boxed{\frac{x^2}{25} + \frac{y^2}{9} = 1}$$

Q50E

Recollect the standard form of the equation of the parabola which has focus $(h+a, k)$, and directrix $x = h - a$ is

$$(y-k)^2 = 4a(x-h)$$

The parabola opens to the right if $a > 0$ and to the left if $a < 0$.

In both cases the graph is symmetric with respect to the *x*-axis, which is the axis of the parabola.

Find the equation of the parabola with focus $(2,1)$ and directrix $x = -4$.

Compare the given equation with the standard form.

$$h + a = 2, k = 1, h - a = -4.$$

Solve the equations $h + a = 2$, and $h - a = -4$ for *h*.

$$\begin{aligned} h + a + h - a &= 2 - 4 \quad \text{Add.} \\ 2h &= -2 \\ h &= -1 \end{aligned}$$

Substitute –1 for h in $h+a=2$.

$$h+a=2$$
$$-1+a=2$$
$$a=2+1$$
$$=3$$

Since $a(=3)>0$, so the parabola opens to the right.

Therefore the equation of the parabola with focus $(2,1)$ and directrix $x=-4$ is given by the following the equation.

$$(y-k)^2=4a(x-h)$$
$$(y-1)^2=4(3)[x-(-1)]$$
$$\boxed{(y-1)^2=12(x+1)}.$$

Q51E

Recollect the standard form of the equation of the hyperbola which has foci $(0,\pm c)$,

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

where $c^2=a^2+b^2$, vertices $(0,\pm a)$ and asymptotes $y=\pm\frac{a}{b}x$

Find the equation of the hyperbola with foci $(0,\pm 4)$ and asymptotes $y=\pm 3x$.

Compare the given equation with the standard form.

$$c=4, \frac{a}{b}=3$$

$$c=4, a=3b.$$

Substitute 4 for c and $3b$ for a in $c^2=a^2-b^2$.

$$c^2=a^2+b^2$$
$$4^2=(3b)^2+b^2$$
$$16=9b^2+b^2$$
$$16=10b^2$$
$$b^2=\frac{16}{10}$$
$$b=\sqrt{\frac{16}{10}}$$
$$=\frac{4}{\sqrt{10}}$$

Substitute $\frac{4}{\sqrt{10}}$ for b in $a = 3b$.

$$
\begin{aligned}
a &= 3b \\
&= 3\left(\frac{4}{\sqrt{10}}\right) \\
&= \frac{12}{\sqrt{10}}
\end{aligned}
$$

Therefore the equation of the hyperbola with foci $(0, \pm 4)$ and asymptotes $y = \pm 3x$ is given by the following the equation.

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$\frac{y^2}{\left(12/\sqrt{10}\right)^2} - \frac{x^2}{\left(4/\sqrt{10}\right)^2} = 1$$

$$\frac{y^2}{(144/10)} - \frac{x^2}{(16/10)} = 1$$

$$\boxed{\frac{y^2}{(72/5)} - \frac{x^2}{(8/5)} = 1}.$$

Q52E

The foci are located at the point $(3, \pm 2)$ and the length of major axis is given to be 8.

If P (x, y) is any point on the ellipse, then by definition of ellipse, we have

$$\sqrt{(x-3)^2 + (y-2)^2} + \sqrt{(x-3)^2 + (y+2)^2} = 8$$

$$\Rightarrow \sqrt{(x-3)^2 + (y-2)^2} = 8 - \sqrt{(x-3)^2 + (y+2)^2}$$

Squaring both the sides

$$(x-3)^2 + (y-2)^2 = 64 + (x-3)^2 + (y+2)^2 - 16\sqrt{(x-3)^2 + (y+2)^2}$$

$$\Rightarrow -8y = 64 - 16\sqrt{(x-3)^2 + (y+2)^2}$$

$$\Rightarrow y + 8 = 2\sqrt{(x-3)^2 + (y+2)^2}$$

Squaring both the sides

$$y^2+16y+64=4(x-3)^2+4y^2+16y+16$$

$$\Rightarrow \boxed{\frac{(x-3)^2}{12}+\frac{y^2}{16}=1}$$

$$\Rightarrow \boxed{\frac{(x-3)^2}{12}+\frac{y^2}{16}=1}$$

This is the required equation of ellipse.

Q53E

Equation of parabola is $x^2+y=100$

$$\Rightarrow x^2=-y+100$$

$$\Rightarrow x^2=-(y-100)$$

Comparing with $(x-h)^2=4p(y-k)$, $h=0,\ k=100$

Vertex of the parabola is $=(0,100)$

So one vertex of the ellipse is $(0,100)$

We have by comparison $4p=-1$

$$\Rightarrow p=-1/4$$

Then focus of the parabola is $=(h,k+p)$

$$=\left(0,100-\frac{1}{4}\right)=\boxed{\left(0,\frac{399}{4}\right)}$$

Thus shared focus of the ellipse is at $\left(0,\frac{399}{4}\right)$

Distance between the foci of the ellipse is

$$=2c=\frac{399}{4}-0 \qquad \text{[Another focus at the origin]}$$

Then $\Rightarrow c=\frac{399}{8}$

So center of the ellipse is $\boxed{\left(0,\frac{399}{8}\right)}$

So $a=100-\frac{399}{8}=\frac{401}{8}$

Since $c^2=a^2-b^2 \quad \Rightarrow b^2=a^2-c^2$

$$\Rightarrow b^2=\frac{401^2}{8^2}-\frac{399^2}{8^2}=25$$

So the equation of the ellipse is $\frac{x^2}{b^2}+\frac{\left(y-\frac{399}{8}\right)^2}{a^2}=1$

$$\Rightarrow \frac{x^2}{25}+\frac{\left(y-\frac{399}{8}\right)^2}{\left(\frac{401}{8}\right)^2}=1$$

Or $\boxed{\frac{x^2}{25}+\frac{(8y-399)^2}{160801}=1}$

Q54E

Let the line $y=mc+c$ be the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ---(1)

Then we put y = mx +c in the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$$

$$\Rightarrow b^2x^2+a^2\left(m^2x^2+c^2+2mcx\right)=a^2b^2$$

$$\Rightarrow x^2\left(a^2m^2+b^2\right)+2ma^2cx+a^2\left(c^2-b^2\right)=0 \qquad ---(2)$$

This is a quadratic equation as $Ax^2+Bx+C=0$

This equation will have equal roots, when $B^2=4AC$

Then $\left(2ma^2c\right)^2=4\left(a^2m^2+b^2\right)a^2\left(c^2-b^2\right)$

$$4m^2a^4c^2=4a^2\left(a^2m^2+b^2\right)\left(c^2-b^2\right)$$

$$\Rightarrow a^2m^2c^2=\left(a^2m^2+b^2\right)\left(c^2-b^2\right)$$

$$\Rightarrow b^2c^2=b^2\left(a^2m^2+b^2\right)$$

$$\Rightarrow c^2=a^2m^2+b^2$$

$$\Rightarrow c=\pm\sqrt{a^2m^2+b^2}$$

Thus for every real value of m, equation of tangent is $\boxed{y=mx\pm\sqrt{a^2m^2+b^2}}$

Q55E

We have eccentricity $e=1/3$

Equation of the directrix is $r=4\sec\theta$

Relation between Cartesian coordinates and polar coordinates

$x=r\cos\theta \qquad y=r\sin\theta$

$\Rightarrow x=4\sec\theta\cos\theta$

$\Rightarrow \boxed{x=4}$ since $\left(\sec\theta=1/\cos\theta\right)$

So equation of directrix in Cartesian coordinates is $\boxed{x=4}$

Then we have $d=4$

The equation of the ellipse as $r = \dfrac{ed}{1+e\cos\theta}$

$$\Rightarrow r = \frac{4/3}{1+\frac{1}{3}\cos\theta}$$

$$\Rightarrow \boxed{r = \frac{4}{3+\cos\theta}}$$

Q56E

Theorem: A polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \qquad \text{or} \qquad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic section with eccentricity e. the conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$.

Show that the angles between the polar axis $y = \pm d$ and the asymptotes of the hyperbola

$$r = \frac{ed}{1 - e\cos\theta}, \ e > 1,$$

are given by $\cos^{-1}\left(\pm\frac{1}{e}\right)$.

Note that $r \to \pm\infty$ when $1 \pm e\cos\theta \to 0^{+}$ or 0^{-}.

And $1 \pm e\cos\theta = 0$ when $\cos\theta = \pm\frac{1}{e}$.

Thus the asymptotes are parallel to the rays $\theta = \cos^{-1}\left(\pm\frac{1}{e}\right)$.

Hence the angles between the polar axis and the asymptotes of the hyperbola are given by

$$\boxed{\cos^{-1}\left(\pm\frac{1}{e}\right)}.$$