DIFFERENTIATION (XII, R. S. AGGARWAL)

EXERCISE 10A (Pg.No.: 370)

Differentiate each of the following w.r.t. x:

1. $y = \sin 4x$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(\sin 4x)}{d(4x)} \times \frac{d(4x)}{dx} \Rightarrow \frac{dy}{dx} = \cos 4x.4 \quad \therefore \frac{dy}{dx} = 4\cos 4x$$

2. $y = \cos 5x$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(\cos 5x)}{d(5x)} \times \frac{d(5x)}{dx} \Rightarrow \frac{dy}{dx} = -\sin 5x.5 \quad \therefore \frac{dy}{dx} = -5\sin 5x$$

3. $y = \tan 3x$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(\tan 3x)}{d(3x)} \times \frac{d(3x)}{dx} \Rightarrow \frac{dy}{dx} = \sec^2 3x.3 \quad \therefore \frac{dy}{dx} = 3\sec^2 3x$$

4. $y = \cos(x^3)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\cos(x^3)\}}{d(x^3)} \times \frac{d(x^3)}{dx} \Rightarrow \frac{dy}{dx} = -\sin(x^3).3x^2 \quad \therefore \frac{dy}{dx} = -3x^2 \sin(x^3)$$

5. $y = \cot^2 x$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(\cot^2 x)}{d(\cot x)} \times \frac{d(\cot x)}{dx} \Rightarrow \frac{dy}{dx} = 2\cot x(-\text{cosec}^2 x) \quad \therefore \frac{dy}{dx} = -2\cot x \,\text{cosec}^2 x$$

6. $y = \tan^3 x$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(\tan^3 x)}{d(\tan x)} \times \frac{d(\tan x)}{dx} \Rightarrow \frac{dy}{dx} = 3\tan^2 x \sec^2 x$$

7. $y = \cot(\sqrt{x})$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\cot(\sqrt{x})\}}{d(\sqrt{x})} \times \frac{d(\sqrt{x})}{dx} \Rightarrow \frac{dy}{dx} = -\text{cosec}^2(\sqrt{x}).\frac{1}{2\sqrt{x}} \quad \therefore \frac{dy}{dx} = -\frac{\text{cosec}^2(\sqrt{x})}{2\sqrt{x}}$$

8. $y = \sqrt{\tan x}$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(\sqrt{\tan x}\right)}{d\left(\tan x\right)}\times\frac{d\left(\tan x\right)}{dx} \quad \Rightarrow \frac{dy}{dx}=\frac{1}{2\sqrt{\tan x}}\sec^2 x \quad \therefore \frac{dy}{dx}=\frac{\sec^2 x}{2\sqrt{\tan x}}$$

9. $y=\left(5+7x\right)^6$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left\{\left(5+7x\right)^6\right\}}{d\left(5+7x\right)}\times\frac{d\left(5+7x\right)}{dx} \quad \Rightarrow \frac{dy}{dx}=6\left(5+7x\right)^{6-1}\left(0+7\right)$$

$$\Rightarrow \frac{dy}{dx}=6\left(5+7x\right)^5.7 \quad \therefore \frac{dy}{dx}=42\left(5+7x\right)^5$$

10. $y=\left(3-4x\right)^5$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left\{\left(3-4x\right)^5\right\}}{d\left(3-4x\right)}\times\frac{d\left(3-4x\right)}{dx} \quad \Rightarrow \frac{dy}{dx}=5\left(3-4x\right)^{5-1}\left\{\frac{d\left(3\right)}{dx}-\frac{d\left(4x\right)}{dx}\right\}$$

$$\Rightarrow \frac{dy}{dx}=5\left(3-4x\right)^4\left\{0-4\right\} \quad \therefore \frac{dy}{dx}=-20\left(3-4x\right)^4$$

11. $y=\left(2x^2-3x+4\right)^5$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left\{\left(2x^2-3x+4\right)^5\right\}}{d\left(2x^2-3x+4\right)}\times\frac{d\left(2x^2-3x+4\right)}{dx} \quad \Rightarrow \frac{dy}{dx}=5\left(2x^2-3x+4\right)^{5-1}\left\{\frac{d\left(2x^2\right)}{dx}-\frac{d\left(3x\right)}{dx}+\frac{d\left(4\right)}{dx}\right\}$$

$$\Rightarrow \frac{dy}{dx}=5\left(2x^2-3x+4\right)^4\left\{2\frac{d\left(x^2\right)}{dx}-3\frac{d\left(x\right)}{dx}+0\right\} \quad \Rightarrow \frac{dy}{dx}=5\left(2x^2-3x+4\right)^4\left(2.2x-3.1\right)$$

$$\Rightarrow \frac{dy}{dx}=5\left(2x^2-3x+4\right)^4\left(4x-3\right) \quad \therefore \frac{dy}{dx}=\left(20x-15\right)\left(2x^2-3x+4\right)^4$$

12. $y=\left(ax^2+bx+c\right)^6$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left\{\left(ax^2+bx+c\right)^6\right\}}{d\left(ax^2+bx+c\right)}\times\frac{d\left(ax^2+bx+c\right)}{dx}$$

$$\Rightarrow \frac{dy}{dx}=6\left(ax^2+bx+c\right)^5\left\{a\frac{d\left(x^2\right)}{dx}+b\frac{d\left(x\right)}{dx}+o\right\} \quad \Rightarrow \frac{dy}{dx}=6\left(ax^2+bx+c\right)^5\left\{a.2x+b.1\right\}$$

$$\Rightarrow \frac{dy}{dx}=6\left(ax^2+bx+c\right)^5\left(2ax+b\right)$$

13. $y=\frac{1}{\left(x^2-3x+5\right)^3}$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left\{\left(x^2-3x+5\right)^{-3}\right\}}{d\left(x^2-3x+5\right)}\times\frac{d\left(x^2-3x+5\right)}{dx} \quad \Rightarrow \frac{dy}{dx}=-3\left(x^2-3x+5\right)^{-3-1}\left\{\frac{d\left(x^2\right)}{dx}-\frac{d\left(3x\right)}{dx}+\frac{d\left(5\right)}{dx}\right\}$$

$$\Rightarrow \frac{dy}{dx} = -3(x^2-3x+5)^{-4}\,(2x-3+0) \quad \Rightarrow \frac{dy}{dx} = -3(x^2-3x+5)^{-4}(2x-3) \quad \therefore \frac{dy}{dx} = \frac{-3(2x-3)}{(x^2-3x+5)^4}$$

14. $y = \sqrt{\frac{a^2-x^2}{a^2+x^2}}$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left(\sqrt{\frac{a^2-x^2}{a^2+x^2}}\right)}{d\left(\frac{a^2-x^2}{a^2+x^2}\right)} \times \frac{d\left(\frac{a^2-x^2}{a^2+x^2}\right)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\frac{a^2-x^2}{a^2+x^2}}}\left\{\frac{(a^2+x^2)\frac{d(a^2-x^2)}{dx} - (a^2-x^2)\frac{d(a^2+x^2)}{dx}}{(a^2+x^2)^2}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{a^2+x^2}}{2\sqrt{a^2-x^2}}\left\{\frac{(a^2+x^2)(0-2x)-(a^2-x^2)(0+2x)}{(a^2+x^2)\sqrt{a^2+x^2}\sqrt{a^2+x^2}}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x(-a^2-x^2-a^2+x^2)}{2\sqrt{a^2-x^2}\,(a^2+x^2)^{3/2}} \quad \Rightarrow \frac{dy}{dx} = \frac{-2a^2x}{\sqrt{(a^2-x^2)}\,(a^2+x^2)^{3/2}} \quad \Rightarrow \frac{dy}{dx} = \frac{-2a^2x}{(a^2+x^2)^{3/2}(a^2-x^2)^{1/2}}$$

15. $y = \sqrt{\frac{1+\sin x}{1-\sin x}}$

Sol. Given, $y = \sqrt{\frac{1+\sin x}{1-\sin x}} = \sqrt{\frac{(1+\sin x)(1+\sin x)}{(1-\sin x)(1+\sin x)}} \quad \Rightarrow y = \sqrt{\frac{(1+\sin x)^2}{1-\sin^2 x}} = \frac{1+\sin x}{\cos x} \Rightarrow y = \sec x + \tan x$

$$\therefore \frac{dy}{dx} = \sec x \tan x + \sec^2 x = \sec x(\tan x + \sec x)$$

16. $y = \cos^2(x^3)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\cos^2(x^3)\}}{d\{\cos(x^3)\}} \times \frac{d(\cos(x^3))}{d(x^3)} \times \frac{d(x^3)}{dx} \quad \Rightarrow \frac{dy}{dx} = 2\cos(x^3).(-\sin x^3).3x^2$$

$$\Rightarrow \frac{dy}{dx} = -3x^2.\{2\cos(x^3).\sin(x^3)\} \quad \Rightarrow \frac{dy}{dx} = -3x^2\sin(2.x^3) \quad \therefore \frac{dy}{dx} = -3x^2\sin(2x^3)$$

17. $y = \sec^3(x^2+1)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\sec^3(x^2+1)\}}{d\{\sec(x^2+1)\}} \times \frac{d\{\sec(x^2+1)\}}{d(x^2+1)} \times \frac{d(x^2+1)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 3\sec^2(x^2+1).\sec(x^2+1)\tan(x^2+1).\left\{\frac{d(x^2)}{dx} + \frac{d(1)}{dx}\right\}$$

$$\Rightarrow \frac{dy}{dx} = 3\sec^3\left(x^2+1\right)\tan\left(x^2+1\right)\left(2x+0\right) \quad \therefore \frac{dy}{dx} = 6x\sec^3\left(x^2+1\right)\tan\left(x^2+1\right)$$

18. $y = \sqrt{\cos 3x}$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left(\sqrt{\cos 3x}\right)}{d\left(\cos 3x\right)} \times \frac{d\left(\cos 3x\right)}{d\left(3x\right)} \times \frac{d\left(3x\right)}{dx} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\cos 3x}}.\left(-\sin 3x\right).3$$

$$\therefore \frac{dy}{dx} = \frac{-3\sin 3x}{2\sqrt{\cos 3x}}$$

19. $y = \sqrt[3]{\sin 2x}$

Sol. $y = \left(\sin 2x\right)^{\frac{1}{3}}$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left\{\left(\sin 2x\right)^{1/3}\right\}}{d\left(\sin 2x\right)} \times \frac{d\left(\sin 2x\right)}{d\left(2x\right)} \times \frac{d\left(2x\right)}{dx} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{3}\left(\sin 2x\right)^{\frac{1}{3}-1}\cos 2x.2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{3}\left(\sin 2x\right)^{-\frac{2}{3}}\cos 2x \quad \therefore \frac{dy}{dx} = \frac{2}{3}\frac{\cos 2x}{\left(\sin 2x\right)^{2/3}}$$

20. $y = \sqrt{1+\cot x}$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left(\sqrt{1+\cot x}\right)}{d\left(1+\cot x\right)} \times \frac{d\left(1+\cot x\right)}{dx} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{1+\cot x}}\left\{\frac{d\left(1\right)}{dx} + \frac{d\left(\cot x\right)}{dx}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{1+\cot x}}\left(0-\cos ec^2 x\right) \quad \therefore \frac{dy}{dx} = -\frac{\text{cosec}^2 x}{2\sqrt{1+\cot x}}$$

21. $y = \text{cosec}^3\left(\frac{1}{x^2}\right)$

Sol. $y = \text{cosec}^3\left(x^{-2}\right)$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left\{\text{cosec}^3\left(x^{-2}\right)\right\}}{d\left\{\left(\text{cosec}\left(x^{-2}\right)\right)\right\}} \times \frac{d\left\{\text{cosec}\left(x^{-2}\right)\right\}}{d\left(x^{-2}\right)} \times \frac{d\left(x^{-2}\right)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 3\,\text{cosec}^{3-1}\left(x^{-2}\right)\left\{-\text{cosec}\left(x^{-2}\right)\cot\left(x^{-2}\right)\right\}\left(-2x^{-2-1}\right)$$

$$\Rightarrow \frac{dy}{dx} = 6\,\text{cosec}^2\left(x^{-2}\right)\text{cosec}\left(x^{-2}\right)\cot\left(x^{-2}\right).x^{-3} \quad \therefore \frac{dy}{dx} = \frac{6\,\text{cosec}^3\left(\frac{1}{x^2}\right).\cot\left(\frac{1}{x^2}\right)}{x^3} = \frac{6}{x^3}\text{cosec}^3\frac{1}{x^2}\cot\frac{1}{x^2}.$$

22. $y = \sqrt{\sin\left(x^3\right)}$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left\{\sqrt{\sin\left(x^3\right)}\right\}}{d\left\{\sin\left(x^3\right)\right\}} \times \frac{d\left\{\sin\left(x^3\right)\right\}}{d\left(x^3\right)} \times \frac{d\left(x^3\right)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\sin(x^3)}}.\cos(x^3).3x^2 \quad \therefore \frac{dy}{dx} = \frac{3x^2\cos(x^3)}{2\sqrt{\sin(x^3)}}$$

23. $y = \sqrt{x\sin x}$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(\sqrt{x\sin x})}{d(x\sin x)} \times \frac{d(x\sin x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x\sin x}}.\left\{x\frac{d(\sin x)}{dx} + \sin x\frac{d(x)}{dx}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x\sin x}}\{x\cos x + \sin x.1\} \quad \therefore \frac{dy}{dx} = \frac{x\cos x + \sin x}{2\sqrt{x\sin x}}$$

24. $y = \sqrt{\cot(\sqrt{x})}$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left\{\sqrt{\cot(\sqrt{x})}\right\}}{d\left\{\cot(\sqrt{x})\right\}} \times \frac{d\left\{\cot(\sqrt{x})\right\}}{d(\sqrt{x})} \times \frac{d(\sqrt{x})}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\cot(\sqrt{x})}}\left\{-\cos ec^2(\sqrt{x})\right\}.\frac{1}{2\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{-\cos ec^2(\sqrt{x})}{4\sqrt{x}\sqrt{\cot(\sqrt{x})}}$$

25. $y = \cot^3(x^2)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left\{\cot^3(x^2)\right\}}{d\left\{\cot(x^2)\right\}} \times \frac{d\left\{\cot(x^2)\right\}}{d(x^2)} \times \frac{d(x^2)}{dx} \Rightarrow \frac{dy}{dx} = 3\cot^2(x^2).\left\{-\cos ec^2(x^2)\right\}2x$$

$$\Rightarrow \frac{dy}{dx} = -6x\cot^2(x^2)\cos ec^2(x^2)$$

26. $y = \cos\left(\sin\sqrt{ax+b}\right)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left\{\cos\left(\sin\sqrt{ax+b}\right)\right\}}{d\left\{\sin\left(\sqrt{ax+b}\right)\right\}} \times \frac{d\left\{\sin\left(\sqrt{ax+b}\right)\right\}}{d\left(\sqrt{ax+b}\right)} \times \frac{d\left(\sqrt{ax+b}\right)}{d(ax+b)} \times \frac{d(ax+b)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin\left(\sin\sqrt{ax+b}\right).\cos\left(\sqrt{ax+b}\right).a}{2\sqrt{ax+b}} \quad \therefore \frac{dy}{dx} = -\frac{a\sin\left(\sin\sqrt{ax+b}\right).\cos\left(\sqrt{ax+b}\right)}{2\sqrt{ax+b}}$$

27. $y = \sqrt{\text{cosec}(x^3+1)}$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left(\sqrt{\text{cosec}(x^3+1)}\right)}{d\left\{\text{cosec}(x^3+1)\right\}} \times \frac{d\left\{\text{cosec}(x^3+1)\right\}}{d(x^3+1)} \times \frac{d(x^3+1)}{dx}$$

$$\Rightarrow \frac{dx}{dx} = \frac{1}{2\sqrt{\text{cosec}(x^3+1)}}\left\{-\text{cosec}(x^3+1).\cot(x^3+1)\right\}.\left\{\frac{d(x^3)}{dx} + \frac{d(1)}{dx}\right\}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-\operatorname{cosec}\left(x^3+1\right)\cot\left(x^3+1\right).\left(3x^2+0\right)}{2\sqrt{\operatorname{cosec}\left(x^3+1\right)}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-3x^2\sqrt{\operatorname{cosec}\left(x^3+1\right)}\sqrt{\operatorname{cosec}\left(x^3+1\right)}.\cot\left(x^3+1\right)}{2\sqrt{\operatorname{cosec}\left(x^3+1\right)}}$$

$$\therefore\ \frac{dy}{dx}=-\frac{3x^2}{2}\sqrt{\operatorname{cosec}\left(x^3+1\right)}.\cot\left(x^3+1\right)$$

28. $y=\sin 5x.\cos 3x$

Sol. $y=\frac{1}{2}(2\sin 5x.\cos 3x)\quad \Rightarrow y=\frac{1}{2}\{\sin(5x+3x)+\sin(5x-3x)\}\quad \Rightarrow y=\frac{1}{2}(\sin 8x+\sin 2x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{1}{2}\left\{\frac{d(\sin 8x)}{d(8x)}\times\frac{d(8x)}{dx}+\frac{d(\sin 2x)}{d(2x)}\times\frac{d(2x)}{dx}\right\}\ \Rightarrow \frac{dy}{dx}=\frac{1}{2}(\cos 8x.8+\cos 2x.2)$$

$$\Rightarrow \frac{dy}{dx}=\frac{2(4\cos 8x+\cos 2x)}{2}\quad \therefore\ \frac{dy}{dx}=4\cos 8x+\cos 2x$$

29. $y=\sin 2x.\sin x$

Sol. $y=\frac{1}{2}(2\sin 2x.\sin x)\quad \Rightarrow y=\frac{1}{2}\{\cos(2x-x)-\cos(2x+x)\}\quad \Rightarrow y=\frac{1}{2}\{\cos x-\cos 3x\}$

Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{1}{2}\left\{\frac{d(\cos x)}{dx}-\frac{d(\cos 3x)}{d(3x)}.\frac{d(3x)}{dx}\right\}\quad \Rightarrow \frac{dy}{dx}=\frac{1}{2}\{-\sin x+\sin 3x.3\}\quad \therefore \frac{dy}{dx}=\frac{3}{2}\sin 3x-\frac{1}{2}\sin x$$

30. $y=\cos 4x.\cos 2x$

Sol. $y=\frac{1}{2}(2\cos 4x.\cos 2x)\quad \Rightarrow y=\frac{1}{2}\{\cos(4x+2x)+\cos(4x-2x)\}\Rightarrow y=\frac{1}{2}(\cos 6x+\cos 2x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{1}{2}\left\{\frac{d(\cos 6x)}{d(6x)}\times\frac{d(6x)}{dx}+\frac{d(\cos(2x))}{d(2x)}\times\frac{d(2x)}{dx}\right\}\quad \Rightarrow \frac{dy}{dx}=\frac{1}{2}(-\sin 6x.6-\sin 2x.2)$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{2}(-3\sin 6x-\sin 2x)\quad \therefore \frac{dy}{dx}=-(3\sin 6x+\sin 2x)$$

Find $\frac{dy}{dx}$, when :

31. $y=\sin\left(\frac{1+x^2}{1-x^2}\right)$

Sol. Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d\left\{\sin\left(\frac{1+x^2}{1-x^2}\right)\right\}}{d\left(\frac{1+x^2}{1-x^2}\right)}\times\frac{d\left(\frac{1+x^2}{1-x^2}\right)}{dx}$

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right)\left\{\frac{(1-x^2)\frac{d(1+x^2)}{dx} - (1+x^2)\frac{d(1-x^2)}{dx}}{(1-x^2)^2}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right)\left\{\frac{(1-x^2)(0+2x)-(1+x^2)(0-2x)}{(1-x^2)^2}\right\}$$

$$\Rightarrow \frac{dy}{dx} = 2x\cos\left(\frac{1+x^2}{1-x^2}\right).\left\{\frac{1-x^2+1+x^2}{(1-x^2)^2}\right\} \quad \Rightarrow \frac{dy}{dx} = \frac{4x}{(1-x^2)^2}\cos\left(\frac{1+x^2}{1-x^2}\right)$$

32. $y = \frac{\sin x + x^2}{\cot 2x}$

Sol. $y = \frac{\sin x}{\cot 2x} + \frac{x^2}{\cot 2x} \quad \Rightarrow y = \sin x.\tan 2x + x^2.\tan 2x$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \left\{\sin x.\frac{d(\tan 2x)}{d(2x)} \times \frac{d(2x)}{dx} + \tan 2x.\frac{d(\sin x)}{dx}\right\} + \left\{x^2.\frac{d(\tan 2x)}{d(2x)}.\frac{d(2x)}{dx} + \tan 2x.\frac{d(x^2)}{dx}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \sin x.\sec^2(2x)2 + \tan 2x.\cos x + x^2\sec^2 2x.2 + \tan 2x.2x$$

$$\Rightarrow \frac{dy}{dx} = 2\sec^2(2x)(\sin x + x^2) + \tan 2x(\cos x + 2x)$$

$$\Rightarrow 2(\sin x + x^2)\sec^2 2x + (\cos x + 2x)\tan 2x$$

33. If $y = \frac{\cos x - \sin x}{\cos x + \sin x}$, prove that $\frac{dy}{dx} + y^2 + 1 = 0$

Sol. We have, $y = \frac{\cos x - \sin x}{\cos x + \sin x}$

Dividing Numerator and Denominator by $\cos x$, we get

$$y = \frac{\frac{\cos x - \sin x}{\cos x}}{\frac{\cos x + \sin x}{\cos x}} \quad \Rightarrow y = \frac{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x} + \frac{\sin x}{\cos x}} \quad \Rightarrow y = \frac{1-\tan x}{1+\tan x}$$

$$\Rightarrow y = \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}.\tan x} \quad \Rightarrow y = \tan\left(\frac{\pi}{4} - x\right) \qquad \ldots(i)$$

Differentiating both sides of (i) with respect to x, $\frac{dy}{dx} = \frac{d\left\{\tan\left(\frac{\pi}{4}-x\right)\right\}}{d\left(\frac{\pi}{4}-x\right)} \times \frac{d\left(\frac{\pi}{4}-x\right)}{dx}$

$$\Rightarrow \frac{dy}{dx} = \sec^2\left(\frac{\pi}{4}-x\right)\{0-1\} \quad \Rightarrow \frac{dy}{dx} = -\sec^2\left(\frac{\pi}{4}-x\right)$$

L.H.S. $\frac{dy}{dx} + y^2 + 1 = -\sec^2\left(\frac{\pi}{4}-x\right) + \tan^2\left(\frac{\pi}{4}-x\right) + 1 = -\sec^2\left(\frac{\pi}{4}-x\right) + \sec^2\left(\frac{\pi}{4}-x\right) = 0. =$ R.H.S

Hence proved

34. If $y=\frac{(\cos x+\sin x)}{(\cos x-\sin x)}$, Prove that $\frac{dy}{dx}=\sec^2\left(x+\frac{\pi}{4}\right)$

Sol. We have, $y=\frac{\cos x+\sin x}{\cos x-\sin x}$, dividing Numerator and Denominator by $\cos x$

$$y=\frac{\frac{\cos x+\sin x}{\cos x}}{\frac{\cos x-\sin x}{\cos x}} \Rightarrow y=\frac{\frac{\cos x}{\cos x}+\frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x}-\frac{\sin x}{\cos x}} \Rightarrow y=\frac{1+\tan x}{1-\tan x}$$

$$\Rightarrow y=\frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}.\tan x} \Rightarrow y=\tan\left(\frac{\pi}{4}+x\right) \qquad \ldots(i)$$

Differentiating both sides of (i) w.r.t. x

$$\frac{dy}{dx}=\frac{d\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}}{d\left(\frac{\pi}{4}+x\right)}\times\frac{d\left(\frac{\pi}{4}+x\right)}{dx} \Rightarrow \frac{dy}{dx}=\sec^2\left(\frac{\pi}{4}+x\right).\left\{\frac{d(\pi/4)}{dx}+\frac{d(x)}{dx}\right\}$$

$$\Rightarrow \frac{dy}{dx}=\sec^2\left(\frac{\pi}{4}+x\right)\{0+1\} \quad \therefore \frac{dy}{dx}=\sec^2\left(\frac{\pi}{4}+x\right)$$

EXERCISE 10B (Pg.no.: 379)

Differentiate each of the following w.r.t. x:

1. (i) e^{4x} (ii) e^{-5x} (iii) e^{x^3}

Sol. **(i)** $y=e^{4x}$, Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(e^{4x}\right)}{d(4x)}\times\frac{d(4x)}{dx} \Rightarrow \frac{dy}{dx}=e^{4x}.4 \therefore \frac{dy}{dx}=4e^{4x}$$

(ii) $y=e^{-5x}$, Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(e^{-5x}\right)}{d(-5x)}\times\frac{d(-5x)}{dx} \Rightarrow \frac{dy}{dx}=e^{-5x}(-5) \therefore \frac{dy}{dx}=-5e^{-5x}$$

(iii) $y=e^{x^3}$, Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(e^{x^3}\right)}{d\left(x^3\right)}\times\frac{d\left(x^3\right)}{dx} \Rightarrow \frac{dy}{dx}=e^{x^3}.3x^2$$

2. (i) $e^{\frac{2}{x}}$ (ii) $e^{\sqrt{x}}$ (iii) $e^{-2\sqrt{x}}$

Sol. **(i)** $y=e^{\frac{2}{x}}$, Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(e^{\frac{2}{x}}\right)}{d\left(\frac{2}{x}\right)}\times\frac{d\left(\frac{2}{x}\right)}{dx} \Rightarrow \frac{dy}{dx}=e^{\frac{2}{x}}.2\frac{d\left(\frac{1}{x}\right)}{dx} \Rightarrow \frac{dy}{dx}=2e^{\frac{2}{x}}\left(-\frac{1}{x^2}\right) \therefore \frac{dy}{dx}=\frac{-2e^{2/x}}{x^2}$$

(ii) $y=e^{\sqrt{x}}$, Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(e^{\sqrt{x}}\right)}{d\left(\sqrt{x}\right)}\times\frac{d\left(\sqrt{x}\right)}{dx} \Rightarrow \frac{dy}{dx}=e^{\sqrt{x}}.\frac{1}{2\sqrt{x}} \therefore \frac{dy}{dx}=\frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

(iii) $y=e^{-2\sqrt{x}}$, Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(e^{-2\sqrt{x}}\right)}{d\left(-2\sqrt{x}\right)}\times\frac{d\left(-2\sqrt{x}\right)}{dx} \Rightarrow \frac{dy}{dx}=e^{-2\sqrt{x}}(-2)\frac{d\left(\sqrt{x}\right)}{dx}$$

$$\Rightarrow \frac{dy}{dx}=e^{-2\sqrt{x}}(-2)\frac{1}{2\sqrt{x}} \quad \therefore \frac{dy}{dx}=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

3. (i) $e^{\cot x}$ (ii) $e^{-\sin 2x}$ (iii) $e^{\sqrt{\sin x}}$

Sol. (i) $y=e^{\cot x}$, Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(e^{\cot x}\right)}{d(\cot x)}\times\frac{d(\cot x)}{dx} \Rightarrow \frac{dy}{dx}=e^{\cot x}\left(-\operatorname{cosec}^2 x\right) \therefore \frac{dy}{dx}=-e^{\cot x}.\operatorname{cosec}^2 x$$

(ii) $y=e^{-\sin 2x}$, Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(e^{-\sin 2x}\right)}{d(-\sin 2x)}\times\frac{d(-\sin 2x)}{d(2x)}\times\frac{d(2x)}{dx} \Rightarrow \frac{dy}{dx}=e^{-\sin 2x}(-\cos 2x)2 \therefore \frac{dy}{dx}=-2e^{-\sin 2x}\cos 2x$$

(iii) $y=e^{\sqrt{\sin x}}$, Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(e^{\sqrt{\sin x}}\right)}{d\left(\sqrt{\sin x}\right)}\times\frac{d\left(\sqrt{\sin x}\right)}{d(\sin x)}\times\frac{d(\sin x)}{dx} \Rightarrow \frac{dy}{dx}=e^{\sqrt{\sin x}}.\frac{1}{2\sqrt{\sin x}}.\cos x \therefore \frac{dy}{dx}=\frac{\cos x.e^{\sqrt{\sin x}}}{2\sqrt{\sin x}}$$

4. (i) $\tan(\log x)$ (ii) $\log\sec x$ (iii) $\log\sin\frac{x}{2}$

Sol. (i) $y=\tan(\log x)$, Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\{\tan(\log x)\}}{d(\log x)}\times\frac{d(\log x)}{dx} \Rightarrow \frac{dy}{dx}=\sec^2(\log x).\frac{1}{x} \therefore \frac{dy}{dx}=\frac{\sec^2(\log x)}{x}$$

(ii) $y=\log(\sec x)$, Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\{\log(\sec x)\}}{d(\sec x)}\times\frac{d(\sec x)}{dx} \Rightarrow \frac{dy}{dx}=\frac{1}{\sec x}.\sec x.\tan x \Rightarrow \frac{dy}{dx}=\tan x$$

(iii) $y=\log\left(\sin\frac{x}{2}\right)$, Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left\{\log\left(\sin\frac{x}{2}\right)\right\}}{d\left(\sin\frac{x}{2}\right)}\times\frac{d\left(\sin\frac{x}{2}\right)}{d\left(\frac{x}{2}\right)}\times\frac{d\left(\frac{x}{2}\right)}{dx} \Rightarrow \frac{dy}{dx}=\frac{1}{\sin\frac{x}{2}}\cos\left(\frac{x}{2}\right).\frac{1}{2}\frac{dx}{dx} \Rightarrow \frac{dy}{dx}=\frac{1}{2}\cot\left(\frac{x}{2}\right)$$

5. (i) $\log_3 x$ (ii) 2^{-x} (iii) 3^{x+2}

Sol. (i) $y=\log_3 x$, Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d(\log_3 x)}{dx} \Rightarrow \frac{dy}{dx}=\frac{1}{x\log 3} \qquad \left[\therefore \frac{d(\log_a x)}{dx}=\frac{1}{x\log a}\right]$$

(ii) $y = 2^{-x}$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left(2^{-x}\right)}{d(-x)} \times \frac{d(-x)}{dx} \quad \Rightarrow \frac{dy}{dx} = 2^{-x}\log 2.(-1) \therefore \frac{dy}{dx} = -2^{-x}\log 2$$

(iii) $y = 3^{x+2}$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left(3^{x+2}\right)}{d(x+2)} \times \frac{d(x+2)}{dx} \quad \Rightarrow \frac{dy}{dx} = 3^{x+2}\log 3.\left\{\frac{d(x)}{dx} + \frac{d(2)}{dx}\right\}$$

$$\Rightarrow \frac{dy}{dx} = 3^x.3^2 \log 3.\{1+0\} \therefore \frac{dy}{dx} = 9.3^x \log 3$$

6. (i) $\log\left(x+\frac{1}{x}\right)$ (ii) $\log\sin 3x$ (iii) $\log\left(x+\sqrt{1+x^2}\right)$

Sol. **(i)** $y = \log\left(x+\frac{1}{x}\right)$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left\{\log\left(x+\frac{1}{x}\right)\right\}}{d\left(x+\frac{1}{x}\right)} \times \frac{d\left(x+\frac{1}{x}\right)}{dx} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{x+\frac{1}{x}}\left\{\frac{d(x)}{dx} + \frac{d\left(\frac{1}{x}\right)}{dx}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{x^2+1}\left\{1+\left(-\frac{1}{x^2}\right)\right\} \quad \Rightarrow \frac{dy}{dx} = \frac{x}{x^2+1}\left\{1-\frac{1}{x^2}\right\} \quad \Rightarrow \frac{dy}{dx} = \frac{x}{x^2+1}\left\{\frac{x^2-1}{x^2}\right\} \quad \therefore \frac{dy}{dx} = \frac{\left(x^2-1\right)}{x\left(x^2+1\right)}$$

(ii) $y = \log(\sin 3x)$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\log(\sin 3x)\}}{d(\sin 3x)} \times \frac{d(\sin 3x)}{d(3x)} \times \frac{d(3x)}{dx} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{\sin 3x}.\cos 3x.3 \quad \Rightarrow \frac{dy}{dx} = 3\cot 3x$$

(iii) $y = \log\left(x+\sqrt{x^2+1}\right)$, Differentiating both sides with respect to x,

$$\frac{dy}{dx} = \frac{d\left\{\log\left(x+\sqrt{x^2+1}\right)\right\}}{d\left(x+\sqrt{x^2+1}\right)} \times \frac{d\left(x+\sqrt{x^2+1}\right)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left\{\frac{d(x)}{dx} + \frac{d\left(\sqrt{x^2+1}\right)}{d\left(x^2+1\right)} \times \frac{d\left(x^2+1\right)}{dx}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left\{1+\frac{1}{2\sqrt{x^2+1}}2x\right\} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left\{\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right\} \quad \therefore \frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$$

7. $y = e^{\sqrt{x}}.\log x$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = e^{\sqrt{x}}.\frac{d(\log x)}{dx} + \log x.\frac{d\left(e^{\sqrt{x}}\right)}{d\left(\sqrt{x}\right)} \times \frac{d\left(\sqrt{x}\right)}{dx} \quad \Rightarrow \frac{dy}{dx} = e^{\sqrt{x}}.\frac{1}{x} + \log x.e^{\sqrt{x}}\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = e^{\sqrt{x}}\left(\frac{1}{x} + \frac{\log x}{2\sqrt{x}}\right) \quad \therefore \frac{dy}{dx} = e^{\sqrt{x}}\left\{\frac{2+\sqrt{x}\log x}{2x}\right\}$$

8. $y=\log\sin\left(\sqrt{x^2+1}\right)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left\{\log\left(\sin\sqrt{x^2+1}\right)\right\}}{d\left(\sin\sqrt{x^2+1}\right)}.\frac{d\left\{\sin\left(\sqrt{x^2+1}\right)\right\}}{d\left(\sqrt{x^2+1}\right)}\times\frac{d\left(\sqrt{x^2+1}\right)}{d\left(x^2+1\right)}\times\frac{d\left(x^2+1\right)}{dx}$$

$$\Rightarrow\frac{dy}{dx}=\frac{1}{\sin\left(\sqrt{x^2+1}\right)}.\cos\left(\sqrt{x^2+1}\right).\frac{1}{2\sqrt{x^2+1}}.2x\;\therefore\;\frac{dy}{dx}=\frac{x}{\sqrt{x^2+1}}.\cot\left(\sqrt{x^2+1}\right)$$

9. $y=e^{2x}\sin 3x$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=e^{2x}\frac{d(\sin 3x)}{d(3x)}\times\frac{d(3x)}{dx}+\sin 3x\frac{d\left(e^{2x}\right)}{d(2x)}\times\frac{d(2x)}{dx}\quad\Rightarrow\frac{dy}{dx}=e^{2x}.\cos 3x.3+\sin 3x.e^{2x}.2$$

$$\therefore\frac{dy}{dx}=e^{2x}(3\cos 3x+2\sin 3x)$$

10. $y=e^{3x}\cos 2x$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(e^{3x}\cos 2x\right)}{dx}\quad\Rightarrow\frac{dy}{dx}=e^{3x}\frac{d(\cos 2x)}{d(2x)}\times\frac{d(2x)}{dx}+\cos 2x\frac{d\left(e^{3x}\right)}{d(3x)}\times\frac{d(3x)}{dx}$$

$$\Rightarrow\frac{dy}{dx}=e^{3x}(-\sin 2x).2+\cos 2x.e^{3x}.3\quad\therefore\quad\frac{dy}{dx}=e^{3x}(-2\sin 2x+3\cos 2x)$$

11. $y=e^{-5x}\cot 4x$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(e^{-5x}\cot 4x\right)}{dx}\quad\Rightarrow\frac{dy}{dx}=e^{-5x}\frac{d(\cot 4x)}{d(4x)}\times\frac{d(4x)}{dx}+\cot 4x\frac{d\left(e^{-5x}\right)}{d(-5x)}\times\frac{d(-5x)}{dx}$$

$$\Rightarrow\frac{dy}{dx}=e^{-5x}\left(-\operatorname{cosec}^2 4x\right).4+\cot 4x.e^{-5x}.(-5)\quad\therefore\quad\frac{dy}{dx}=-e^{-5x}\left(5\cot 4x+4\operatorname{cosec}^2 4x\right)$$

12. $y=e^x\log(\sin 2x)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=e^x\frac{d\{\log(\sin 2x)\}}{d(\sin 2x)}\times\frac{d(\sin 2x)}{d(2x)}\times\frac{d(2x)}{dx}+\log(\sin 2x)\times\frac{d\left(e^x\right)}{dx}$$

$$\Rightarrow\frac{dy}{dx}=e^x.\frac{1}{\sin 2x}.\cos 2x.2+\log(\sin 2x).e^x\quad\therefore\frac{dy}{dx}=e^x\{2\cot 2x+\log(\sin 2x)\}$$

13. $y=\log(\operatorname{cosec}x-\cot x)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\{\log(\operatorname{cosec}x-\cot x)\}}{d(\operatorname{cosec}x-\cot x)}\times\frac{d(\operatorname{cosec}x-\cot x)}{dx}\Rightarrow\frac{dy}{dx}=\frac{1}{\operatorname{cosec}x-\cot x}\left\{\frac{d(\operatorname{cosec}x)}{dx}-\frac{d(\cot x)}{dx}\right\}$$

$$\Rightarrow\frac{dy}{dx}=\frac{1}{\operatorname{cosec}x-\cot x}\left\{-\operatorname{cosec}x\cot x+\operatorname{cosec}^2x\right\}\;\Rightarrow\frac{dy}{dx}=\frac{\operatorname{cosec}x(\operatorname{cosec}x-\cot x)}{(\operatorname{cosec}x-\cot x)}\;\therefore\frac{dy}{dx}=\operatorname{cosec}x$$

14. $y=\log\left(\sec\frac{x}{2}+\tan\frac{x}{2}\right)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left\{\log\left(\sec\frac{x}{2}+\tan\frac{x}{2}\right)\right\}}{d\left(\sec\frac{x}{2}+\tan\frac{x}{2}\right)}\times\frac{d\left(\sec\frac{x}{2}+\tan\frac{x}{2}\right)}{dx}$$

$$\Rightarrow\frac{dy}{dx}=\frac{1}{\sec\frac{x}{2}+\tan\frac{x}{2}}\left\{\frac{d\left(\sec\frac{x}{2}\right)}{d\left(\frac{x}{2}\right)}\times\frac{d\left(\frac{x}{2}\right)}{dx}+\frac{d\left(\tan\frac{x}{2}\right)}{d\left(\frac{x}{2}\right)}\times\frac{d\left(\frac{x}{2}\right)}{dx}\right\}$$

$$\Rightarrow\frac{dy}{dx}=\frac{1}{\sec\frac{x}{2}+\tan\frac{x}{2}}\left\{\sec\frac{x}{2}\tan\frac{x}{2}.\frac{1}{2}+\sec^2\frac{x}{2}.\frac{1}{2}\right\}\Rightarrow\frac{dy}{dx}=\frac{\frac{1}{2}\sec\frac{x}{2}\left\{\tan\frac{x}{2}+\sec\frac{x}{2}\right\}}{\left\{\sec\frac{x}{2}+\tan\frac{x}{2}\right\}}\ \therefore\frac{dy}{dx}=\frac{1}{2}\sec\frac{x}{2}$$

15. $y=\sqrt{\frac{1+e^x}{1-e^x}}$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(\sqrt{\frac{1+e^x}{1-e^x}}\right)}{d\left(\frac{1+e^x}{1-e^x}\right)}\times\frac{d\left(\frac{1+e^x}{1-e^x}\right)}{dx}\quad\Rightarrow\frac{dy}{dx}=\frac{1}{2\sqrt{\frac{1+e^x}{1-e^x}}}\left\{\frac{\left(1-e^x\right)\frac{d\left(1+e^x\right)}{dx}-\left(1+e^x\right)\frac{d\left(1-e^x\right)}{dx}}{\left(1-e^x\right)^2}\right\}$$

$$\Rightarrow\frac{dy}{dx}=\frac{\sqrt{1-e^x}}{2\sqrt{1+e^x}}\left\{\frac{\left(1-e^x\right)\left(0+e^x\right)-\left(1+e^x\right)\left(o-e^x\right)}{\left(1-e^x\right)\sqrt{1-e^x}\sqrt{1-e^x}}\right\}\quad\Rightarrow\frac{dy}{dx}=\frac{e^x\left(1-e^x+1+e^x\right)}{2\sqrt{1+e^x}\sqrt{1-e^x}\left(1-e^x\right)}$$

$$\therefore\frac{dy}{dx}=\frac{2e^x}{\left(1-e^x\right)\sqrt{1+e^x}}$$

16. $y=\frac{e^x+e^{-x}}{e^x-e^{-x}}$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)}{dx}\quad\Rightarrow\frac{dy}{dx}=\frac{\left(e^x-e^{-x}\right)\frac{d\left(e^x+e^{-x}\right)}{dx}-\left(e^x+e^{-x}\right)\frac{d\left(e^x-e^{-x}\right)}{dx}}{\left(e^x-e^{-x}\right)^2}$$

$$\Rightarrow\frac{dy}{dx}=\frac{\left(e^x-e^{-x}\right)\left(e^x-e^{-x}\right)-\left(e^x+e^{-x}\right)\left(e^x+e^{-x}\right)}{\left(e^x-e^{-x}\right)^2}$$

$$\Rightarrow\frac{dy}{dx}=\frac{\left(e^x-e^{-x}\right)^2-\left(e^x+e^{-x}\right)^2}{\left(e^x-e^{-x}\right)^2}\quad\Rightarrow\frac{dy}{dx}=\frac{\left(e^{2x}+e^{-2x}-2\right)-\left(e^{2x}+e^{-2x}+2\right)}{\left(e^x-e^{-x}\right)^2}$$

$$\Rightarrow\frac{dy}{dx}=\frac{e^{2x}+e^{-2x}-2-e^{2x}-e^{-2x}-2}{\left(e^x-e^{-x}\right)^2}\quad\therefore\frac{dy}{dx}=\frac{-4}{\left(e^x-e^{-x}\right)^2}$$

17. $y = xe^{\sqrt{\sin x}}$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left(x.e^{\sqrt{\sin x}}\right)}{dx} \Rightarrow \frac{dy}{dx} = x.\frac{d\left(e^{\sqrt{\sin x}}\right)}{d\left(\sqrt{\sin x}\right)} \times \frac{d\left(\sqrt{\sin x}\right)}{d(\sin x)} \times \frac{d(\sin x)}{dx} + e^{\sqrt{\sin x}}.\frac{d(x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = x.e^{\sqrt{\sin x}}.\frac{1}{2\sqrt{\sin x}}.\cos x + e^{\sqrt{\sin x}}. \quad \therefore \frac{dy}{dx} = e^{\sqrt{\sin x}}.\left\{\frac{x\cos x}{2\sqrt{\sin x}} + 1\right\}$$

18. $y = e^{\sin x}.\sin\left(e^x\right)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = e^{\sin x}.\frac{d\left\{\sin\left(e^x\right)\right\}}{d\left(e^x\right)} \times \frac{d\left(e^x\right)}{dx} + \sin\left(e^x\right)\frac{d\left(e^{\sin x}\right)}{d(\sin x)} \times \frac{d(\sin x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin x}.\cos\left(e^x\right)e^x + \sin\left(e^x\right).e^{\sin x}.\cos x \quad \therefore \frac{dy}{dx} = e^{\sin x}.\left\{e^x\cos\left(e^x\right) + \cos x.\sin e^x\right\}$$

19. $y = e^{\sqrt{1-x^2}}\tan x$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left(e^{\sqrt{1-x^2}}.\tan x\right)}{dx} \Rightarrow \frac{dy}{dx} = e^{\sqrt{1-x^2}}\frac{d(\tan x)}{dx} + \tan x.\frac{d\left(e^{\sqrt{1-x^2}}\right)}{d\left(\sqrt{1-x^2}\right)} \times \frac{d\left(\sqrt{1-x^2}\right)}{d\left(1-x^2\right)} \times \frac{d\left(1-x^2\right)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^{\sqrt{1-x^2}}.\sec^2 x + \tan x.e^{\sqrt{1-x^2}}.\frac{1}{2\sqrt{1-x^2}}.(0-2x)$$

$$\Rightarrow \frac{dy}{dx} = e^{\sqrt{1-x^2}}\sec^2 x + \tan x.\frac{e^{\sqrt{1-x^2}}.(-2x)}{2\sqrt{1-x^2}} \quad \therefore \frac{dy}{dx} = e^{\sqrt{1-x^2}}\left\{\sec^2 x - \frac{x.\tan x}{\sqrt{1-x^2}}\right\}$$

20. $y = \dfrac{e^x}{1+\cos x}$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left(\frac{e^x}{1+\cos x}\right)}{dx} \Rightarrow \frac{dy}{dx} = \frac{(1+\cos x)\frac{d\left(e^x\right)}{dx} - e^x\frac{(1+\cos x)}{dx}}{(1+\cos x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+\cos x).e^x - e^x(0-\sin x)}{(1+\cos x)^2} \quad \therefore \frac{dy}{dx} = \frac{e^x(1+\cos x+\sin x)}{(1+\cos x)^2}$$

21. $y = x^3e^x\cos x$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left(x^3e^x\cos x\right)}{dx} \Rightarrow \frac{dy}{dx} = x^3.\frac{d\left(e^x\cos x\right)}{dx} + \left(e^x\cos x\right)\frac{d\left(x^3\right)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = x^3.\left\{e^x\sin x + \cos x.e^x\right\} + e^x\cos x.3x^2 \Rightarrow \frac{dy}{dx} = e^xx^2\left(-x\sin x + x\cos + 3\cos x\right)$$

22. $y = e^{x\cos x}$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(e^{x\cos x}\right)}{d\left(x\cos x\right)}\times\frac{d\left(x\cos x\right)}{dx} \quad \Rightarrow \frac{dy}{dx}=e^{x\cos x}\left\{x\frac{d\left(\cos x\right)}{dx}+\cos x.\frac{d\left(x\right)}{dx}\right\}$$

$$\Rightarrow \frac{dy}{dx}=e^{x\cos x}.\left\{-x\sin x+\cos x.1\right\} \quad \therefore \frac{dy}{dx}=e^{x\cos x}\left(\cos x-x\sin x\right)$$

EXERCISE 10C (Pg.No.: 388)

Differentiate each of the following w.r.t.x:

1. $y=\cos^{-1}\left(2x\right)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left\{\cos^{-1}\left(2x\right)\right\}}{d\left(2x\right)}\times\frac{d\left(2x\right)}{dx} \quad \Rightarrow \frac{dy}{dx}=-\frac{1}{\sqrt{1-\left(2x\right)^2}}.2\frac{dx}{dx} \quad \therefore \frac{dy}{dx}=\frac{-2}{\sqrt{1-4x^2}}$$

2. $y=\tan^{-1}\left(x^2\right)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left\{\tan^{-1}\left(x^2\right)\right\}}{d\left(x^2\right)}\times\frac{d\left(x^2\right)}{dx} \quad \Rightarrow \frac{dy}{dx}=\frac{1}{1+\left(x^2\right)^2}.2x \quad \therefore \frac{dy}{dx}=\frac{2x}{1+x^4}$$

3. $y=\sec^{-1}\left(\sqrt{x}\right)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left\{\sec^{-1}\left(\sqrt{x}\right)\right\}}{d\left(\sqrt{x}\right)}\times\frac{d\left(\sqrt{x}\right)}{dx} \quad \Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{x}\sqrt{\left(\sqrt{x}\right)^2-1}}.\frac{1}{2\sqrt{x}} \quad \therefore \frac{dy}{dx}=\frac{1}{2x\sqrt{x-1}}$$

4. $y=\sin^{-1}\left(\frac{x}{a}\right)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left\{\sin^{-1}\left(\frac{x}{a}\right)\right\}}{d\left(\frac{x}{a}\right)}\times\frac{d\left(\frac{x}{a}\right)}{dx} \quad \Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}.\frac{1}{a}.\frac{d\left(x\right)}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}.\frac{1}{a}.1 \quad \Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}}.\frac{1}{a} \quad \Rightarrow \frac{dy}{dx}=\frac{a}{\sqrt{a^2-x^2}}.\frac{1}{a} \quad \therefore \frac{dy}{dx}=\frac{1}{\sqrt{a^2-x^2}}$$

5. $y=\tan^{-1}\left(\log x\right)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left\{\tan^{-1}\left(\log x\right)\right\}}{d\left(\log x\right)}\times\frac{d\left(\log x\right)}{dx} \quad \Rightarrow \frac{dy}{dx}=\frac{1}{1+\left(\log x\right)^2}.\frac{1}{x} \quad \therefore \frac{dy}{dx}=\frac{1}{x\left\{1+\left(\log x\right)^2\right\}}$$

6. $y=\cot^{-1}\left(e^x\right)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left\{\cot^{-1}\left(e^x\right)\right\}}{d\left(e^x\right)}\times\frac{d\left(e^x\right)}{dx} \quad \Rightarrow \frac{dy}{dx}=-\frac{1}{1+\left(e^x\right)^2}.e^x \quad \therefore \frac{dy}{dx}=-\frac{e^x}{1+e^{2x}}$$

7. $y=\log\left(\tan^{-1}x\right)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left\{\log\left(\tan^{-1}x\right)\right\}}{d\left(\tan^{-1}x\right)}\times\frac{d\left(\tan^{-1}x\right)}{dx}\quad\therefore\ \frac{dy}{dx}=\frac{1}{\tan^{-1}x}.\frac{1}{\left(1+x^2\right)}$$

8. $y=\cot^{-1}\left(x^3\right)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left\{\cot^{-1}\left(x^3\right)\right\}}{d\left(x^3\right)}\times\frac{d\left(x^3\right)}{dx}\quad\Rightarrow\frac{dy}{dx}=-\frac{1}{1+\left(x^3\right)^2}.3x^2\quad\therefore\frac{dy}{dx}=\frac{-3x^2}{1+x^6}$$

9. $y=\sin^{-1}\left(\cos x\right)$

Sol. Given $y=\sin^{-1}\left(\cos x\right)=\sin^{-1}\sin\left(\frac{\pi}{2}-x\right)=\frac{\pi}{2}-x$

Differentiating both sides with respect to x, we get, $\frac{dy}{dx}=0-1=-1$.

10. $y=\left(1+x^2\right)\tan^{-1}x$

Sol. Differentiating both sides with respect to x, $\frac{dy}{dx}=\left(1+x^2\right)\frac{d\left(\tan^{-1}x\right)}{dx}+\tan^{-1}x\frac{d\left(1+x^2\right)}{dx}$

$$\Rightarrow\frac{dy}{dx}=\left(1+x^2\right).\frac{1}{1+x^2}+\tan^{-1}x.\left(0+2x\right)\quad\Rightarrow\frac{dy}{dx}=1+2x.\tan^{-1}x$$

11. $y=\tan^{-1}\left(\cot x\right)$

Sol. Given $y=\tan^{-1}\left(\cot x\right)=\tan^{-1}\tan\left(\frac{\pi}{2}-x\right)=\frac{\pi}{2}-x$

Differentiating both sides with respect to x, we get, $\frac{dy}{dx}=0-1=-1$

12. $y=\log\left(\sin^{-1}x^4\right)$

Sol. Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d\left\{\log\left(\sin^{-1}x^4\right)\right\}}{d\left(\sin^{-1}x^4\right)}\times\frac{d\left(\sin^{-1}x^4\right)}{d\left(x^4\right)}\times\frac{d\left(x^4\right)}{dx}$

$$\Rightarrow\frac{dy}{dx}=\frac{1}{\sin^{-1}x^4}.\frac{1}{\sqrt{1-\left(x^4\right)^2}}.4x^3\quad\Rightarrow\frac{dy}{dx}=\frac{4x^3}{\sin^{-1}\left(x^4\right)}.\frac{1}{\sqrt{1-x^8}}$$

13. $y=\left\{\cot^{-1}\left(x^2\right)\right\}^3$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left[\left\{\cot^{-1}\left(x^2\right)\right\}^3\right]}{d\left\{\cot^{-1}\left(x^2\right)\right\}}\times\frac{d\left\{\cot^{-1}\left(x^2\right)\right\}}{d\left(x^2\right)}\times\frac{\left(x^2\right)}{dx}\quad\Rightarrow\frac{dy}{dx}=3\left\{\cot^{-1}\left(x^2\right)\right\}^2.\left\{-\frac{1}{1+\left(x^2\right)^2}\right\}.2x$$

$$\therefore\ \frac{dy}{dx}=\frac{-6x}{1+x^4}\left\{\cot^{-1}\left(x^2\right)\right\}^2=\frac{-6x\left(\cot^{-1}x^2\right)^2}{\left(1+x^4\right)}$$

14. $y=\tan^{-1}\left(\cos\sqrt{x}\right)$

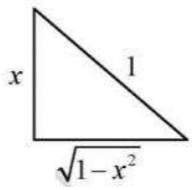
Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left\{\tan^{-1}\left(\cos\sqrt{x}\right)\right\}}{d\left(\cos\sqrt{x}\right)}\times\frac{d\left\{\cos\left(\sqrt{x}\right)\right\}}{d\left(\sqrt{x}\right)}\times\frac{d\left(\sqrt{x}\right)}{dx}$$

$$\Rightarrow\frac{dy}{dx}=\frac{1}{1+\left(\cos\sqrt{x}\right)^2}.-\sin\left(\sqrt{x}\right).\frac{1}{2\sqrt{x}}\quad\therefore\frac{dy}{dx}=\frac{-\sin\left(\sqrt{x}\right)}{2\sqrt{x}\left(1+\cos^2\sqrt{x}\right)}$$

15. $y=\tan\left(\sin^{-1}x\right)$

Sol. Let $\sin^{-1}x=t$, $\sin t=x$ …(i)



$$=\tan t=\frac{x}{\sqrt{1-x^2}}\qquad\Rightarrow t=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

Putting value of t in equation (i), $y=\tan\left(\tan^{-1}\frac{x}{\sqrt{1-x^2}}\right)\quad\Rightarrow y=\frac{x}{\sqrt{1-x^2}}$

Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(\frac{x}{\sqrt{1-x^2}}\right)}{dx}\quad\Rightarrow\frac{dy}{dx}=\left\{\frac{\left(\sqrt{1-x^2}\right)\frac{d(x)}{dx}-x\frac{d\left(\sqrt{1-x^2}\right)}{d\left(1-x^2\right)}\times\frac{d\left(1-x^2\right)}{dx}}{\left(\sqrt{1-x^2}\right)^2}\right\}$$

$$\Rightarrow\frac{dy}{dx}=\left\{\frac{\sqrt{1-x^2}.1-x.\frac{1}{2\sqrt{1-x^2}}(-2x)}{\left(1-x^2\right)}\right\}\quad\Rightarrow\frac{dy}{dx}=\frac{\sqrt{1-x^2}\sqrt{1-x^2}+x^2}{\sqrt{1-x^2}}.\frac{1}{1-x^2}$$

$$\Rightarrow\frac{dy}{dx}=\frac{1-x^2+x^2}{\left(1-x^2\right)\sqrt{1-x^2}}\quad\therefore\frac{dy}{dx}=\frac{1}{\left(1-x^2\right)^{3/2}}$$

16. $y=e^{\tan^{-1}\left(\sqrt{x}\right)}$

Sol. Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d\left\{e^{\tan^{-1}\left(\sqrt{x}\right)}\right\}}{d\left\{\tan^{-1}\left(\sqrt{x}\right)\right\}}\times\frac{d\left\{\tan^{-1}\left(\sqrt{x}\right)\right\}}{d\left(\sqrt{x}\right)}\times\frac{d\left(\sqrt{x}\right)}{dx}$

$$\Rightarrow\frac{dy}{dx}=e^{\tan^{-1}}\left(\sqrt{x}\right).\frac{1}{1+\left(\sqrt{x}\right)^2}.\frac{1}{2\sqrt{x}}\quad\Rightarrow\frac{dy}{dx}=\frac{e^{\tan^{-1}\left(\sqrt{x}\right)}}{2\sqrt{x}\left(1+x\right)}$$

17. $y=\sqrt{\sin^{-1}\left(x^2\right)}$

Sol. Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d\left(\sqrt{\sin^{-1}x^2}\right)}{d\left(\sin^{-1}x^2\right)}\times\frac{d\left(\sin^{-1}x^2\right)}{d\left(x^2\right)}\times\frac{d\left(x^2\right)}{dx}$

$$\Rightarrow\frac{dy}{dx}=\frac{1}{2\sqrt{\sin^{-1}x^2}}.\frac{1}{\sqrt{1-\left(x^2\right)^2}}.2x\quad\therefore\frac{dy}{dx}=\frac{x}{\sqrt{\sin^{-1}\left(x^2\right)}}\frac{1}{\sqrt{1-x^4}}$$

18. If $y=\sin^{-1}(\cos x)+\cos^{-1}(\sin x)$ prove that $\frac{dy}{dx}=-2$

Sol. Given $y=\sin^{-1}(\cos x)+\cos^{-1}(\sin x)$

$$\Rightarrow y=\sin^{-1}\left\{\sin\left(\frac{\pi}{2}-x\right)\right\}+\cos^{-1}\left\{\cos\left(\frac{\pi}{2}-x\right)\right\} \Rightarrow y=\frac{\pi}{2}-x+\frac{\pi}{2}-x \Rightarrow y=\pi-2x$$

Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d(\pi)}{dx}-\frac{d(2x)}{dx} \Rightarrow \frac{dy}{dx}=0-2 \therefore \frac{dy}{dx}=-2$$

19. Prove that $\frac{d}{dx}\left\{2x.\tan^{-1}x-\log\left(1+x^2\right)\right\}=2.\tan^{-1}x$

Sol. $y=2x.\tan^{-1}x-\log\left(1+x^2\right)$, Differentiating both sides with respect to x

$$\frac{dy}{dx}=\left[2x\frac{d\left(\tan^{-1}x\right)}{dx}+\tan^{-1}x\frac{d(2x)}{dx}\right]-\left[\frac{d\left\{\log\left(1+x^2\right)\right\}}{d\left(1+x^2\right)}\times\frac{d\left(1+x^2\right)}{dx}\right]$$

$$\Rightarrow \frac{dy}{dx}=\left(2x.\frac{1}{1+x^2}+\tan^{-1}x.2\right)-\left(\frac{1}{1+x^2}.2x\right) \quad \therefore \frac{dy}{dx}=2\tan^{-1}x$$

EXERCISE 10D (Pg.No.: 402)

1. $y=\sin^{-1}\left(\sqrt{\frac{1-\cos x}{2}}\right)$

Sol. $y=\sin^{-1}\left\{\sqrt{\frac{2\sin^2\frac{x}{2}}{2}}\right\} \Rightarrow y=\sin^{-1}\left(\sin\frac{x}{2}\right) \Rightarrow y=\frac{x}{2}$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d\left(\frac{x}{2}\right)}{dx} \Rightarrow \frac{dy}{dx}=\frac{1}{2}\frac{d(x)}{dx} \therefore \frac{dy}{dx}=\frac{1}{2}$

2. $y=\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$

Sol. $y=\tan^{-1}\left(\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right) \Rightarrow y=\tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right) \Rightarrow y=\tan^{-1}\left(\tan\frac{x}{2}\right) \Rightarrow y=\frac{x}{2}$

Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(\frac{x}{2}\right)}{dx} \Rightarrow \frac{dy}{dx}=\frac{1}{2}\frac{d(x)}{dx} \Rightarrow \frac{dy}{dx}=\frac{1}{2}.1 \therefore \frac{dy}{dx}=\frac{1}{2}$$

3. $y=\cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

Sol. $y=\cot^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right) \Rightarrow y=\cot^{-1}\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right) \Rightarrow y=\cot^{-1}\left(\cot\frac{x}{2}\right) \Rightarrow y=\frac{x}{2}$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d\left(\frac{x}{2}\right)}{dx} \Rightarrow \frac{dy}{dx}=\frac{1}{2}\frac{d(x)}{dx} \quad \therefore \frac{dy}{dx}=\frac{1}{2}$

4. $y=\cot^{-1}\left(\sqrt{\frac{1+\cos x}{1-\cos x}}\right)$

Sol. $y=\cot^{-1}\left(\sqrt{\frac{2\cos^2\frac{x}{2}}{2\sin^2\frac{x}{2}}}\right) \Rightarrow y=\cot^{-1}\left(\cot\frac{x}{2}\right) \Rightarrow y=\frac{x}{2}$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d\left(\frac{x}{2}\right)}{dx} \Rightarrow \frac{dy}{dx}=\frac{1}{2}\frac{d(x)}{dx} \Rightarrow \frac{dy}{dx}=\frac{1}{2}$

5. $y=\tan^{-1}\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)$

Sol. $y=\tan^{-1}\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)$, Dividing Numerator and Denominator by $\cos x$

$$\Rightarrow y=\tan^{-1}\left(\frac{\frac{\cos x+\sin x}{\cos x}}{\frac{\cos x-\sin x}{\cos x}}\right) \Rightarrow y=\tan^{-1}\left(\frac{\frac{\cos x}{\cos x}+\frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x}-\frac{\sin x}{\cos x}}\right) \Rightarrow y=\tan^{-1}\left(\frac{1+\tan x}{1-\tan x}\right)$$

$$\Rightarrow y=\tan^{-1}\left(\frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}\right) \Rightarrow y=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}+x\right)\right\} \Rightarrow y=\frac{\pi}{4}+x$$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d(\pi/4)}{dx}+\frac{d(x)}{dx} \Rightarrow \frac{dy}{dx}=0+1 \therefore \frac{dy}{dx}=1$

6. $y=\cot^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$

Sol. $y=\cot^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$, Dividing Numerator and Denominator by $\cos x$

$$\Rightarrow y=\cot^{-1}\left(\frac{\frac{\cos x-\sin x}{\cos x}}{\frac{\cos x+\sin x}{\cos x}}\right) \Rightarrow y=\cot^{-1}\left(\frac{\frac{\cos x}{\cos x}-\frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x}+\frac{\sin x}{\cos x}}\right) \Rightarrow y=\cot^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$\Rightarrow y=\cot^{-1}\left\{\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right\} \Rightarrow y=\cot^{-1}\left\{\tan\left(\frac{\pi}{4}-x\right)\right\} \Rightarrow y=\cot^{-1}\left[\cot\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}-x\right)\right\}\right]$$

$$\Rightarrow y=\frac{\pi}{2}-\frac{\pi}{4}+x \Rightarrow y=\frac{\pi}{4}+x$$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d(\pi/4)}{dx}+\frac{d(x)}{2x} \Rightarrow \frac{dy}{dx}=1+0 \therefore \frac{dy}{dx}=1$

7. $y=\cot^{-1}\left(\sqrt{\frac{1+\cos 3x}{1-\cos 3x}}\right)$

Sol. $y=\cot^{-1}\left(\sqrt{\frac{1+\cos 3x}{1-\cos 3x}}\right) \Rightarrow y=\cot^{-1}\left(\sqrt{\frac{2\cos^2\frac{3x}{2}}{2\sin^2\frac{3x}{2}}}\right)$

$\Rightarrow y=\cot^{-1}\left(\sqrt{\cot^2\frac{3x}{2}}\right) \Rightarrow y=\cot^{-1}\left(\cot\frac{3x}{2}\right) \Rightarrow y=\frac{3x}{2}$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d\left(\frac{3}{2}x\right)}{dx} \Rightarrow \frac{dy}{dx}=\frac{3}{2}\frac{d(x)}{dx} \ \therefore \frac{dy}{dx}=\frac{3}{2}$

8. $y=\sec^{-1}\left(\frac{1+\tan^2 x}{1-\tan^2 x}\right)$

Sol. $y=\sec^{-1}\left(\frac{1+\tan^2 x}{1-\tan^2 x}\right) \Rightarrow y=\sec^{-1}\left(\frac{1}{\frac{1-\tan^2 x}{1+\tan^2 x}}\right) \Rightarrow y=\sec^{-1}\left(\frac{1}{\cos 2x}\right)$

$\Rightarrow y=\sec^{-1}(\sec 2x) \Rightarrow y=2x$

Differentiating both sides with respect to x

$\frac{dy}{dx}=\frac{d(2x)}{dx} \Rightarrow \frac{dy}{dx}=2\frac{d(x)}{dx} \Rightarrow \frac{dy}{dx}=2.1 \ \therefore \frac{dy}{dx}=2$

9. $y=\sin^{-1}\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)$

Sol. $y=\sin^{-1}\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right) \Rightarrow y=\sin^{-1}(\cos 2x) \Rightarrow y=\sin^{-1}\left\{\sin\left(\frac{\pi}{2}-2x\right)\right\} \Rightarrow y=\frac{\pi}{2}-2x$

Differentiating both sides with respect to x

$\Rightarrow \frac{dy}{dx}=\frac{d\left(\frac{\pi}{2}\right)}{dx}-\frac{d(2x)}{2x} \Rightarrow \frac{dy}{dx}=0-2 \ \therefore \frac{dy}{dx}=-2$

10. $y=\operatorname{cosec}^{-1}\left(\frac{1+\tan^2 x}{2\tan x}\right)$

Sol. $y=\operatorname{cosec}^{-1}\left(\frac{1+\tan^2 x}{2\tan x}\right) \Rightarrow y=\operatorname{cosec}^{-1}\left(\frac{1}{\frac{2\tan x}{1+\tan^2 x}}\right)$

$\Rightarrow y=\operatorname{cosec}^{-1}\left(\frac{1}{\sin 2x}\right) \Rightarrow y=\operatorname{cosec}^{-1}(\operatorname{cosec} 2x) \Rightarrow y=2x$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d(2x)}{dx} \Rightarrow \frac{dy}{dx}=2\frac{dx}{dx} \ \therefore \frac{dy}{dx}=2$

11. $y=\cot^{-1}(\operatorname{cosec}x+\cot x)$

Sol. $y=\cot^{-1}(\operatorname{cosec}x+\cot x)\Rightarrow y=\cot^{-1}\left(\frac{1}{\sin x}+\frac{\cos x}{\sin x}\right)\Rightarrow y=\cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

$\Rightarrow y=\cot^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)\Rightarrow y=\cot^{-1}\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right)\Rightarrow y=\cot^{-1}\left(\cot\frac{x}{2}\right)\Rightarrow y=\frac{x}{2}$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d(x/2)}{dx}\Rightarrow\frac{dy}{dx}=\frac{1}{2}.\frac{d(x)}{dx}\Rightarrow\frac{dy}{dx}=\frac{1}{2}$

12. $y=\tan^{-1}(\cot x)+\cot^{-1}(\tan x)$

Sol. $y=\tan^{-1}(\cot x)+\cot^{-1}(\tan x)\Rightarrow y=\tan^{-1}\left\{\tan\left(\frac{\pi}{2}-x\right)\right\}+\cot^{-1}\left\{\cot\left(\frac{\pi}{2}-x\right)\right\}$

$\Rightarrow y=\frac{\pi}{2}-x+\frac{\pi}{2}-x\Rightarrow y=\pi-2x$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d(\pi)}{dx}-\frac{d(2x)}{dx}\Rightarrow\frac{dy}{dx}=0-2\ \therefore\frac{dy}{dx}=-2$

13. $y=\sin^{-1}\left(\sqrt{1-x^2}\right)$

Sol. $y=\sin^{-1}\left(\sqrt{1-x^2}\right)$, Let $x=\cos\theta\ \therefore\theta=\cos^{-1}(x)$

$\Rightarrow y=\sin^{-1}\left(\sqrt{1-\cos^2\theta}\right)\Rightarrow y=\sin^{-1}\left(\sqrt{\sin^2\theta}\right)\Rightarrow y=\sin^{-1}(\sin\theta)\Rightarrow y=\theta$

Putting the value of θ, $y=\cos^{-1}(x)$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d\{\cos^{-1}(x)\}}{dx}\ \therefore\ \frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}}$

14. $y=\sin^{-1}\left(\sqrt{\frac{1-x}{2}}\right)$

Sol. $y=\sin^{-1}\left(\sqrt{\frac{1-x}{2}}\right)$, Let $x=\cos\theta\ \therefore\theta=\cos^{-1}(x)\Rightarrow y=\sin^{-1}\left(\sqrt{\frac{1-\cos\theta}{2}}\right)$

$\Rightarrow y=\sin^{-1}\left(\frac{\sqrt{2\sin^2\theta/2}}{2}\right)\Rightarrow y=\sin^{-1}\left(\sin\frac{\theta}{2}\right)\Rightarrow y=\frac{\theta}{2}$

Putting the value of $\theta\Rightarrow y=\frac{1}{2}.\cos^{-1}(x)$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{1}{2}\frac{d\{\cos^{-1}(x)\}}{dx}\Rightarrow\frac{dy}{dx}=-\frac{1}{2\sqrt{1-x^2}}$

15. $y = \cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$

Sol. $y = \cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$, Let $x = \cos\theta \;\therefore \theta = \cos^{-1}(x)$

$\Rightarrow y = \cos^{-1}\left(\sqrt{\frac{1+\cos\theta}{2}}\right) \quad \Rightarrow y = \cos^{-1}\left(\sqrt{\frac{2\cos^2\frac{\theta}{2}}{2}}\right)$

$\Rightarrow y = \cos^{-1}\left(\sqrt{\cos^2\theta/2}\right) \quad \Rightarrow y = \cos^{-1}(\cos\theta/2) \quad \Rightarrow y = \theta/2$

Putting the value of θ , $y = \frac{1}{2}\cos^{-1}(x)$

Differentiating both sides with respect to x, $\frac{dy}{dx} = \frac{1}{2}\frac{d\left(\cos^{-1}x\right)}{dx} \quad \Rightarrow \frac{dy}{dx} = -\frac{1}{2\sqrt{1-x^2}}$

16. $y = \cos^{-1}\left(\sqrt{1-x^2}\right)$

Sol. $y = \cos^{-1}\left(\sqrt{1-x^2}\right)$, Let $x = \sin\theta \;\therefore\; \theta = \sin^{-1}x$

$\Rightarrow y = \cos^{-1}\left(\sqrt{1-\sin^2\theta}\right) \quad \Rightarrow y = \cos^{-1}\left(\sqrt{\cos^2\theta}\right) \quad \Rightarrow y = \cos^{-1}(\cos\theta),\; y = \theta$

Putting the value of θ, $y = \sin^{-1}x$

Differentiating both sides with respect to x, $\frac{dy}{dx} = \frac{d\left(\sin^{-1}x\right)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

17. $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

Sol. $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$, Let $x = \sin\theta \;\therefore \theta = \sin^{-1}(x)$

$y = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right) \quad \Rightarrow y = \sin^{-1}(2\sin\theta.\cos\theta) \quad \Rightarrow y = \sin^{-1}(\sin 2\theta) \Rightarrow y = 2\theta$

Putting the value of θ, $y = 2\sin^{-1}(x)$

Differentiating both sides with respect to x, $\frac{dy}{dx} = 2\frac{d\left(\sin^{-1}x\right)}{dx} \quad \Rightarrow \frac{dy}{dx} = 2.\frac{1}{\sqrt{1-x^2}} \quad \therefore \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$

18. $y = \sin^{-1}\left(3x-4x^3\right)$

Sol. $y = \sin^{-1}\left(3x-4x^3\right)$, Let $x = \sin\theta \;\therefore\; \theta = \sin^{-1}(x) \quad \Rightarrow y = \sin^{-1}\left(3\sin\theta - 4\sin^3\theta\right)$

$\Rightarrow y = \sin^{-1}(\sin 3\theta) \quad \Rightarrow y = 3\theta$

Putting the value of θ, $y = 3\sin^{-1}x$

Differentiating both sides with respect to x, $\frac{dy}{dx} = 3\frac{d\left(\sin^{-1}x\right)}{dx}$

$\Rightarrow \frac{dy}{dx} = 3.\frac{1}{\sqrt{1-x^2}} \quad \therefore \frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$

19. $y=\sin^{-1}\left(1-2x^2\right)$

Sol. $y=\sin^{-1}\left(1-2x^2\right)$, Let $x=\sin\theta \therefore \theta=\sin^{-1}x$

$y=\sin^{-1}\left(1-2\sin^2\theta\right) \Rightarrow y=\sin^{-1}\left(\cos 2\theta\right) \Rightarrow y=\sin^{-1}\left\{\sin\left(\frac{\pi}{2}-2\theta\right)\right\}$

$\Rightarrow y=\frac{\pi}{2}-2\theta$, Putting the value of θ, $y=\frac{\pi}{2}-2\sin^{-1}x$

Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(\frac{\pi}{2}\right)}{dx}-2\frac{d\left(\sin^{-1}x\right)}{dx} \Rightarrow \frac{dy}{dx}=0-2.\frac{1}{\sqrt{1-x^2}} \quad \therefore \frac{dy}{dx}=\frac{-2}{\sqrt{1-x^2}}$$

20. $y=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

Sol. $y=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$, Let $x=\sin\theta \therefore \theta=\sin^{-1}x$

$y=\sec^{-1}\left(\frac{1}{\sqrt{1-\sin^2\theta}}\right) \Rightarrow y=\sec^{-1}\left(\frac{1}{\sqrt{\cos^2\theta}}\right)$

$\Rightarrow y=\sec^{-1}\left(\frac{1}{\cos\theta}\right) \Rightarrow y=\sec^{-1}\left(\sec\theta\right) \Rightarrow y=\theta$

Putting the value of θ, $y=\sin^{-1}x$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d\left(\sin^{-1}x\right)}{dx} \quad \therefore \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$

21. $y=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Sol. $y=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, Let $x=\sin\theta \therefore \theta=\sin^{-1}(x)$

$y=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right) \Rightarrow y=\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) \Rightarrow y=\tan^{-1}\left(\tan\theta\right) \Rightarrow y=\theta$

Putting the value of θ, $y=\sin^{-1}x$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d\left(\sin^{-1}x\right)}{dx} \therefore \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$

22. $y=\tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$

Sol. $y=\tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$, Let $x=\sin\theta \therefore \theta=\sin^{-1}(x)$

$$y=\tan^{-1}\left(\frac{\sin\theta}{1+\sqrt{1-\sin^2\theta}}\right) \Rightarrow y=\tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right) \Rightarrow y=\tan^{-1}\left(\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right)$$

$$\Rightarrow\ y=\tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)\quad \Rightarrow\ y=\tan^{-1}\left(\tan\frac{\theta}{2}\right)\quad \Rightarrow\ y=\frac{\theta}{2}$$

Putting the value of θ, $y=\frac{1}{2}\sin^{-1}x$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{1}{2}\frac{d\left(\sin^{-1}x\right)}{dx}\quad \Rightarrow \frac{dy}{dx}=\frac{1}{2}.\frac{1}{\sqrt{1-x^2}}\ \therefore \frac{dy}{dx}=\frac{1}{2\sqrt{1-x^2}}$

23. $y=\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

Sol. $y=\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$, Let $x=\sin\theta\ \therefore\theta=\sin^{-1}(x)$

$$y=\cot^{-1}\left(\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right)\quad \Rightarrow\ y=\cot^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)\quad \Rightarrow\ y=\cot^{-1}\left(\cot\theta\right)\quad \Rightarrow\ y=\theta$$

Putting the value of θ, $y=\sin^{-1}x$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d\left(\sin^{-1}x\right)}{dx}\ \therefore\ \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$

24. $y=\sec^{-1}\left(\frac{1}{1-2x^2}\right)$

Sol. $y=\sec^{-1}\left(\frac{1}{1-2x^2}\right)$, Let $x=\sin\theta\ \therefore\theta=\sin^{-1}(x)$

$$y=\sec^{-1}\left(\frac{1}{1-2\sin^2\theta}\right)\quad \Rightarrow y=\sec^{-1}\left(\frac{1}{\cos 2\theta}\right)\quad \Rightarrow y=\sec^{-1}\left(\sec 2\theta\right)\quad \Rightarrow y=2\theta$$

Putting the value of θ, $y=2\sin^{-1}(x)$

Differentiating both sides with respect to x, $\frac{dy}{dx}=2\frac{d\left(\sin^{-1}x\right)}{dx}\quad \Rightarrow \frac{dy}{dx}=2.\frac{1}{\sqrt{1-x^2}}\ \therefore \frac{dy}{dx}=\frac{2}{\sqrt{1-x^2}}$

25. $y=\sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

Sol. $y=\sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$, Let $x=\cot\theta\ \therefore\theta=\cot^{-1}(x)$

$$y=\sin^{-1}\left(\frac{1}{\sqrt{1+\cot^2\theta}}\right)\quad \Rightarrow\ y=\sin^{-1}\left(\frac{1}{\sqrt{\cos ec^2\theta}}\right)\quad \Rightarrow\ y=\sin^{-1}\left(\frac{1}{\cos ec\theta}\right)$$

$$\Rightarrow\ y=\sin^{-1}\left(\sin\theta\right)\quad \Rightarrow\ y=\theta$$

Putting the value of θ, $y=\cot^{-1}(x)$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d\left\{\cot^{-1}(x)\right\}}{dx}\quad \therefore\ \frac{dy}{dx}=-\frac{1}{1+x^2}$

26. $y=\tan^{-1}\left(\frac{1+x}{1-x}\right)$

Sol. $y=\tan^{-1}\left(\frac{1+x}{1-x}\right)$, Let $x=\tan\theta \;\therefore \theta=\tan^{-1}(x)$

$$y=\tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right) \Rightarrow y=\tan^{-1}\left(\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}\right) \Rightarrow y=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\theta\right)\right\} \Rightarrow y=\frac{\pi}{4}+\theta$$

Putting the value of θ, $y=\frac{\pi}{4}+\tan^{-1}(x)$

Differentiating both sides with respect to x. $\frac{dy}{dx}=\frac{d\left(\frac{\pi}{4}\right)}{dx}+\frac{d\left(\tan^{-1}x\right)}{dx}$

$$\Rightarrow \frac{dy}{dx}=0+\frac{1}{1+x^2} \quad \therefore \frac{dy}{dx}=\frac{1}{1+x^2}$$

27. $y=\cot^{-1}\left(\frac{1+x}{1-x}\right)$

Sol. $y=\cot^{-1}\left(\frac{1+x}{1-x}\right)$, Let $x=\tan\theta \;\therefore \theta=\tan^{-1}(x)$

$$y=\cot^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right) \Rightarrow y=\cot^{-1}\left(\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}\right) \Rightarrow y=\cot^{-1}\left\{\tan\left(\frac{\pi}{4}+\theta\right)\right\}$$

$$\Rightarrow y=\cot^{-1}\left\{\cot\left(\frac{\pi}{2}-\left(\frac{\pi}{4}+\theta\right)\right)\right\} \Rightarrow y=\frac{\pi}{2}-\frac{\pi}{4}-\theta \Rightarrow y=\frac{\pi}{4}-\theta$$

Putting the value of 'θ', $y=\frac{\pi}{4}-\tan^{-1}(x)$

Differentiating both sides with respect to x

$$\Rightarrow \frac{dy}{dx}=\frac{d\left(\frac{\pi}{4}\right)}{dx}-\frac{d\left(\tan^{-1}x\right)}{dx} \Rightarrow \frac{dy}{dx}=0-\frac{1}{1+x^2} \therefore \frac{dy}{dx}=-\frac{1}{1+x^2}$$

28. $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

Sol. $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$, Let $x=\tan\theta \;\therefore \theta=\tan^{-1}(x)$

$$y=\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right) \Rightarrow y=\tan^{-1}(\tan 3\theta) \Rightarrow y=3\theta$$

Putting the value of 'θ', $y=3\tan^{-1}(x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx}=3\frac{d\left(\tan^{-1}x\right)}{dx} \Rightarrow \frac{dy}{dx}=3.\frac{1}{1+x^2} \therefore \frac{dy}{dx}=\frac{3}{1+x^2}$$

29. $y = \operatorname{cosec}^{-1}\left(\frac{1+x^2}{2x}\right)$

Sol. $y = \operatorname{cosec}^{-1}\left(\frac{1+x^2}{2x}\right)$, Let $x = \tan\theta \therefore \theta = \tan^{-1}(x)$

$$y = \operatorname{cosec}^{-1}\left(\frac{1+\tan^2\theta}{2\tan\theta}\right) \Rightarrow y = \operatorname{cosec}^{-1}\left(\frac{1}{\frac{2\tan\theta}{1+\tan^2\theta}}\right)$$

$$\Rightarrow y = \operatorname{cosec}^{-1}\left(\frac{1}{\sin 2\theta}\right) \Rightarrow y = \operatorname{cosec}^{-1}(\operatorname{cosec} 2\theta) \Rightarrow y = 2\theta$$

Putting the value of 'θ', $y = 2\tan^{-1}(x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(2.\tan^{-1}x)}{dx} \Rightarrow \frac{dy}{dx} = 2.\frac{1}{1+x^2} \quad \therefore \frac{dy}{dx} = \frac{2}{1+x^2}$$

30. $y = \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$

Sol. $y = \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$, Let $x = \tan\theta \therefore \theta = \tan^{-1}(x)$

$$y = \sec^{-1}\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right) \Rightarrow y = \sec^{-1}\left(\frac{1}{\frac{1-\tan^2\theta}{1+\tan^2\theta}}\right)$$

$$y = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right) \Rightarrow y = \sec^{-1}(\sec 2\theta) \Rightarrow y = 2\theta$$

Putting the value of 'θ', $y = 2\tan^{-1}(x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = 2\frac{d(\tan^{-1}x)}{dx} \Rightarrow \frac{dy}{dx} = 2.\frac{1}{1+x^2} \therefore \frac{dy}{dx} = \frac{2}{1+x^2}$$

31. $y = \sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

Sol. $y = \sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$, Let $x = \tan\theta \therefore \theta = \tan^{-1}(x)$

$$\Rightarrow y = \sin^{-1}\left(\frac{1}{\sqrt{1+\tan^2\theta}}\right) \Rightarrow y = \sin^{-1}\left(\frac{1}{\sec\theta}\right) \Rightarrow y = \sin^{-1}(\cos\theta)$$

$$\Rightarrow y = \sin^{-1}\left\{\sin\left(\frac{\pi}{2}-\theta\right)\right\} \Rightarrow y = \frac{\pi}{2}-\theta$$

Putting the value of 'θ', $y = \frac{\pi}{2} - \tan^{-1}(x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(\frac{\pi}{2}\right)}{dx}-\frac{d\left(\tan^{-1}x\right)}{dx}\Rightarrow\frac{dy}{dx}=0-\frac{1}{1+x^2}\quad\Rightarrow\frac{dy}{dx}=-\frac{1}{1+x^2}$$

32. $y=\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

Sol. $y=\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)\quad\Rightarrow y=\sec^{-1}\left\{\frac{-\left(1+x^2\right)}{\left(1-x^2\right)}\right\}\quad\Rightarrow y=\pi-\sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$

Let $x=\tan\theta\ \therefore\theta=\tan^{-1}(x),\ y=\pi-\sec^{-1}\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right)$

$$\Rightarrow y=\pi-\sec^{-1}\left(\frac{1}{\frac{1-\tan^2\theta}{1+\tan^2\theta}}\right)\quad\Rightarrow y=\pi-\sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$$

$$\Rightarrow y=\pi-\sec^{-1}(\sec 2\theta)\quad\Rightarrow y=\pi-2\theta$$

Putting the value of 'θ', $y=\pi-2\tan^{-1}(x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d(\pi)}{dx}-2\frac{d\left(\tan^{-1}x\right)}{dx}\quad\Rightarrow\frac{dy}{dx}=0-2.\frac{1}{1+x^2}\quad\Rightarrow\frac{dy}{dx}=-\frac{2}{1+x^2}$$

33. $y=\cos^{-1}\left(\frac{1-x^{2n}}{1+x^{2n}}\right)$

Sol. $y=\cos^{-1}\left(\frac{1-x^{2n}}{1+x^{2n}}\right)\quad\Rightarrow y=\cos^{-1}\left\{\frac{1-\left(x^n\right)^2}{1+\left(x^n\right)^2}\right\}$, Let $\tan\theta=x^n\ \therefore\theta=\tan^{-1}\left(x^n\right)$

$$y=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)\quad\Rightarrow\quad y=\cos^{-1}(\cos 2\theta)\Rightarrow y=2\theta$$

Putting the value of 'θ', $y=2\tan^{-1}\left(x^n\right)$

Differentiating both sides with respect to x

$$\frac{dy}{dx}=2\frac{d\left\{\tan^{-1}\left(x^n\right)\right\}}{d\left(x^n\right)}\times\frac{d\left(x^n\right)}{dx}\quad\Rightarrow\frac{dy}{dx}=2.\frac{1}{1+\left(x^n\right)^2}.nx^{n-1}\quad\therefore\quad\frac{dy}{dx}=\frac{2nx^{n-1}}{1+x^{2n}}$$

34. $y=\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$

Sol. $y=\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$, Let $x=a\sin\theta\quad\Rightarrow\frac{x}{a}=\sin\theta\quad\therefore\theta=\sin^{-1}\left(\frac{x}{a}\right)$

$$\Rightarrow y=\tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)\quad\Rightarrow y=\tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2\left(1-\sin^2\theta\right)}}\right)$$

$$\Rightarrow y=\tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)\quad\Rightarrow y=\tan^{-1}(\tan\theta)\quad\Rightarrow y=\theta$$

Putting the value of θ, $y=\sin^{-1}\left(\frac{x}{a}\right)$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d\left\{\sin^{-1}\left(\frac{x}{a}\right)\right\}}{d\left(\frac{x}{a}\right)}\times d\frac{\left(\frac{x}{a}\right)}{dx}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}.\frac{1}{a}\frac{d(x)}{dx}$ $\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}.\frac{1}{a}$ $\Rightarrow \frac{dy}{dx}=\frac{a}{\sqrt{a^2-x^2}}.\frac{1}{a}$ $\therefore \frac{dy}{dx}=\frac{1}{\sqrt{a^2-x^2}}$

35. $y=\sin^{-1}\left\{2ax\sqrt{1-a^2x^2}\right\}$

Sol. $y=\sin^{-1}\left\{2ax\sqrt{1-a^2x^2}\right\}$, Let $ax=\sin\theta$ $\therefore \theta=\sin^{-1}(ax)$

$\Rightarrow y=\sin^{-1}\left\{2\sin\theta\sqrt{1-\sin^2\theta}\right\}$ $\Rightarrow y=\sin^{-1}\{\sin 2\theta\}$,

Putting the value of θ, $y=2\sin^{-1}ax$

Differentiating both sides with respect to x

$\frac{dy}{dx}=2\frac{d\left\{\sin^{-1}(ax)\right\}}{d(ax)}\times\frac{d(ax)}{dx}$ $\Rightarrow \frac{dy}{dx}=2.\frac{1}{\sqrt{1-(ax)^2}}\times a$ $\Rightarrow \frac{dy}{dx}=\frac{2a}{\sqrt{1-a^2x^2}}$

36. $y=\tan^{-1}\left(\frac{\sqrt{1+a^2x^2}-1}{ax}\right)$

Sol. $y=\tan^{-1}\left(\frac{\sqrt{1+(ax)^2}-1}{ax}\right)$, Let $ax=\tan\theta$ $\therefore \theta=\tan^{-1}(ax)$

$y=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$ $\Rightarrow y=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$ $\Rightarrow y=\tan^{-1}\left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right)$

$\Rightarrow y=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$ $\Rightarrow y=\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$ $\Rightarrow y=\tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)$

$\Rightarrow y=\tan^{-1}\left(\tan\frac{\theta}{2}\right)$ $\Rightarrow y=\frac{\theta}{2}$

Putting the value of θ, $y=\frac{1}{2}\tan^{-1}(ax)$

Differentiating both sides with respect to x

$\frac{dy}{dx}=\frac{1}{2}\frac{d\left\{\tan^{-1}(ax)\right\}}{d(ax)}\times\frac{d(ax)}{dx}$ $\Rightarrow \frac{dy}{dx}=\frac{1}{2}.\frac{1}{1+(ax)^2}.a$ $\therefore$ $\frac{dy}{dx}=\frac{a}{2\left(1+a^2x^2\right)}$

37. $y=\sin^{-1}\left(\dfrac{x^2}{\sqrt{x^4+a^4}}\right)$

Sol. $y=\sin^{-1}\left(\dfrac{x^2}{\sqrt{x^4+a^4}}\right)$, Let $x^2=a^2\cot\theta \Rightarrow \dfrac{x^2}{a^2}=\cot\theta \quad \therefore \theta=\cot^{-1}\left(\dfrac{x^2}{a^2}\right)$

$$y=\sin^{-1}\left(\frac{a^2\cot\theta}{\sqrt{a^4\cot^2\theta+a^4}}\right) \quad \Rightarrow y=\sin^{-1}\left(\frac{a^2\cot\theta}{\sqrt{a^4\left(\cot^2\theta+1\right)}}\right)$$

$$\Rightarrow y=\sin^{-1}\left(\frac{a^2\cot\theta}{a^2\operatorname{cosec}\theta}\right) \quad \Rightarrow y=\sin^{-1}\left(\frac{\cos\theta}{\sin\theta}\times\frac{\sin\theta}{1}\right)$$

$$\Rightarrow y=\sin^{-1}\left(\cos\theta\right) \quad \Rightarrow y=\sin^{-1}\left\{\sin\left(\frac{\pi}{2}-\theta\right)\right\} \Rightarrow y=\frac{\pi}{2}-\theta$$

Putting the value of θ. $y=\dfrac{\pi}{2}-\cot^{-1}\left(\dfrac{x^2}{a^2}\right)$

Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left(\frac{\pi}{2}\right)}{dx}-\frac{d\left\{\cot^{-1}\left(\frac{x^2}{a^2}\right)\right\}}{d\left(\frac{x^2}{a^2}\right)}\times\frac{d\left(\frac{x^2}{a^2}\right)}{dx} \quad \Rightarrow \frac{dy}{dx}=0+\frac{1}{1+\left(\frac{x^2}{a^2}\right)^2}\cdot\frac{1}{a^2}.2x$$

$$\Rightarrow \frac{dy}{dx}=\frac{a^4}{a^4+x^4}\cdot\frac{1}{a^2}.2x \quad \Rightarrow \frac{dy}{dx}=\frac{2a^2x}{a^4+x^4}$$

38. $y=\tan^{-1}\left(\dfrac{e^{2x}+1}{e^{2x}-1}\right)$

Sol. $y=\tan^{-1}\left(\dfrac{e^{2x}+1}{e^{2x}-1}\right) \quad \Rightarrow y=\tan^{-1}\left\{\dfrac{e^{2x}+1}{-\left(1-e^{2x}\right)}\right\} \quad \Rightarrow y=-\tan^{-1}\left(\dfrac{1+e^{2x}}{1-e^{2x}}\right)$

Let $e^{2x}=\tan\theta \;\therefore \theta=\tan^{-1}\left(e^{2x}\right)$

$$y=-\tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right) \quad \Rightarrow y=-\tan^{-1}\left(\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}+\tan\theta}\right)$$

$$\Rightarrow y=-\tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\theta\right)\right\} \quad \Rightarrow y=-\frac{\pi}{4}-\theta$$

Putting the value of θ, $y=-\dfrac{\pi}{4}-\tan^{-1}\left(e^{2x}\right)$

Differentiating both sides with respect to x, $\dfrac{dy}{dx}=-\dfrac{d\left(\frac{\pi}{4}\right)}{dx}-\dfrac{d\left\{\tan^{-1}\left(e^{2x}\right)\right\}}{d\left(e^{2x}\right)}\times\dfrac{d\left(e^{2x}\right)}{d(2x)}\times\dfrac{d(2x)}{dx}$

$$\Rightarrow \frac{dy}{dx}=0-\frac{1}{1+\left(e^{2x}\right)^2}.e^{2x}.2 \quad \therefore \frac{dy}{dx}=\frac{-2e^{2x}}{1+e^{4x}}$$

39. $y=\cos^{-1}(2x)+2\cos^{-1}\left(\sqrt{1-4x^2}\right)$

Sol. $y=\cos^{-1}(2x)+2\cos^{-1}\left(\sqrt{1-4x^2}\right)$, Let $2x=\cos\theta \;\therefore\; \theta=\cos^{-1}(2x)$

$$y=\cos^{-1}(\cos\theta)+2\cos^{-1}\left(\sqrt{1-\cos^2\theta}\right)$$

$$\Rightarrow y=\theta+2\cos^{-1}(\sin\theta) \quad \Rightarrow y=\theta+2\cos^{-1}\left\{\cos\left(\frac{\pi}{2}-\theta\right)\right\} \quad \Rightarrow y=\theta+2\left(\frac{\pi}{2}-\theta\right)$$

$$\Rightarrow y=\theta+\pi-2\theta \quad \Rightarrow y=\pi-\theta$$

Putting the value of θ, $y=\pi-\cos^{-1}(2x)$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d(\pi)}{dx}-\frac{d\{\cos^{-1}(2x)\}}{d(2x)}\times\frac{d(2x)}{dx}$

$$\Rightarrow \frac{dy}{dx}=0+\frac{1}{\sqrt{1-(2x)^2}}.2 \quad \therefore \frac{dy}{dx}=\frac{2}{\sqrt{1-4x^2}}$$

40. $y=\tan^{-1}\left(\frac{a-x}{1+ax}\right)$

Sol. $y=\tan^{-1}\left(\frac{a-x}{1+ax}\right)=tan^{-1}a-tan^{-1}x \Rightarrow \frac{dy}{dx}=0-\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{-1}{1+x^2}$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{d\{\tan^{-1}(a)\}}{d(a)}\times\frac{d(a)}{dx}-\frac{d\left(\tan^{-1}x\right)}{dx}$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{1+a^2}.0-\frac{1}{1+x^2} \;\therefore\; \frac{dy}{dx}=-\frac{1}{1+x^2}$$

41. $y=\tan^{-1}\left(\frac{\sqrt{x}-x}{1+x^{3/2}}\right)$

Sol. $y=\tan^{-1}\left(\frac{\sqrt{x}-x}{1+x^{3/2}}\right)$, $y=\tan^{-1}\left(\frac{\sqrt{x}-x}{1+\sqrt{x}\,x}\right)=\tan^{-1}\sqrt{x}-\tan^{-1}x$

Differentiating both sides with respect to x, $\frac{dy}{dx}=\frac{1}{1+\left(\sqrt{x}\right)^2}.\frac{1}{2\sqrt{x}}-\frac{1}{1+x^2}$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2\sqrt{x}(1+x)}-\frac{1}{1+x^2}$$

42. $y=\tan^{-1}\left(\frac{\sqrt{a}+\sqrt{x}}{1-\sqrt{ax}}\right)$

Sol. $y=\tan^{-1}\left\{\frac{\sqrt{a}+\sqrt{x}}{1-\sqrt{a}\sqrt{x}}\right\} \quad \Rightarrow y=\tan^{-1}\left(\sqrt{a}\right)+\tan^{-1}\left(\sqrt{x}\right) \Rightarrow \frac{dy}{dx}=0+\frac{d}{dx}\left(\tan^{-1}\sqrt{x}\right)=\frac{1}{2\sqrt{x}(1+x)}$

Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\left\{\tan^{-1}\left(\sqrt{a}\right)\right\}}{d\left(\sqrt{a}\right)}\times\frac{d\left(\sqrt{a}\right)}{dx}+\frac{d\left\{\tan^{-1}\left(\sqrt{x}\right)\right\}}{d\left(\sqrt{x}\right)}\times\frac{d\left(\sqrt{x}\right)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 0 + \frac{1}{2\sqrt{x}(1+x)} \quad \therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}(1+x)}$$

43. $y = \tan^{-1}\left(\frac{3-2x}{1+6x}\right)$

Sol. $y = \tan^{-1}\left(\frac{3-2x}{1+6x}\right) \quad \Rightarrow y = \tan^{-1}(3) - \tan^{-1}(2x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\tan^{-1}(3)\}}{dx} - \frac{d\{\tan^{-1}(2x)\}}{d(2x)} \times \frac{d(2x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 0 - \frac{1}{1+(2x)^2}.2 \quad \therefore \frac{dy}{dx} = -\frac{2}{1+4x^2}$$

44. $y = \tan^{-1}\left(\frac{5x}{1-6x^2}\right)$

Sol. $y = \tan^{-1}\left(\frac{5x}{1-6x^2}\right) \quad \Rightarrow y = \tan^{-1}\left(\frac{3x+2x}{1-3x.2x}\right)$

Let $3x = \tan A \quad \therefore A = \tan^{-1}(3x)$ & $2x = \tan B \quad \therefore B = \tan^{-1}(2x)$

$$y = \tan^{-1}\left(\frac{\tan A + \tan B}{1 - \tan A.\tan B}\right) \quad \Rightarrow y = \tan^{-1}\{\tan(A+B)\} \quad \Rightarrow y = A + B$$

Putting the value of $A \& B$, $y = \tan^{-1}(3x) + \tan^{-1}(2x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\tan^{-1}(3x)\}}{d(3x)} \times \frac{d(3x)}{dx} + \frac{d\{\tan^{-1}(2x)\}}{d(2x)} \times \frac{d(2x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+(3x)^2}.3 + \frac{1}{1+(2x)^2} \quad \therefore \frac{dy}{dx} = \frac{3}{1+9x^2} + \frac{2}{1+4x^2}$$

45. $y = \tan^{-1}\left(\frac{2x}{1+15x^2}\right)$

Sol. $y = \tan^{-1}\left(\frac{2x}{1+15x^2}\right) \quad \Rightarrow y = \tan^{-1}\left(\frac{5x-3x}{1+5x.3x}\right) \quad \Rightarrow y = \tan^{-1}(5x) - \tan^{-1}(3x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\tan^{-1}(5x)\}}{d(5x)} \times \frac{d(5x)}{dx} - \frac{d\{\tan^{-1}(3x)\}}{d(3x)} \times \frac{d(3x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+(5x)^2}.5 - \frac{1}{1+(3x)^2}.3 \quad \therefore \frac{dy}{dx} = \frac{5}{1+25x^2} - \frac{3}{1+9x^2}$$

46. If $y = \tan^{-1}\left(\frac{ax-b}{a+bx}\right)$, prove that $\frac{dy}{dx} = \frac{1}{1+x^2}$

Sol. $y = \tan^{-1}\left(\frac{ax-b}{a+bx}\right)$

Dividing nominator and denominator by a, $y=\tan^{-1}\left(\dfrac{\frac{ax-b}{a}}{\frac{a+bx}{a}}\right) \Rightarrow y=\tan^{-1}\left(\dfrac{\frac{ax}{a}-\frac{b}{a}}{\frac{a}{a}+\frac{b}{a}.x}\right)$

$$\Rightarrow y=\tan^{-1}\left(\frac{x-\frac{b}{a}}{1+\frac{b}{a}.x}\right) \Rightarrow y=\tan^{-1}(x)-\tan^{-1}(b/a)$$

Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\{\tan^{-1}(x)\}}{dx}-\frac{d}{dx}\left(\tan^{-1}\frac{b}{a}\right) \Rightarrow \frac{dy}{dx}=\frac{1}{1+x^2} \quad \therefore \frac{dy}{dx}=\frac{1}{1+x^2}$$

47. If $y=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)+\sec^{-1}\left(\dfrac{1+x^2}{1-x^2}\right)$ show that $\dfrac{dy}{dx}=\dfrac{4}{1+x^2}$

Sol. $y=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)+\sec^{-1}\left(\dfrac{1+x^2}{1-x^2}\right)$, Let $x=\tan\theta \;\therefore \theta=\tan^{-1}(x)$

$$y=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)+\sec^{-1}\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right)$$

$$\Rightarrow y=\sin^{-1}(\sin 2\theta)+\sec^{-1}(\sec 2\theta) \Rightarrow y=2\theta+2\theta \Rightarrow y=4\theta$$

Putting the value of θ, $y=4\tan^{-1}(x)$

Differentiating both sides with respect to x $\dfrac{dy}{dx}=4\dfrac{d(\tan^{-1}x)}{dx} \Rightarrow \dfrac{dy}{dx}=4.\dfrac{1}{1+x^2} \quad \therefore \dfrac{dy}{dx}=\dfrac{4}{1+x^2}$

48. If $y=\sec^{-1}\left(\dfrac{x+1}{x-1}\right)+\sin^{-1}\left(\dfrac{x-1}{x+1}\right)$ show that $\dfrac{dy}{dx}=0$

Sol. $y=\sec^{-1}\left(\dfrac{x+1}{x-1}\right)+\sin^{-1}\left(\dfrac{x-1}{x+1}\right) \Rightarrow y=\cos^{-1}\left(\dfrac{x-1}{x+1}\right)+\sin^{-1}\left(\dfrac{x-1}{x+1}\right) \quad \left[\therefore \sin^{-1}x+\cos^{-1}x=\dfrac{\pi}{2}\right]$

$$\Rightarrow y=\frac{\pi}{2} \Rightarrow \frac{dy}{dx}=\frac{d\left(\frac{\pi}{2}\right)}{dx} \quad \therefore \frac{dy}{dx}=0$$

49. If $y=\sin\left\{2\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}\right\}$ show that $\dfrac{dy}{dx}=-\dfrac{x}{\sqrt{1-x^2}}$

Sol. $y=\sin\left\{2\tan^{-1}\left(\sqrt{\dfrac{1-x}{1+x}}\right)\right\}$

Let $x=\cos\theta \;\therefore \theta=\cos^{-1}(x)$, $y=\sin\left\{2\tan^{-1}\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}\right\} \Rightarrow y=\sin\left\{2.\tan^{-1}\left(\sqrt{\dfrac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\right)\right\}$

$$y=\sin\left\{2\tan^{-1}\left(\tan\frac{\theta}{2}\right)\right\} \Rightarrow y=\sin\left\{2\times\frac{\theta}{2}\right\} \Rightarrow y=\sin\theta$$

Putting the value of θ, $y=\sin(\cos^{-1}x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d\sin\left(\cos^{-1}x\right)}{d\left(\cos^{-1}x\right)}\times\frac{d\left(\cos^{-1}x\right)}{dx} \Rightarrow \frac{dy}{dx}=-\cos\left(\cos^{-1}x\right).\frac{1}{\sqrt{1-x^2}} \quad \therefore \frac{dy}{dx}=-\frac{x}{\sqrt{1-x^2}}$$

50. If $y=\tan^{-1}\left\{\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right\}$ prove that $\frac{dy}{dx}=\frac{1}{2\sqrt{1-x^2}}$

Sol. $y=\tan^{-1}\left\{\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right\}$, Let $x=\cos 2\theta \Rightarrow 2\theta=\cos^{-1}(x) \quad \therefore \theta=\frac{1}{2}\cos^{-1}(x)$

$$\Rightarrow y=\tan^{-1}\left\{\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right\} \Rightarrow y=\tan^{-1}\left\{\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right\}$$

$$\Rightarrow y=\tan^{-1}\left\{\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right\} \Rightarrow y=\tan^{-1}\left\{\frac{\sqrt{2}(\cos\theta-\sin\theta)}{\sqrt{2}(\cos\theta+\sin\theta)}\right\} \Rightarrow y=\tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)$$

Dividing nminator and denominator by $\cos\theta$

$$y=\tan^{-1}\left(\frac{\frac{\cos\theta-\sin\theta}{\cos\theta}}{\frac{\cos\theta+\sin\theta}{\cos\theta}}\right) \Rightarrow y=\tan^{-1}\left(\frac{\frac{\cos\theta}{\cos\theta}-\frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}\right) \Rightarrow y=\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$\Rightarrow y=\tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right) \Rightarrow y=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\theta\right)\right\} \Rightarrow y=\frac{\pi}{4}-\theta$$

Putting the value of θ, $y=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}(x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx}=\frac{d(\pi/4)}{dx}-\frac{1}{2}\frac{d\left(\cos^{-1}x\right)}{dx} \Rightarrow \frac{dy}{dx}=0+\frac{1}{2}.\frac{1}{\sqrt{1-x^2}} \quad \therefore \frac{dy}{dx}=\frac{1}{2\sqrt{1-x^2}}$$

51. Differentiate $\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$ w.r.t. x

Sol. $y=\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right) \Rightarrow y=\sin^{-1}\left(\frac{2^x.2^1}{1+\left(2^2\right)^x}\right) \Rightarrow y=\sin^{-1}\left(\frac{2.2^x}{1+\left(2^x\right)^2}\right)$

Let $2^x=\tan\theta \therefore \theta=\tan^{-1}\left(2^x\right)$, $y=\sin^{-1}\left(\frac{2.\tan\theta}{1+\tan^2\theta}\right) \Rightarrow y=\sin^{-1}(\sin 2\theta) \Rightarrow y=2\theta$

Putting the value of θ, $y=2\tan^{-1}\left(2^x\right)$

Differentiating both sides with respect to x

$$\frac{dy}{dx}=2\frac{d\left\{\tan^{-1}\left(2^x\right)\right\}}{d\left(2^x\right)}\times\frac{d\left(2^x\right)}{dx} \Rightarrow \frac{dy}{dx}=2.\frac{1}{1+\left(2^x\right)^2}2^x\log 2 \Rightarrow \frac{dy}{dx}=\frac{2^{1+x}\log 2}{1+4^x}$$

Exercise 10 E

Find $\frac{dy}{dx}$, when:

1. $x^2+y^2=4$

Sol. Differentiating both sides with respect to x, $\frac{d(x^2)}{dx}+\frac{d(y^2)}{dy}.\frac{dy}{dx}=\frac{d(4)}{dx}$

$\Rightarrow 2x.1+2y.\frac{dy}{dx}=0 \quad \Rightarrow 2y\frac{dy}{dx}=-2x \quad \Rightarrow \frac{dy}{dx}=\frac{-2x}{2y} \therefore \frac{dy}{dx}=-\frac{x}{y}$

2. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Sol. Differentiating both sides with respect to x, $\frac{1}{a^2}\frac{d(x^2)}{dx}+\frac{1}{b^2}\frac{d(y^2)}{dy}.\frac{dy}{dx}=\frac{d(1)}{dx}$

$\Rightarrow \frac{1}{a^2}.2x.1+\frac{1}{b^2}.2y.\frac{dy}{dx}=0 \quad \Rightarrow \frac{2y}{b^2}=\frac{-2x}{a^2} \therefore \frac{dy}{dx}=-\frac{b^2x}{a^2y}$

3. $\sqrt{x}+\sqrt{y}=\sqrt{a}$

Sol. Differentiating both sides with respect to x

$\frac{d(\sqrt{x})}{dx}+\frac{d(\sqrt{y})}{dy}.\frac{dy}{dx}=\frac{d(\sqrt{a})}{da} \quad \Rightarrow \frac{1}{2\sqrt{x}}.1+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$

$\Rightarrow \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0 \quad \Rightarrow \frac{1}{2\sqrt{y}}\frac{dy}{dx}=-\frac{1}{2\sqrt{x}} \quad \Rightarrow \frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}} \therefore \frac{dy}{dx}=-\sqrt{\frac{y}{x}}$

4. $x^{2/3}+y^{2/3}=a^{2/3}$

Sol. Differentiating both sides with respect to x, $\frac{d(x^2/3)}{dx}.\frac{dx}{dx}+\frac{d(y^{2/3})}{dy}.\frac{dy}{dx}=\frac{d(a^{2/3})}{da}.\frac{da}{dx}$

$\Rightarrow \frac{2}{3}x^{\frac{2}{3}-1}.1+\frac{2}{3}y^{\frac{2}{3}-1}\frac{dy}{dx}=\frac{2}{3}a^{\frac{2}{3}-1}.0 \quad \Rightarrow \frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx}=0$

$\Rightarrow \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx}=-\frac{2}{3}x^{\frac{-1}{3}} \quad \Rightarrow \frac{dy}{dx}=-\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} \quad \therefore \frac{dy}{dx}=-\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$

5. $xy=c^2$

Sol. $xy=c^2$ ...(i)

Differentiating both sides with respect to x

$x\frac{dy}{dx}+y\frac{dx}{dx}=\frac{d(c^2)}{dc}.\frac{dc}{dx} \quad \Rightarrow x\frac{dy}{dx}+y.1=2c.0 \quad \Rightarrow x\frac{dy}{dx}+y=0$

$\Rightarrow x\frac{dy}{dx}=-y \Rightarrow \frac{dy}{dx}=-\frac{y}{x}$ ...(ii)

From equation (i), $xy=c^2 \Rightarrow y=\frac{c^2}{x}$

Putting the value of y in equation (ii), $\frac{dy}{dx}=-\frac{c^2}{x^2}$

6. $x^2+y^2-3xy=1$

Sol. Differentiating both sides with respect to x, $\frac{d(x^2)}{dx}+\frac{d(y^2)}{dy}\frac{dy}{dx}-3\left[x\frac{dy}{dx}+y\frac{dx}{dx}\right]=\frac{d(1)}{dx}$

$\Rightarrow 2x.1+2y\frac{dy}{dx}-3\left[x\frac{dy}{dx}+y.1\right]=0 \quad \Rightarrow 2x+2y\frac{dy}{dx}-3x\frac{dy}{dx}-3y=0$

$\Rightarrow 2y\frac{dy}{dx}-3x\frac{dy}{dx}=3y-2x \quad \Rightarrow \frac{dy}{dx}(2y-3x)=3y-2x$

$\Rightarrow \frac{dy}{dx}=\frac{3y-2x}{2y-3x} \quad \Rightarrow \frac{dy}{dx}=\frac{-(2x-3y)}{-(3x-2y)} \quad \therefore \frac{dy}{dx}=\frac{2x-3y}{3x-2y}$

7. $xy^2-x^2y-5=0$

Sol. Differentiating both sides with respect to x, $\frac{d(xy^2)}{dx}-\frac{d(x^2y)}{dx}-\frac{d(5)}{dx}=0$

$\Rightarrow \left\{x\frac{d(y^2)}{dy}.\frac{dy}{dx}+y^2\frac{d(x)}{dx}\right\}-\left\{x^2\frac{dy}{dx}+y\frac{d(x^2)}{dx}\right\}-0=0$

$\Rightarrow \left(x.2y\frac{dy}{dx}+y^2.1\right)-\left(x^2\frac{dy}{dx}+2xy\right)=0 \quad \Rightarrow 2xy\frac{dy}{dx}-x^2\frac{dy}{dx}=2xy-y^2$

$\Rightarrow \frac{dy}{dx}(2xy-x^2)=2xy-y^2 \quad \Rightarrow \frac{dy}{dx}=\frac{-(y^2-2xy)}{-(x^2-2xy)} \quad \therefore \frac{dy}{dx}=\frac{y^2-2xy}{x^2-2xy}$

8. $(x^2+y^2)^2=xy$

Sol. Differentiating both sides with respect to x, $\frac{d\{(x^2+y^2)^2\}}{d(x^2+y^2)}\times\frac{d(x^2+y^2)}{dx}=\frac{d(xy)}{dx}$

$\Rightarrow 2(x^2+y^2)\times\left\{\frac{d(x^2)}{dx}+\frac{d(y^2)}{dy}.\frac{dy}{dx}\right\}=x\frac{dy}{dx}+y\frac{dx}{dx}$

$\Rightarrow 2(x^2+y^2).\left\{2x.1+2y\frac{dy}{dx}\right\}=x\frac{dy}{dx}+y.1 \quad \Rightarrow 2(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right)=x\frac{dy}{dx}+y$

$\Rightarrow 4x(x^2+y^2)+4y(x^2+y^2)\frac{dy}{dx}=x\frac{dy}{dx}+y \quad \Rightarrow 4y(x^2+y^2)\frac{dy}{dx}-x\frac{dy}{dx}=y-4x(x^2+y^2)$

$\Rightarrow \frac{dy}{dx}(4x^2y+4y^3-x)=y-4x^3-4xy^2 \quad \therefore \frac{dy}{dx}=\frac{y-4xy^2-4x^3}{4y^3+4x^2y-x}$

9. $x^2+y^2=\log(xy)$

Sol. Differentiating both sides with respect to x, $\frac{d(x^2)}{dx}.\frac{dx}{dx}+\frac{d(y^2)}{dy}.\frac{dy}{dx}=\frac{d\{\log(xy)\}}{d(xy)}\times\frac{d(xy)}{dx}$

$\Rightarrow 2x.1+2y\frac{dy}{dx}=\frac{1}{xy}\left\{x\frac{dy}{dx}+y\frac{dx}{dx}\right\} \quad \Rightarrow 2x+2y\frac{dy}{dx}=\frac{1}{xy}\left(x\frac{dy}{dx}+y\right)$

$\Rightarrow 2x+2y\frac{dy}{dx}=\frac{1}{xy}.x\frac{dy}{dx}+\frac{1}{xy}.y \quad \Rightarrow 2x+2y\frac{dy}{dx}=\frac{1}{y}\frac{dy}{dx}+\frac{1}{x}$

$$\Rightarrow 2y\frac{dy}{dx}-\frac{1}{y}\frac{dy}{dx}=\frac{1}{x}-2x \quad \Rightarrow \frac{dy}{dx}\left(2y-\frac{1}{y}\right)=\left(\frac{1}{x}-2x\right)$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{2y^2-1}{y}\right)=\left(\frac{1-2x^2}{x}\right) \quad \Rightarrow \frac{dy}{dx}=\frac{y(1-2x^2)}{x(2y^2-1)}$$

10. $x^n+y^n=a^n$

Sol. Differentiating both sides with respect to x, $\frac{d(x^n)}{dx}.\frac{dx}{dx}+\frac{d(y^n)}{dy}.\frac{dy}{dx}=\frac{d(a^n)}{da}.\frac{da}{dx}$

$$\Rightarrow nx^{n-1}+ny^{n-1}\frac{dy}{dx}=na^{n-1}.0 \quad \Rightarrow nx^{n-1}+ny^{n-1}\frac{dy}{dx}=0$$

$$\Rightarrow ny^{n-1}\frac{dy}{dx}=-nx^{n-1} \quad \Rightarrow \frac{dy}{dx}=-\frac{nx^{n-1}}{ny^{n-1}} \quad \therefore \frac{dy}{dx}=-\frac{x^{n-1}}{y^{n-1}}$$

11. $x\sin 2y=y\cos 2x$

Sol. Differentiating both sides with respect to x

$$x\frac{d(\sin 2y)}{d(2y)}\times\frac{d(2y)}{dx}+\sin 2y\frac{d(x)}{dx}=y\frac{d(\cos 2x)}{d(2x)}\times\frac{d(2x)}{dx}+\cos 2x\frac{d(y)}{dx}$$

$$\Rightarrow x\cos 2y.2\frac{dy}{dx}+\sin 2y.1=-y\sin 2x.2+\cos 2x\frac{dy}{dx}$$

$$\Rightarrow 2x\cos 2y\frac{dy}{dx}-\cos 2x\frac{dy}{dx}=-2y\sin 2x-\sin 2y \quad \Rightarrow \frac{dy}{dx}(2x\cos 2y-\cos 2x)=-(2y\sin 2x+\sin 2y)$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(2y\sin 2x+\sin 2y)}{-(\cos 2x-2x\cos 2y)} \quad \therefore \frac{dy}{dx}=\frac{2y\sin 2x+\sin 2y}{\cos 2x-2x\cos 2y}$$

12. $\sin^2 x+2\cos y+xy=0$

Sol. Differentiating both sides with respect to x, $\frac{d(\sin^2 x)}{d(\sin x)}\times\frac{d(\sin x)}{dx}+2\frac{d(\cos y)}{dy}.\frac{dy}{dx}+\frac{d(xy)}{dx}=0$

$$\Rightarrow 2\sin x\cos x+2(-\sin y)\frac{dy}{dx}+\left(x\frac{dy}{dx}+y\frac{dx}{dx}\right)=0$$

$$\Rightarrow 2\sin 2x-2\sin y\frac{dy}{dx}+x\frac{dy}{dx}+y=0 \quad \Rightarrow -2\sin y\frac{dy}{dx}+x\frac{dy}{dx}=-y-\sin 2x$$

$$\Rightarrow \frac{dy}{dx}(-2\sin y+x)=-(y+\sin 2x) \quad \Rightarrow \frac{dy}{dx}=\frac{-(y+\sin 2x)}{-(2\sin y-x)} \quad \therefore \frac{dy}{dx}=\frac{y+\sin 2x}{2\sin y-x}$$

13. $y\sec x+\tan x+x^2y=0$

Sol. Differentiating both sides with respect to x, $y\frac{d(\sec x)}{dx}+\sec x\frac{dy}{dx}+\frac{d(\tan x)}{dx}+\left(x^2\frac{dy}{dx}+y\frac{d(x^2)}{dx}\right)=0$

$$\Rightarrow y\sec x.\tan x+\sec x\frac{dy}{dx}+\sec^2 x+x^2\frac{dy}{dx}+y.2x=0$$

$$\Rightarrow \sec x\frac{dy}{dx}+x^2\frac{dy}{dx}=-2xy-\sec^2 x-y\sec x+\tan x$$

$$\Rightarrow \frac{dy}{dx}(\sec x+x^2)=-(2xy+\sec^2 x+y\sec x\tan x) \quad \therefore \frac{dy}{dx}=-\frac{(2xy+\sec^2 x+y\sec x\tan x)}{\sec x+x^2}$$

14. $\cot(xy)+xy=y$

Sol. Differentiating both sides with respect to x, $\frac{d\{\cot(xy)\}}{d(xy)}\times\frac{d(xy)}{dx}+\frac{d(xy)}{dx}=\frac{d(y)}{dx}$

$$\Rightarrow -\operatorname{cosec}^2(xy)\left\{x\frac{dy}{dx}+y\frac{dx}{dx}\right\}+\left\{x\frac{dy}{dx}+y\frac{dx}{dx}\right\}=\frac{dy}{dx}$$

$$\Rightarrow -\operatorname{cosec}^2(xy)\left\{x\frac{dy}{dx}+y\right\}+\left\{x\frac{dy}{dx}+y\right\}=\frac{dy}{dx}$$

$$\Rightarrow -x\operatorname{cosec}^2(xy)\frac{dy}{dx}-y\operatorname{cosec}^2(xy)+x\frac{dy}{dx}+y=\frac{dy}{dx}$$

$$\Rightarrow -x\operatorname{cosec}^2(xy)\frac{dy}{dx}+x\frac{dy}{dx}-\frac{dy}{dx}=-y+y\operatorname{cosec}^2(xy)$$

$$\Rightarrow \frac{dy}{dx}\left\{-x\operatorname{cosec}^2(xy)+x-1\right\}=y\left\{\operatorname{cosec}^2(xy)-1\right\}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y\cot^2(xy)}{-1-x\left\{\operatorname{cosec}^2(xy)-1\right\}} \quad \Rightarrow \frac{dy}{dx}=\frac{y\cot^2(xy)}{-1-x\cot^2(xy)} \quad \therefore \frac{dy}{dx}=\frac{-y\cot^2(xy)}{1+x\cot^2(xy)}$$

15. $y\tan x-y^2\cos x+2x=0$

Sol. Differentiating both sides with respect to x

$$\left\{y\frac{d(\tan x)}{dx}+\tan x\frac{dy}{dx}\right\}-\left\{y^2\frac{d(\cos x)}{dx}+\cos x\frac{d(y^2)}{dy}.\frac{dy}{dx}\right\}+\frac{d(2x)}{dx}=0$$

$$\Rightarrow y\sec^2 x+\tan x\frac{dy}{dx}+y^2\sin x-\cos x.2y\frac{dy}{dx}+2=0$$

$$\Rightarrow \tan x\frac{dy}{dx}-2y\cos x\frac{dy}{dx}=-2-y^2\sin x-y\sec^2 x$$

$$\Rightarrow \frac{dy}{dx}(\tan x-2y\cos x)=-\left(2+y^2\sin x+y\sec^2 x\right)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{\left(2+y^2\sin x+y\sec^2 x\right)}{-(2y\cos x-\tan x)} \quad \therefore \frac{dy}{dx}=\frac{2+y^2\sin x+y\sec x}{2y\cos x-\tan x}$$

16. $e^x\log y=\sin^{-1}x+\sin^{-1}y$

Sol. Differentiating both sides with respect to x

$$e^x\frac{d(\log y)}{dy}.\frac{dy}{dx}+\log y\frac{d(e^x)}{dx}=\frac{d(\sin^{-1}x)}{dx}+\frac{d(\sin^{-1}y)}{dy}.\frac{dy}{dx}$$

$$\Rightarrow e^x.\frac{1}{y}\frac{dy}{dx}+\log y.e^x=\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} \quad \Rightarrow \frac{e^x}{y}\frac{dy}{dx}-\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\log y.e^x$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{e^x}{y}-\frac{1}{\sqrt{1-y^2}}\right)=\frac{1}{\sqrt{1-x^2}}-\log y.e^x \quad \Rightarrow \frac{dy}{dx}\left(\frac{e^x\sqrt{1-y^2}-y}{y\sqrt{1-y^2}}\right)=\left(\frac{1-\log y.e^x\sqrt{1-x^2}}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{y\sqrt{1-y^2}\left(1-\log y.e^x.\sqrt{1-x^2}\right)}{\sqrt{1-x^2}\left(e^x\sqrt{1-y^2}-y\right)} \quad \therefore \frac{dy}{dx}=y\sqrt{\frac{1-y^2}{1-x^2}}\left\{\frac{1-\log y.e^x.\sqrt{1-x^2}}{e^x\sqrt{1-y^2}-y}\right\}$$

17. $xy\log(x+y)=1$

Sol. $xy\log(x+y)=1 \qquad \ldots(1)$

Differentiating both sides with respect to x

$$xy\frac{d\{\log(x+y)\}}{d(x+y)}\times\frac{d(x+y)}{dx}+\log(x+y)\frac{d(xy)}{dx}=\frac{d(1)}{dx}$$

$$\Rightarrow xy.\frac{1}{x+y}\left(\frac{dx}{dx}+\frac{dy}{dx}\right)+\log(x+y)\left\{x\frac{dy}{dx}+y\frac{dx}{dx}\right\}=0$$

$$\Rightarrow \frac{xy}{x+y}\left(1+\frac{dy}{dx}\right)+\log(x+y)\left(x\frac{dy}{dx}+y\right)=0$$

$$\Rightarrow \frac{xy}{x+y}+\frac{xy}{x+y}\frac{dy}{dx}+x\log(x+y)\frac{dy}{dx}+y\log(x+y)=0$$

$$\Rightarrow \frac{xy}{x+y}\frac{dy}{dx}+x\log(x+y)\frac{dy}{dx}=-y\log(x+y)-\frac{xy}{x+y}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{xy}{x+y}+x\log(x+y)\right)=-\left(y\log(x+y)+\frac{xy}{x+y}\right)$$

$$\Rightarrow \frac{dy}{dx}\left\{\frac{xy+x(x+y)\log(x+y)}{x+y}\right\}=-\left\{\frac{y(x+y)\log(x+y)+xy}{(x+y)}\right\}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{y\{(x+y)\log(x+y)+x\}}{x\{y+(x+y)\log(x+y)\}} \qquad \ldots(\text{ii})$$

From equation (i), $xy\log(x+y)=1 \;\Rightarrow \log(x+y)=\frac{1}{xy}$

Putting the value of $\log(x+y)$ in equation (ii),

$$\frac{dy}{dx}=-\frac{y\left\{(x+y).\frac{1}{xy}+x\right\}}{x\{y+(x+y)\log(x+y)\}} \quad \Rightarrow \frac{dy}{dx}=-\frac{y\left\{\frac{x+y+x^2y}{xy}\right\}}{x\{y+(x+y)\log(x+y)\}}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{(x+y+x^2y)}{x^2\{y+(x+y)\log(x+y)\}}$$

18. $\tan(x+y)+\tan(x-y)=1$

Sol. Differentiating both sides with respect to x

$$\frac{d\{\tan(x+y)\}}{d(x+y)}\times\frac{d(x+y)}{dx}+\frac{d\{\tan(x-y)\}}{d(x-y)}\times\frac{d(x-y)}{dx}=\frac{d(1)}{dx}$$

$$\Rightarrow \sec^2(x+y)\left\{\frac{dx}{dx}+\frac{dy}{dx}\right\}+\sec^2(x-y)\left\{\frac{dx}{dx}-\frac{dy}{dx}\right\}=0$$

$$\Rightarrow \sec^2(x+y)\left\{1+\frac{dy}{dx}\right\}+\sec^2(x-y)\left\{1-\frac{dy}{dx}\right\}=0$$

$$\Rightarrow \sec^2(x+y)+\sec^2(x+y)\frac{dy}{dx}+\sec^2(x-y)-\sec^2(x-y)\frac{dy}{dx}=0$$

$$\Rightarrow \sec^2(x+y)\frac{dy}{dx}-\sec^2(x-y)\frac{dy}{dx}=-\sec^2(x-y)-\sec^2(x+y)$$

$$\Rightarrow \frac{dy}{dx}\left\{\sec^2(x+y)-\sec^2(x-y)\right\}=-\left\{\sec^2(x-y)+\sec^2(x+y)\right\}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-\left\{\sec^2(x-y)+\sec^2(x+y)\right\}}{-\left\{\sec^2(x-y)-\sec^2(x+y)\right\}} \quad \therefore \frac{dy}{dx}=\frac{\sec^2(x+y)+\sec^2(x-y)}{\sec^2(x-y)-\sec^2(x+y)}$$

19. $\log\sqrt{x^2+y^2}=\tan^{-1}\left(\frac{y}{x}\right)$

Sol. Given, $\log\sqrt{x^2+y^2}=\tan^{-1}\left(\frac{y}{x}\right) \quad \Rightarrow \frac{1}{2}\log\left(x^2+y^2\right)=\tan^{-1}\frac{y}{x}$

Differentiating both sides with respect to x

$$\frac{1}{2\left(x^2+y^2\right)}\cdot\left\{\frac{d\left(x^2\right)}{dx}+\frac{d\left(y^2\right)}{dy}\frac{dy}{dx}\right\}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left\{\frac{x\frac{dy}{dx}-y\frac{dx}{dx}}{(x)^2}\right\}$$

$$\Rightarrow \frac{1}{2\left(x^2+y^2\right)}\cdot\left(2x+2y.\frac{dy}{dx}\right)=\frac{x^2}{x^2+y^2}\cdot\frac{\left(x\frac{dy}{dx}-y\right)}{x^2} \quad \Rightarrow \frac{2\left(x+y\frac{dy}{dx}\right)}{2\left(x^2+y^2\right)}=\frac{\left(x\frac{dy}{dx}-y\right)}{x^2+y^2}$$

$$\Rightarrow x+y\frac{dy}{dx}=x\frac{dy}{dx}-y \quad \Rightarrow x+y=x\frac{dy}{dx}-y\frac{dy}{dx} \quad \Rightarrow x+y=\frac{dy}{dx}(x-y) \quad \therefore \frac{dy}{dx}=\frac{x+y}{x-y}$$

20. If $y=x\sin y$, prove that $x.\frac{dy}{dx}=\frac{y}{1-x\cos y}$

Sol. $y=x\sin y \qquad \ldots$(i)

Differentiating both sides of (i) with respect to x, $\frac{dy}{dx}=x\frac{d(\sin y)}{dx}.\frac{dy}{dx}+\sin y\frac{d(x)}{dx}$

$$\Rightarrow \frac{dy}{dx}=x\cos y\frac{dy}{dx}+\sin y.1 \quad \Rightarrow \frac{dy}{dx}-x\cos y\frac{dy}{dx}=\sin y$$

$$\Rightarrow \frac{dy}{dx}(1-x\cos y)=\sin y \quad \Rightarrow \frac{dy}{dx}=\frac{\sin y}{1-x\cos y} \quad \ldots\text{(ii)}$$

From equation (i), $y=x\sin y \quad \therefore \sin y=\frac{y}{x}$

Putting the value of $\sin y$ inequation (ii), $\frac{dy}{dx}=\frac{y}{x(1-x\cos y)} \quad \therefore x\frac{dy}{dx}=\frac{y}{1-x\cos y}$

21. If $xy=\tan(xy)$ show that $\frac{dy}{dx}=-\frac{y}{x}$

Sol. $xy=\tan(xy) \qquad \ldots$ (i)

Differentiating both sides of (i) with respect to x, $x\frac{dy}{dx}+y\frac{dx}{dx}=\frac{d\{\tan(xy)\}}{d(xy)}\times\frac{d(xy)}{dx}$

$$\Rightarrow x\frac{dy}{dx}+y=\sec^2(xy).\left\{x\frac{dy}{dx}+y\frac{dx}{dx}\right\} \quad \Rightarrow x\frac{dy}{dx}+y=\sec^2(xy)\left\{x\frac{dy}{dx}+y\right\}$$

$$\Rightarrow x\frac{dy}{dx}+y=x\sec^2(xy)\frac{dy}{dx}+y\sec^2(xy) \quad \Rightarrow y-y\sec^2(xy)=x\sec^2(xy)\frac{dy}{dx}-x\frac{dy}{dx}$$

$$\Rightarrow y\{1-\sec^2(xy)\} = x\frac{dy}{dx}\{\sec^2(xy)-1\} \quad \Rightarrow \frac{dy}{dx} = -\frac{y\{\sec^2(xy)-1\}}{x\{\sec^2(xy)-1\}} \therefore \frac{dy}{dx} = \frac{-y}{x}$$

22. If $y\log x = x - y$ prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

Sol. $y\log x = x - y$ ...(i)

Differentiating both sides of (i) with respect to x

$$y\frac{d(\log x)}{dx} + \log x\frac{dy}{dx} = \frac{d(x)}{dx} - \frac{d(y)}{dx} \quad \Rightarrow y.\frac{1}{x} + \log x\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \log x\frac{dy}{dx} + \frac{dy}{dx} = 1 - \frac{y}{x} \quad \Rightarrow \frac{dy}{dx}(\log x + 1) = \frac{x-y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x-y}{x(1+\log x)} \quad \therefore \frac{dy}{dx} = \frac{y\log x}{x(1+\log x)} \quad \text{...(ii)}$$

From equation (i), $y\log x = x - y \Rightarrow y\log x + y = x \Rightarrow y(\log x + 1) = x \Rightarrow \frac{y}{x} = \frac{1}{(1+\log x)}$

Putting the value of $\frac{y}{x}$ in equation (ii), $\frac{dy}{dx} = \frac{\log x}{(1+\log x)(1+\log x)} \therefore \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

23. If $\cos y = x\cos(y+a)$ prove that $\frac{dy}{dx} = \frac{\cos^2(y+a)}{\sin a}$

Sol. $\cos y = x\cos(y+a)$ ...(i)

Differentiating both sides with respect to x, $\frac{\cos y}{\cos(y+a)} = x$

$$\Rightarrow \frac{\cos(y+a).\frac{d\cos y}{dx} - \cos y.\frac{d\cos(y+a)}{dx}}{\cos^2(y+a)} = 1$$

$$\Rightarrow \frac{\cos(y+a)(-\sin y)\frac{dy}{dx} + \cos y.\sin(a+y)\frac{dy}{dx}}{\cos^2(y+a)} = 1$$

$$\Rightarrow \frac{dy}{dx}(\sin(a+y)\cos y - \sin y\cos(y+a)) = \cos^2(y+a)$$

$$\Rightarrow \frac{dy}{dx}(\sin(y+a-y)) = \cos^2(a+y) \quad \therefore \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

Putting the value of x in equation (ii),

$$\frac{dy}{dx} = \frac{\cos(y+a)}{-\sin y + \frac{\cos y}{\cos(y+a)}.\sin(y+a)} \quad \Rightarrow \frac{dy}{dx} = \frac{\cos^2(y+a)}{-\sin y\cos(y+a) + \cos y.\sin(y+a)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(y+a)}{\sin(y+a-y)} \quad \therefore \frac{dy}{dx} = \frac{\cos^2(y+a)}{\sin a} \quad [\therefore \sin(A-B) = \sin A\cos B - \cos A\sin B]$$

24. If $\cos^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right)=\tan^{-1}(a)$ prove that $\frac{dy}{dx}=\frac{y}{x}$.

Sol. $\cos^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right)=\tan^{-1}(a) \Rightarrow \frac{x^2-y^2}{x^2+y^2}=\cos\left\{\tan^{-1}(a)\right\}$

Differentiating both sides with respect to x

$$\frac{\left(x^2+y^2\right)\frac{d\left(x^2-y^2\right)}{dx}-\left(x^2-y^2\right)\frac{d\left(x^2+y^2\right)}{dx}}{\left(x^2+y^2\right)^2}=\frac{d\left\{\cos\left(\tan^{-1}a\right)\right\}}{dx}$$

$$\Rightarrow \frac{\left(x^2+y^2\right)\left(2x-2y\frac{dy}{dx}\right)-\left(x^2-y^2\right)\left(2x+2y\frac{dy}{dx}\right)}{\left(x^2+y^2\right)^2}=0$$

$$\Rightarrow \frac{2\left(x^2+y^2\right)\left(x-y\frac{dy}{dx}\right)-2\left(x^2-y^2\right)\left(x+y\frac{dy}{dx}\right)}{\left(x^2+y^2\right)^2}=0$$

$$\Rightarrow 2\left(x^2+y^2\right)\left(x-y\frac{dy}{dx}\right)-2\left(x^2-y^2\right)\left(x+y\frac{dy}{dx}\right)=0$$

$$\Rightarrow \left(x^2+y^2\right)\left(x-y\frac{dy}{dx}\right)=\left(x^2-y^2\right)\left(x+y\frac{dy}{dx}\right)$$

$$\Rightarrow x^3-x^2y\frac{dy}{dx}+y^2x-y^3\frac{dy}{dx}=x^3+x^2y\frac{dy}{dx}-y^2x-y^3\frac{dy}{dx}$$

$$\Rightarrow -x^2y\frac{dy}{dx}-x^2y\frac{dy}{dx}=-y^2x-y^2x \quad \Rightarrow -2x^2y\frac{dy}{dx}=-2y^2x \quad \Rightarrow \frac{dy}{dx}=\frac{-2y^2x}{-2x^2y} \quad \therefore \frac{dy}{dx}=\frac{y}{x}$$

EXERCISE 10F (Pg.no.: 425)

Find $\frac{dy}{dx}$, when:

1. $y = x^{\frac{1}{x}}$

Sol. $y = x^{\frac{1}{x}}$, Taking log both sides we get, $\log y = \log\left(x^{\frac{1}{x}}\right) \Rightarrow \log y = \frac{1}{x}\log(x)$

Differentiating both sides with respect to x, $\frac{d(\log y)}{dy}.\frac{dy}{dx} = \frac{1}{x}.\frac{d(\log x)}{dx} + \log x\frac{d\left(\frac{1}{x}\right)}{dx}$

$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \frac{1}{x}.\frac{1}{x} + \log x\left(-\frac{1}{x^2}\right) \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2} - \frac{\log x}{x^2} \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1-\log x}{x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x^2}(1-\log x) \quad \therefore \frac{dy}{dx} = \frac{x^{\frac{1}{x}}(1-\log x)}{x^2}$

2. $y = x^{\sqrt{x}}$

Sol. $y = x^{\sqrt{x}}$, Taking log both sides we get , $\log y = \log\left(x^{\sqrt{x}}\right) \Rightarrow \log y = \sqrt{x}\log x$

Differentiating both sides with respect to x

$\frac{d(\log y)}{dy}.\frac{dy}{dx} = \sqrt{x}\frac{d(\log x)}{dx} + \log x.\frac{d\left(\sqrt{x}\right)}{dx} \Rightarrow \frac{1}{y}.\frac{dy}{dx} = \sqrt{x}.\frac{1}{x} + \log x.\frac{1}{2\sqrt{x}}$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{\sqrt{x}} + \frac{\log x}{2\sqrt{x}} \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{2+\log x}{2\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{y(2+\log x)}{2\sqrt{x}} \quad \therefore \frac{dy}{dx} = \frac{x^{\sqrt{x}}(2+\log x)}{2\sqrt{x}}$

3. $y = (\log x)^x$

Sol. $y = (\log x)^x$, Taking log both sides we get , $\log y = \log\left\{(\log x)^x\right\} \Rightarrow \log y = x\log(\log x)$

Differentiating both sides with respect to x

$\frac{d(\log y)}{dy}.\frac{dy}{dx} = x\frac{d\{\log(\log x)\}}{d(\log x)} \times \frac{d(\log x)}{dx} + \log(\log x).\frac{d(x)}{dx}$

$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = x.\frac{1}{\log x}.\frac{1}{x} + \log(\log x).1 \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{\log x} + \log(\log x) \Rightarrow \frac{dy}{dx} = y\left\{\frac{1}{\log x} + \log(\log x)\right\}$

$\therefore \frac{dy}{dx} = (\log x)^x\left\{\frac{1}{\log x} + \log(\log x)\right\}$

4. $y = x^{\sin x}$

Sol. $y = x^{\sin x}$, Taking log both sides we get, $\log y = \log\left(x^{\sin x}\right) \Rightarrow \log y = \sin x.\log x$

Differentiating both sides with respect to x

$\frac{d(\log y)}{dy}.\frac{dy}{dx} = \sin x.\frac{d(\log x)}{dx} + \log x\frac{d(\sin x)}{dx} \Rightarrow \frac{1}{y}\frac{dy}{dx} = \sin x.\frac{1}{x} + \log x.\cos x$

$\Rightarrow \frac{dy}{dx} = y\left(\frac{\sin x}{x} + \log x.\cos x\right) \quad \therefore \frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \log x.\cos x\right)$

5. $y = x^{\cos^{-1}(x)}$

Sol. $y = x^{\cos^{-1}(x)}$, Taking log both sides we get, $\log y = \log\left(x^{\cos^{-1}x}\right) \Rightarrow \log y = \cos^{-1} x.\log x$

Differentiating both sides with respect to x

$$\frac{d(\log y)}{dy}.\frac{dy}{dx} = \cos^{-1} x.\frac{d(\log x)}{dx} + \log x.\frac{d(\cos^{-1} x)}{dx} \quad \Rightarrow \frac{1}{y}.\frac{dy}{dx} = \cos^{-1} x.\frac{1}{x} + \log x\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{\cos^{-1} x}{x} - \frac{\log x}{\sqrt{1-x^2}}\right) \quad \therefore \frac{dy}{dx} = x^{\cos^{-1}x}\left(\frac{\cos^{-1} x}{x} - \frac{\log x}{\sqrt{1-x^2}}\right)$$

6. $y = (\tan x)^{\frac{1}{x}}$

Sol. $y = (\tan x)^{\frac{1}{x}}$, Taking log both sides we get, $\log y = \log\left\{(\tan x)^{\frac{1}{x}}\right\} \Rightarrow \log y = \frac{1}{x}\log(\tan x)$

Differentiating both sides with respect to x

$$\frac{d(\log y)}{dy}.\frac{dy}{dx} = \frac{1}{x}.\frac{d\{\log(\tan x)\}}{d(\tan x)} \times \frac{d(\tan x)}{dx} + \log(\tan x)\frac{d\left(\frac{1}{x}\right)}{dx}$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \frac{1}{x}.\frac{1}{\tan x}.\sec^2 x + \log(\tan x).\left(-\frac{1}{x^2}\right) \quad \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}.\frac{2}{2\sin x.\cos x} - \frac{\log(\tan x)}{x^2}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{2\operatorname{cosec}2x}{x} - \frac{\log(\tan x)}{x^2} \quad \Rightarrow \frac{dy}{dx} = y\left\{\frac{2x(\operatorname{cosec} 2x) - \log(\tan x)}{x^2}\right\}$$

$$\therefore \frac{dy}{dx} = (\tan x)^{\frac{1}{x}}\left\{\frac{2x\operatorname{cosec}2x - \log(\tan x)}{x^2}\right\}$$

7. $y = (\sin x)^{\cos x}$

Sol. $y = (\sin x)^{\cos x}$, Taking log both sides we get, $\log y = \log\left\{(\sin x)^{\cos x}\right\} \Rightarrow \log y = \cos x.\log(\sin x)$

Differentiating both sides with respect to x

$$\frac{d\log(y)}{dy}.\frac{dy}{dx} = \cos x.\frac{d\{\log(\sin x)\}}{d(\sin x)} \times \frac{d(\sin x)}{dx} + \log(\sin x)\frac{d(\cos x)}{dx}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos x.\frac{1}{\sin x}.\cos x + \log(\sin x).(-\sin x) \quad \Rightarrow \frac{dy}{dx} = y(\cos x.\cot x - \log(\sin x).\sin x)$$

$$\therefore \frac{dy}{dx} = (\sin x)^{\cos x}(\cot x\cos x - \sin x.\log(\sin x))$$

8. $y = (\log x)^{\sin x}$

Sol. $y = (\log x)^{\sin x}$, Taking log both sides we get, $\log y = \log\left\{(\log x)^{\sin x}\right\} \Rightarrow \log y = \sin x.\log(\log x)$

Differentiating both sides with respect to x

$$\frac{d(\log y)}{dy}.\frac{dy}{dx} = \sin x.\frac{d\{\log(\log x)\}}{d(\log x)} \times \frac{d(\log x)}{dx} + \log(\log x)\frac{d(\sin x)}{dx}$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \sin x.\frac{1}{\log x}.\frac{1}{x} + \log(\log x).\cos x \quad \Rightarrow \frac{dy}{dx} = y\left\{\frac{\sin x}{x\log x} + \cos x.\log(\log x)\right\}$$

$$\therefore \frac{dy}{dx} = (\log x)^{\sin x}\left\{\frac{\sin x}{x\log x} + \cos x.\log(\log x)\right\}$$

9. $y = (\cos x)^{\log x}$

Sol. $y = (\cos x)^{\log x}$, Taking log both sides we get, $\log y = \log\left\{(\cos x)^{\log x}\right\} \Rightarrow \log y = \log x.\log(\cos x)$

On differentiating both sides with respect x

$$\frac{d(\log y)}{dy}.\frac{dy}{dx} = \log x.\frac{d\{\log(\cos x)\}}{d(\cos x)} \times \frac{d(\cos x)}{dx} + \log(\cos x)\frac{d(\log x)}{dx}$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \log x.\frac{1}{\cos x}(-\sin x) + \log(\cos x).\frac{1}{x} \quad \Rightarrow \frac{dy}{dx} = y\left(-\log x.\tan x + \frac{\log(\cos x)}{x}\right)$$

$$\therefore \frac{dy}{dx} = (\cos x)^{\log x}\left\{\frac{\log(\cos x)}{x} - \log x.\tan x\right\}$$

10. $y = (\tan x)^{\sin x}$

Sol. $y = (\tan x)^{\sin x}$, Taking log both sides we get, $\log y = \log\left\{(\tan x)^{\sin x}\right\} \Rightarrow \log y = \sin x.\log(\tan x)$

On differentiating both sides with respect x

$$\frac{d(\log y)}{dy}.\frac{dy}{dx} = \sin x.\frac{d\{\log(\tan x)\}}{d(\tan x)} \times \frac{d(\tan x)}{dx} + \log(\tan x).\frac{d(\sin x)}{dx}$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \sin x.\frac{1}{\tan x}.\sec^2 x + \log(\tan x).\cos x$$

$$\Rightarrow \frac{dy}{dx} = y\{\sec x + \cos x.\log(\tan x)\} \quad \therefore \frac{dy}{dx} = (\tan x)^{\sin x}\{\cos x.\log(\tan x) + \sec x\}$$

11. $y = (\cos x)^{\cos x}$

Sol. $y = (\cos x)^{\cos x}$, Taking log both sides we get, $\log y = \log\left\{(\cos x)^{\cos x}\right\} \Rightarrow \log y = \cos x.\log(\cos x)$

On differentiating both sides with respect to x

$$\frac{d(\log y)}{dy}.\frac{dy}{dx} = \cos x.\frac{d(\log(\cos x))}{d(\cos x)} \times \frac{d(\cos x)}{dx} + \log(\cos x).\frac{d(\cos x)}{dx}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos x.\frac{1}{\cos x}.(-\sin x) + \log(\cos x)(-\sin x) \quad \Rightarrow \frac{dy}{dx} = -y(\sin x + \sin x.\log(\cos x))$$

$$\therefore \frac{dy}{dx} = -(\cos x)^{\cos x}.\sin x(1 + \log\cos x)$$

12. $y = (\tan x)^{\cot x}$

Sol. $y = (\tan x)^{\cot x}$, Taking log both sides we get, $\log\ y = \log\left\{\tan x^{\cot x}\right\} \Rightarrow \log y = \cot x.\log(\tan x)$

On differentiating both sides with respect to x

$$\frac{d(\log y)}{dy} \times \frac{dy}{dx} = \cot x.\frac{d\{\log(\tan x)\}}{d(\tan x)} \times \frac{d(\tan x)}{dx} + \log(\tan x).\frac{d(\cot x)}{dx}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cot x.\frac{1}{\tan x}.\sec^2 x + \log(\tan x)\left(-\cos ec^2 x\right) \quad \Rightarrow \frac{dy}{dx} = y\left\{\cos ec^2 x - \log(\tan x)\cos ec^2 x\right\}$$

$$\Rightarrow \frac{dy}{dx}=(\tan x)^{\cot x}.\operatorname{cosec}^2 x\{1-\log(\tan x)\}$$

13. $y=x^{\sin 2x}$

Sol. $y=x^{\sin 2x}$, Taking long both sides we get, $\log y=\log\left(x^{\sin 2x}\right) \Rightarrow \log y=\sin 2x.\log(x)$

On differentiating both sides with respect to x

$$\frac{d(\log y)}{dy}.\frac{dy}{dx}=\sin 2x.\frac{d(\log x)}{dx}+\log x.\frac{d(\sin 2x)}{d(2x)}\times\frac{d(2x)}{dx}$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\sin 2x.\frac{1}{x}+\log x.\cos 2x.2 \quad \Rightarrow \frac{dy}{dx}=y\left(\frac{\sin 2x}{x}+\log x.\cos 2x.2\right)$$

$$\therefore \frac{dy}{dx}=x^{\sin 2x}\left\{\frac{\sin 2x}{2}+(2\cos 2x)\log x\right\}$$

14. $y=\left(\sin^{-1}x\right)^x$

Sol. $y=\left(\sin^{-1}x\right)^x$, Taking log both sides we get, $\log y=\log\left\{\left(\sin^{-1}x\right)^x\right\} \Rightarrow \log y=x\log\left(\sin^{-1}x\right)$

On differentiating both sides with respect to x

$$\frac{d(\log y)}{dy}.\frac{dy}{dx}=\frac{x\cdot d\left\{\log\left(\sin^{-1}x\right)\right\}}{d\left(\sin^{-1}x\right)}\times\frac{d\left(\sin^{-1}x\right)}{dx}+\log\left(\sin^{-1}x\right)\frac{d(x)}{dx}$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=x.\frac{1}{\sin^{-1}x}.\frac{1}{\sqrt{1-x^2}}+\log\left(\sin^{-1}x\right).1 \quad \Rightarrow \frac{dy}{dx}=y\left\{\frac{x}{\sin^{-1}x.\sqrt{1-x^2}}+\log\left(\sin^{-1}x\right)\right\}$$

$$\therefore \frac{dy}{dx}=\left(\sin^{-1}x\right)^x\left\{\log\left(\sin^{-1}x\right)+\frac{x}{\sin^{-1}x.\sqrt{1-x^2}}\right\}$$

15. $y=\sin\left(x^x\right)$

Sol. $y=\sin\left(x^x\right)$, Let $x^x=t \Rightarrow y=\sin(t)$ ...(i)

On differentiating both sides of (i) with respect to x

$$\frac{dy}{dx}=\frac{d(\sin t)}{dt}.\frac{d(t)}{dx} \quad ...(ii)$$

$\Rightarrow t=x^x$, taking log both sides we get, $\log t=\log\left(x^x\right) \Rightarrow \log t=x\log x$ ...(iii)

On differentiating both sides of (iii) with respect to x

$$\frac{d(\log t)}{dt}\frac{dt}{dx}=x\frac{d(\log x)}{dx}+\log x\frac{d(x)}{dx} \quad \Rightarrow \frac{1}{t}.\frac{dt}{dx}=x.\frac{1}{x}+\log x.1$$

$$\Rightarrow \frac{1}{t}.\frac{dt}{dx}=1+\log x \quad \Rightarrow \frac{dt}{dx}=t(1+\log x) \quad \Rightarrow \frac{dt}{dx}=x^x(1+\log x)$$

Putting the value of $\frac{dt}{dx}$ in equation (ii), $\frac{dy}{dx}=\cos\left(x^x\right).x^x(1+\log x)$

16. $y=(3x+5)^{2x-3}$

Sol. $y=(3x+5)^{2x-3}$

Taking log both sides we get, $\log y=\log\left\{(3x+5)^{(2x-3)}\right\} \Rightarrow \log y=(2x-3)\log(3x+5)$

On differentiating both sides with respect to x

$$\frac{1}{y}.\frac{dy}{dx}=(2x-3)\frac{d\{\log(3x+5)\}}{d(3x+5)}\times\frac{d(3x+5)}{dx}+\log(3x+5)\frac{d(2x-3)}{dx}$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=(2x-3).\frac{1}{3x+5}.3+\log(3x+5).2 \quad \Rightarrow \frac{dy}{dx}=y\left\{\frac{3(2x-3)}{3x+5}+2\log(3x+5)\right\}$$

$$\therefore \frac{dy}{dx}=(3x+5)^{2x-3}\left\{\frac{3(2x-3)}{3x+5}+2\log(3x+5)\right\}$$

17. $y=(x+1)^3(x+2)^4(x+3)^5$

Sol. $y=(x+1)^3(x+2)^4(x+3)^5$, Taking long both sides we get, $\log y=\log\left\{(x+1)^3(x+2)^4(x+3)^5\right\}$

$$\Rightarrow \log y=\log\left\{(x+1)^3\right\}+\log\left\{(x+2)^4\right\}+\log\left\{(x+3)^5\right\}$$

$$\Rightarrow \log y=3\log(x+1)+4\log(x+2)+5\log(x+3) \quad \ldots(i)$$

On differentiating both sides of (i) with respect to x

$$\frac{1}{y}.\frac{dy}{dx}=3.\frac{1}{x+1}.1+4.\frac{1}{x+2}.1+5.\frac{1}{x+3}.1 \quad \Rightarrow \frac{1}{y}.\frac{dy}{dx}=\frac{3}{x+1}+\frac{4}{x+2}+\frac{5}{x+3}$$

$$\Rightarrow \frac{dy}{dx}=y\left(\frac{3}{x+1}+\frac{4}{x+2}+\frac{5}{x+3}\right) \quad \therefore \frac{dy}{dx}=(x+1)^3(x+2)^4(x+3)^5\left(\frac{3}{x+1}+\frac{4}{x+2}+\frac{5}{x+3}\right)$$

18. $y=\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Sol. $y=\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} \quad \Rightarrow y=\left\{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right\}^{\frac{1}{2}}$

Taking long both sides we get,

$$\log y=\log\left\{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right\}^{\frac{1}{2}} \quad \Rightarrow \log y=\frac{1}{2}\log\left\{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right\}$$

$$\Rightarrow \log y=\frac{1}{2}\left[\log\{(x-1)(x-2)\}-\log\{(x-3)(x-4)(x-5)\}\right]$$

$$\Rightarrow \log y=\frac{1}{2}\left[\log(x-1)+\log(x-2)-\log(x-3)-\log(x-4)-\log(x-5)\right] \quad \ldots(i)$$

On differentiating both sides of (i) with respect to x

$$\frac{1}{y}.\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right] \quad \Rightarrow \frac{dy}{dx}=\frac{y}{2}\left\{\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right\}$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$$

19. $y=(2-x)^3(3+2x)^5$

Sol. $y=(2-x)^3(3+2x)^5$, Taking log both sides we get, $\log y=\log\left\{(2-x)^3(3+2x)^5\right\}$

$$\Rightarrow \log y=\log\left\{(2-x)^3\right\}+\log\left\{(3+2x)^5\right\} \quad \Rightarrow \log y = 3\log(2-x)+5\log(3+2x) \qquad \ldots(i)$$

On differentiating both sides of (i) with respect to x

$$\frac{d(\log y)}{dy}.\frac{dy}{dx}=3\frac{d\{\log(2-x)\}}{d(2-x)}\times\frac{d(2-x)}{dx}+5\frac{d\{\log(3+2x)\}}{d(3+2x)}\times\frac{d(3+2x)}{dx}$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=3.\frac{1}{2-x}.(-1)+5.\frac{1}{3+2x}.(2) \quad \Rightarrow \frac{dy}{dx}=y\left(\frac{-3}{2-x}+\frac{10}{3+2x}\right)$$

$$\therefore \frac{dy}{dx}=(2-x)^3(3+2x)^5\left(\frac{10}{3+2x}-\frac{3}{2-x}\right)$$

20. $y=\cos x.\cos 2x.\cos 3x$

Sol. $y=\cos x.\cos 2x.\cos 3x$, Taking log both sides we get, $\log y=\log(\cos x.\cos 2x.\cos 3x)$

$$\Rightarrow \log y=\log(\cos x)+\log(\cos 2x)+\log(\cos 3x) \quad \ldots(i)$$

On differentiating both sides of (i) with respect to x

$$\frac{1}{y}.\frac{dy}{dx}=\left\{\frac{1}{\cos x}(-\sin x)+\frac{1}{\cos 2x}(-\sin 2x).2+\frac{1}{\cos 3x}(-\sin 3x).3\right\}$$

$$\Rightarrow \frac{dy}{dx}=y(-\tan x-2\tan 2x-3\tan 3x) \quad \therefore \frac{dy}{dx}=-\cos x\cos 2x\cos 3x(\tan x+2\tan 2x+3\tan 3x)$$

21. $y=\dfrac{x^5\sqrt{x+4}}{(2x+3)^2}$

Sol. $y=\dfrac{x^5\sqrt{x+4}}{(2x+3)^2}$, Taking log both sides we get, $\log y=\log\left\{\dfrac{x^5(x+4)^{\frac{1}{2}}}{(2x+3)^2}\right\}$

$$\Rightarrow \log y=\log\left\{x^5.(x+4)^{\frac{1}{2}}\right\}-\log\left\{(2x+3)^2\right\} \quad \Rightarrow \log y=\log(x^5)+\log\left\{(x+4)^{\frac{1}{2}}\right\}-\log\left\{(2x+3)^2\right\}$$

$$\Rightarrow \log y=5\log x+\frac{1}{2}\log(x+4)-2\log(2x+3)$$

On differentiating both sides with respect to x, $\dfrac{1}{y}.\dfrac{dy}{dx}=5.\dfrac{1}{x}+\dfrac{1}{2}.\dfrac{1}{x+4}-2.\dfrac{1}{2x+3}.2$

$$\Rightarrow \frac{dy}{dx}=y\left(\frac{5}{x}+\frac{1}{2(x+4)}-\frac{4}{2x+3}\right) \quad \therefore \frac{dy}{dx}=\frac{x^5\sqrt{(x+4)}}{(2x+3)^2}\left(\frac{5}{x}+\frac{1}{2(x+4)}-\frac{4}{2x+3}\right)$$

22. $y=\dfrac{(x+1)^2\sqrt{x-1}}{(x+4)^3e^x}$

Sol. $y=\dfrac{(x+1)^2\sqrt{x-1}}{(x+4)^3e^x}$, Taking log both sides we get, $\log y=\log\left\{\dfrac{(x+1)^2(x-1)^{\frac{1}{2}}}{(x+4)^3e^x}\right\}$

$$\Rightarrow \log y=\log\left\{(x+1)^2(x-1)^{\frac{1}{2}}\right\}-\log\left\{(x+4)^3.e^x\right\}$$

$$\Rightarrow \log y=\log\left\{(x+1)^2\right\}+\log\left\{(x-1)^{\frac{1}{2}}\right\}-\log\left\{(x+4)^3\right\}-\log e^x$$

$$\Rightarrow \log y = 2\log(x+1)+\frac{1}{2}\log(x-1)-3\log(x+4)-x\log e$$

$$\Rightarrow \log y = 2\log(x+1)+\frac{1}{2}\log(x-1)-3\log(x+4)-x \qquad \ldots(i)$$

On differentiating both sides of (i) with respect to x

$$\frac{1}{y}.\frac{dy}{dx}=2.\frac{1}{x+1}+\frac{1}{2}.\frac{1}{x-1}-3.\frac{1}{x+4}-1 \quad \Rightarrow \frac{dy}{dx}=y\left(\frac{2}{x+1}+\frac{1}{2(x-1)}-\frac{3}{x+4}-1\right)$$

$$\therefore \frac{dy}{dx}=\frac{(x+1)^2\sqrt{x-1}}{(x+4)^3e^x}\left(\frac{2}{x+1}+\frac{1}{2(x-1)}-\frac{3}{x+4}-1\right)$$

23. $y=\dfrac{\sqrt{x}(3x+5)^2}{\sqrt{x+1}}$

Sol. $y=\dfrac{\sqrt{x}(3x+5)^2}{\sqrt{x+1}}$, Taking log both sides we get, $\log y=\log\left(\dfrac{x^{\frac{1}{2}}(3x+5)^2}{(x+1)^{\frac{1}{2}}}\right)$

$$\Rightarrow \log y=\log\left\{x^{\frac{1}{2}}(3x+5)^2\right\}-\log\left\{(x+1)^{\frac{1}{2}}\right\} \quad \Rightarrow \log y=\log\left(x^{\frac{1}{2}}\right)+\log\left\{(3x+5)^2\right\}-\log\left\{(x+1)^{\frac{1}{2}}\right\}$$

$$\Rightarrow \log y=\frac{1}{2}\log x+2\log(3x+5)-\frac{1}{2}\log(x+1) \qquad \ldots(i)$$

Differentiating both sides of (i) with respect to x ,

$$\frac{1}{y}.\frac{dy}{dx}=\frac{1}{2}.\frac{1}{x}+2.\frac{1}{3x+5}.3-\frac{1}{2}.\frac{1}{x+1}.1 \quad \Rightarrow \frac{dy}{dx}=y\left(\frac{1}{2x}+\frac{6}{3x+5}-\frac{1}{2(x+1)}\right)$$

$$\therefore \frac{dy}{dx}=\frac{\sqrt{x}(3x+5)^2}{\sqrt{x+1}}\left(\frac{1}{2x}+\frac{6}{3x+5}-\frac{1}{2(x+1)}\right)$$

24. $y=\dfrac{x^2\sqrt{1+x}}{(1+x^2)^{\frac{3}{2}}}$

Sol. $y=\dfrac{x^2\sqrt{1+x}}{(1+x^2)^{\frac{3}{2}}}$, Taking log both sides we get, $\log y=\log\left(\dfrac{x^2(x+1)^{\frac{1}{2}}}{(1+x^2)^{\frac{3}{2}}}\right)$

$$\Rightarrow \log y=\log\left\{x^2.(x+1)^{\frac{1}{2}}\right\}-\log\left\{(1+x^2)^{\frac{3}{2}}\right\} \quad \Rightarrow \log y=\log(x^2)+\log\left\{(1+x)^{\frac{1}{2}}\right\}-\log\left\{(1+x^2)^{\frac{3}{2}}\right\}$$

$$\Rightarrow \log y=2\log x+\frac{1}{2}\log(1+x)-\frac{3}{2}\log(1+x^2) \quad \Rightarrow \frac{1}{y}.\frac{dy}{dx}=2.\frac{1}{x}+\frac{1}{2}.\frac{1}{1+x}-\frac{3}{2}.\frac{1}{1+x^2}.2x$$

$$\Rightarrow \frac{dy}{dx}=y\left(\frac{2}{x}+\frac{1}{2(1+x)}-\frac{3x}{1+x^2}\right) \quad \therefore \frac{dy}{dx}=\frac{x^2\sqrt{(1+x)}}{(1+x^2)^{\frac{3}{2}}}\left(\frac{2}{x}+\frac{1}{2(1+x)}-\frac{3x}{1+x^2}\right)$$

25. $y=\sqrt{(x-2)(2x-3)(3x-4)}$

Sol. $y=\sqrt{(x-2)(2x-3)(3x-4)}$, Taking log both sides we get,

$$\log y=\log\left\{(x-2)(2x-3)(3x-4)\right\}^{\frac{1}{2}} \Rightarrow \log y=\frac{1}{2}\log\left\{(x-2)(2x-3)(3x-4)\right\}$$

$$\Rightarrow \log y=\frac{1}{2}\left[\log(x-2)+\log(2x-3)+\log(3x-4)\right] \quad \ldots(i)$$

On differentiation of (i) with respect to x,

$$\frac{1}{y}.\frac{dy}{dx}=\frac{1}{2}\left\{\frac{1}{x-2}.1+\frac{1}{2x-3}.2+\frac{1}{3x-4}.3\right\} \Rightarrow \frac{dy}{dx}=\frac{y}{2}\left[\frac{1}{x-2}+\frac{2}{2x-3}+\frac{3}{3x-4}\right]$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\sqrt{(x-2)(2x-3)(3x-4)}\left[\frac{1}{x-2}+\frac{2}{2x-3}+\frac{3}{3x-4}\right]$$

26. $y=\sin 2x\sin 3x\sin 4x$

Sol. $y=\sin 2x\sin 3x\sin 4x \quad \ldots(i)$

Taking log both sides we get, $\log y=\log(\sin 2x.\sin 3x.\sin 4x)$

$$\Rightarrow \log y=\log(\sin 2x)+\log(\sin 3x)+\log(\sin 4x) \quad \ldots(ii)$$

On differentiating both sides of (ii) w.r.t. x,

$$\frac{1}{y}.\frac{dy}{dx}=\frac{1}{\sin 2x}\cos 2x.2+\frac{1}{\sin 3x}.\cos 3x.3+\frac{1}{\sin 4x}.\cos 4x.4$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=2\cot 2x+3\cot 3x+4\cot 4x \Rightarrow \frac{dy}{dx}=y(2\cot 2x+3\cot 3x+4\cot 4x)$$

$$\Rightarrow \frac{dy}{dx}=\sin 2x.\sin 3x.\sin 4x(2\cot 2x+3\cot 3x+4\cot 4x)$$

27. $y=\dfrac{x^3\sin x}{e^x}$

Sol. $y=\dfrac{x^3\sin x}{e^x} \quad ..(i)$

Taking log both sides we get, $\log y=\log\left(\dfrac{x^3\sin x}{e^x}\right) \Rightarrow \log y=\log(x^3.\sin x)-\log(e^x)$

$$\Rightarrow \log y=\log(x^3)+\log(\sin x)-x\log e \Rightarrow \log y=3\log x+\log(\sin x)-x \quad \ldots(ii)$$

On differentiating both sides of (ii) w.r.t. x,

$$\frac{1}{y}.\frac{dy}{dx}=3.\frac{1}{x}+\frac{1}{\sin x}.\cos x-1 \Rightarrow \frac{dy}{dx}=y\left(\frac{3}{x}+\cot x-1\right) \therefore \frac{dy}{dx}=\frac{x^3\sin x}{e^x}\left(\frac{3}{x}+\cot x-1\right)$$

28. $y=\dfrac{e^x\log x}{x^2}$

Sol. $y=\dfrac{e^x\log x}{x^2} \quad \ldots(i)$

Taking log both sides we get, $\log y=\log\left(\dfrac{e^x.\log x}{x^2}\right) \Rightarrow \log y=\log(e^x.\log x)-\log(x^2)$

$$\Rightarrow \log y=x\log e+\log(\log x)-2\log x \Rightarrow \log y=x+\log(\log x)-2\log x \quad \ldots(ii)$$

On differentiating both sides of (ii) w.r.t. x, $\dfrac{1}{y}.\dfrac{dy}{dx}=1+\dfrac{1}{\log x}.\dfrac{1}{x}-2.\dfrac{1}{x}$

$$\Rightarrow \frac{dy}{dx} = y\left(1+\frac{1}{x\log x}-\frac{2}{x}\right) \quad \Rightarrow \frac{dy}{dx} = \frac{e^x \log x}{x^2}\left(1+\frac{1}{x\log x}-\frac{2}{x}\right)$$

29. $y = \frac{x\cos^{-1}x}{\sqrt{1-x^2}}$

Sol. $y = \frac{x\cos^{-1}x}{\sqrt{1-x^2}}$ ...(i)

Taking log both sides we get, $\log y = \log\left(\frac{x\cos^{-1}x}{\sqrt{1-x^2}}\right)$

$$\Rightarrow \log y = \log\left(x\cos^{-1}x\right) - \log\left\{\left(1-x^2\right)^{\frac{1}{2}}\right\} \quad \Rightarrow \log y = \log x + \log\left(\cos^{-1}x\right) - \frac{1}{2}\log\left(1-x^2\right) \quad \text{...(ii)}$$

On differentiating both sides (ii) w.r.t. x, $\frac{1}{y}.\frac{dy}{dx} = \frac{1}{x} + \frac{1}{\cos^{-1}x}\left(-\frac{1}{\sqrt{1-x^2}}\right) - \frac{1}{2}.\frac{1}{1-x^2}(-2x)$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{x} - \frac{1}{\sqrt{1-x^2}\cos^{-1}(x)} + \frac{x}{1-x^2}\right) \quad \Rightarrow \frac{dy}{dx} = \frac{x\cos^{-1}x}{\sqrt{1-x^2}}\left(\frac{1}{x} - \frac{1}{\cos^{-1}x\sqrt{1-x^2}} + \frac{x}{1-x^2}\right)$$

30. $y = (1+x)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^6\right)$

Sol. $y = (1+x)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^6\right)$ ...(i)

Taking log both sides we get, $\log y = \log\left\{(1+x)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^6\right)\right\}$

$$\Rightarrow \log y = \log(1+x) + \log\left(1+x^2\right) + \log\left(1+x^4\right) + \log\left(1+x^6\right) \quad \text{...(ii)}$$

On differentiating both sides of (ii) w.r.t. x,

$$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{1+x}.1 + \frac{1}{1+x^2}.2x + \frac{1}{1+x^4}.4x^3 + \frac{1}{1+x^6}.6x^5$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{6x^5}{1+x^6}\right)$$

$$\therefore \frac{dy}{dx} = (1+x)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^6\right)\left(\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{6x^5}{1+x^6}\right)$$

31. $y = x^x - 2^{\sin x}$

Sol. $y = x^x - 2^{\sin x}$, Let $x^x = u$ & $2^{\sin x} = v$

$y = u - v$ ...(i)

On differentiating both sides of (i) w.r.t.x, $\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$ ...(ii)

$u = x^x$

Taking log both sides we get, $\log u = \log\left(x^x\right) \quad \Rightarrow \log(u) = x\log x$

On differentiating both sides w.r.t.x, $\frac{1}{u}.\frac{du}{dx} = x.\frac{1}{x} + \log x.1 \quad \Rightarrow \frac{du}{dx} = u(1+\log x)$

$\therefore \frac{du}{dx} = x^x(1+\log x)$

$v = 2^{\sin x}$

On differentiating both sides w.r.t. x, $\frac{dv}{dx}=\frac{d\left(2^{\sin x}\right)}{d(\sin x)}.\frac{d(\sin x)}{dx}$ $\Rightarrow \frac{dv}{dx}=2^{\sin x}.\log 2.\cos x$

Putting the value of $\frac{du}{dx},\frac{dv}{dx}$ in equation of (ii), $\frac{dy}{dx}=x^x(1+\log x)-2^{\sin x}.\log 2(\cos x)$

32. $y=(\log x)^x+x^{(\log x)}$

Sol. $y=(\log x)^x+x^{(\log x)}$, Let $u=(\log x)^x, v=x^{\log x}$

$y=u+v$ ...(i)

On differentiating both sides of (i) w.r.t.x.

$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ ...(ii)

$u=(\log x)^x$

Taking long both sides we get, $\log u=\log\left\{(\log x)^x\right\}$ $\Rightarrow \log u=x\log(\log x)$

On differentiating both sides w.r.t. x, $\frac{1}{u}.\frac{du}{dx}=x.\frac{1}{\log x}.\frac{1}{x}+\log(\log x).1$

$\Rightarrow \frac{du}{dx}=u\left(\frac{1}{\log x}+\log(\log x)\right)$ $\Rightarrow \frac{dy}{dx}=(\log x)^x\left(\frac{1}{\log x}+\log(\log x)\right)$

$v=x^{\log x}$

Taking log both sides we get, $\log v=\log\left(x^{\log x}\right)$ $\Rightarrow \log v=\log x.\log x$

On differentiating both sides w.r.t. x,, $\frac{1}{v}.\frac{dv}{dx}=\log x.\frac{1}{x}+\log x.\frac{1}{x}$

$\Rightarrow \frac{dv}{dx}=v.\frac{2}{x}\log x$ $\Rightarrow \frac{dv}{dx}=v.\frac{2}{x}\log x$ $\Rightarrow \frac{dv}{dx}=x^{\log x}.\frac{2\log x}{x}$

Putting the value of $\frac{dy}{dv}\&\frac{dv}{dx}$ in equation (ii), $\frac{dy}{dx}=(\log x)^x\left\{\frac{1}{\log x}+\log(\log x)\right\}+x^{\log x}.\frac{2\log x}{x}$

33. $y=x^{\sin x}+(\sin x)^{\cos x}$

Sol. $y=x^{\sin x}+(\sin x)^{\cos x}$, Let $u=x^{\sin x}, v=(\sin x)^{\cos x}$

$y=u+v$ ...(i)

On differentiating both sides of (i) w.r.t.x

$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ ...(ii)

$u=x^{\sin x}$ and $v=(\sin x)^{\cos x}$

$u=x^{\sin x}$, Taking log both sides we get, $\log u=\log\left(x^{\sin x}\right)$ $\Rightarrow \log u=\sin x.\log x$

Differentiating both sides with respect to x, $\frac{d(\log u)}{du}.\frac{du}{dx}=\sin x.\frac{d(\log x)}{dx}+\log x\frac{d(\sin x)}{dx}$

$\Rightarrow \frac{1}{u}\frac{du}{dx}=\sin x.\frac{1}{x}+\log x.\cos x$ $\Rightarrow \frac{du}{dx}=u\left(\frac{\sin x}{x}+\log x.\cos x\right)$ $\Rightarrow \frac{du}{dx}=x^{\sin x}\left(\frac{\sin x}{x}+\log x.\cos x\right)$

and $v=(\sin x)^{\cos x}$, Taking log both sides we get, $\log v=\log\left\{(\sin x)^{\cos x}\right\}$ $\Rightarrow \log v=\cos x.\log(\sin x)$

Differentiating both sides with respect to x,

$$\frac{d\log(v)}{dv}\cdot\frac{dv}{dx}=\cos x.\frac{d\{\log(\sin x)\}}{d(\sin x)}\times\frac{d(\sin x)}{dx}+\log(\sin x)\frac{d(\cos x)}{dx}$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\cos x.\frac{1}{\sin x}.\cos x+\log(\sin x).(-\sin x) \quad \Rightarrow \frac{dv}{dx}=v(\cos x.\cot x-\log(\sin x).\sin x)$$

$$\Rightarrow \frac{dv}{dx}=(\sin x)^{\cos x}(\cot x\cos x-\sin x.\log(\sin x))$$

Putting the value of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in equation (ii),

$$\frac{dy}{dx}=x^{\sin x}\left(\frac{\sin x}{x}+\log x.\cos x\right)+(\sin x)^{\cos x}(\cot x\cos x-\sin x.\log(\sin x))$$

34. $y=(x\cos x)^x+(x\sin x)^{\frac{1}{x}}$

Sol. $y=(x\cos x)^x+(x\sin x)^{\frac{1}{x}}$, Let $u=(x\cos x)^x$, $v=(x\sin x)^{\frac{1}{x}}$

$y=u+v$ ...(i)

On differentiating both sides w.r.t.x in equation (i)

$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ ...(ii)

$u=(x\cos x)^x$

Taking log both sides we get, $\log u=\log\{(x\cos x)^x\}$

$\Rightarrow \log u=x\log(x\cos x) \quad \Rightarrow \log u=x\{\log x+\log(\cos x)\}$ ...(iii)

On differentiating both sides of (iii) w.r.t. x, $\frac{1}{u}\cdot\frac{du}{dx}=x\left\{\frac{1}{x}+\frac{1}{\cos x}(-\sin x)\right\}+\{\log x+\log\cos x\}$

$$\Rightarrow \frac{dy}{dx}=(x\cos x)^x(1-x\tan x+\log x+\log\cos x)$$

$v=(x\sin x)^{\frac{1}{x}}$

Taking log both sides we get, $\log v=\log\left\{(x\sin x)^{\frac{1}{x}}\right\}$

$\Rightarrow \log v=\frac{1}{x}\log(x\sin x) \quad \Rightarrow \log v=\frac{1}{x}[\log x+\log\sin x]$ ...(iv)

On differentiating both sides of (iv) w.r.t. x,

$$\frac{1}{v}\cdot\frac{dv}{dx}=\frac{1}{x}\left\{\frac{1}{x}+\frac{1}{\sin x}\cos x\right\}+\{\log x+\log(\sin x)\}\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow \frac{dv}{dx}=v\left[\frac{1}{x^2}+\frac{1}{x}\cot x-\frac{\log x}{x^2}-\frac{\log(\sin x)}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx}=(x\sin x)^{\frac{1}{x}}\left[\frac{1}{x^2}+\frac{\cot x}{x}-\frac{\log x}{x^2}-\frac{\log(\sin x)}{x^2}\right]$$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii),

$\therefore\ \frac{dy}{dx}=(x\cos x)^x[1-x\tan x+\log x+\log\cos x]+(x\sin x)^{\frac{1}{x}}\left[\frac{1}{x^2}+\frac{\cot x}{x}-\frac{\log x}{x^2}-\frac{\log(\sin x)}{x^2}\right]$

35. $y=(\sin x)^x+\sin^{-1}(\sqrt{x})$

Sol. Let $u=(\sin^{-1}x)^x$, $v=\sin^{-1}(\sqrt{x})$

$y=u+v$ ...(i)

On differentiating both sides of (i) w.r.t.x

$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ ...(ii)

$u=(\sin^{-1}x)^x$, Taking log both sides we get, $\log u=\log\left\{(\sin^{-1}x)^x\right\}$

$\Rightarrow \log u=x\log(\sin^{-1}x)$ ...(iii)

On differentiating both sides of (iii) w.r.t. x, $\frac{1}{u}.\frac{du}{dx}=x.\frac{1}{\sin^{-1}x}.\frac{1}{\sqrt{1-x^2}}+\log(\sin^{-1}x).1$

$\Rightarrow \frac{du}{dx}=u\left\{\frac{x}{\sin^{-1}x.\sqrt{1-x^2}}+\log(\sin^{-1}x)\right\}$

$v=\sin^{-1}(\sqrt{x})$ ...(iv)

On differentiating both sides of (iv) w.r.t. x, $\frac{dv}{dx}=\frac{1}{\sqrt{1-(\sqrt{x})^2}}.\frac{1}{2\sqrt{x}}\ \Rightarrow \frac{dv}{dx}=\frac{1}{\sqrt{1-x}}.\frac{1}{2\sqrt{x}}$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii),

$\frac{dy}{dx}=(\sin x)^x\left\{\frac{x}{\sin^{-1}x\sqrt{1-x^2}}+\log(\sin^{-1}x)\right\}+\frac{1}{2\sqrt{x}.\sqrt{1-x}}$

$\therefore\ \frac{dy}{dx}=(\sin x)^x\left(\frac{x}{\sin^{-1}x\sqrt{1-x^2}}+\log(\sin^{-1}x)\right)+\frac{1}{2\sqrt{x-x^2}}$

36. $y=x^{x\cos x}+\frac{x^2+1}{x^2-1}$

Sol. $y=x^{x\cos x}+\frac{x^2+1}{x^2-1}$, Let $u=x^{x\cos x}, v=\frac{x^2+1}{x^2-1}$

$y=u+v$ ...(i)

On differentiating both sides of (i) w.r.t.x

$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ ...(ii)

$u=x^{x\cos x}$, Taking log both sides we get,

$\log u=\log(x^{x\cos x})\ \Rightarrow \log u=(x\cos x).\log x$ ...(iii)

On differentiating both sides of w.r.t.x, $\frac{1}{u}.\frac{du}{dx}=(x\cos x).\frac{1}{x}+\log x.(-x\sin x+\cos x.1)$

$\Rightarrow \frac{du}{dx}=u\{\cos x-x\log x\sin x+\log x\cos x\}$

$$\Rightarrow \frac{du}{dx} = x^{x\cos x}\{\cos x - x\log x \sin x + \log x.\cos x\}$$

$$v = \frac{x^2+1}{x^2-1} \qquad \ldots(iv)$$

On differentiating both sides of (iv) w.r.t. x,

$$\frac{dv}{dx} = \frac{(x^2-1)\frac{d(x^2+1)}{dx} - (x^2+1)\frac{d(x^2-1)}{dx}}{(x^2-1)^2} \Rightarrow \frac{dv}{dx} = \frac{(x^2-1)2x - (x^2+1).2x}{(x^2-1)^2}$$

$$\Rightarrow \frac{dv}{dx} = \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2} \Rightarrow \frac{dv}{dx} = \frac{-4x}{(x^2-1)^2}$$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii), $\frac{dy}{dx} = x^{\cos x}\{\cos x - x\log x.\sin x + \log x.\cos x\} - \frac{4x}{(x^2-1)^2}$

37. $y = e^x \sin^3 x \cos^4 x$

Sol. $y = e^x \sin^3 x \cos^4 x \qquad \ldots(i)$

Taking log both sides, we get, $\log y = \log\{e^x.\sin^3 x.\cos^4 x\}$

$$\Rightarrow \log y = \log(e^x) + \log(\sin^3 x) + \log(\cos^4 x) \Rightarrow \log y = x\log e + 3\log(\sin x) + 4\log(\cos x)$$

$$\Rightarrow \log y = x + 3\log(\sin x) + 4\log(\cos x) \qquad \ldots(ii)$$

On differentiating both sides of (ii) w.r.t. x, $\frac{1}{y}.\frac{dy}{dx} = 1 + 3.\frac{1}{\sin x}.\cos x + 4.\frac{1}{\cos x}(-\sin x)$

$$\Rightarrow \frac{dy}{dx} = y(1 + 3\cot x - 4\tan x) \quad \therefore \frac{dy}{dx} = e^x \sin^3 x \cos^4 x(1 + 3\cot x - 4\tan x)$$

38. $y = 2^x.e^{3x}\sin 4x$

Sol. $y = 2^x.e^{3x}\sin 4x \qquad \ldots(i)$

Taking log both sides we get, $\log y = \log(2^x.e^{3x}.\sin 4x)$

$$\Rightarrow \log y = \log(2^x) + \log(e^{3x}) + \log(\sin 4x) \Rightarrow \log y = x\log 2 + 3x\log e + \log(\sin 4x)$$

$$\Rightarrow \log y = x\log 2 + 3x\log e + \log(\sin 4x) \Rightarrow \log y = x\log 2 + 3x + \log(\sin 4x) \qquad \ldots(ii)$$

On differentiating both sides of (iv) w.r.t.x,

$$\frac{1}{y}.\frac{dy}{dx} = \log 2.1 + 3 + \frac{1}{\sin 4x}.\cos 4x.4 \Rightarrow \frac{dy}{dx} = 2^x.e^{3x}\sin 4x(\log 2 + 3 + 4\cot 4x)$$

39. $y = x^x.e^{(2x+5)}$

Sol. $y = x^x.e^{(2x+5)} \qquad \ldots(i)$

Taking log both sides, we get, $\log y = \log\{x^x.e^{2x+5}\} \Rightarrow \log y = \log(x^x) + \log(e^{2x+5})$

$$\Rightarrow \log y = x\log x + (2x+5)\log e \Rightarrow \log y = x\log x + (2x+5) \qquad \ldots(ii)$$

On differentiating both sides of (ii). w.r.t. x,

$$\frac{1}{y}.\frac{dy}{dx} = x.\frac{1}{x} + \log x.1 + 2 \Rightarrow \frac{dy}{dx} = y(1 + \log x + 2) \quad \therefore \frac{dy}{dx} = x^x.e^{2x+5}(3 + \log x)$$

40. $y = (2x+3)^5(3x-5)^7(5x-1)^3$

Sol. $y=(2x+3)^5(3x-5)^7(5x-1)^3$ ...(i)

Taking log both sides, we get, $\log y=\log\left\{(2x+3)^5(3x-5)^7(5x-1)^3\right\}$

$\Rightarrow \log y=\log\left\{(2x+3)^5+\log(3x-5)^7\right\}+\log\left\{(5x-1)^3\right\}$

$\Rightarrow \log y=5\log(2x+3)+7\log(3x-5)+3\log(5x-1)$ ...(ii)

On differentiating both sides of (ii) w.r.t. x, $\frac{1}{y}.\frac{dy}{dx}=5.\frac{1}{2x+3}.2+7.\frac{1}{3x-5}.3+3.\frac{1}{5x-1}.5$

$\Rightarrow \frac{dy}{dx}=y\left(\frac{10}{2x+3}+\frac{21}{3x-5}+\frac{15}{5x-1}\right)$ $\therefore \frac{dy}{dx}=(2x+3)^5(3x-5)^7(5x-1)^3\left\{\frac{10}{2x+3}+\frac{21}{3x-5}+\frac{15}{5x-1}\right\}$

41. $(\cos x)^y=(\cos y)^x$

Sol. $(\cos x)^y=(\cos y)^x$, Taking log both sides, we get, $\log\left\{(\cos x)^y\right\}=\log\left\{(\cos y)^x\right\}$

$\Rightarrow y\log(\cos x)=x\log(\cos y)$ ...(i)

On differentiating both sides of (i) w.r.t. x,

$$y.\frac{1}{\cos x}(-\sin x)+\log(\cos x).\frac{dy}{dx}=x.\frac{1}{\cos y}(-\sin y)\frac{dy}{dx}+\log(\cos y).1$$

$\Rightarrow -y\tan x+\log(\cos x)\frac{dy}{dx}=-x\tan y\frac{dy}{dx}+\log(\cos y)$

$\Rightarrow \log(\cos x)\frac{dy}{dx}+x\tan y\frac{dy}{dx}=\log(\cos y)+y\tan x$

$\Rightarrow \frac{dy}{dx}\left\{\log(\cos x)+x\tan y\right\}=\left\{\log(\cos y)+y\tan x\right\}$ $\therefore \frac{dy}{dx}=\frac{\log(\cos y)+y\tan x}{\log\cos x+x\tan y}$

42. $(\tan x)^y=(\tan y)^x$

Sol. $(\tan x)^y=(\tan y)^x$ ...(i)

Taking both sides, we get, $\log\left\{(\tan x)^y\right\}=\log\left\{(\tan y)^x\right\}$

$\Rightarrow y\log(\tan x)=x\log(\tan y)$ ...(ii)

On differentiating both sides of (ii) w.r.t. x,

$$y.\frac{1}{\tan x}.\sec^2 x+\log(\tan x)\frac{dy}{dx}=x.\frac{1}{\tan y}\sec^2 y\frac{dy}{dx}+\log(\tan y).1$$

$\Rightarrow y.\frac{2}{\sin 2x}+\log(\tan x)\frac{dy}{dx}=x.\frac{2}{\sin 2y}\frac{dy}{dx}+\log(\tan y)$

$\Rightarrow 2y\cos ec2x+\log(\tan x)\frac{dy}{dx}=2x\cos ec2y\frac{dy}{dx}+\log(\tan y)$

$\Rightarrow \log(\tan x)\frac{dy}{dx}-2x\cos ec2y\frac{dy}{dx}=\log(\tan y)-2y\cos ec2x$

$\Rightarrow \frac{dy}{dx}\left(\log(\tan x)-2x\cos ec2y\right)=\log(\tan x)-2y\cos ec2x$ $\therefore \frac{dy}{dx}=\frac{\log(\tan y)-2y\cos ec2x}{\log(\tan x)-2x\cos ec2y}$

43. $y=(\log x)^x+x^{(\log x)}$

Sol. $y=(\log x)^x+x^{(\log x)}$, Let $u=(\log x)^x$, $v=x^{\log x}$

$y = u + v \qquad \ldots(i)$

On differentiating both sides of (ii) w.r.t.x, $\frac{dy}{dx} = \frac{dy}{dx} + \frac{dv}{dx} \qquad \ldots(ii)$

$u = (\log x)^x$

Taking log both sides, we get, $\log u = \log\left\{(\log x)^x\right\} \Rightarrow \log u = x\log(\log x) \qquad \ldots(iii)$

On differentiating both sides of (iii) w.r.t. x, $\frac{1}{u}.\frac{du}{dx} = x.\frac{1}{\log x}.\frac{1}{x} + \log(\log x).1$

$\Rightarrow \frac{du}{dx} = u\left\{\frac{1}{\log x} + \log(\log x)\right\} \Rightarrow \frac{du}{dx} = (\log x)^x\left\{\frac{1}{\log x} + \log(\log x)\right\}$

$\Rightarrow v = x^{\log x}$

Taking log both sides we get , $\log v = \log\left(x^{\log x}\right) \Rightarrow \log v = \log x.\log x \qquad \ldots(iv)$

On differentiating both sides of (iv) w.r.t. x,

$$\frac{1}{v}.\frac{dv}{dx} = \log x.\frac{1}{x} + \log x.\frac{1}{x} \Rightarrow \frac{dv}{dx} = v\left(\frac{2\log x}{x}\right) \Rightarrow \frac{dv}{dx} = x^{\log x}\left(\frac{2\log x}{x}\right)$$

Putting the value differentiation $\frac{du}{dx} \& \frac{dv}{dx}$ in equation (ii),

$$\frac{dy}{dx} = (\log x)^x\left\{\frac{1}{\log x} + \log(\log x)\right\} + x^{\log x}.\frac{2}{x}(\log x)$$

44. If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ prove that $(1-x^2)\frac{dy}{dx} = xy + 1$

Sol. $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}} \qquad \ldots(i)$

On differentiating both sides of w.r.t. x,

$$\frac{dy}{dx} = \frac{\left(\sqrt{1-x^2}\right)\frac{d\left(\sin^{-1} x\right)}{dx} - \sin^{-1} x\frac{d\left(\sqrt{1-x^2}\right)}{d\left(1-x^2\right)} \times \frac{d\left(1-x^2\right)}{dx}}{\left(\sqrt{1-x^2}\right)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\left(\sqrt{1-x^2}\right).\frac{1}{\sqrt{1-x^2}} - \sin^{-1} x.\frac{1}{2\sqrt{1-x^2}}.(-2x)}{\left(1-x^2\right)} \Rightarrow \left(1-x^2\right)\frac{dy}{dx} = 1 + \frac{x\sin^{-1} x}{\sqrt{1-x^2}}$$

$\Rightarrow \left(1-x^2\right)\frac{dy}{dx} = (1+xy)$. Proved.

45. If $y = \sqrt{x+y}$ prove that $\frac{dy}{dx} = \frac{1}{2y-1}$

Sol. $y = \sqrt{x+y} \qquad \ldots(i)$

Squaring both sides we get, $(y)^2 = \left(\sqrt{x+y}\right)^2$, $y^2 = x + y \qquad \ldots(ii)$

On differentiation of both sides at (ii) w.r.t. x,

$$2y\frac{dy}{dx} = 1 + \frac{dy}{dx} \Rightarrow 2y\frac{dy}{dx} - \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx}(2y-1) = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y-1}$$

46. If $x^a y^b = (x+y)^{a+b}$ prove that $\frac{dy}{dx} = \frac{y}{x}$

Sol. $x^a y^b = (x+y)^{a+b}$ ...(i)

Taking log both sides of (i) w.r.t. x, $\log(x^a . y^b) = \log\{(x+y)^{a+b}\}$

$\Rightarrow \log(x^a) + \log(y^b) = (a+b)\log(x+y)$

$\Rightarrow a\log x + b\log y = (a+b)\log(x+y)$ ...(ii)

On differentiating both sides of (ii) w.r.t. x, $a.\frac{1}{x} + b.\frac{1}{y}.\frac{dy}{dx} = (a+b).\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)$

$\Rightarrow \frac{a}{x} + \frac{b}{y}\frac{dy}{dx} = \frac{a+b}{x+y} + \frac{a+b}{x+y}\frac{dy}{dx} \Rightarrow \frac{b}{y}\frac{dy}{dx} - \left(\frac{a+b}{x+y}\right)\frac{dy}{dx} = \frac{a+b}{x+y} - \frac{a}{x}$

$\Rightarrow \frac{dy}{dx}\left(\frac{b}{y} - \frac{a+b}{x+y}\right) = \frac{a+b}{x+y} - \frac{a}{x} \Rightarrow \frac{dy}{dx}\left(\frac{b(x+y) - y(a+b)}{y(x+y)}\right) = \frac{(a+b)x - a(x+y)}{(x+y)x}$

$\Rightarrow \frac{dy}{dx} = \frac{y(ax+bx-ax-ay)}{x(bx+by-ay-by)} \Rightarrow \frac{dy}{dx} = \frac{y}{x}\left(\frac{bx-ay}{bx-ay}\right) \quad \therefore \frac{dy}{dx} = \frac{y}{x}$

47. If $x^x + y^x = 1$ show that $\frac{dy}{dx} = -\left\{\frac{x^x(1+\log x) + y^x \log y}{xy^{x-1}}\right\}$

Sol. $x^x + y^x = 1$

Let $u = x^x$, $v = y^x$

$u + v = 1$ ...(i)

On differentiating both sides w.r.t. x, $\frac{du}{dx} + \frac{dv}{dx} = \frac{d(1)}{dx}$, $\frac{du}{dx} + \frac{dv}{dx} = 0$ ...(ii)

$u = x^x$, Taking log both sides we get , $\log u = \log(x^x) \Rightarrow \log u = x\log x$ ...(iii)

On differentiating both sides of (iii) w.r.t. x,

$\frac{1}{u}.\frac{du}{dx} = x.\frac{1}{x} + \log x.1 \Rightarrow \frac{du}{dx} = u(1+\log x) \Rightarrow \frac{du}{dx} = x^x(1+\log x)$

$v = y^x$, Taking log both sides we get , $\log v = \log(y^x) \Rightarrow \log v = x\log y$ ...(iv)

On differentiating both sides of (iv) w.r.t. x,

$\frac{1}{v}.\frac{dv}{dx} = x.\frac{1}{y}\frac{dy}{dx} + \log y.1 \Rightarrow \frac{dv}{dx} = v\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) \Rightarrow \frac{dv}{dx} = y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right)$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii),

$x^x(1+\log x) + y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = 0 \Rightarrow x^x(1+\log x) + \frac{y^x.x}{y}\frac{dy}{dx} + y^x\log y = 0$

$\Rightarrow xy^{x-1}\frac{dy}{dx} = -\{y^x\log y + x^x(1+\log x)\} \quad \therefore \frac{dy}{dx} = -\frac{\{x^x + (1+\log x) + y^x\log y\}}{xy^{x-1}}$

48. If $y = e^{\sin x} + (\tan x)^x$ prove that $\frac{dy}{dx} = e^{\sin x}\cos x + (\tan x)^2\,[2x\cos ec\ 2x + \log\tan x]$

Sol. $y = e^{\sin x} + (\tan x)^x$

Let $u=e^{\sin x},\ v=(\tan x)^{x},\ y=u+v$ ...(i)

Differentiating both sides of (i) w.r.t. x, $\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ ...(ii)

$u=e^{\sin x}$ ...(iii)

Differentiating both sides of (iii) w.r.t. x, $\frac{du}{dx}=e^{\sin x}.\cos x$

$v=(\tan x)^{x}$, Taking log both sides we get ,

$\log v=\log\{(\tan x)^{x}\}\ \Rightarrow\ \log v=x\log(\tan x)$ ...(iv)

On differentiating both sides of (iv) w.r.t. x, $\frac{1}{v}.\frac{dv}{dx}=x.\frac{1}{\tan x}.\sec^2 x+\log(\tan x).1$

$\Rightarrow\ \frac{dv}{dx}=v\{2x\operatorname{cosec}2x+\log(\tan x)\}\ \Rightarrow\ \frac{dv}{dx}=(\tan x)^{x}(2x\operatorname{cosec}2x+\log(\tan x))$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii), $\frac{dy}{dx}=e^{\sin x}.\cos x+(\tan x)^{x}\left[2x\operatorname{cosec}2x+\log(\tan x)\right]$

49. If $y=\log\left(x+\sqrt{1+x^2}\right)$ prove that $\frac{dy}{dx}=\frac{1}{\sqrt{1+x^2}}$

Sol. $y=\log\left(x+\sqrt{1+x^2}\right)$ ...(i)

On differentiating both sides of (i) w.r.t. x, $\frac{dy}{dx}=\frac{1}{x+\sqrt{1+x^2}}\left\{1+\frac{1}{2\sqrt{1+x^2}}.2x\right\}$

$\Rightarrow\ \frac{dy}{dx}=\frac{1}{x+\sqrt{1+x^2}}.\left\{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right\}\ \Rightarrow\ \frac{dy}{dx}=\frac{1}{\sqrt{1+x^2}}$. Proved.

50. If $y=\log\sin\left(\sqrt{x^2+1}\right)$ Proved that $\frac{dy}{dx}=\frac{x\cot\sqrt{x^2+1}}{\sqrt{x^2+1}}$

Sol. $y=\log\left\{\sin\left(\sqrt{x^2+1}\right)\right\}$ ...(i)

On differentiating both sides of (i) w.r.t. x, $\frac{dy}{dx}=\frac{1}{\sin\sqrt{x^2+1}}.\cos\left(\sqrt{x^2+1}\right).\frac{1}{2\sqrt{x^2+1}}.2x$

$\Rightarrow\ \frac{dy}{dx}=\cot\left(\sqrt{x^2+1}\right).\frac{x}{\sqrt{x^2+1}}\quad\therefore\ \frac{dy}{dx}=\frac{x\cot\left(\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}$

51. If $y=\log\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$ show that $\frac{dy}{dx}=\operatorname{cosec}x$

Sol. $y=\log\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)\ \Rightarrow\ y=\log\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)\ \Rightarrow\ y=\log\left(\tan\frac{x}{2}\right)$ ...(i)

On differentiating both sides of (ii) w.r.t. x,

$\frac{dy}{dx}=\frac{1}{\tan\frac{x}{2}}.\sec^2\frac{x}{2}.\frac{1}{2}\ \Rightarrow\ \frac{dy}{dx}=\frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}\ \Rightarrow\ \frac{dy}{dx}=\frac{1}{\sin x}\ \Rightarrow\ \frac{dy}{dx}=\operatorname{cosec}x$

52. If $y=\log\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$ show that $\frac{dy}{dx}=\sec x$.

Sol. $y=\log\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$ ...(i)

On differentiating both sides of (i) w.r.t. x,

$$\frac{dy}{dx}=\frac{1}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)}.\sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right).\frac{1}{2} \Rightarrow \frac{dy}{dx}=\frac{1}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sin\left\{2\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}} \Rightarrow \frac{dy}{dx}=\frac{1}{\sin\left(\frac{\pi}{2}+x\right)} \Rightarrow \frac{dy}{dx}=\frac{1}{\cos x} \Rightarrow \frac{dy}{dx}=\sec x$$

53. If $y=\sqrt{\frac{1-\sin 2x}{1+\sin 2x}}$ show that $\frac{dy}{dx}+\sec^2\left(\frac{\pi}{4}-x\right)=0$

Sol. $y=\sqrt{\frac{1-\sin 2x}{1+\sin 2x}}$ ...(i)

$$y=\sqrt{\frac{(\cos x-\sin x)^2}{(\cos x+\sin x)^2}} \Rightarrow y=\frac{\cos x-\sin x}{\cos x+\sin x}$$

Dividing nominator and denominator $\cos x$, $y=\frac{\frac{\cos x-\sin x}{\cos x}}{\frac{\cos x+\sin x}{\cos x}} \Rightarrow y=\frac{\frac{\cos x}{\cos x}-\frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x}+\frac{\sin x}{\cos x}}$

$$\Rightarrow y=\frac{1-\tan x}{1+\tan x} \Rightarrow y=\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x} \Rightarrow y=\tan\left(\frac{\pi}{4}-x\right) \quad ...(ii)$$

On differentiating both sides of (ii) w.r.t. x, $\frac{dy}{dx}=\sec^2\left(\frac{\pi}{4}-x\right)(-1)$ $\therefore \frac{dy}{dx}=-\sec^2\left(\frac{\pi}{4}-x\right)$

L.H.S. $=\frac{dy}{dx}+\sec^2\left(\frac{\pi}{4}-x\right)=-\sec^2\left(\frac{\pi}{4}-x\right)+\sec^2\left(\frac{\pi}{4}-x\right)=0$

54. If $y=\log\sqrt{\frac{1+\cos^2 x}{1-e^{2x}}}$ show that $\frac{dy}{dx}=\frac{e^{2x}}{1-e^{2x}}-\frac{\sin x\cos x}{1+\cos^2 x}$

Sol. $y=\log\left(\sqrt{\frac{1+\cos^2 x}{1-e^{2x}}}\right)$ ...(i)

On differentiating both sides of (i) w.r.t. x,

$$\frac{dy}{dx}=\frac{1}{\sqrt{\frac{1+\cos^2 x}{1-e^{2x}}}}.\frac{1}{2\sqrt{\frac{1+\cos^2 x}{1-e^{2x}}}}\left\{\frac{\left(1-e^{2x}\right)\frac{d\left(1+\cos^2 x\right)}{dx}-\left(1+\cos^2 x\right)\frac{d\left(1-e^{2x}\right)}{dx}}{\left(1-e^{2x}\right)^2}\right\}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2\left(\frac{1+\cos^2 x}{1-e^{2x}}\right)}\left\{\frac{\left(1-e^{2x}\right)(-2)\sin x\cos x-\left(1+\cos^2 x\right)e^{2x}(-2)}{\left(1-e^{2x}\right)^2}\right\}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1-e^{2x}}{1+\cos^2 x}.\left\{\frac{\left(1-e^{2x}\right)\sin x\cos x+e^{2x}\left(1+\cos^2 x\right)}{\left(1-e^{2x}\right)^2}\right\}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-\left(1-e^{2x}\right)\sin x\cos x+e^{2x}\left(1+\cos^2 x\right)}{\left(1+\cos^2 x\right)\left(1-e^{2x}\right)}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{\sin x\cos x}{1+\cos^2 x}+\frac{e^{2x}}{1-e^{2x}} \quad \therefore \frac{dy}{dx}=\frac{e^2 x}{1-e^{2x}}-\frac{\sin x\cos x}{1+\cos^2 x}$$

55. If $y=x^{\cos x}+(\sin x)^{\tan x}$ prove that $\frac{dy}{dx}=x\cos x\left\{\frac{\cos x}{x}-(\sin x)\log x\right\}+(\sin x)^{\tan x}\left\{1+\log(\sin x).\sec^2 x\right\}$

Sol. $y=x^{\cos x}+(\sin x)^{\tan x}$, Let $u=x^{\cos x}, v=(\sin x)^{\tan x}$

$y=u+v \quad \ldots$ (i)

On differentiating both sides of (i) w.r.t.x

$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \quad \ldots$(ii)

$u=x^{\cos x}$, Taking log both sides, we get, $\log u=\log\left(x^{\cos x}\right) \Rightarrow \log u=\cos x.\log x \quad \ldots$(iii)

On differentiating both sides of (ii) w.r.t. x, $\frac{1}{u}.\frac{du}{dx}=\cos x.\frac{1}{x}+\log x(-\sin x)$

$$\Rightarrow \frac{du}{dx}=u\left(\frac{\cos x}{x}-\log x\sin x\right) \quad \Rightarrow \frac{du}{dx}=x^{\cos x}\left(\frac{\cos x}{x}-\log x.\sin x\right)$$

$v=(\sin x)^{\tan x}$, Taking log both sides we get,

$\log v=\log\left\{\sin^{\tan x}\right\} \Rightarrow \log v=\tan x.\log(\sin x) \quad \ldots$(iv)

On differentiating both sides of (iv) w.r.t. x, $\frac{1}{v}.\frac{dv}{dx}=\tan x.\frac{1}{\sin x}.\cos x+\log(\sin x).\sec^2 x$

$$\Rightarrow \frac{dv}{dx}=v\left\{1+\log(\sin x).\sec^2 x\right\} \quad \Rightarrow \frac{dv}{dx}=(\sin x)^{\tan x}\left(1+\log(\sin x)\sec^2 x\right)$$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii),

$$\frac{dy}{dx}=x^{\cos x}\left(\frac{\cos x}{x}-\log x.\sin x\right)+(\sin x)^{\tan x}\left(1+\log(\sin x).\sec^2 x\right)$$

56. If $y=(\sin x)^{\cos x}+(\cos x)^{\sin x}$ prove that

$$\frac{dy}{dx}=(\sin x)^{\cos x}\left[\cot x\cos x-\sin x\log(\sin x)\right]+(\cos x)^{\sin x}\left[\cos x\log(\cos x)-\sin x\tan x\right]$$

Sol. $y=(\sin x)^{\cos x}+(\cos x)^{\sin x}$, Let $u=(\sin x)^{\cos x}, v=(\cos x)^{\sin x}$

$y=u+v \quad \ldots$ (i)

On differentiating both sides of (i) w.r.t.x

$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \quad \ldots$ (ii)

$v=(\sin x)^{\cos x}$

Taking log both sides, we get, $\log v=\log\left\{(\sin x)^{\cos x}\right\} \Rightarrow \log v=\cos x.\log(\sin x) \quad \ldots$(iii)

On differentiating both sides of (iii) w.r.t. x, $\frac{1}{v}.\frac{du}{dx}=\cos x.\frac{1}{\sin x}.\cos x+\log(\sin x)(-\sin x)$

$\Rightarrow \frac{dv}{dx}=v(\cot x\cos x-\log(\sin x).\sin x) \quad \Rightarrow \frac{du}{dx}=(\sin x)^{\cos x}\{\cot x\cos x-\sin x.\log(\sin x)\}$

$v=(\cos x)^{\sin x}$

Taking log both sides, we get , $\log v=\log\{(\cos x)^{\sin x}\} \quad \Rightarrow \log v=\sin x.\log(\cos x)$ ...(iv)

On differentiating both sides of (iv) w.r.t. x, $\frac{1}{v}.\frac{dv}{dx}=\sin x.\frac{1}{\cos x}.(-\sin x)+\log(\cos x)\cos x$

$\Rightarrow \frac{dv}{dx}=v(-\sin x\tan x+\cos x.\log(\cos x)) \quad \Rightarrow \frac{dv}{dx}=(\cos x)^{\sin x}(\cos x.\log(\cos x)-\sin x.\tan x)$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (i),

$$\frac{dy}{dx}=(\sin x)^{\cos x}[\cot x\cos x-\sin x\log(\sin x)]+(\cos x)^{\sin x}[\cos x.\log(\cos x)-\sin x\tan x]$$

57. If $y=(\tan x)^{\cot x}+(\cot x)^{\tan x}$ prove that

$\frac{dy}{dx}=(\tan x)^{\cot x}\cos ec^2x\left(1-\log(\tan x)+(\cot x)^{\tan x}\sec^2 x\right)[\log(\cot x)-1]$

Sol. $y=(\tan x)^{\cot x}+(\cot x)^{\tan x}$, Let $u=(\tan x)^{\cot x}$, $v=(\cot x)^{\tan x}$

$y=u+v$ ...(i)

On differentiating both sides of (ii) w.r.t. x

$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ ...(ii)

$u=(\tan x)^{\cot x}$

Taking log both sides, we get, $\log u=\log\{(\tan x)^{\cot x}\} \quad \Rightarrow \log u=\cot x.\log(\tan x)$ ...(iii)

On differentiating both sides of (iii) w.r.t. x,

$\frac{1}{u}.\frac{du}{dx}=\cot x.\frac{1}{\tan x}.\sec^2 x+\log(\tan x).(-\cos ec^2x)$

$\Rightarrow \frac{1}{u}\frac{du}{dx}=\cos ec^2x-\log(\tan x).\cos ec^2x \quad \Rightarrow \frac{dv}{dx}=u\cos ec^2x(1-\log(\tan x))$

$\Rightarrow \frac{du}{dx}=(\tan x)^{\cot x}.\cos ec^2x(1-\log(\tan x))$, $\quad v=(\cot x)^{\tan x}$

Taking log both sides we get, $\log v=\log\{(\cot x)^{\tan x}\} \quad \Rightarrow \log v=\tan x.\log(\cot x)$ ...(iv)

On differentiating both sides of (iv) w.r.t. x,

$\frac{1}{v}.\frac{dv}{dx}=\tan x.\frac{1}{\cot x}.(-\cos ec^2x)+\log(\cot x)\sec^2 x$

$\Rightarrow \frac{dv}{dx}=v(-\sec^2 x+\log(\cot x)\sec^2 x) \quad \Rightarrow \frac{dv}{dx}=(\cot x)^{\tan x}\sec^2 x\{\log(\cot x)-1\}$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii),

$$\frac{dy}{dx}=(\tan x)^{\cot x}\cos ec^2x(1-\log\tan x)+(\cot x)^{\tan x}.\sec^2 x\{\log(\cot x)-1\}$$

58. If $y = x^{\cos x} + (\cos x)^x$ prove that

$$\frac{dy}{dx} = x^{\cos x}\left\{\frac{\cos x}{x} - (\sin x).\log x\right\} + (\cos x)^x\left[\log(\cos x) - x\tan x\right]$$

Sol. $y = x^{\cos x} + (\cos x)^x$, Let $u = x^{\cos x}$, $v = (\cos x)^x$

$y = u + v$ ...(i)

On differentiating both sides of (i) w.r.t.x

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{...(ii)}$$

$u = x^{\cos x}$

Taking log both sides, we get, $\log u = \log\left(x^{\cos x}\right) \Rightarrow \log u = \cos x.\log x$ ...(iii)

On differentiating both sides of (iii) w.r.t. x, $\frac{1}{u}.\frac{du}{dx} = \cos x.\frac{1}{x} + \log x.(-\sin x)$

$$\Rightarrow \frac{du}{dx} = u\left(\frac{\cos x}{x} - \log x.\sin x\right) \Rightarrow \frac{du}{dx} = x^{\cos x}\left(\frac{\cos x}{x} - \sin x.\log x\right)$$

$v = (\cos x)^x$

Taking log both sides, we get, $\log v = \log\left(\cos x^x\right) \Rightarrow \log v = x\log(\cos x)$ ...(iv)

On differentiating both sides of (iii) w.r.t.x, $\frac{1}{v}.\frac{dv}{dx} = x.\frac{1}{\cos x}.(-\sin x) + \log(\cos x).1$

$$\Rightarrow \frac{dv}{dx} = (\cos x)^x\left[\log(\cos x) - x\tan x\right]$$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii),

$$\frac{dy}{dx} = x^{\cos x}\left[\frac{\cos x}{x} - \sin x.\log x\right] + (\cos x)^x\left[\log\cos x - x\tan x\right]$$

59. If $y = x^{\log x} + (\log x)^x$ prove that $\frac{dy}{dx} = x^{\log x}\left\{\frac{2\log x}{x}\right\} + (\log x)^x\left\{\frac{1}{\log x} + \log(\log x)\right\}$

Sol. Let $u = (\log x)^x$, $v = x^{\log x}$

$y = u + v$ ...(i)

On differentiating both sides of (i) w.r.t.x,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{...(ii)}$$

$u = (\log x)^x$

Taking long both sides we get, $\log u = \log\left\{(\log x)^x\right\} \Rightarrow \log u = x\log(\log x)$

On differentiating both sides w.r.t. x, $\frac{1}{u}.\frac{du}{dx} = x.\frac{1}{\log x}.\frac{1}{x} + \log(\log x).1$

$$\Rightarrow \frac{dy}{dx} = u\left(\frac{1}{\log x} + \log(\log x)\right) \Rightarrow \frac{dy}{dx} = (\log x)^x\left(\frac{1}{\log x} + \log(\log x)\right)$$

$v = x^{\log x}$, Taking log both sides we get, $\log v = \log\left(x^{\log x}\right) \Rightarrow \log v = \log x.\log x$

On differentiating both sides w.r.t. x, $\frac{1}{v}.\frac{dv}{dx}=\log x.\frac{1}{x}+\log x.\frac{1}{x}$

$\Rightarrow \frac{dv}{dx}=v.\frac{2}{x}\log x \quad \Rightarrow \frac{dv}{dx}=v.\frac{2}{x}\log x \quad \Rightarrow \frac{dv}{dx}=x^{\log x}.\frac{2\log x}{x}$

Putting the value of $\frac{du}{dv}$ & $\frac{dv}{dx}$ in equation (ii), $\frac{dy}{dx}=(\log x)^x\left\{\frac{1}{\log x}+\log(\log x)\right\}+x^{\log x}.\frac{2\log x}{x}$

60. If $y=x^{(x^2-3)}+(x-3)^{x^2}$, find $\frac{dy}{dx}$

Sol. $y=x^{(x^2-3)}+(x-3)^{x^2}$, Let $u=x^{(x^2-3)}, v=(x-3)^{x^2}$

$y=u+v \qquad \ldots(i)$

On differentiating both sides of (i) w.r.t. x

$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad \ldots(ii)$

$u=x^{(x^2-3)}$, taking log both sides, we get, $\log u=\log\left\{x^{(x^2-3)}\right\} \Rightarrow \log u=(x^2-3)\log x \quad \ldots(iii)$

On differentiating both sides of (iii) w.r.t. x, $\frac{1}{u}.\frac{du}{dx}=(x^2-3).\frac{1}{x}+\log x.2x$

$\Rightarrow \frac{du}{dx}=u\left(\frac{x^2-3}{x}+2x\log x\right) \Rightarrow \frac{du}{dx}=x^{(x^2-3)}\left\{\frac{x^2-3}{x}+2x\log x\right\}$

$v=(x-3)^{x^2}$

Taking log both sides we get, $\log v=\log\left\{(x-3)^{x^2}\right\} \Rightarrow \log v=x^2\log(x-3) \qquad \ldots(iv)$

On differentiating both sides of (iv) w.r.t. x, $\frac{1}{v}.\frac{dv}{dx}=x^2.\frac{1}{x-3}+\log(x-3).2x$

$\Rightarrow \frac{dv}{dx}=v\left\{\frac{x^2}{x-3}+2x\log(x-3)\right\} \Rightarrow \frac{dv}{dx}=(x-3)^{x^2}\left\{\frac{x^2}{x-3}+2x\log(x-3)\right\}$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii),

$$\frac{dy}{dx}=x^{(x^2-3)}\left\{\frac{x^2-3}{x}+2x\log x\right\}+(x-3)^{x^2}\left\{\frac{x^2}{x-3}+2x\log(x-3)\right\}$$

61. If $f(x)=\left(\frac{3+x}{1+x}\right)^{2+3x}$ find $f'(0)$.

Sol. $f(x)=\left(\frac{3+x}{1+x}\right)^{2+3x} \qquad \ldots(i)$

Taking log both sides we get, $\log f(x)=\log\left\{\left(\frac{3+x}{1+x}\right)^{2+3x}\right\}$

$\Rightarrow \log f(x)=(2+3x)\log\left(\frac{3+x}{1+x}\right) \Rightarrow \log f(x)=(2+3x)\{\log(3+x)-\log(1+x)\} \qquad \ldots(ii)$

On differentiating both sides of w.r.t. x,

$$\frac{1}{f(x)}f'(x)=(2+3x).\left(\frac{1}{3+x}-\frac{1}{1+x}\right)+(\log(3+x)-\log(1+x)).3$$

$$\Rightarrow f'(x)=f(x).\left[(2+3x)\left\{\frac{1+x-3-x}{(1+x)(3+x)}\right\}+3\{\log(3+x)-\log(1+x)\}\right]$$

$$\Rightarrow f'(x)=\left(\frac{3+x}{1+x}\right)^{2+3x}\left[(-2)\frac{(2+3x)}{(1+x)(3+x)}+3\{\log(3+x)-\log(1+x)\}\right]$$

At $x=0$, $f'(0)=\left(\frac{3+0}{1+0}\right)^{2+3.0}\left\{(-2)\frac{(2+3.0)}{(1+0)(3+0)}+3\{\log(3+0)-\log(1+0)\}\right\}$

$$\Rightarrow f'(0)=3^2\left\{\frac{(-4)}{3}3\log 3\right\} \Rightarrow f'(0)=-12+27\log 3 \quad \therefore f'(0)=27\log 3-12$$

EXERCISE 10G (Pg.No.: 435)

1. If $y=(\sin x)^{(\sin x)^{(\sin x)\ldots\ldots\infty}}$ Prove that $\frac{dy}{dx}=\frac{y^2\cot x}{(1-y\log+\tan x)}$

Sol. $y=(\sin x)^{(\sin x)^{(\sin x)\ldots\ldots\infty}} \Rightarrow y=(\sin x)^y$

Taking log both sides, we get, $\log y=\log\{(\sin x)^y\} \Rightarrow \log y=y\log(\sin x)$ ...(i)

On differentiating both sides of (i) w.r.t. x,

$$\frac{d(\log y)}{dy}.\frac{dy}{dx}=y.\frac{d\{\log(\sin x)\}}{d(\sin x)}\times\frac{d(\sin x)}{dx}+\log(\sin x)\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=y.\frac{1}{\sin x}\cos x+\log(\sin x)\frac{dy}{dx} \quad \Rightarrow \frac{1}{y}\frac{dy}{dx}-\log(\sin x)\frac{dy}{dx}=y\cot x$$

$$\Rightarrow \frac{dy}{dx}\left\{\frac{1}{y}-\log(\sin x)\right\}=y\cot x \quad \Rightarrow \frac{dy}{dx}\left\{\frac{1-y\log(\sin x)}{y}\right\}=y\cot x \quad \therefore \frac{dy}{dx}=\frac{y^2\cot x}{1-y\log(\sin x)}$$

2. If $y=(\cos x)^{(\cos x)^{\cos x\ldots\infty}}$, prove that $\frac{dy}{dx}=\frac{-y^2\tan x}{(1-y\log\cos x)}$

Sol. $y=(\cos x)^{(\cos x)^{(\cos x)\ldots\infty}} \Rightarrow y=(\cos x)^y$

Taking log both sides, we get, $\log y=\log\{(\cos x)^y\} \Rightarrow \log y=y\log(\cos x)$ ...(i)

On differentiating both sides of (i) w.r.t. x,

$$\frac{d(\log y)}{dy}.\frac{dy}{dx}=y\frac{d\{\log(\cos x)\}}{d(\cos x)}\times\frac{d(\cos x)}{dx}+\log(\cos x)\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=y.\frac{1}{\cos x}(-\sin x)+\log(\cos x)\frac{dy}{dx} \quad \Rightarrow \frac{1}{y}\frac{dy}{dx}-\log(\cos x)\frac{dy}{dx}=-y\tan x$$

$$\Rightarrow \frac{dy}{dx}\left\{\frac{1}{y}-\log(\cos x)\right\}=-y\tan x \quad \Rightarrow \frac{dy}{dx}\left\{\frac{1-y\log(\cos x)}{y}\right\}=-y\tan x \quad \therefore \frac{dy}{dx}=\frac{-y^2.\tan x}{1-y\log(\cos x)}$$

3. If $y=\sqrt{x+\sqrt{x+\sqrt{x+...\infty}}}$, prove that $\frac{dy}{dx}=\frac{1}{2y-1}$.

Sol. $y=\sqrt{x+\sqrt{x+\sqrt{x+...\infty}}} \Rightarrow y=\sqrt{x+y}$

Squaring both sides, we get, $(y)^2=\left(\sqrt{x+y}\right)^2 \Rightarrow y^2=x+y$

On differentiating both sides of (i) w.r.t. x, $\frac{d(y^2)}{dy}.\frac{dy}{dx}=\frac{d(x)}{dx}+\frac{dy}{dx}$

$\Rightarrow 2y.\frac{dy}{dx}=1+\frac{dy}{dx} \Rightarrow 2y\frac{dy}{dx}-\frac{dy}{dx}=1 \Rightarrow \frac{dy}{dx}(2y-1)=1 \quad \therefore \frac{dy}{dx}=\frac{1}{2y-1}$

4. If $y=\sqrt{\cos x+\sqrt{\cos x+\sqrt{\cos x+...\infty}}}$, prove that $\frac{dy}{dx}=\frac{\sin x}{1-2y}$.

Sol. $y=\sqrt{\cos x+\sqrt{\cos x+\sqrt{\cos x+....\infty}}} \Rightarrow y=\sqrt{\cos x+y}$

Squaring both sides, we get, $(y)^2=\left(\sqrt{\cos x+y}\right)^2 \Rightarrow y^2=\cos x+y \quad ...(i)$

On differentiating both sides of (i) w.r.t. x, $\frac{d(y^2)}{dy}.\frac{dy}{dx}=\frac{d(\cos x)}{dx}+\frac{dy}{dx}$

$\Rightarrow 2y.\frac{dy}{dx}-\frac{dy}{dx}=-\sin x \Rightarrow \frac{dy}{dx}(2y-1)=-\sin x \Rightarrow \frac{dy}{dx}=\frac{-\sin x}{-(1-2y)} \quad \therefore \frac{dy}{dx}=\frac{\sin x}{1-2y}$

5. If $y=\sqrt{\tan x+\sqrt{\tan x+\sqrt{\tan x+...\infty}}}$, prove that $\frac{dy}{dx}=\frac{\sec^2 x}{2y-1}$.

Sol. $y=\sqrt{\tan x+\sqrt{\tan x+\sqrt{\tan x+...\infty}}} \Rightarrow y=\sqrt{\tan x+y}$

Squaring both sides we get, $(y)^2=\left(\sqrt{\tan x+y}\right)^2 \Rightarrow y^2=(\tan x+y) \quad ... \text{(i)}$

On differentiating both sides of (i) w.r.t. x,

$\frac{d(y^2)}{dy}.\frac{dy}{dx}=\frac{d(\tan x)}{dx}+\frac{dy}{dx} \Rightarrow 2y.\frac{dy}{dx}=\sec^2 x+\frac{dy}{dx} \Rightarrow 2y\frac{dy}{dx}-\frac{dy}{dx}=\sec^2 x$

$\Rightarrow \frac{dy}{dx}(2y-1)=\sec^2 x \quad \therefore \frac{dy}{dx}=\frac{\sec^2 x}{2y-1}$

6. If $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+...\infty}}}$ show that $(2y-1)\frac{dy}{dx}=\frac{1}{x}$

Sol. $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+...\infty}}} \Rightarrow y=\sqrt{\log x+y}$

Squaring both sides we get, $(y)^2=\left(\sqrt{\log x+y}\right)^2 \Rightarrow y^2=\log x+y$

On differentiating both sides of (i) w.r.t. x, $\frac{d(y^2)}{dy}.\frac{dy}{dx}=\frac{d(\log x)}{dx}+\frac{dy}{dx}$

$\Rightarrow 2y\frac{dy}{dx}=\frac{1}{x}+\frac{dy}{dx} \Rightarrow 2y\frac{dy}{dx}-\frac{dy}{dx}=\frac{1}{x} \Rightarrow \frac{dy}{dx}(2y-1)=\frac{1}{x} \quad \therefore (2y-1)\frac{dy}{dx}=\frac{1}{x}$

7. If $y=a^{x^{a^{x^{a^{x...\infty}}}}}$ Prove that of $\frac{dy}{dx}=\frac{y^2\log y}{x(1-y\log x\log y)}$

Sol. $y=a^{x^{a^{x^{a^{x...\infty}}}}} \Rightarrow y=a^{x^y}$

Taking both sides, we get, $\log y=\log\left(a^{x^y}\right) \Rightarrow \log y=x^y\log(a)$

Again taking log both sides, we get, $\log(\log y)=\log\left(x^y.\log a\right)$

$\Rightarrow \log(\log y)=\log\left(x^y\right)+\log(\log a) \Rightarrow \log(\log y)=y\log x+\log(\log a)$

$\Rightarrow \log(\log y)=y\log x+\log(\log a)$ ...(i)

On differentiating both sides of (i) w.r.t. x,

$$\frac{d\{\log(\log y)\}}{d(\log y)}\times\frac{d(\log y)}{dy}.\frac{dy}{dx}=y\frac{d(\log x)}{dx}+\log x\frac{dy}{dx}+\frac{d\{\log(\log a)\}}{d(\log a)}\times\frac{d(\log a)}{da}.\frac{d(a)}{dx}$$

$\Rightarrow \frac{1}{\log y}.\frac{1}{y}\frac{dy}{dx}=y.\frac{1}{x}+\log x\frac{dy}{dx}+\frac{1}{\log a}.\frac{1}{a}.0 \Rightarrow \frac{1}{y\log y}\frac{dy}{dx}-\log x\frac{dy}{dx}=\frac{y}{x}$

$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y\log y}-\log x\right)=\frac{y}{x} \Rightarrow \frac{dy}{dx}=\frac{y^2\log y}{x(1-y\log x\log y)}$

8. If $y=x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+...\infty}}}$ prove that $\frac{dy}{dx}=\frac{y}{2y-x}$.

Sol. $y=x+\frac{1}{y} \Rightarrow y=\frac{xy+1}{y} \Rightarrow y^2=xy+1$ ...(i)

On differentiating both sides of (i) w.r.t. x, $\frac{d\left(y^2\right)}{dy}.\frac{dy}{dx}=x\frac{dy}{dx}+y\frac{dx}{dx}+\frac{d(1)}{dx}$

$\Rightarrow 2y\frac{dy}{dx}=x\frac{dy}{dx}+y.1+0 \Rightarrow 2y\frac{dy}{dx}=x\frac{dy}{dx}+y \Rightarrow 2y\frac{dy}{dx}-x\frac{dy}{dx}=y$

$\Rightarrow \frac{dy}{dx}(2y-x)=y \quad \therefore \frac{dy}{dx}=\frac{y}{2y-x}$

EXERCISE 10H (Pg.No.: 438)

1. Differentiation x^6 with respect to $\left(\frac{1}{\sqrt{x}}\right)$.

Sol. Let $u=x^6$, $v=\frac{1}{\sqrt{x}}$, find $\frac{du}{dv}$

$u=x^6$ ...(i)

On differentiating both sides of (i) w.r.t.x, $\frac{du}{dx}=\frac{d\left(x^6\right)}{dx} \Rightarrow \frac{du}{dx}=6x^{6-1} \therefore \frac{du}{dx}=6x^5$

$v=\frac{1}{\sqrt{x}}$ ...(ii)

On differentiating both sides of (ii) w.r.t.x, $\frac{dv}{dx}=\frac{d\left(x^{\frac{-1}{2}}\right)}{dx} \Rightarrow \frac{dv}{dx}=-\frac{1}{2}x^{\frac{-3}{2}}$

$$\therefore \quad \frac{du}{dv}=\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{6x^5}{-\frac{1}{2}x^{-3/2}} \quad \Rightarrow \frac{du}{dv}=-12x^{\frac{13}{2}}$$

2. Differentiation $(\log x)$ with respect to $\cot x$.

Sol. Let $u=\log x,\ v=\cot x$, find $\frac{du}{dv}$

$u=\log x$ ...(i)

On differentiating both sides of (i) w.r.t. x, $\frac{du}{dx}=\frac{1}{x}$

$v=\cot x$ ...(ii)

On differentiating both sides of (ii) w.r.t. x, $\frac{dv}{dx}=-\operatorname{cosec}^2 x$

$$\therefore \quad \frac{du}{dv}=\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{\frac{1}{x}}{-\operatorname{cosec}^2 x} \quad \Rightarrow \frac{du}{dx}=-\frac{1}{x}\cdot\frac{1}{x\operatorname{cosec}^2 x} \quad \Rightarrow \frac{du}{dv}=\frac{-\operatorname{cosec}^2 x}{x}$$

3. Differentiation $e^{\sin x}$ with respect to $\cos x$.

Sol. Let $u=e^{\sin x},\ v=\cos x$ find $\frac{du}{dv}$

$u=e^{\sin x}$ ...(i)

On differentiating both sides of (i) w.r.t. x, $\frac{du}{dx}=\frac{d\left(e^{\sin x}\right)}{d(\sin x)}\times\frac{d(\sin x)}{dx} \quad \Rightarrow \frac{du}{dx}=e^{\sin x}.\cos x$

$v=\cos x$ ...(ii)

On differentiating both sides of (ii) w.r.t. x, $\frac{dv}{dx}=\frac{d(\cos x)}{dx} \quad \Rightarrow \frac{dv}{dx}=-\sin x$

$$\therefore \quad \frac{du}{dv}=\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{e^{\sin x}.\cos x}{-\sin x} \quad \Rightarrow \frac{du}{dv}=-e^{\sin x}\cot x$$

4. Differentiation $\tan^{-1}\sqrt{\frac{1-x^2}{1+x^2}}$ with respect to $\cos^{-1}\left(x^2\right)$.

Sol. Let $u=\tan^{-1}\sqrt{\frac{1-x^2}{1+x^2}},\ v=\cos^{-1}\left(x^2\right)$ find $\frac{du}{dv}$

$u=\tan^{-1}\left(\sqrt{\frac{1-x^2}{1+x^2}}\right)$, Let $x^2=\cos 2\theta \Rightarrow 2\theta=\cos^{-1}\left(x^2\right) \Rightarrow \theta=\frac{1}{2}\cos^{-1}\left(x^2\right)$

$u=\tan^{-1}\left(\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\right) \Rightarrow u=\tan^{-1}\left(\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\right) \Rightarrow u=\tan^{-1}\left(\sqrt{\tan^2\theta}\right)$

$\Rightarrow u=\tan^{-1}(\tan\theta) \quad \Rightarrow u=\theta$

Putting the value of θ, $u=\frac{1}{2}\left\{\cos^{-1}\left(x^2\right)\right\}$

Differentiating both sides with respect to x,

$$\frac{du}{dx}=\frac{1}{2}\frac{d\left\{\cos^{-1}\left(x^2\right)\right\}}{d\left(x^2\right)}\times\frac{d\left(x^2\right)}{dx} \quad \Rightarrow \frac{du}{dx}=-\frac{1}{2}.\frac{1}{\sqrt{1-\left(x^2\right)^2}}.2x \quad \Rightarrow \frac{du}{dx}=-\frac{x}{\sqrt{1-x^4}}$$

$v=\cos^{-1}\left(x^2\right)$

Differentiating both sides with respect to x, $\frac{dv}{dx}=\frac{d\left\{\cos^{-1}\left(x^2\right)\right\}}{d\left(x^2\right)}\times\frac{d\left(x^2\right)}{dx}$

$$\Rightarrow \frac{dv}{dx}=-\frac{1}{\sqrt{1-\left(x^2\right)^2}}.2x \quad \Rightarrow \frac{dv}{dx}=-\frac{2x}{\sqrt{1-x^4}} \quad \Rightarrow \frac{du}{dv}=\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{\frac{-x}{\sqrt{1-x^4}}}{\frac{-2x}{\sqrt{1-x^4}}} \quad \therefore \frac{du}{dv}=\frac{1}{2}$$

5. Differentiation $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Sol. Let $u=\tan^{-1}\left(\frac{2x}{1-x^2}\right), v=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ find $\frac{du}{dx}$

$v=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$; Let $x=\tan\theta \therefore \theta=\tan^{-1}(x)$

$u=\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) \quad \Rightarrow u=\tan^{-1}(\tan 2\theta) \quad \Rightarrow u=2\theta$

Putting the value of θ, $u=2\tan^{-1}(x)$

Differentiating both sides with respect to x, $\frac{du}{dx}=2\frac{d\left\{\tan^{-1}(x)\right\}}{dx} \quad \Rightarrow \frac{du}{dx}=2.\frac{1}{1+x^2} \quad \therefore \frac{du}{dx}=\frac{2}{1+x^2}$

$v=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$; Let $x=\tan\theta \quad \therefore \theta=\tan^{-1}(x)$

$\Rightarrow v=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \quad \Rightarrow v=\sin^{-1}(\sin 2\theta) \quad \Rightarrow v=2\theta$

Putting the value of θ, $v=2\tan^{-1}(x)$

Differentiating both sides with respect to x, $\frac{dv}{dx}=2\frac{d\left\{\tan^{-1}(x)\right\}}{dx} \quad \Rightarrow \frac{dv}{dx}=2.\frac{1}{1+x^2}$

$$\Rightarrow \frac{dv}{dx}=\frac{2}{1+x^2} \quad \therefore \frac{du}{dv}=\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}}=1$$

6. Differentiation $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ with respect to $\cos^{-1}\left(2x^2-1\right)$.

Sol. Let $u=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right), v=\cos^{-1}\left(2x^2-1\right)$ find $\frac{du}{dv}$

$u=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, Let $x=\cos\theta \quad \therefore \theta=\cos^{-1}(x)$

$$u=\tan^{-1}\left(\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}\right) \Rightarrow u=\tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right) \Rightarrow u=\tan^{-1}(\cot\theta)$$

$$\Rightarrow u=\tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\theta\right)\right\} \Rightarrow u=\frac{\pi}{2}-\theta$$

Putting the value of θ, $u=\frac{\pi}{2}-\cos^{-1}(x)$

Differentiating both sides with respect to x, $\frac{du}{dx}=\frac{d\left(\frac{\pi}{2}\right)}{dx}-\frac{d\left(\cos^{-1}x\right)}{dx}$

$$\Rightarrow \frac{du}{dx}=0+\frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{du}{dx}=\frac{1}{\sqrt{1-x^2}}$$

$v=\cos^{-1}\left(2x^2-1\right)$; Let $x=\cos\theta \;\therefore \theta=\cos^{-1}(x)$

$$\Rightarrow v=\cos^{-1}\left(2\cos^2\theta-1\right) \Rightarrow v=\cos^{-1}(\cos 2\theta) \Rightarrow v=2\theta$$

Putting the value of θ, $v=2\cos^{-1}(x)$

Differentiating both sides with respect to x,

$$\frac{dv}{dx}=-2.\frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{dv}{dx}=\frac{-2}{\sqrt{1-x^2}} \quad \therefore \frac{du}{dv}=\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{\frac{1}{\sqrt{1-x^2}}}{-\frac{2}{\sqrt{1-x^2}}}=-\frac{1}{2}$$

7. Differentiation $\sin^3 x$ with respect to $\cos^3 x$.

Sol. Let $u=\sin^3 x$, $v=\cos^3 x$, find $\frac{du}{dv}$

$u=\sin^3 x$, Differentiating both sides with respect to x,

$$\frac{du}{dx}=\frac{d\left(\sin^3 x\right)}{d(\sin x)}.\frac{d(\sin x)}{dx} \Rightarrow \frac{du}{dx}=3\sin^2 x\cos x$$

$v=\cos^3 x$

Differentiating both sides with respect to x, $\frac{dv}{dx}=\frac{d\left(\cos^3 x\right)}{d(\cos x)}\times\frac{d(\cos x)}{dx} \Rightarrow \frac{dv}{dx}=-3\cos^2 x\sin x$

$$\therefore \frac{du}{dv}=\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{3\sin^2 x\cos x}{-3\cos^2 x\sin x}=-\frac{\sin x}{\cos x} \Rightarrow \frac{du}{dv}=-\tan x$$

8. Differentiation $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ with respect to $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$.

Sol. Let $u=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $v=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ find $\frac{du}{dv}$

$u=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, Let $x=\tan\theta \;\therefore \theta=\tan^{-1}(x)$

$$u=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) \Rightarrow u=\cos^{-1}(\cos 2\theta) \Rightarrow u=2\theta$$

Putting the value of θ, $u=2\tan^{-1}(x)$

Differentiating both sides with respect to x, $\frac{du}{dx}=2\frac{d(\tan^{-1}x)}{dx}$ $\Rightarrow\frac{du}{dx}=\frac{2}{1+x^2}$

$v=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$, Let $x=\tan\theta$ $\therefore\theta=\tan^{-1}(x)$

$\Rightarrow v=\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$ $\Rightarrow v=\tan^{-1}(\tan 3\theta)$ $\Rightarrow v=3\theta$

Putting the value of θ, $v=3\tan^{-1}x$

Differentiating both sides with respect to x,

$\frac{dv}{dx}=3\frac{d\{\tan^{-1}(x)\}}{dx}$ $\Rightarrow\frac{dv}{dx}=3.\frac{1}{1+x^2}$ $\Rightarrow\frac{dv}{dx}=\frac{3}{1+x^2}$

$\therefore\ \frac{du}{dv}=\frac{du/dx}{dv/dx}=\frac{\frac{2}{1+x^2}}{\frac{3}{1+x^2}}=\frac{2}{3}$ $\Rightarrow\frac{du}{dv}=\frac{2}{3}$

9. Differentiation $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Sol. Let $u=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, $v=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ find $\frac{du}{dv}$

$u=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, Let $x=\tan\theta$ $\therefore\theta=\tan^{-1}(x)$

$\Rightarrow u=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$ $\Rightarrow u=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$ $\Rightarrow u=\tan^{-1}\left(\frac{\frac{1}{\cos\theta}-\frac{1}{1}}{\frac{\sin\theta}{\cos\theta}}\right)$

$\Rightarrow u=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$ $\Rightarrow u=\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$

$\Rightarrow u=\tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)$ $\Rightarrow u=\tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)$ $\Rightarrow u=\tan^{-1}\left(\tan\frac{\theta}{2}\right)$ $\Rightarrow u=\frac{\theta}{2}$

Putting the value of θ, $u=\frac{1}{2}\tan^{-1}(x)$

Differentiating both sides with respect to x,

$\frac{du}{dx}=\frac{1}{2}\frac{d\{\tan^{-1}(x)\}}{dx}$ $\Rightarrow\frac{du}{dx}=\frac{1}{2}.\frac{1}{1+x^2}$ $\Rightarrow\frac{du}{dx}=\frac{1}{2(1+x^2)}$

$v=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, Let $x=\tan\theta$ $\therefore\theta=\tan^{-1}(x)$

$$v=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\quad\Rightarrow v=\sin^{-1}(\sin 2\theta)\quad\Rightarrow v=2\theta$$

Putting the value of θ, $v=2\tan^{-1}(x)$

Differentiating both sides with respect to x, $\frac{dv}{dx}=2\frac{d\left(\tan^{-1}x\right)}{dx}$

$$\Rightarrow\frac{dv}{dx}=2.\frac{1}{1+x^2}\quad\Rightarrow\frac{dv}{dx}=\frac{2}{1+x^2}\quad\Rightarrow\frac{du}{dx}=2\frac{1}{\left(1+x^2\right)}$$

$$\therefore\ \frac{du}{dv}=\frac{du/dx}{dv/dx}=\frac{\frac{1}{2\left(1+x^2\right)}}{\frac{2}{\left(1+x^2\right)}}=\frac{1}{4}\quad\Rightarrow\frac{du}{dv}=\frac{1}{4}$$

10. Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ w.r.t. $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$, when $x\neq 0$

Sol. Put $x=\cos\theta$ so that $\theta=\cos^{-1}x$

Let $u=\tan^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right]=\tan^{-1}\left[\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right]=\tan^{-1}\left[\frac{\sin\theta}{\cos\theta}\right]=\tan^{-1}(\tan\theta)=\theta=\cos^{-1}x$

$$\Rightarrow\frac{du}{dx}=\frac{d}{dx}\cos^{-1}x=\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

and let $v=\cos^{-1}\left(2x\sqrt{1-x^2}\right)=\cos^{-1}\left\{2\cos\theta\sqrt{1-\cos^2\theta}\right\}$

$$=\cos^{-1}\{2\cos\theta\sin\theta\}=\cos^{-1}\{\sin 2\theta\}=\cos^{-1}\left\{\cos\left(\frac{\pi}{2}-2\theta\right)\right\}=\frac{\pi}{2}-2\theta=\frac{\pi}{2}-2\cos^{-1}x$$

$$\Rightarrow\frac{dv}{dx}=0-2\left(\frac{-1}{\sqrt{1-x^2}}\right)=\frac{2}{\sqrt{1-x^2}}\quad\therefore\ \frac{du}{dv}=\frac{du/dx}{dv/dx}=\frac{\left(\frac{-1}{\sqrt{1-x^2}}\right)}{\left(\frac{2}{\sqrt{1-x^2}}\right)}=\left(\frac{-1}{2}\right)$$

EXERCISE 10 I (Pg.No.: 442)

Find $\frac{dy}{dx}$, when:

1. $x=at^2$, $y=2at$

Sol. $x=at^2$ ...(i)

Differentiating both sides of (i) w.r.t. t, $\frac{dx}{dt}=a\frac{d\left(t^2\right)}{dt}\quad\Rightarrow\frac{dx}{dt}=a.2t\quad\Rightarrow\frac{dx}{dt}=2at$

$y=2at$ ...(ii)

Differentiating both sides of (ii) w.r.t. t, $\frac{dy}{dt}=2a\frac{d(t)}{dt}\quad\Rightarrow\frac{dy}{dt}=2a.1\quad\Rightarrow\frac{dy}{dt}=2a$

$$\therefore\ \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2a}{2at}=\frac{1}{t}$$

2. $x = a\cos\theta,\quad y = b\sin\theta$

Sol. $x = a\cos\theta$ ...(i)

Differentiating both sides of (i) w.r.t. θ, $\dfrac{dx}{d\theta} = a\dfrac{d(\cos\theta)}{d\theta} \Rightarrow \dfrac{dx}{d\theta} = -a\sin\theta$

$y = b\sin\theta$ ...(ii)

Differentiating both sides of (i) w.r.t. θ, $\dfrac{dy}{d\theta} = b\dfrac{d(\sin\theta)}{d\theta} \Rightarrow \dfrac{dy}{d\theta} = b\cos\theta$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cos\theta}{-a\sin\theta} \Rightarrow \frac{dy}{dx} = \frac{-b}{a}\cot\theta$$

3. $x = a\cos^2\theta,\quad y = b\sin^2\theta\ \dfrac{dy}{dx}$

Sol. $x = a\cos^2\theta$ ...(i)

Differentiating both sides of (i) w.r.t. θ, $\dfrac{dx}{d\theta} = a2\cos\theta.(-\sin\theta) \Rightarrow \dfrac{dx}{d\theta} = -a\sin 2\theta$

$y = b\sin^2\theta$ ...(ii)

Differentiating both sides of (ii) w.r.t. θ, $\dfrac{dy}{d\theta} = b.2\sin\theta.\cos\theta$

$$\Rightarrow \frac{dy}{d\theta} = b\sin 2\theta \quad \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\sin^2\theta}{-a\sin^2\theta} \quad \therefore \frac{dy}{dx} = -\frac{b}{a}$$

4. $x = a\cos^3\theta,\ y = a\sin^3\theta$

Sol. $x = a\cos^3\theta$ ...(i)

Differentiating both sides of (i) w.r.t. θ, $\dfrac{dx}{d\theta} = a.3\cos^2\theta(-\sin\theta) \Rightarrow \dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$

$y = a\sin^3\theta$ ...(ii)

Differentiating both sides of (ii) w.r.t. θ, $\dfrac{dy}{d\theta} = a\sin^2\theta.\cos\theta$

$$\Rightarrow \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta \quad \therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} \quad \Rightarrow \frac{dy}{dx} = -\frac{\sin\theta}{\cos\theta} \quad \Rightarrow \frac{dy}{dx} = -\tan\theta$$

5. $x = a(1-\cos\theta), y = a(\theta+\sin\theta)$

Sol. $x = a(1-\cos\theta)$ ...(i)

Differentiating both sides of (i) w.r.t. θ

$$\frac{dx}{d\theta} = a\left\{\frac{d(1)}{d\theta} - \frac{d(\cos\theta)}{d\theta}\right\} \quad \Rightarrow \frac{dx}{d\theta} = a\{0+\sin\theta\} \quad \Rightarrow \frac{dx}{d\theta} = a\sin\theta$$

$y = a(\theta+\sin\theta)$ ...(ii)

Differentiating both sides of (ii) w.r.t. θ, $\dfrac{dy}{d\theta} = a\left\{\dfrac{d(\theta)}{d\theta} + \dfrac{d(\sin\theta)}{d\theta}\right\} \Rightarrow \dfrac{dy}{d\theta} = a\{1+\cos\theta\}$

$$\therefore \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{a(1+\cos\theta)}{a\sin\theta} \Rightarrow \frac{dy}{dx}=\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \Rightarrow \frac{dy}{dx}=\cot\frac{\theta}{2}$$

6. $x=a\log t,\ y=b\sin t$

Sol. $x=a\log t$ ...(i)

Differentiating both sides of (i) w.r.t t, $\frac{dx}{dt}=a.\frac{1}{t} \Rightarrow \frac{dx}{dt}=\frac{a}{t}$

$y=b\sin t$ ...(ii)

Differentiating both sides of (ii) w.r.t t, $\frac{dy}{dt}=b\cos t$

$$\therefore \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{b\cos t}{\frac{a}{t}} \Rightarrow \frac{dy}{dx}=\frac{bt\cos t}{a}$$

7. $x=(\log t+\cos t),\ y=e^t+\sin t$ find $\frac{dy}{dx}$

Sol. $x=\log t+\cos t$ ...(i)

Differentiating both sides of (i) w.r.t t, $\frac{dx}{dt}=\frac{1}{t}+(-\sin t) \Rightarrow \frac{dx}{dt}=\frac{1}{t}-\sin t \Rightarrow \frac{dx}{dt}=\frac{1-t\sin t}{t}$

$y=e^t+\sin t$ ...(ii)

Differentiating both sides of (ii) w.r.t t, $\frac{dy}{dt}=e^t+\cos t$

$$\therefore \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{e^t+\cos t}{\frac{1-t\sin t}{t}} \Rightarrow \frac{dy}{dx}=\frac{t(e^t+\cos t)}{1-t\sin t}$$

8. $x=\cos\theta+\cos 2\theta, y=\sin\theta+\sin 2\theta$

Sol. $x=\cos\theta+\cos 2\theta$ ...(i)

Differentiating both sides of (i) w.r.t. θ, $\frac{dx}{d\theta}=-\sin\theta+(-\sin 2\theta).2 \Rightarrow \frac{dx}{d\theta}=-(\sin\theta+2\sin 2\theta)$

$y=\sin\theta+\sin 2\theta$ ...(ii)

Differentiating both sides of (ii) w.r.t. θ, $\frac{dy}{d\theta}=\cos\theta+\cos 2\theta.2$

$$\Rightarrow \frac{dy}{d\theta}=\cos\theta+2\cos 2\theta \Rightarrow \frac{dy}{d\theta}=\cos\theta+2\cos 2\theta$$

$$\therefore \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{\cos\theta+2\cos 2\theta}{-(\sin\theta+2\sin 2\theta)} \Rightarrow \frac{dy}{dx}=-\frac{(\cos\theta+2\cos 2\theta)}{(\sin\theta+2\sin 2\theta)}$$

9. $x=\sqrt{\sin 2\theta},\ y=\sqrt{\cos 2\theta}$

Sol. $x=\sqrt{\sin 2\theta}$ ...(i)

Differentiating both sides of (i) w.r.t. θ, $\frac{dx}{d\theta}=\frac{1}{2\sqrt{\sin 2\theta}}.\cos 2\theta.2 \Rightarrow \frac{dx}{d\theta}=\frac{\cos 2\theta}{\sqrt{\sin 2\theta}}$

$y=\sqrt{\cos 2\theta}$ ...(ii)

Differentiating both sides of (ii) w.r.t. θ, $\frac{dy}{d\theta}=\frac{1}{2\sqrt{\cos 2\theta}}\{-(\sin 2\theta).2\}$ $\Rightarrow \frac{dy}{d\theta}=-\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}$

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{-\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}}{\frac{\cos 2\theta}{\sqrt{\sin 2\theta}}} \quad \Rightarrow \frac{dy}{dx}=-(\tan 2\theta)^{\frac{3}{2}}$$

10. $x=e^{\theta}(\sin\theta+\cos\theta), y=e^{\theta}(\sin\theta-\cos\theta)$

Sol. $x=e^{\theta}(\sin\theta+\cos\theta)$ ...(i)

Differentiating both sides of (i) w.r.t. θ, $\frac{dx}{d\theta}=e^{\theta}\left\{\frac{d(\sin\theta+\cos\theta)}{d\theta}\right\}+\left\{(\sin\theta+\cos\theta)\frac{d(e^{\theta})}{d\theta}\right\}$

$$\Rightarrow \frac{dx}{d\theta}=e^{\theta}\{(\cos\theta-\sin\theta)+(\sin\theta+\cos\theta)\} \quad \Rightarrow \frac{dx}{d\theta}=2e^{\theta}\cos\theta$$

$y=e^{\theta}(\sin\theta-\cos\theta)$ ...(ii)

Differentiating both sides of (ii) w.r.t. θ, $\frac{dy}{d\theta}=e^{\theta}\frac{d(\sin\theta-\cos\theta)}{d\theta}+(\sin\theta-\cos\theta)\frac{d(e^{\theta})}{d\theta}$

$$\Rightarrow \frac{dy}{d\theta}=e^{\theta}(\cos\theta+\sin\theta)+(\sin\theta-\cos\theta)e^{\theta} \quad \Rightarrow \frac{dy}{d\theta}=e^{\theta}(\cos\theta+\sin\theta+\sin\theta-\cos\theta)$$

$$\Rightarrow \frac{dy}{d\theta}=2e^{\theta}\sin\theta \quad \Rightarrow \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{2e^{\theta}\sin\theta}{2e^{\theta}\cos\theta} \quad \therefore \frac{dy}{dx}=\tan\theta$$

11. $x=a(\cos\theta+\theta\sin\theta), y=a(\sin\theta-\theta\cos\theta)$

Sol. $x=a(\cos\theta+\theta\sin\theta)$ ...(i)

Differentiating both sides of (i) w.r.t. θ, $\frac{dx}{d\theta}=a\left\{\frac{d(\cos\theta)}{d\theta}+\frac{d(\theta\sin\theta)}{d\theta}\right\}$

$$\Rightarrow \frac{dx}{d\theta}=a\{-\sin\theta+\theta\cos\theta+\sin\theta.1\} \quad \Rightarrow \frac{dx}{d\theta}=a\theta\cos\theta$$

$y=a(\sin\theta-\theta\cos\theta)$ ...(ii)

Differentiating both sides of (ii) w.r.t. θ, $\frac{dy}{d\theta}=a\left\{\frac{d(\sin\theta)}{d\theta}-\frac{d(\theta\cos\theta)}{d\theta}\right\}$

$$\Rightarrow \frac{dy}{d\theta}=a\left[\cos\theta-\left\{\theta\frac{d(\cos\theta)}{d\theta}+\cos\theta\frac{d\theta}{d\theta}\right\}\right] \quad \Rightarrow \frac{dy}{d\theta}=a\{\cos\theta+\theta\sin\theta-\cos\theta\} \quad \Rightarrow \frac{dy}{d\theta}=a\theta\sin\theta$$

$$\therefore \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{a\theta\sin\theta}{a\theta\cos\theta} \quad \Rightarrow \frac{dy}{dx}=\tan\theta$$

12. $x=\frac{3at}{1+t^2}, \; y=\frac{3at^2}{1+t^2}$

Sol. $x=\frac{3at}{1+t^2}$ ...(i)

Differentiating both sides of (i) w.r.t t, $\frac{dx}{dt}=\left\{\frac{\left(1+t^2\right)\frac{d(3at)}{dt}-3at\frac{d\left(1+t^2\right)}{dt}}{\left(1+t^2\right)^2}\right\}$

$\Rightarrow \frac{dx}{dt}=\left\{\frac{\left(1+t^2\right)3a-3at.2t}{\left(1+t^2\right)^2}\right\}$ $\Rightarrow \frac{dx}{dt}=\left\{\frac{3a+3t^2a-6at^2}{\left(1+t^2\right)^2}\right\}$

$\Rightarrow \frac{dx}{dt}=\left\{\frac{3a-3at^2}{\left(1+t^2\right)^2}\right\}$ $\Rightarrow \frac{dx}{dt}=\frac{3a\left(1-t^2\right)}{\left(1+t^2\right)^2}$

$y=\frac{3at^2}{1+t^2}$ ...(ii)

Differentiating both sides of (ii) w.r.t t, $\frac{dy}{dt}=\left\{\frac{\left(1+t^2\right)\frac{d\left(3at^2\right)}{dt}-3at^2\frac{d\left(1+t^2\right)}{dt}}{\left(1+t^2\right)^2}\right\}$

$\Rightarrow \frac{dy}{dt}=\left\{\frac{\left(1+t^2\right)2t3a-3at^2.2t}{\left(1+t^2\right)^2}\right\}$ $\Rightarrow \frac{dy}{dt}=\left\{\frac{6at+6t^3a-6t^3a}{\left(1+t^2\right)^2}\right\}$ $\Rightarrow \frac{dy}{dt}=\frac{6at}{\left(1+t^2\right)^2}$

$\therefore \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\frac{6at}{\left(1+t^2\right)^2}}{\frac{3a\left(1-t^2\right)}{\left(1+t^2\right)^2}}$ $\Rightarrow \frac{dy}{dx}=\frac{2t}{1-t^2}$

13. $x=\frac{1-t^2}{1+t^2},\ y=\frac{2t}{1+t^2}$

Sol. $x=\frac{1-t^2}{1+t^2}$ ...(i)

Differentiating both sides of (i) w.r.t t, $\frac{dx}{dt}=\left\{\frac{\left(1+t^2\right)\frac{d\left(1-t^2\right)}{dt}-\left(1-t^2\right)\frac{d\left(1+t^2\right)}{dt}}{\left(1+t^2\right)^2}\right\}$

$\Rightarrow \frac{dx}{dt}=\left\{\frac{\left(1+t^2\right)(-2t)-\left(1-t^2\right)2t}{\left(1+t^2\right)^2}\right\}$ $\Rightarrow \frac{dx}{dt}=\frac{2t\left\{-1-t^2-1+t^2\right\}}{\left(1+t^2\right)^2}$ $\Rightarrow \frac{dx}{dt}=-\frac{-4t}{\left(1+t^2\right)^2}$

$y=\frac{2t}{1+t^2}$ ...(ii)

Differentiating both sides of (ii) w.r.t t, $\frac{dy}{dt}=\left\{\frac{\left(1+t^2\right)\frac{d(2t)}{dt}-2t\frac{d\left(1+t^2\right)}{dt}}{\left(1+t^2\right)^2}\right\}$

$$\Rightarrow \frac{dy}{dt}=\left\{\frac{(1+t^2).2-2t.2t}{(1+t^2)^2}\right\} \Rightarrow \frac{dy}{dt}=\left\{\frac{2+2t^2-4t^2}{(1+t^2)^2}\right\} \Rightarrow \frac{dy}{dt}=\frac{2-2t^2}{(1+t^2)^2} \Rightarrow \frac{dy}{dt}=\frac{2(1-t^2)}{(1+t^2)^2}$$

$$\therefore \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\frac{2(1-t^2)}{(1+t^2)^2}}{\frac{-4t}{(1+t^2)^2}}=\frac{2(1-t^2)}{-4t} \Rightarrow \frac{dy}{dx}=\frac{t^2-1}{2t}$$

14. $x=\cos^{-1}\left(\frac{1}{\sqrt{1+t^2}}\right),\ y=\sin^{-1}\left(\frac{t}{\sqrt{1+t^2}}\right)$

Sol. $x=\cos^{-1}\left(\frac{1}{\sqrt{1+t^2}}\right)$, Let $t=\tan\theta \therefore \theta=\tan^{-1}(t)$

$$x=\cos^{-1}\left(\frac{1}{\sqrt{1+\tan^2\theta}}\right) \Rightarrow x=\cos^{-1}\left(\frac{1}{\sec\theta}\right) \Rightarrow x=\cos^{-1}(\cos\theta) \Rightarrow x=\theta$$

Putting the value of θ, $x=\tan^{-1}(t)$ ...(i)

On differentiating both sides of (i) w.r.t. t, $\frac{dx}{dt}=\frac{1}{1+t^2}$

$y=\sin^{-1}\left(\frac{t}{\sqrt{1+t^2}}\right)$, Let $t=\tan\theta \ \therefore \theta=\tan^{-1}(t)$

$$\Rightarrow y=\sin^{-1}\left(\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}\right) \Rightarrow y=\sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right) \Rightarrow y=\sin^{-1}(\sin\theta) \Rightarrow y=\theta$$

Putting the value of θ, $y=\tan^{-1}(t)$ ..(ii)

On differentiating both sides of (i) w.r.t. t, $\frac{dy}{dt}=\frac{1}{1+t^2} \ \therefore \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\frac{1}{1+t^2}}{\frac{1}{1+t^2}}=1$

15. If $x=3\cos t-2\cos^3 t,\ y=3\sin t-2\sin^3 t$ show that $\frac{dy}{dx}=\cot t$

Sol. $x=3\cos t-2\cos^3 t$ ...(i)

Differentiating both sides of (i) w.r.t. t, $\frac{dx}{dt}=-3\sin t-2.3\cos^2 t(-\sin t)$

$$\Rightarrow \frac{dx}{dt}=-3\sin t+6\cos^2 t.\sin t \Rightarrow \frac{dx}{dt}=-3\sin t(1-2\cos^2 t)$$

$y=3\sin t-2\sin^3 t$ ...(ii)

Differentiating both sides of (ii) w.r.t. t, $\frac{dy}{dt}=3\cos t-2.3\sin^2 t.\cos t$

$$\Rightarrow \frac{dy}{dt}=3\cos t\{1-2\sin^2 t\} \Rightarrow \frac{dy}{dx}=3\cos t\{1-2(1-\cos^2 t)\} \Rightarrow \frac{dy}{dt}=3\cos t\{1-2+2\cos^2 t\}$$

$$\Rightarrow \frac{dy}{dt}=3\cos t\{-1+2\cos^2 t\} \Rightarrow \frac{dy}{dt}=-3\cos t\{1-2\cos^2 t\}$$

$\therefore\ \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-3\cos t\left\{1-2\cos^2 t\right\}}{-3\sin t\left(1-2\cos^2 t\right)} \quad \Rightarrow \frac{dy}{dx}=\cot t$

16. If $x=\frac{1+\log t}{t^2}$ and $y=\frac{3+2\log t}{t}$ show that $\frac{dy}{dt}=t$

Sol. $x=\frac{1+\log t}{t^2}$ ...(i)

Differentiating both sides of (ii) w.r.t. t, $\frac{dx}{dt}=\left\{\frac{t^2\frac{d(1+\log t)}{dt}-(1+\log t)\frac{d(t^2)}{dt}}{(t^2)^2}\right\}$

$\Rightarrow \frac{dx}{dt}=\left\{\frac{t^2.\frac{1}{t}-(1+\log t).2t}{t^4}\right\} \quad \Rightarrow \frac{dx}{dt}=\left\{\frac{t-2t-2t\log t}{t^4}\right\} \quad \Rightarrow \frac{dx}{dt}=\frac{-t-2t\log t}{t^4}$

$\Rightarrow \frac{dx}{dt}=\frac{-t(1+2\log t)}{t^4} \quad \Rightarrow \frac{dx}{dt}=-\frac{t(1+2\log t)}{t^4} \quad \Rightarrow \frac{dx}{dt}=\frac{-(1+2\log t)}{t^3}$

$y=\frac{3+2\log t}{t}$ ...(ii)

Differentiating both sides of (ii) w.r.t. t, $\frac{dy}{dx}=\left\{\frac{t.\frac{d(3+2\log t)}{dt}-(3+2\log t)\frac{d(t)}{dt}}{t^2}\right\}$

$\Rightarrow \frac{dy}{dt}=\left\{\frac{t.2.\frac{1}{t}-(3+2\log t).1}{t^2}\right\} \quad \Rightarrow \frac{dy}{dt}=\left\{\frac{2-3-2\log t}{t^2}\right\} \quad \Rightarrow \frac{dy}{dt}=\frac{-1-2\log t}{t^2}$

$\Rightarrow \frac{dy}{dt}=\frac{-(1+2\log t)}{t^2} \qquad \therefore\ \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\frac{-(1+2\log t)}{t^2}}{\frac{-(1+2\log t)}{t^3}} \quad \Rightarrow \frac{dy}{dx}=t$

17. If $x=a(\theta-\sin\theta), y=a(1-\cos\theta)$, find $\frac{dy}{dx}$ at $\theta=\frac{\pi}{2}$

Sol. $x=a(\theta-\sin\theta)$ ...(i)

Differentiating both sides of (i) w.r.t. θ, $\frac{dx}{d\theta}=a\left\{\frac{d(\theta)}{d\theta}-\frac{d(\sin\theta)}{d\theta}\right\} \quad \Rightarrow \frac{dx}{d\theta}=a\{1-\cos\theta\}$

$y=a(1-\cos\theta)$ ...(ii)

Differentiating both sides of (ii) w.r.t. θ, $\frac{dy}{d\theta}=a\left\{\frac{d(1)}{d\theta}-\frac{d(\cos\theta)}{d\theta}\right\}$

$\Rightarrow \frac{dy}{d\theta}=a\{0+\sin\theta\} \quad \Rightarrow \frac{dy}{d\theta}=a\sin\theta$

$\therefore \dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}=\dfrac{a\sin\theta}{a(1-\cos\theta)} \Rightarrow \dfrac{dy}{dx}=\dfrac{\sin\theta}{1-\cos\theta} \Rightarrow \dfrac{dy}{dx}_{\left(\theta=\frac{\pi}{2}\right)}=\dfrac{\sin\frac{\pi}{2}}{1-\cos\frac{\pi}{2}}=\dfrac{1}{1-0} \Rightarrow \dfrac{dy}{dx}=1$

18. If $x=2\cos\theta-\cos 2\theta$ and $y=2\sin\theta-\sin 2\theta$ show that $\dfrac{dy}{dx}=-1$ at $\theta=\dfrac{\pi}{2}$

Sol. $x=2\cos\theta-\cos 2\theta$ … (i)

Differentiating both sides of (i) w.r.t. θ, $\dfrac{dx}{d\theta}=2\dfrac{d(\cos\theta)}{d\theta}-\dfrac{d(\cos 2\theta)}{d\theta}$

$\Rightarrow \dfrac{dx}{d\theta}=-2\sin\theta+\sin 2\theta.2 \Rightarrow \dfrac{dx}{d\theta}=-2\sin\theta+2\sin 2\theta \Rightarrow \dfrac{dx}{d\theta}=2(\sin 2\theta-\sin\theta)$

$y=2\sin\theta-\sin 2\theta$ … (ii)

Differentiating both sides of (ii) w.r.t. θ, $\dfrac{dy}{d\theta}=2\dfrac{d(\sin\theta)}{d\theta}-\dfrac{d(\sin 2\theta)}{d\theta}$

$\Rightarrow \dfrac{dy}{d\theta}=2\cos\theta-\cos 2\theta.2 \Rightarrow \dfrac{dy}{d\theta}=2(\cos\theta-\cos 2\theta)$

$\therefore \dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}=\dfrac{2(\cos\theta-\cos 2\theta)}{2(\sin 2\theta-\sin\theta)} \Rightarrow \dfrac{dy}{dx}=\dfrac{\cos\theta-\cos 2\theta}{\sin 2\theta-\sin\theta}$

$\Rightarrow \dfrac{dy}{dx}$ at $\left(\theta=\dfrac{\pi}{2}\right)=\dfrac{\cos\frac{\pi}{2}-\cos 2.\frac{\pi}{2}}{\sin\left(2.\frac{\pi}{2}\right)-\sin\frac{\pi}{2}}=\dfrac{0-(-1)}{0-1}=\dfrac{1}{-1} \Rightarrow \dfrac{dy}{dx}=-1$

19. If $x=\dfrac{\sin^3 t}{\sqrt{\cos 2t}}$, $y=\dfrac{\cos^3 t}{\sqrt{\cos 2t}}$ find $\dfrac{dy}{dx}$.

Sol. $x=\dfrac{\sin^3 t}{\sqrt{\cos 2t}}$ …(i)

Taking log both sides we get , $\log x=\log\left(\dfrac{\sin^3 t}{\sqrt{\cos 2t}}\right)$

$\Rightarrow \log x=\log(\sin^3 t)-\log(\sqrt{\cos 2t}) \Rightarrow \log x=3\log(\sin t)-\dfrac{1}{2}\log(\cos 2t)$ …(ii)

On differentiating both sides (ii) w.r.t. t, $\dfrac{1}{x}\dfrac{dx}{dt}=3.\dfrac{1}{\sin t}.\cos t-\dfrac{1}{2}.\dfrac{1}{\cos 2t}.(-\sin 2t).2$

$\Rightarrow \dfrac{dx}{dt}=x(3\cot+\tan 2t) \Rightarrow \dfrac{dx}{dt}=\dfrac{\sin^3 t}{\sqrt{\cos 2t}}(3\cot t+\tan 2t)$

$y=\dfrac{\cos^3 t}{\sqrt{\cos 2t}}$ …(iii)

Taking log both sides we get , $\log y=\log\left(\dfrac{\cos^3 t}{\sqrt{\cos 2t}}\right)$

$\Rightarrow \log y=\log(\cos^3 t)-\log(\sqrt{\cos 2t}) \Rightarrow \log y=3\log(\cos t)-\dfrac{1}{2}\log(\cos 2t)$ …(iv)

On differentiating both sides (iv) w.r.t. t, $\dfrac{1}{y}.\dfrac{dy}{dt}=3.\dfrac{1}{\cos t}.(-\sin t)-\dfrac{1}{2}.\dfrac{1}{\cos 2t}.(-\sin 2t).2$

$$\Rightarrow \frac{dy}{dt} = y(-3\tan t + \tan 2t) \quad \Rightarrow \frac{dy}{dt} = \frac{\cos^3 t}{\sqrt{\cos 2t}}(-3\tan t + \tan 2t)$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{\cos^3 t}{\sqrt{\cos 2t}}(-3\tan t + \tan 2t)}{\frac{\sin^3 t}{\sqrt{\cos 2t}}(3\cot(t) + \tan 2t)} = \frac{\cot^3 t\left(-3\tan t + \frac{2\tan t}{1-\tan^2 t}\right)}{\left(\frac{3}{\tan t} + \frac{2\tan t}{1-\tan^2 t}\right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(1-3\tan^2 t)}{3\tan t - \tan^3 t} = -\frac{1}{\tan 3t} \quad \Rightarrow \frac{dy}{dx} = -\cot 3t$$

20. If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$ find $\frac{d^2y}{dx^2}, \theta = \frac{\pi}{2}$.

Sol. $x = 2\cos\theta - \cos 2\theta \quad \ldots(i)$

On differentiating both sides of (i) w.r.t. θ, $\frac{dx}{d\theta} = -2\sin\theta + \sin 2\theta.2 \quad \Rightarrow \frac{dx}{d\theta} = 2(\sin 2\theta - \sin\theta)$

$y = 2\sin\theta - \sin 2\theta \quad \ldots(ii)$

On differentiating both sides of (ii) w.r.t. θ, $\frac{dy}{d\theta} = 2\cos\theta - \cos 2\theta.2 \quad \Rightarrow \frac{dy}{d\theta} = 2(\cos\theta - \cos 2\theta)$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2(\cos\theta - \cos 2\theta)}{2(\sin 2\theta - \sin\theta)} \quad \therefore \frac{dy}{dx} = \frac{\cos\theta - \cos 2\theta}{\sin 2\theta - \sin\theta} \quad \ldots(iii)$$

On differentiating both sides of (iii) w.r.t. θ

$$\frac{d^2y}{dx^2} = \left\{\frac{(\sin 2\theta - \sin\theta)\frac{d(\cos\theta - \cos 2\theta)}{d\theta} - (\cos\theta - \cos 2\theta)\frac{d(\sin 2\theta - \sin\theta)}{d\theta}}{(\sin 2\theta - \sin\theta)^2}\right\} \times \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left\{\frac{(\sin 2\theta - \sin\theta)(-\sin\theta + \sin 2\theta.2) - (\cos\theta - \cos 2\theta)(\cos 2\theta.2 - \cos\theta)}{(\sin 2\theta - \sin\theta)^2}\right\} \times \frac{1}{2(\sin 2\theta - \sin\theta)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{(\sin 2\theta - \sin\theta)(2\sin 2\theta - \sin\theta) - (\cos\theta - \cos 2\theta)(2\cos 2\theta - \cos 2\theta)}{2(\sin 2\theta - \sin\theta)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2}\left(\theta = \frac{\pi}{2}\right) = -\frac{3}{2} \quad \therefore \frac{d^2y}{dx^2} = -\frac{3}{2}$$

21. If $x = a(\theta - \sin\theta), y = a(1 + \cos\theta)$, find $\frac{d^2y}{dx^2}$

Sol. $x = a(\theta - \sin\theta)$

Diff. both side w.r.t. $\frac{dx}{d\theta} = a(1 - \cos\theta)$

Now, $y = a(1 + \cos\theta)$

Diff both side w.r..t θ , we get $\frac{dy}{d\theta} = a(0 + (-\sin\theta))$

$$\frac{dy}{d\theta} = -a\sin\theta$$

$\because \frac{dy}{dx}=\frac{-a\sin\theta}{a(1-\cos\theta)}=\frac{-2\sin\frac{\theta}{2}\geq\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$

$\frac{dy}{dx}=-\cot\frac{\theta}{2}$

Again differentiating w.r..t x we get $\frac{d^2y}{dx^2}=-\frac{d}{dx}\left(\cot\frac{\theta}{2}\right)$

$\frac{d^2y}{dx^2}=\text{cosec}^2\frac{\theta}{2}\times\frac{1}{2}\times\frac{d\theta}{dx}$

$\frac{d^2y}{dx^2}=\frac{1}{2}\text{cosec}^2\frac{\theta}{2}\times\frac{1}{2a\sin^2\frac{\theta}{2}}$

$\frac{d^2y}{dx^2}=\frac{1}{4a}\cdot\text{cosec}^4\frac{\theta}{2}$

EXERCISE 10 J (Pg.No.: 449)

1. Find the second derivative of

(i) x^{11} (ii) 5^x (iii) $y=\tan x$ (iv) $y=\cos^{-1}x$

Sol. **(i)** $y=x^{11}$ ...(i)

Differentiating both sides of (1), w.r.t. x, $\frac{dy}{dx}=\frac{d\left(x^{11}\right)}{dx}\Rightarrow\frac{dy}{dx}=11x^{10}$ ...(ii)

Again Differentiating both sides of (ii) w.r.t. x, $\frac{d^2y}{dx^2}=11\frac{d\left(x^{10}\right)}{dx}$ $\therefore\frac{d^2y}{dx^2}=110x^9$

(ii) $y=5^x$ ...(i)

Differentiating both sides of (i) w.r.t.x; $\frac{dy}{dx}=\frac{d\left(5^x\right)}{dx}\Rightarrow\frac{dy}{dx}=5^x.\log 5$ ...(ii)

Again Differentiating both sides of (ii) w.r.t.x; $\frac{d^2y}{dx^2}=\log 5.5^x\log 5$ $\therefore\frac{d^2y}{dx^2}=5^x(\log 5)^2$

(iii) $y=\tan x$ ...(i)

Differentiating both sides of (i) w.r.t.x; $\frac{dy}{dx}=\frac{d(\tan x)}{dx}\Rightarrow\frac{dy}{dx}=\sec^2 x$ ...(ii)

Again Differentiating both sides of (ii) w.r.t. x,

$\frac{d^2y}{dx^2}=2\sec x.(\tan x.\sec x)$ $\therefore\frac{d^2y}{dx^2}=2\sec^2 x\tan x$

(iv) $y=\cos^{-1}(x)$ ...(i)

Differentiating both sides of (i) w.r.t. x, $\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}}$ ...(ii)

Again Differentiating both sides of (ii) w.r.t. x, $\frac{d^2y}{dx^2}=-\left\{\frac{\sqrt{1-x^2}\frac{d(1)}{dx}-\frac{d\left(\sqrt{1-x^2}\right)}{d\left(1-x^2\right)}\times\frac{d\left(1-x^2\right)}{dx}}{\left(\sqrt{1-x^2}\right)^2}\right\}$

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{\sqrt{1-x^2}.0 - \frac{1}{2\sqrt{1-x^2}}(-2x)}{(1-x^2)}\right] \quad \Rightarrow \frac{d^2y}{dx^2} = -\frac{x}{\left(\sqrt{1-x^2}\right)(1-x^2)} \quad \Rightarrow \frac{d^2y}{dx^2} = \frac{-x}{(1-x^2)^{3/2}}$$

2. Find the second derivative of

(i) $x\sin x$ (ii) $e^{2x}\cos 3x$ (iii) $x^3\log x$

Sol. (i) $y = x\sin x$ ...(i)

Differentiating both sides of (i) w.r.t. x, $\frac{dy}{dx} = x\frac{d(\sin x)}{dx} + \sin x\frac{d(x)}{dx} \Rightarrow \frac{dy}{dx} = x\cos x + \sin x.1$

$\Rightarrow \frac{dy}{dx} = x\cos x + \sin x$ ...(ii)

Again Differentiating both sides of (ii) w.r.t. x, $\frac{d^2y}{dx^2} = x\frac{d(\cos x)}{dx} + \cos x.\frac{d(x)}{dx} + \frac{d(\sin x)}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = -x\sin x + \cos x.1 + \cos x \quad \therefore \frac{d^2y}{dx^2} = -x\sin x + 2\cos x$

(ii) $y = e^{2x}\cos 3x$ ...(i)

Differentiating both sides of (i) w.r.t. x, $\frac{dy}{dx} = e^{2x}(-\sin 3x).3 + \cos 3x.e^{2x}.2$

$\Rightarrow \frac{dy}{dx} = e^{2x}(-3\sin 3x + 2\cos 3x)$ ...(ii)

Again Differentiating both sides of (ii) w.r.t. x,

$\frac{d^2y}{dx^2} = e^{2x}\frac{d(-3\sin 3x + 2\cos 3x)}{dx} + (-3\sin 3x + 2\cos 3x).\frac{d(e^{2x})}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = e^{2x}\{(-3\cos 3x.3 + 2(-\sin 3x).3)\} + (-3\sin 3x + 2\cos 3x).e^{2x}.2$

$\Rightarrow \frac{d^2y}{dx^2} = e^{2x}(-9\cos 3x - 6\sin 3x - 6\sin 3x + 4\cos 3x) \Rightarrow \frac{d^2y}{dx^2} = e^{2x}(-5\cos 3x - 12\sin 3x)$

$\therefore \frac{d^2y}{dx^2} = -e^{2x}(5\cos 3x + 12\sin 3x)$

(iii) $y = x^3\log x$ ...(i)

Differentiating both sides of (i) w.r.t. x,

$\frac{dy}{dx} = x^3.\frac{1}{x} + \log x.3x^2 \Rightarrow \frac{dy}{dx} = x^2 + 3\log x.x^2 \Rightarrow \frac{dy}{dx} = x^2(1 + 3\log x)$ ...(ii)

Again Differentiating both sides of (ii) w.r.t. x, $\frac{d^2y}{dx^2} = x^2\frac{d(1+3\log x)}{dx} + (1+3\log x)\frac{d(x^2)}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = x^2.\left(3.\frac{1}{x}\right) + (1+3\log x).2x \Rightarrow \frac{d^2y}{dx^2} = 3x + 2x + 6x\log x \quad \therefore \frac{d^2y}{dx^2} = 5x + 6x\log x$

3. If $y = x + \tan x$ show that $\cos^2 x.\frac{d^2y}{dx^2} - 2y + 2x = 0$.

Sol. $y = x + \tan x$ ...(i)

Differentiating both sides of (i) w.r.t. x, $\frac{dy}{dx} = 1 + \sec^2 x \Rightarrow \frac{dy}{dx} = 1 + \frac{1}{\cos^2 x}$

$\Rightarrow \frac{dy}{dx} = \frac{\cos^2 x + 1}{\cos^2 x} \quad \Rightarrow \cos^2 x \frac{dy}{dx} = \cos^2 x + 1 \quad \ldots(ii)$

Again Differentiating both sides of (ii) w.r.t.x, $\cos^2 x.\frac{d^2y}{dx^2} + \frac{dy}{dx}.2\cos x(-\sin x) = 2\cos x(-\sin x)$

$\Rightarrow \cos^2 x.\frac{d^2y}{dx^2} = -\sin 2x + \sin 2x \frac{dy}{dx} \quad \Rightarrow \cos^2 x.\frac{d^2y}{dx^2} = \sin 2x\left(-1 + \frac{dy}{dx}\right)$

$\Rightarrow \cos^2 x.\frac{d^2y}{dx^2} = \sin 2x\left\{-1 + \left(1 + \sec^2 x\right)\right\} \quad \Rightarrow \cos^2 x.\frac{d^2y}{dx^2} = \sin 2x.\sec^2 x$

$\Rightarrow \cos^2 x.\frac{d^2y}{dx^2} = 2\tan x \quad \Rightarrow \cos^2 x.\frac{d^2y}{dx^2} - 2\tan x - 2x + 2x = 0$

$\Rightarrow \cos^2 x.\frac{d^2y}{dx^2} - 2(\tan x + x) + 2x = 0 \quad \therefore \cos^2 x.\frac{d^2y}{dx^2} - 2y + 2x = 0$

4. If $y = 2\sin x + 3\cos x$ show that $y + \frac{d^2y}{dx^2} = 0$.

Sol. $y = 2\sin x + 3\cos x \quad \ldots (i)$

Differentiating both sides of (i) w.r.t.x, $\frac{dy}{dx} = 2\cos x - 3\sin x \quad \ldots(ii)$

Again Differentiating both sides of (ii) w.r.t.x, $\frac{d^2y}{dx^2} = -2\sin x - 3\cos x$

$\Rightarrow \frac{d^2y}{dx^2} = -(2\sin x + 3\cos x) \quad \Rightarrow \frac{d^2y}{dx^2} = -y \quad [\text{From (i)}] \quad \therefore \frac{d^2y}{dx^2} + y = 0$

5. If $y = 3\cos(\log x) + 4\sin(\log x)$ prove that $x^2 y_2 + xy_1 + y = 0$

Sol. $y = 3\cos(\log x) + 4\sin(\log x) \quad \ldots(i)$

Differentiating both sides of (i) w.r.t. x,

$\frac{dy}{dx} = -3\sin(\log x).\frac{1}{x} + 4\cos(\log x).\frac{1}{x} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{x}\{-3\sin(\log x) + 4\cos(\log x)\}$

$\Rightarrow x\frac{dy}{dx} = -3\sin(\log x) + 4\cos(\log x) \quad \ldots(ii)$

Again Differentiating both sides of (ii) w.r.t.x, $x\frac{d^2y}{dx^2} + \frac{dy}{dx}.1 = -3\cos(\log x).\frac{1}{x} - 4\sin(\log x).\frac{1}{x}$

$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{1}{x}\{3\cos(\log x) + 4\sin(\log x)\} \quad \Rightarrow x^2\frac{d^2y}{dx} + x\frac{dy}{dx} = -y \quad$ [From (i)]

$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \quad \therefore x^2 y_2 + xy_1 + y = 0$

6. If $y = e^{-x}\cos x$ show that $\frac{d^2y}{dx^2} = 2e^{-x}\sin x$

Sol. $y = e^{-x}\cos x \quad \ldots(i)$

Differentiating both sides of (i) w.r.t. x, $\frac{dy}{dx} = e^{-x}(-\sin x) + \cos x.(e^{-x})(-1)$

$\Rightarrow \frac{dy}{dx} = -e^{-x}(\sin x + \cos x) \quad \ldots(ii)$

Again Differentiating both sides of (ii) w.r.t. x, $\frac{d^2y}{dx^2} = -\left[e^{-x}(\cos x - \sin x) + (\sin x + \cos x)e^{-x}(-1)\right]$

$\Rightarrow \frac{d^2y}{dx^2} = -e^{-x}[\cos x - \sin x - \sin x - \cos x] \quad \Rightarrow \frac{d^2y}{dx^2} = 2e^{-x}\sin x$

7. If $y = \sec x - \tan x$ show that $\cos x.\frac{d^2y}{dx^2} = y^2$

Sol. $y = \sec x - \tan x \qquad \ldots(i)$

Differentiating both sides of (i) w.r.t. x, $\frac{dy}{dx} = \sec x \tan x - \sec^2 x \quad \Rightarrow \frac{dy}{dx} = \sec x(\tan x - \sec x)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos x}\{-(\sec x - \tan x)\} \quad \Rightarrow \cos x.\frac{dy}{dx} = -y \quad [\text{From (i)}] \qquad \ldots(ii)$

Again Differentiating both sides of (ii) w.r.t.x, $\cos x.\frac{d^2y}{dx^2} + \frac{dy}{dx}(-\sin x) = -\frac{dy}{dx}$

$\Rightarrow \cos x.\frac{d^2y}{dx^2} = -\frac{dy}{dx} + \sin x\frac{dy}{dx} \quad \Rightarrow \cos x.\frac{d^2y}{dx^2} = \frac{dy}{dx}(-1 + \sin x)$

$\Rightarrow \cos x.\frac{d^2y}{dx^2} = \{-(1 - \sin x)\}.\left\{-\frac{(\sec x - \tan x)}{\cos x}\right\} \quad \Rightarrow \cos x.\frac{d^2y}{dx^2} = \left(\frac{1 - \sin x}{\cos x}\right)(\sec x - \tan x)$

$\Rightarrow \cos x.\frac{d^2y}{dx^2} = (\sec x - \tan x)(\sec x - \tan x) \quad \therefore \cos x.\frac{d^2y}{dx^2} = y^2$

8. If $y = \operatorname{cosec} x + \cot x$ prove that $\sin x.\frac{d^2y}{dx^2} - y^2 = 0$

Sol. $y = \operatorname{cosec} x + \cot x \qquad \ldots(i)$

Differentiating both sides of (i) w.r.t.x, $\frac{dy}{dx} = -\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x$

$\Rightarrow \frac{dy}{dx} = -\operatorname{cosec} x(\cot x + \operatorname{cosec} x) \quad \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin x}(\cot x + \operatorname{cosec} x)$

$\Rightarrow \sin x\frac{dy}{dx} = -(\cot x + \operatorname{cosec} x) \quad \Rightarrow \sin x.\frac{dy}{dx} = -y \quad [\text{From (i)}] \qquad \ldots(ii)$

Again Differentiating both sides of (ii) w.r.t.x, $\sin x\frac{d^2y}{dx^2} + \frac{dy}{dx}.\cos x = -\frac{dy}{dx}$

$\Rightarrow \sin x.\frac{d^2y}{dx^2} = -\frac{dy}{dx} - \cos x.\frac{dy}{dx} \quad \Rightarrow \sin x.\frac{d^2y}{dx^2} = -\frac{dy}{dx}(1 + \cos x)$

$\Rightarrow \sin x.\frac{d^2y}{dx^2} = +\frac{1}{\sin x}(\operatorname{cosec} x + \cot x)(1 + \cos x)$

$\Rightarrow \sin x.\frac{d^2y}{dx^2} = (\operatorname{cosec} x + \cot x)(\operatorname{cosec} x + \cot x) \quad \Rightarrow \sin x.\frac{d^2y}{dx^2} = y^2 \quad \Rightarrow \sin x.\frac{d^2y}{dx^2} - y^2 = 0$

9. If $y=\tan^{-1}x$ show that $\left(1+x^2\right)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=0$.

Sol. $y=\tan^{-1}x$ ...(i)

Differentiating both sides of (i) w.r.t. x, $\frac{dy}{dx}=\frac{1}{1+x^2}$ $\Rightarrow\left(1+x^2\right)\frac{dy}{dx}=1$ ...(ii)

Again Differentiating both sides of (ii) w.r.t. x, $\left(1+x^2\right)\frac{d^2y}{dx^2}+\frac{dy}{dx}.2x=0$ $\therefore\left(1+x^2\right)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=0$

10. If $y=\sin(\sin x)$ prove that $\frac{d^2y}{dx^2}+(\tan x)\frac{dy}{dx}+y\cos^2x=0$.

Sol. $y=\sin(\sin x)$ ...(i)

Differentiation both sides of (i) w.r.t.x, $\frac{dy}{dx}=\cos(\sin x).\cos x$ ...(ii)

Again Differentiating both sides of (ii) w.r.t.x, $\frac{d^2y}{dx^2}=\cos(\sin x)(-\sin x)+\cos x\{-\sin(\sin x)\cos x\}$

$\Rightarrow\frac{d^2y}{dx^2}=-\{\sin x.\cos(\sin x)+\cos^2x\sin(\sin x)\}$ $\Rightarrow\frac{d^2y}{dx^2}=-\left\{\sin x.\frac{1}{\cos x}\frac{dy}{dx}+\cos^2x.y\right\}$

$\therefore\ \frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2x=0$. Hence proved.

11. If $y=a\cos(\log x)+b\sin(\log x)$ prove that $x^2y_2+xy_1+y=0$.

Sol. $y=a\cos(\log x)+b\sin(\log x)$ ...(i)

Differentiating both sides of (i) w.r.t.x, $\frac{dy}{dx}=-a\sin(\log x).\frac{1}{x}+b\cos(\log x).\frac{1}{x}$

$\Rightarrow\frac{dy}{dx}=\frac{1}{x}\{-a\sin(\log x)+b\cos(\log x)\}$ $\Rightarrow x\frac{dy}{dx}=-a\sin(\log x)+b\cos(\log x)$ ...(ii)

Again differentiating both sides of eq.(ii) w.r.t. x, we get,

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}.1=a\cos(\log x).\frac{1}{x}-b\sin(\log x).\frac{1}{x}$$

$\Rightarrow x\frac{d^2y}{dx^2}+\frac{dy}{dx}=-\frac{1}{x}\{a\cos(\log x)+b\sin(\log x)\}$ $\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-y$ [From equation (i)]

$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0\ \therefore x^2y_2+xy_1+y=0$

12. Find the second derivative of $e^{3x}\sin 4x$.

Sol. $y=e^{3x}\sin 4x$ ...(i)

Differentiating both sides of (i) w.r.t.x, $\frac{dy}{dx}=e^{3x}\cos 4x.4+\sin 4x.e^{3x}.3$

$\Rightarrow\frac{dy}{dx}=e^{3x}(4\cos 4x+3\sin 4x)$ ...(ii)

Again Differentiating both sides of (ii) w.r.t.x

$$\frac{d^2y}{dx^2}=e^{3x}(-4\sin x4x.4+3\cos 4x.4)+(4\cos 4x+3\sin 4x)\ e^{3x}.3$$

$$\Rightarrow\frac{d^2y}{dx^2}=4e^{3x}(-4\sin 4x+3\cos 4x)+3e^{3x}(4\cos 4x+3\sin 4x)$$

$$\Rightarrow \frac{d^2y}{dx^2}=e^{3x}\left(-16\sin 4x+12\cos 4x+12\cos 4x+9\sin 4x\right) \quad \Rightarrow \frac{d^2y}{dx^2}=e^{3x}\left(24\cos 4x-7\sin 4x\right)$$

13. Find the second derivative of $\sin 3x.\cos 5x$

Sol. $y=\sin 3x.\cos 5x \qquad \ldots(i)$

$$\Rightarrow y=\frac{1}{2}\left(2\sin 3x.\cos 5x\right) \quad \Rightarrow y=\frac{1}{2}\left[\sin\left(3x+5x\right)+\sin\left(3x-5x\right)\right] \quad \Rightarrow y=\frac{1}{2}\left[\sin 8x+\sin\left(-2x\right)\right]$$

$$y=\frac{1}{2}\left[\sin 8x-\sin 2x\right] \qquad \ldots(ii)$$

Differentiating both sides of (ii) w.r.t. x, $\frac{dy}{dx}=\frac{1}{2}\left[\cos 8x.8-\cos 2x.2\right]$

$$\Rightarrow \frac{dy}{dx}=4\cos 8x-\cos 2x \qquad \ldots(ii)$$

Again Differentiating both sides of (ii) w.r.t. x, $\frac{d^2y}{dx^2}=-4\sin 8x.8+\sin 2x.2$

$$\Rightarrow \frac{d^2y}{dx^2}=-32\sin 8x+2\sin 2x \quad \therefore \frac{d^2y}{dx^2}=\left(2\sin 2x-32\sin 8x\right)$$

14. If $y=e^{\tan x}$ prove that $\left(\cos^2 x\right)\frac{d^2y}{dx^2}-\left(1+\sin 2x\right)\frac{dy}{dx}=0$

Sol. $y=e^{\tan x} \qquad \ldots(i)$

Differentiating both sides of (i) w.r.t. x, $\frac{dy}{dx}=e^{\tan x}.\sec^2 x \Rightarrow \frac{dy}{dx}=\frac{e^{\tan x}}{\cos^2 x}$

$$\Rightarrow \cos^2 x\frac{dy}{dx}=e^{\tan x} \quad \Rightarrow \cos^2 x\frac{dy}{dx}=y \quad \left[\text{From equation (i)}\right] \qquad \ldots(ii)$$

Again Differentiating both sides of (ii) w.r.t. x, $\cos^2 x.\frac{d^2y}{dx^2}+\frac{dy}{dx}.2\cos x\left(-\sin x\right)=\frac{dy}{dx}$

$$\Rightarrow \cos^2 x.\frac{d^2y}{dx^2}-\sin 2x\frac{dy}{dx}=\frac{dy}{dx} \quad \Rightarrow \cos^2 x\frac{d^2y}{dx^2}-\left(1+\sin 2x\right)\frac{dy}{dx}=0$$

15. If $y=\frac{\log x}{x}$ show that $\frac{d^2y}{dx^2}=\frac{2\log x-3}{x^3}$

Sol. $y=\frac{\log x}{x} \qquad \ldots(i)$

Differentiating both sides of (i) w.r.t.x, $\frac{dy}{dx}=\left\{\frac{x.\frac{1}{x}-\log x}{\left(x\right)^2}\right\} \Rightarrow \frac{dy}{dx}=\frac{1-\log x}{x^2} \qquad \ldots(ii)$

Again Differentiating both sides of (ii) w.r.t.x, $\frac{d^2y}{dx^2}=\left\{\frac{x^2\left(0-\frac{1}{x}\right)-\left(1-\log x\right).2x}{\left(x^2\right)^2}\right\}$

$$\Rightarrow \frac{d^2y}{dx^2}=\left\{\frac{-x-2x\left(1-\log x\right)}{x^4}\right\} \quad \Rightarrow \frac{d^2y}{dx^2}=\frac{-\left(1+2-2\log x\right)}{x^3} \quad \Rightarrow \frac{d^2y}{dx^2}=\frac{2\log x-3}{x^3}$$

16. If $y=e^{ax}\cos bx$ show that $\frac{d^2y}{dx^2}-2a\frac{dy}{dx}+\left(a^2+b^2\right)y=0$

Sol. $y = e^{ax}\cos bx$ ...(i)

Differentiating both sides of (i) w.r.t. x, $\frac{dy}{dx} = -e^{ax}\sin bx.b + \cos bx.e^{ax}.a$

$\Rightarrow \frac{dy}{dx} = e^{ax}\left(-b\sin bx + a\cos bx\right)$ ...(ii)

Again Differentiating both sides of (ii) w.r.t. x,

$\frac{d^2y}{dx^2} = e^{ax}\left(-b\cos bx.b - a\sin bx.b\right) + \left(-b\sin bx + a\cos bx\right)e^{ax}.a$

$\Rightarrow \frac{d^2y}{dx^2} = e^{ax}\left\{-b^2\cos bx - ab\sin bx - ab\sin bx + a^2\cos bx\right\}$

$\Rightarrow \frac{d^2y}{dx^2} = e^{ax}\left\{\cos bx\left(a^2 - b^2\right) - 2ab\sin bx\right\}$ $\Rightarrow \frac{d^2y}{dx^2} = \left(a^2 - b^2\right)e^{ax}\cos bx - 2ab\,e^{ax}\sin bx$

$\Rightarrow \frac{d^2y}{dx^2} = \left(a^2 - b^2\right)y - 2abe^{ax}\sin bx$ ...(iii)

From equation (ii), $\frac{dy}{dx} = -be^{ax}\sin bx + ae^{ax}\cos bx$ $\Rightarrow be^{ax}\sin bx = \left(ae^{ax}\cos bx - \frac{dy}{dx}\right)$

From equation (iii), $\frac{d^2y}{dx^2} = \left(a^2 - b^2\right) - 2a\left(ae^{ax}\cos bx - \frac{dy}{dx}\right)$

$\Rightarrow \frac{d^2y}{dx^2} = \left(a^2 - b^2\right) - 2a^2e^{ax}\cos bx + 2a\frac{dy}{dx}$ $\Rightarrow \frac{d^2y}{dx^2} = \left(a^2 - b^2\right)y - 2a^2y + 2a\frac{dy}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = -\left(a^2 + b^2\right) + 2a\frac{dy}{dx}$ $\therefore \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + \left(a^2 + b^2\right)y = 0$. Hence proved.

17. If $y = e^{a\cos^{-1}x}, -1 \le x \le 1$ show that $\left(1 - x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

Sol. $y = e^{a\cos^{-1}x}$ ...(i)

Differentiating both sides of (i) w.r.t. x, $\frac{dy}{dx} = e^{a\cos^{-1}x}.\ a\left(-\frac{1}{\sqrt{1 - x^2}}\right)$

$\Rightarrow \sqrt{1 - x^2}\frac{dy}{dx} = -ay$ [From equation (i)] ...(ii)

Squaring both sides we get , $\left(\sqrt{1 - x^2}\frac{dy}{dx}\right)^2 = \left(-ay\right)^2$ $\Rightarrow \left(1 - x^2\right)\left(\frac{dy}{dx}\right)^2 = a^2y^2$ ...(iii)

Again Differentiating both sides of (iii) w.r.t. x,

$\left(1 - x^2\right)2\frac{dy}{dx}.\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\left(-2x\right) = a^2.2y\frac{dy}{dx}$ $\Rightarrow 2\frac{dy}{dx}\left\{\left(1 - x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx}\right\} = a^2 2y\frac{dy}{dx}$

$\Rightarrow \left(1 - x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = a^2y$ $\therefore \left(1 - x^2\right)\frac{d^2y}{dx^2} - \frac{x\,dy}{dx} - a^2y = 0$

18. If $x = at^2, y = 2at$ find $\frac{d^2y}{dx^2}$ at $t = 2$

Sol. $x = at^2$ ...(i), $\frac{dx}{dt} = a.2t$; $y = 2at$ ...(ii), $\frac{dy}{dt} = 2a$

$\therefore \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{2a}{2at} \Rightarrow \frac{dy}{dx}=\frac{1}{t}$ ...(iii)

Again differentiation w.r.t.x, $\frac{d^2y}{dx^2}=\frac{d\left(\frac{1}{t}\right)}{dt}.\frac{dt}{dx} \Rightarrow \frac{d^2y}{dx^2}=-\frac{1}{t^2}.\frac{1}{2at} \Rightarrow \frac{d^2y}{dx^2}=-\frac{1}{2at^3}$

$\Rightarrow \frac{d^2y}{dx^2}_{(t=2)}=-\frac{1}{2a(2)^3} \quad \therefore \frac{d^2y}{dx^2}=-\frac{1}{16a}$

19. If $x=a(\theta-\sin\theta)$, and $y=a(1-\cos\theta)$ find $\frac{d^2y}{dx^2}$ at $\theta=\pi$

Sol. $x=a(\theta-\sin\theta)$ ...(i)

Differentiating both sides of (i) w.r.t. x, $\frac{dx}{d\theta}=a(1-\cos\theta)$

$y=a(1-\cos\theta)$ ...(ii)

Differentiating both sides of (ii) w.r.t. x, $\frac{dy}{d\theta}=a(0+\sin\theta) \Rightarrow \frac{dy}{d\theta}=a\sin\theta$

$\therefore \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{a\sin\theta}{a(1-\cos\theta)} \Rightarrow \frac{dy}{dx}=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} \Rightarrow \frac{dy}{dx}=\cot\frac{\theta}{2}$ ...(ii)

Again Differentiating both sides of (ii) w.r.t. x, $\frac{d^2y}{dx^2}=-\operatorname{cosec}^2\frac{\theta}{2}.\frac{1}{2}.\frac{d\theta}{dx}$

$\Rightarrow \frac{d^2y}{dx^2}=-\frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}.\frac{1}{a(1-\cos\theta)} \Rightarrow \frac{d^2y}{dx^2}=-\frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}.\frac{1}{a.2\sin^2\frac{\theta}{2}}$

$\Rightarrow \frac{d^2y}{dx^2}=-\frac{1}{4a}\operatorname{cosec}^4\left(\frac{\theta}{2}\right) \Rightarrow \frac{d^2y}{dx^2}_{\left(\theta=\frac{\pi}{2}\right)}=-\frac{1}{4a}\operatorname{cosec}^4\left(\frac{\pi}{2}\right)=-\frac{1}{4a}$

20. If $y=\sin(\log x)$ Prove that $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$

Sol. $y=\sin(\log x)$ ...(i)

Differentiating both sides of (ii) w.r.t.x, $\frac{dy}{dx}=\cos(\log x).\frac{1}{x}$

$x\frac{dy}{dx}=\cos(\log x)$ ...(ii)

Again Differentiating both sides of (ii) w.r.t. x, $x\frac{d^2y}{dx^2}+\frac{dy}{dx}.1=-\sin(\log x).\frac{1}{x}$

$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-y \quad \therefore x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$

21. If $y=\frac{\sin^{-1}x}{\sqrt{1-x^2}}$, show that $(1-x^2)\frac{d^2y}{dx^2}-3x\frac{dy}{dx}-y=0$

Sol. Differentate the function 'y' w.r.t x

$\frac{dy}{dx}=\sqrt{1-x^2}\times\frac{d}{dx}\sin^{-1}x-\sin^{-1}x\times\frac{d}{dx}\sqrt{1-x^2}$

$$\frac{dy}{dx}=\frac{\sqrt{1-x^2}\times\frac{1}{\sqrt{1-x^2}}-\sin^{-1}x\times\frac{1}{2\sqrt{1-x^2}}\times -2x}{\left(1-x^2\right)}$$

$$\frac{dy}{dx}=\frac{1+\frac{x+\sin^{-1}x}{\sqrt{1-x^2}}}{\left(1-x^2\right)}=\frac{1}{\left(1-x^2\right)}+\frac{x\sin^{-1}x}{\left(1-x^2\right)^{3/2}}$$

Again Differentiate $\frac{d^2y}{dx^2}=\frac{\left(1-x^2\right)\frac{d}{dx}[1]-1\frac{d}{dx}\left(1-x^2\right)}{\left(1-x^2\right)^2}+\frac{\left(1-x^2\right)^{3/2}\frac{d}{dx}\left(x\sin^{-1}x\right)-x\sin^{-1}x\frac{d}{dx}\left(1-x^2\right)^{3/2}}{\left\{\left(1-x^2\right)^{3/2}\right\}^2}$

$$\frac{d^2y}{dx^2}=\frac{0-(-2x)}{\left(1-x^2\right)^2}+\frac{\left(1-x^2\right)^{3/2}\times\left[\frac{x}{\sqrt{1-x^2}}+\sin^{-1}x\right]-x\sin^{-1}x\times\frac{3}{2}\left(1-x^2\right)^{1/2}\times -2x}{\left\{\left(1-x^2\right)^{3/2}\right\}^2}$$

$$\frac{d^2y}{dx^2}=\frac{2x}{\left(1-x^2\right)}+\frac{\left(1-x^2\right)\sqrt{1-x^2}\left[\frac{x+\sqrt{1-x^2}\sin^{-1}x}{\sqrt{1-x^2}}\right]+3x^2\sin^{-1}x\sqrt{1-x^2}}{\left(1-x^2\right)^3}$$

$$\frac{d^2y}{dx^2}=\frac{2x}{\left(1-x^2\right)^2}+\frac{\left(1-x^2\right)\left[x+\sqrt{1-x^2}\sin^{-1}x\right]+3x^2\sin^{-1}x\sqrt{1-x^2}}{\left(1-x^2\right)^3}$$

$$\frac{d^2y}{dx^2}=\frac{2x}{\left(1-x^2\right)^2}+\frac{\left(1-x^2\right)\left[x+\sqrt{1-x^2}\sin^{-1}x\right]}{\left(1-x^2\right)^3}+\frac{3x^2\sin^{-1}x\sqrt{1-x^2}^{\,3}}{\left(1-x^2\right)}$$

$$\frac{d^2y}{dx^2}=\left[\frac{2x}{\left(1-x^2\right)^2}+\frac{x}{\left(1-x^2\right)^2}+\frac{\sqrt{1-x^2}\sin^{-1}x}{\left(1-x^2\right)^2}+\frac{3x^2\sin^{-1}x\sqrt{1-x^2}}{\left(1-x^2\right)^3}\right]$$

Now, L.H.S. $\left(1-x^2\right)\frac{d^2y}{dx^2}-3x\frac{dy}{dx}-y=0$

$$\Rightarrow\left(1-x^2\right)\left[\frac{2x}{\left(1-x^2\right)^2}+\frac{x}{\left(1-x^2\right)^2}+\frac{\sqrt{1-x^2}\sin^{-1}x}{\left(1-x^2\right)^2}+\frac{3x^2\sin^{-1}x\sqrt{1-x^2}}{\left(1-x^2\right)^3}-3x\left[\frac{1}{\left(1-x^2\right)}+\frac{x\sin^{-1}x}{\left(1-x^2\right)^{3/2}}\right]-\frac{\sin^{-1}x}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\frac{3x}{\left(1-x^2\right)}+\frac{\sqrt{1-x^2}\sin^{-1}x}{\left(1-x^2\right)}+\frac{3x^2\sin^{-1}x\sqrt{1-x^2}}{\left(1-x^2\right)^2}-\frac{3x}{\left(1-x^2\right)}-\frac{3x^2\sin^{-1}x}{\sqrt{1-x^2}\left(1-x^2\right)}-\frac{\sin^{-1}x}{\sqrt{1-x^2}}$$

$$\Rightarrow\frac{\sin^{-1}x}{\sqrt{1-x^2}}-\frac{\sin^{-1}x}{\sqrt{1-x^2}}+\frac{3x^2\sin^{-1}x\sqrt{1-x^2}}{\left(1-x^2\right)\sqrt{1-x^2}}-\frac{3x^2\sin^{-1}x}{\left(1-x^2\right)\sqrt{1-x^2}}$$

$$\Rightarrow 0=R.H.S$$

22. If $y = e^x \sin x$, prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

Sol. $\frac{dy}{dx} = e^x \times \frac{d}{dx}(\sin x) + \sin x \times \frac{d}{dx}(e^x)$

Again diff. the function $\frac{d^2y}{dx^2} = e^x \frac{d}{dx}(\cos x) + \cos x \times \frac{d}{dx}(e^x) + \frac{d}{dx}(\sin x \cdot e^x)$

$\frac{d^2y}{dx^2} = e^x - \sin x + \cos x \times e^x + \sin x \cdot e^x + e^x \cos x$

$\frac{d^2t}{dx^2} = -e^x \sin x + \cos x \cdot e^x + e^x \sin e^x + \cos xe^x$

$\frac{d^2y}{dx^2} = 2e^x \cos x$

Now. L.H.S

$\Rightarrow 2e^x \cos x - 2(e^x \cos x + \sin e^x) + 2(e^x \sin x)$

$\Rightarrow 2e^x \cos x - 2e^x \cos x + 2e^x \sin x - 2e^x \sin x \quad \Rightarrow O = R.H.S$

23. If $x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$ and y $a\sin\theta$, show that the value of $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$ is $\frac{4}{a}$

Sol. Differentiate the function $'x'$ w.r.t. θ

$$\frac{dr}{d\theta} = a\left[-\sin\theta + \frac{1}{\tan\frac{\theta}{2}}\sec^2\frac{\theta}{2} \times \frac{1}{2}\right]$$

$$\frac{dx}{d\theta} = a\left[-\sin\theta + \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} \cdot \frac{1}{\cos^2\frac{\theta}{2}} \cdot \frac{1}{2}\right]$$

$$\frac{dx}{d\theta} = a\left[-\sin\theta + \frac{1}{\sin\theta}\right] = a\left[-\frac{\sin^2\theta + 1}{\sin\theta}\right] = a\left[\frac{\cos^2\theta}{\sin\theta}\right] = a\cos\theta \cdot \cot\theta$$

$$\frac{dy}{d\theta} = a\cos\theta$$

$\therefore$ Now diff the function y w.r.t. x

$$\frac{dy}{dx} = \frac{a\cos\theta}{a\cos\theta \cdot \cot\theta} = \tan\theta$$

$$\frac{dy}{dx} = \sec^2\theta \cdot \frac{d\theta}{dx} = \sec^2\theta \times \frac{1}{a\cos\theta \cdot \cot\theta} = \frac{1}{a\cos^2\theta \cdot \cos\theta \cdot \frac{\cos\theta}{\sin\theta}}$$

$$\frac{d^2y}{dx^2} = \frac{\sin\theta}{a\cos^4\theta}$$

$$\therefore \left.\frac{d^2y}{dx^2}\right|_{\theta=\frac{\pi}{4}} = \frac{\sin\frac{\pi}{4}}{a\left(\cos\frac{\pi}{4}\right)^4} = \frac{\frac{1}{\sqrt{2}}}{a\left(\frac{1}{\sqrt{2}}\right)^4} = \frac{\frac{1}{\sqrt{2}}}{a \times \frac{1}{4}} = \frac{4}{\sqrt{2}a}$$ proved

24. If $x = \cot + \log\tan\frac{t}{2}, y = \sin t$ then find the values of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$

Sol. Diff the function 'x' w.r.t. t

$$\frac{dx}{dt} = -\sin t + \frac{1}{\tan\frac{t}{2}} \times \sec^2\frac{t}{2} \times \frac{t}{2}$$

Again, diff the function 'y' w.r.t. t we get $\frac{dy}{dt} = \cos t$

Again differentate w.r.t. to t we get $\frac{d^2y}{dt^2} = -\sin t$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{4}} = -\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$$

Now, $\frac{dx}{dt} = -\sin t + \frac{1}{\tan\frac{t}{2}} \times \sec^2\frac{t}{2} \times \frac{1}{2}$

$$\frac{dx}{dt} = -\sin t + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} \times \frac{1}{\cos^2\frac{t}{2}} \times \frac{1}{2}$$

$$\frac{dx}{dt} = -\sin t + \frac{1}{\sin t} = \frac{-\sin^2 t + 1}{\sin t} = \frac{\cos^2 t}{\sin t} = \cos t \cdot \cot t$$

$$\therefore \frac{dy}{dx} = \frac{\cos t}{\cot \cdot \cot t} = \frac{1}{\cot t} = \tan t$$

$$\frac{d^2y}{dx^2} = \sec^2 t\frac{dt}{dx} = \sec^2 t \times \frac{1}{\cos t \cdot \cot t} = \frac{\sin t}{\cos^4 t}$$

Now $\left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{4}} = \frac{\sin\frac{\pi}{4}}{\left(\cos\frac{\pi}{4}\right)^2} = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{4}} = \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = 2\sqrt{2}$

25. if $y = x^x$ prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$

Sol. Diff both side w.r.t. x

$$\frac{dy}{dx} = x^x(1 + \log x)$$

Again differentate $\frac{d^2y}{dx^2} = x^x\left(\frac{1}{x}\right) + (1 + \log x)\left\{x^2(1 + \log x)\right\}$

L.H.s $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$

$$\Rightarrow \frac{x^x}{x} + x^x(1 + \log x)^2 - \frac{1}{x^x}(1 + \log x)^2 - \frac{x^x}{x}$$

$$\Rightarrow x^x(1 + \log x)^2 - x^x(1 + \log x)^2 \Rightarrow 0$$

$\therefore$ L.H.S = R.H.S

26. if $y=\left(\cot^{-1}x\right)^2$, then show that $\left(x^2+1\right)^2\frac{d^2y}{dx^2}+2x\left(x^2+1\right)\frac{dy}{dx}=2$

Sol. Diff. both side w.r.t. x

$$\frac{dy}{dx}=2\cot^{-1}x\left(\frac{-1}{1+x^2}\right)=\frac{-2\cot^{-1}x}{1+x^2}$$

$$\frac{d^2y}{dx^2}=-2\left[\frac{\left(1+x^2\right)-\frac{1}{1+x^2}-\cot^{-1}x(2x)}{\left(1+x^2\right)^2}\right]$$

$$\frac{d^2y}{dx^2}=-2\left[\frac{-1-2x\cot^{-1}x}{\left(1+x^2\right)^2}\right]=\frac{2+4x\cdot\cot^{-1}x}{\left(1+x^2\right)^2}$$

$$\text{L.H.S.}=\left(x^2+1\right)\left\{\frac{2+4x\cot^{-1}x}{\left(1+x^2\right)^2}\right\}+2x\left(x^2+1\right)\left\{\frac{-2\cot^{-1}x}{\left(1+x^2\right)}\right\}$$

$$=2+4x\cot^{-1}x-4x\cot^{-1}x=2=R.H.S$$

27. If $y=\left\{x+\sqrt{x^2+1}\right\}^m$, then show that $\left(x^2+1\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-m^2y=0$

Sol. Diff the function w.r.t. x

$$\frac{dy}{dx}=m\left\{x+\sqrt{x^2+1}\right\}^{m-1}\left[1+\frac{1}{2\sqrt{x^2+1}}2x\right]$$

$$\frac{dy}{dx}=m\frac{\left\{x+\sqrt{x^2+1}\right\}^m}{\left\{x+\sqrt{x^2+1}\right\}}\times\left[1+\frac{x}{\sqrt{x^2+1}}\right]$$

$$\frac{dy}{dx}=m\frac{\left\{x+\sqrt{x^2+1}\right\}^m}{\left\{x+\sqrt{x^2+1}\right\}}\times\left[\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right]$$

$$\frac{dy}{dx}=m\frac{\left\{x+\sqrt{x^2+1}\right\}^m}{\sqrt{x^2+1}}$$

$$\frac{d^2y}{dx^2}=m\left[\frac{\sqrt{x^2+1}\left\{\frac{m\left(x+\sqrt{x^2+1}\right)^m}{\sqrt{x^2+1}}\right\}-\left\{x+\sqrt{x^2+1}\right\}^m\frac{1}{2\sqrt{x^2+1}}2x}{\left(\sqrt{x^2+1}\right)^2}\right]$$

$$\frac{d^2y}{dx^2}=\frac{m^2\left\{x+\sqrt{x^2+1}\right\}^m-\left(x+\sqrt{x^2+1}\right)^m\frac{x}{\sqrt{x^2+1}}}{x^2+1}$$

$$\text{L.H.S } \left(x^2+1\right)\left[\frac{m^2\left\{x+\sqrt{x^2+1}\right\}^m - m\left(x+\sqrt{x^2+1}\right)^m \frac{x}{\sqrt{x^2+1}}}{x^2+1}\right]+x$$

$$+\frac{xm\left\{x+\sqrt{x^2+1}\right\}^m}{\sqrt{x^2+1}} - m^2\left\{x+\sqrt{x^2+1}\right\}^m$$

$\Rightarrow 0$

28. if $y=\log\left[x+\sqrt{x^2+a^2}\right]$, then prove that $\left(x^2+a^2\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$

Sol. Diff both side w.r..t x

$$\frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+a^2}}\times\left[1+\frac{1}{2\sqrt{x^2+a^2}}2x\right]$$

$$\frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+a^2}}\left[\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right]=\frac{1}{\sqrt{x^2+a^2}}$$

Again differentate w.r..t x

$$\frac{d^2y}{dx^2}=\frac{\sqrt{x^2+a^2}\frac{d}{dx}[1]-1\frac{d}{dx}\left[\sqrt{x^2+a^2}\right]}{\left(\sqrt{x^2+a^2}\right)^2}$$

$$\frac{d^2y}{dx^2}=\frac{-\frac{1}{2\sqrt{x^2+a^2}}\times 2x}{\left(x^2+a^2\right)}=\frac{-x}{\left(x^2+a^2\right)\sqrt{x^2+a^2}}$$

$$\text{L.H.S} \Rightarrow \left(x^2+a^2\right)\frac{(x)}{\left(x^2+a^2\right)\sqrt{x^2+a^2}}+x\frac{1}{\sqrt{x^2+a^2}}=0=R.H.S$$

29. If $x=a(\cos\theta+\theta\sin\theta)$ and $y=a(\sin\theta-\theta\cos\theta)$, show that $\frac{d^2y}{dx^2}=\frac{1}{a}\left(\frac{\sec^3\theta}{0}\right)$

Sol. Differentate the function 'x' w.r..t θ

$$\frac{dx}{d\theta}=a\left[-a\sin\theta+\theta\cos\theta+\sin\theta\right]=(a\theta\cos\theta)$$

Again diff the function 'y' w.r.t. θ

$$\frac{dy}{d\theta}=a\left[\cos\theta-\left(\theta(-\sin\theta)+\cos\theta\right)\right]$$

$$\frac{dy}{d\theta}=a\left[\cos\theta+\theta\sin\theta-\cos\theta\right]$$

$$\frac{dy}{d\theta}=a\theta\sin\theta$$

$$\therefore \frac{dy}{dx}=\frac{a\theta\sin\theta}{a\theta\cos\theta}=\tan\theta$$

$$\therefore \frac{d^2y}{dx^2}=\sec^2\theta\frac{d\theta}{dx}=\sec^2\theta\times\frac{1}{a\theta\cos\theta}=\frac{1}{a\theta}\sec^2\theta\cdot\sec\theta=\left(\frac{\sec^3\theta}{a\theta}\right)$$

30. If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$ show that $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

Sol. Diff the 'x' w.r.t.

$$\frac{dx}{d\theta} = -a\sin\theta + b\cos\theta$$

$$\frac{dx}{d\theta} = -[a\sin\theta - b\cos\theta]$$

$$\frac{dx}{d\theta} = -y$$

Again diff 'y' w.r.t. θ

$$\frac{dy}{d\theta} = a\cos\theta + b\sin\theta$$

$$\frac{dy}{d\theta} = x$$

$$\therefore \frac{dy}{dx} = \frac{x}{-y} = \frac{-x}{y}$$

Again diff $\frac{d^2y}{dx^2} = \left[\frac{y\cdot 1 - x\frac{dy}{dx}}{y^2}\right] \Rightarrow -\left[\frac{y - x\left(-\frac{x}{y}\right)}{y^2}\right]$

$$\frac{d^2y}{dx^2} = -\left[\frac{y + \frac{x^2}{y}}{y^2}\right]$$

L.H.S $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

$$\Rightarrow -y^2\left[\frac{y - \frac{x^2}{y}}{y^2}\right] - x\left(-\frac{x}{y}\right) + y \Rightarrow -y - \frac{x^2}{y} + \frac{x^2}{y} + y = 0 \text{ proved}$$