Scalar Product of Vectors

Q.1. Find $\vec{a} \cdot \vec{b}$ if $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $|\vec{a} \times \vec{b}| = 8$.

Solution : 1

$|a \times b| = |a|\ |b| \sin\theta \Rightarrow \sin\theta = 8/[2\times5] = 4/5$, $\cos\theta = 3/5$,

Therefore , $a \cdot b = |a|\ |b| \sin\theta = 2 \times 5 \times (3/5) = 6$.

Q.2. If $\vec{a}$ and $\vec{b}$ are unit vectors such that $2\vec{a} - 4\vec{b}$ and $10\vec{a} + 8\vec{b}$ are perpendicular to each other. Find the angle between vectors $\vec{a}$ and $\vec{b}$.

Solution : 2

The vectors $2\vec{a} - 4\vec{b}$ and $10\vec{a} + 8\vec{b}$ are perpendicular,

Therefore, $(2\vec{a} - 4\vec{b}).(10\vec{a} + 8\vec{b}) = 0$

Or, $20 + 16\,\vec{a}.\,\vec{b} - 40\vec{b}.\,\vec{a} - 32 = 0$

Or, $-24\,\vec{a}.\,\vec{b} = 12$

Or, $\vec{a}.\,\vec{b} = -1/2$.

As, $\cos\theta = [\vec{a}.\,\vec{b}]/|\vec{a}|\,|\vec{b}|$ and $|\vec{a}| = |\vec{b}| = 1$.

Therefore, $\cos\theta = -1/2$ [where θ is angle between $\vec{a}$ and $\vec{b}$]

Or, $\cos\theta = -\cos\pi/3 = \cos(\pi - \pi/3) = \cos 2\pi/3$

Therefore, $\theta = 2\pi/3$.

Q.3. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a} + \vec{b}| = |\vec{a}|$ then show that vector $(2\vec{a} + \vec{b})$ is perpendicular to $\vec{b}$.

Solution : 3

Let $(2\vec{a} + \vec{b}).\vec{b} = 0$

Or, $2\vec{a}.\vec{b} + \vec{b}.\ \vec{b} = 0$

Adding $|a^2|$ to both sides we get

$2\vec{a}.\vec{b} + |b^2| + |a^2| = |a^2|$

Or, $|\vec{a} + \vec{b}|2 = |\vec{a}|2$

Or, $|\vec{a} + \vec{b}| = |\vec{a}|$.

Q.4. The vectors $\vec{a} = 3i + xj - k$ and $\vec{b} = 2i + j + yk$ are mutually perpendicular. Given that $|\vec{a}| = |\vec{b}|$. Find the values of x and y.

Solution : 4

$|\vec{a}| = \sqrt{\{3^2 + x^2 + (-1)^2\}} = \sqrt{(10 + x^2)}$

$|\vec{b}| = \sqrt{\{2^2 + 1^2 + y^2\}} = \sqrt{\{5 + y^2\}}$

$|\vec{a}| = |\vec{b}|$

Or, $\sqrt{(10 + x^2)} = \sqrt{(5 + y^2)}$

Or, $10 + x^2 = 5 + y^2 \Rightarrow y^2 - x^2 = 5$ -------------- (1)

Vectors $\vec{a}$ and $\vec{b}$ are perpendicular, hence $\vec{a}.\ \vec{b} = 0$

Or, $(3i + xj - k).(2i + j + yk) = 0$

Or, $6 + x - y = 0 \Rightarrow y = x + 6$ --------------- (2)

Putting in (1), we get $(x + 6)^2 - x^2 = 5$

Or, $x^2 + 12x + 36 - x^2 = 5$

Or, $12x = -31 \Rightarrow x = -31/12$

And $y = 6 + (-31/12) = 41/12$.

Hence, $x = -31/12$, $y = 41/12$.

Q.5. Prove by vector method that the diameter of a circle will subtend a right angle at a point on its circumference.

Solution : 5

Fig. Solution – 11(b)(i)/Page – 365 [10 yrs.]

Let $\vec{OB} = \vec{b}$, $\vec{OA} = -\vec{b}$ and $\vec{OC} = \vec{c}$

Now $\vec{CA} = -\vec{b} - \vec{c}$ and $\vec{CB} = \vec{b} - \vec{c}$

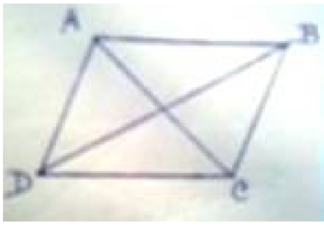
$\vec{CA}.\vec{CB} = -(\vec{b} + \vec{c}).(\vec{b} - \vec{c})$

$= -\{|\vec{b}|^2 - |\vec{c}|^2\} = 0$ [As, $|\vec{b}| = |\vec{c}|$ = radii]

Therefore, LACB = 90º.

Q.6. Show that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals of the parallelogram.

Solution : 6



Fig

Let ABCD be the parallelogram.

$\vec{BA} + \vec{BC} = \vec{BD}$ ------------- (i) [By parallelogram law]

$\vec{BC} - \vec{BA} = \vec{AC}$ ------------- (ii) [By triangle law]

Squaring (i) and (ii) and then adding we get,

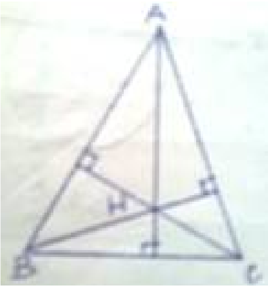
$(\vec{BA} + \vec{BC})^2 + (\vec{BC} - \vec{BA})^2 = |\vec{BD}|^2 + |\vec{AC}|^2$

Or, $|\vec{BA}|^2 + |\vec{BC}|^2 + |\vec{BC}|^2 + |\vec{BA}|^2 + 2\vec{BA}.\vec{BC} - 2\vec{BC}.\vec{BA} = |\vec{BD}|^2 + |\vec{AC}|^2$ Or, $|\vec{BA}|^2 + |\vec{CD}|^2 + |\vec{BC}|^2 + |\vec{AD}|^2 = |\vec{BD}|^2 + |\vec{AC}|^2$

[As, $|\vec{BA}| = |\vec{CD}|$ & $|\vec{BC}| = |\vec{AD}|$]

Q.7. Using vectors, show that the perpendiculars from the vertices to the opposite sides of the triangle ABC are concurrent.

Solution : 7



Fig

Let H be the point of intersection of the perpendiculars from A, B to the opposite sides and let $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{h}$ be the position vectors of A, B, C, H respectively.

As AH is perpendicular to BC,

Therefore, $(\vec{h} - \vec{a}).(\vec{c} - \vec{b}) = 0$

Or, $\vec{h}.\vec{c} - \vec{a}.\vec{c} + \vec{a}.\vec{b} - \vec{h}.\vec{b} = 0$ ------------ (1)

Again BH is perpendicular to CA

Therefore, $(\vec{h} - \vec{b}).(\vec{a} - \vec{c}) = 0$

Or, $\vec{h}.\vec{a} - \vec{h}.\vec{c} - \vec{b}.\vec{a} + \vec{b}.\vec{c} = 0$ ------------ (2)

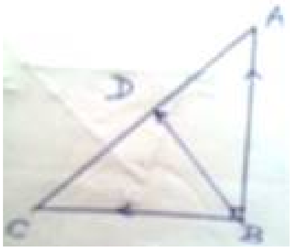
Adding (1) and (2) we get $\vec{h}.\vec{a} - \vec{h}.\vec{b} - \vec{c}.\vec{a} + \vec{c}.\vec{b} = 0$

Or, $(\vec{h} - \vec{c}).(\vec{a} - \vec{b}) = 0$

This shows that CH is perpendicular to BA. This porved the result.

Q. 8. Prove by thee vector method that the middle-point of the hypotenuse of a right-angled triangle is at equi-distant from the vertices of the triangle.

Solution : 8



Fig

Taking B as origin in the right-angled triangle ABC and position vectors of A and C be $\vec{BA} = \vec{a}$ and $\vec{BC} = \vec{b}$.

Let the mid-point of the hypotenuse AC be D.

Then position vector of D = $\vec{BD} = \vec{BA} - \vec{DA}$

$= \vec{BA} - 1/2\ \vec{CA}$

$= \vec{BA} - 1/2\ (\vec{BA} - \vec{BC})$

$= 1/2\ (\vec{BA} + \vec{BC})$

Point D is in CA and it is the middle point of CA, hence D is equi-distant from A and C.

That is DA = DC

$\vec{BA} = \vec{BD} + \vec{DA}$, $\vec{BC} = \vec{BD} + \vec{DC}$

And $\vec{BA} . \vec{BC} = (\vec{BD} + \vec{DA}) . (\vec{BD} + \vec{DC})$

$= \vec{BD} . \vec{BD} + \vec{BD} . \vec{DC} + \vec{DA} . \vec{BD} + \vec{DA} . \vec{DC}$

But $\vec{DA}$ and $\vec{DC}$ are opposite vectors of equal magnitude. Then $\vec{DA} = - \vec{DC}$

Therefore, $\vec{BA} . \vec{BC} = |\vec{BD}|2 + \vec{BD} . \vec{DC} - \vec{BD} . \vec{DC} + \vec{DA} . (- \vec{DA})$

$= |\vec{BD}|^2 - |\vec{DA}|^2 = \vec{BD}^2 - \vec{DA}^2$

But $\vec{BA}$ and $\vec{BC}$ are at right angle.

Therefore, $\vec{BA} . \vec{BC} = 0$

Hence, $\vec{BD}^2 - \vec{DA}^2 = 0$ => BD = DA and also DA = DC

Thus BD = DA = DC.

Therefore, D is equidistant from A, B and C.