Mathematics

(Chapter – 5) (Continuity and Differentiability) (Exercise 5.1)
(Class – XII)

Question 1:
Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.

Answer 1:
Given function $f(x) = 5x - 3$

At $x = 0, f(0) = 5(0) - 3 = -3$

LHL = $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (5x - 3) = -3$

RHL = $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (5x - 3) = -3$

Here, at $x = 0$, LHL = RHL = $f(0) = -3$

Hence, the function f is continuous at $x = 0$.

At $x = -3, f(-3) = 5(-3) - 3 = -18$

LHL = $\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (5x - 3) = -18$

RHL = $\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (5x - 3) = -18$

Here, at $x = -3$, LHL = RHL = $f(-3) = -18$

Hence, the function f is continuous at $x = -3$.

At $x = 5, f(5) = 5(5) - 3 = 22$

LHL = $\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (5x - 3) = 22$

RHL = $\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (5x - 3) = 22$

Here, at $x = 5$, LHL = RHL = $f(5) = 22$

Hence, the function f is continuous at $x = 5$.

Question 2:
Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.

Answer 2:
Given function $f(x) = 2x^2 - 1$. At $x = 3, f(3) = 2(3)^2 - 1 = 17$

LHL = $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (2x^2 - 1) = 17$

RHL = $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (2x^2 - 1) = 17$

Here, at $x = 3$, LHL = RHL = $f(3) = 17$

Hence, the function f is continuous at $x = 3$.

Question 3:
Examine the following functions for continuity:

(a) $f(x) = x - 5$

(b) $f(x) = \frac{1}{x-5}, x \neq 5$

(c) $f(x) = \frac{x^2-25}{x+5}, x \neq -5$

(d) $f(x) = |x - 5|$

Answer 3:
(a) Given function $f(x) = x - 5$

Let, k be any real number. At $x = k, f(k) = k - 5$

LHL = $\lim_{x \to k^-} f(x) = \lim_{x \to k^-} (x - 5) = k - 5$

RHL = $\lim_{x \to k^+} f(x) = \lim_{x \to k^+} (x - 5) = k - 5$

At, $x = k$, LHL = RHL = $f(k) = k - 5$

Hence, the function f is continuous for all real numbers.

(b) Given function $f(x) = \frac{1}{x-5}, x \neq 5$

Let, k $(k \neq 5)$ be any real number. At $x = k$, $f(k) = \frac{1}{k-5}$

LHL = $\lim_{x \to k^-} f(x) = \lim_{x \to k^-} \left(\frac{1}{x-5}\right) = \frac{1}{k-5}$

RHL = $\lim_{x \to k^+} f(x) = \lim_{x \to k^+} \left(\frac{1}{x-5}\right) = \frac{1}{k-5}$

At, $x = k$, LHL = RHL = $f(k) = \frac{1}{k-5}$

Hence, the function f is continuous for all real numbers (except 5).

(c) Given function $f(x) = \frac{x^2-25}{x+5}, x \neq -5$

Let, k $(k \neq -5)$ be any real number.

At $x = k$, $f(k) = \frac{k^2-25}{k+5} = \frac{(k+5)(k-5)}{(k+5)} = (k-5)$

LHL = $\lim_{x \to k^-} f(x) = \lim_{x \to k^-} \left(\frac{x^2-25}{x+5}\right) = \lim_{x \to k^-} \left(\frac{(k+5)(k-5)}{(k+5)}\right) = k - 5$

RHL = $\lim_{x \to k^+} f(x) = \lim_{x \to k^+} \left(\frac{x^2-25}{x+5}\right) = \lim_{x \to k^+} \left(\frac{(k+5)(k-5)}{(k+5)}\right) = k - 5$

At, $x = k$, LHL = RHL = $f(k) = k - 5$

Hence, the function f is continuous for all real numbers (except – 5).

(d) Given function $f(x) = |x-5| = \begin{cases} 5-x, & x < 5 \\ x-5, & x \geq 5 \end{cases}$

Let, k be any real number. According to question, $k < 5$ or $k = 5$ or $k > 5$.

First case: If, $k < 5$,

$f(k) = 5 - k$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(5 - x) = 5 - k$, Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers less than 5.

Second case: If, $k = 5$,

$f(k) = k - 5$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(x - 5) = k - 5$, Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous at x = 5.

Third case: If, $k > 5$,

$f(k) = k - 5$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(x - 5) = k - 5$, Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers greater than 5.

Hence, the function f is continuous for all real numbers.

Question 4:

Prove that the function $f(x) = x^n$, is continuous at $x = n$, where n is a positive integer.

Answer 4:

Given function $f(x) = x^n$.

At $x = n$, $f(n) = n^n$

$$\lim_{x \to n} f(x) = \lim_{x \to n}(x^n) = n^n$$

Here, at $x = n$, $\lim_{x \to n} f(x) = f(n) = n^n$

Hence, the function f is continuous at $x = n$, where n is positive integer.

Question 5:

Is the function *f* defined by $f(x) = \begin{cases} x, & x \leq 1 \\ 5, & x > 1 \end{cases}$ continuous at $x = 0$? At $x = 1$? At $x = 2$?

Answer 5:

Given function $f(x) = \begin{cases} x, & x \leq 1 \\ 5, & x > 1 \end{cases}$

At $x = 0, f(0) = 0$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0}(x) = 0$$

Here, $x = 0, \lim_{x \to 0} f(x) = f(0) = 0$

Hence, the function f is discontinuous at $x = 0$.

At $x = 1, f(1) = 1$

LHL = $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x) = 1$

RHL = $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (5) = 5$

Here, at $x = 1$, LHL $\neq$ RHL. Hence, the function f is discontinuous at x = 1.

At $x = 2, f(2) = 5$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2}(5) = 5$$

Here, at $x = 2, \lim_{x \to 2} f(x) = f(2) = 5$

Hence, the function f is continuous at $x = 2$.

Find all points of discontinuity of *f*, where *f* is defined by

Question 6:

$$f(x) = \begin{cases} 2x + 3, & \text{If } x \leq 2 \\ 2x - 3, & \text{If } x > 2 \end{cases}$$

Answer 6:

Let, k be any real number. According to question, $k < 2$ or $k = 2$ or $k > 2$

First case: यदि, $k < 2$,

$f(k) = 2k + 3$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(2x + 3) = 2k + 3$, Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers smaller than 2.

Second case: If, $k = 2, f(2) = 2k + 3$

LHL = $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x + 3) = 7$

RHL = $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x - 3) = 1$

Here, at $x = 2$, LHL $\neq$ RHL. Hence, the function f is discontinuous at $x = 2$.

Third case: If, $k > 2$,

$f(k) = 2k - 3$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(2x - 3) = 2k - 3$, Here, $\lim_{x \to k} f(x) = f(k)$

Therefore, the function f is continuous for all real numbers greater than 2.

Hence, the function f is discontinuous only at $x = 2$.

Question 7:

$$f(x) = \begin{cases} |x| + 3, & \text{If } x \leq -3 \\ -2x, & \text{If } -3 < x < 3 \\ 6x + 2, & \text{If } x \geq 3 \end{cases}$$

Answer 7:

Let, k be any real number. According to question,

$k < -3$ or $k = -3$ or $-3 < k < 3$ or $k = 3$ or $k > 3$

First case: If, $k < -3$,
$f(k) = -k + 3$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(-x + 3) = -k + 3$. Here, $\lim_{x \to k} f(x) = f(k)$
Hence, the function f is continuous for all real numbers less than -3.

Second case: If, $k = -3$, $f(-3) = -(-3) + 3 = 6$
LHL = $\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-}(-x + 3) = -(-3) + 3 = 6$
RHL = $\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+}(-2x) = -2(-3) = 6$. Here, $\lim_{x \to k} f(x) = f(k)$
Hence, the function f is continuous at $x = -3$.

Third case: If, $-3 < k < 3$,
$f(k) = -2k$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(-2x) = -2k$. Here, $\lim_{x \to k} f(x) = f(k)$
Hence, the function f is continuous at $-3 < x < 3$.

Fourth case: If $k = 3$,
LHL = $\lim_{x \to k^-} f(x) = \lim_{x \to k^-}(-2x) = -2k$
RHL = $\lim_{x \to k^+} f(x) = \lim_{x \to k^+}(6x + 2) = 6k + 2$,
Here, at $x = 3$, LHL $\neq$ RHL. Hence, the function f is discontinuous at $x = 3$.

Fifth case: If, $k > 3$,
$f(k) = 6k + 2$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(6x + 2) = 6k + 2$. Here, $\lim_{x \to k} f(x) = f(k)$
Hence, the function f is continuous for all numbers greater than 3.
Hence, the function f is discontinuous only at $x = 3$.

Question 8:

$$f(x) = \begin{cases} \frac{|x|}{x}, & \text{If } x \neq 0 \\ 0, & \text{If } x = 0 \end{cases}$$

Answer 8:

After redefining the function f, we get

$$f(x) = \begin{cases} -\frac{x}{x} = -1, & \text{If } x < 0 \\ 0, & \text{If } x = 0 \\ \frac{x}{x} = 1, & \text{If } x > 0 \end{cases}$$

Let, k be any real number. According to question, $k < 0$ or $k = 0$ or $k > 0$.
First case: If, $k < 0$,
$f(k) = -\frac{k}{k} = -1$ and $\lim_{x \to k} f(x) = \lim_{x \to k}\left(-\frac{x}{x}\right) = -1$. Here, $\lim_{x \to k} f(x) = f(k)$
Hence, the function f is continuous for all real numbers smaller than 0.

Second case: If, $k = 0$, $f(0) = 0$
LHL = $\lim_{x \to k^-} f(x) = \lim_{x \to k^-}\left(-\frac{x}{x}\right) = -1$ and RHL = $\lim_{x \to k^+} f(x) = \lim_{x \to k^+}\left(\frac{x}{x}\right) = 1$,

Here, at $x = 0$, LHL $\neq$ RHL. Hence, the function f is discontinuous at $x = 0$.

Third case: If, $k > 0$,

$f(k) = \frac{k}{k} = 1$ and $\lim_{x \to k} f(x) = \lim_{x \to k}\left(\frac{x}{x}\right) = 1$. Here, $\lim_{x \to k} f(x) = f(k)$
Hence, the function f is continuous for all real numbers greater than 0.
Therefore, the function f is discontinuous only at $x = 0$.

Question 9:

$$f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{If } x < 0 \\ -1, & \text{If } x \geq 0 \end{cases}$$

Answer 9:

Redefining the function, we get

$$f(x) = \begin{cases} \dfrac{x}{|x|} = \dfrac{x}{-x} = -1, & \text{If } x < 0 \\ -1, & \text{If } x \geq 0 \end{cases}$$

Here, $\lim\limits_{x \to k} f(x) = f(k) = -1$, where k is a real number.

Hence, the function f is continuous for all real numbers.

Question 10:

$$f(x) = \begin{cases} x + 1, & \text{If } \geq 1 \\ x^2 + 1, & \text{If } x < 1 \end{cases}$$

Answer 10:

Let, k be any real number. According to question, $k < 1$ or $k = 1$ or $k > 1$

First case: If, $k < 1$,

$f(k) = k^2 + 1$ and $\lim\limits_{x \to k} f(x) = \lim\limits_{x \to k} (x^2 + 1) = k^2 + 1$. Here, $\lim\limits_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers smaller than 1.

Second case: If, $k = 1$, $f(1) = 1 + 1 = 2$

LHL = $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x^2 + 1) = 1 + 1 = 2$

RHL = $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x + 1) = 1 + 1 = 2$,

Here, at $x = 1$, LHL = RHL = $f(1)$. Hence, the function f is continuous at $x = 1$.

Third case: If, $k > 1$,

$f(k) = k + 1$ and $\lim\limits_{x \to k} f(x) = \lim\limits_{x \to k} (x + 1) = k + 1$. Here, $\lim\limits_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers greater than 1.

Therefore, the function f is continuous for all real numbers.

Question 11:

$$f(x) = \begin{cases} x^3 - 3, & \text{If } x \leq 2 \\ x^2 + 1, & \text{If } x > 2 \end{cases}$$

Answer 11:

Let, k be any real number. According to question, $k < 2$ or $k = 2$ or $k > 2$

First case: If, $k < 2$,

$f(k) = k^3 - 3$ and $\lim\limits_{x \to k} f(x) = \lim\limits_{x \to k} (x^3 - 3) = k^3 - 3$. Here, $\lim\limits_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers less than 2.

Second case: If, $k = 2$, $f(2) = 2^3 - 3 = 5$

LHL = $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x^3 - 3) = 2^3 - 3 = 5$

RHL = $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (x^2 + 1) = 2^2 + 1 = 5$,

Here, at $x = 2$, LHL = RHL = $f(2)$

Hence, the function f is continuous at $x = 2$.

Third case: If, $k > 2$,

$f(k) = k^2 + 1$ and $\lim\limits_{x \to k} f(x) = \lim\limits_{x \to k} (x^2 + 1) = k^2 + 1$. Here, $\lim\limits_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for real numbers greater than 2.

Hence, the function f is continuous for all real numbers.

Question 12:

$$f(x) = \begin{cases} x^{10} - 1, & \text{If } x \leq 1 \\ x^2, & \text{If } x > 1 \end{cases}$$

Answer 12:

Let, k be any real number. According to question, $k < 1$ or $k = 1$ or $k > 1$

First case: If, $k < 1$,

$f(k) = k^{10} - 1$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(x^{10} - 1) = k^{10} - 1$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers less than 1.

Second case: If, $k = 1$, $f(1) = 1^{10} - 1 = 0$

LHL = $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^{10} - 1) = 0$

RHL = $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2) = 1$,

Here, at $x = 1$, LHL $\neq$ RHL. Hence, the function f is discontinuous at $x = 1$.

Third case: If, $k > 1$,

$f(k) = k^2$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(x^2) = k^2$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real values greater than 1.

Hence, the function f is discontinuous only at $x = 1$.

Question 13:

Is the function defined by $f(x) = \begin{cases} x + 5, & \text{If } x \leq 1 \\ x - 5, & \text{If } x > 1 \end{cases}$ a continuous function?

Answer 13:

Let, k be any real number. According to question, $k < 1$ or $k = 1$ or $k > 1$

First case: If, $k < 1$,

$f(k) = k + 5$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(x + 5) = k + 5$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers less than 1.

Second case: If, $k = 1$, $f(1) = 1 + 5 = 6$

LHL = $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x + 5) = 6$

RHL = $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x - 5) = -4$,

Here, at $x = 1$, LHL $\neq$ RHL. Hence, the function f is discontinuous at $x = 1$.

Third case: If, $k > 1$,

$f(k) = k - 5$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(x - 5) = k - 5$.

Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers greater than 1.

Hence, the function f is discontinuous only at $x = 1$.

Discuss the continuity of the function *f*, where *f* is defined by:

Question 14:

$$f(x) = \begin{cases} 3, & \text{If } 0 \leq x \leq 1 \\ 4, & \text{If } 1 < x < 3 \\ 5, & \text{If } 3 \leq x \leq 10 \end{cases}$$

Answer 14:

Let, k be any real number. According to question,

$0 \leq k \leq 1$ or $k = 1$ or $1 < k < 3$ or $k = 3$ or $3 \leq k \leq 10$

First case: If, $0 \leq k \leq 1$,
$f(k) = 3$ and $\lim_{x\to k} f(x) = \lim_{x\to k}(3) = 3$. Here, $\lim_{x\to k} f(x) = f(k)$
Hence, the function f is continuous for $0 \leq x \leq 1$.

Second case: If, $k = 1$, $f(1) = 3$
LHL = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(3) = 3$
RHL = $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(4) = 4$,
Here, at $x = 1$, LHL ≠ RHL. Hence, the function f is discontinuous at $x = 1$.

Third case: If, $1 < k < 3$,
$f(k) = 4$ and $\lim_{x\to k} f(x) = \lim_{x\to k}(4) = 4$. Here, $\lim_{x\to k} f(x) = f(k)$
Hence, the function f is continuous for $1 < x < 3$.

Fourth case: If $k = 3$,
LHL = $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(4) = 4$ and RHL = $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(5) = 5$,
Here, at $x = 3$, LHL ≠ RHL. Hence, the function f is discontinuous at $x = 3$.

Fifth case: If, $3 \leq k \leq 10$,
$f(k) = 5$ and $\lim_{x\to k} f(x) = \lim_{x\to k}(5) = 5$. Here, $\lim_{x\to k} f(x) = f(k)$
Hence, the function f is continuous for $3 \leq x \leq 10$.
Hence, the function f is discontinuous only at $x = 1$ and $x = 3$.

Question 15:

$$f(x) = \begin{cases} 2x, & \text{If } x < 0 \\ 0, & \text{If } 0 \leq x \leq 1 \\ 4x, & \text{If } x > 1 \end{cases}$$

Answer 15:

Let, k be any real number. According to question,
$k < 0$ or $k = 0$ or $0 \leq k \leq 1$ or $k = 1$ or $k > 1$
First case: If, $k < 0$,
$f(k) = 2k$ and $\lim_{x\to k} f(x) = \lim_{x\to k}(2x) = 2k$. Here, $\lim_{x\to k} f(x) = f(k)$
Hence, the function f is continuous for all real numbers less than 0.

Second case: If, $k = 0$, $f(0) = 0$
LHL = $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(2x) = 0$
RHL = $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(0) = 0$. Here, $\lim_{x\to k} f(x) = f(k)$
Hence, the function f is continuous at x = 0.

Third case: If, $0 \leq k \leq 1$,
$f(k) = 0$ and $\lim_{x\to k} f(x) = \lim_{x\to k}(0) = 0$. Here, $\lim_{x\to k} f(x) = f(k)$
Hence, the function f is continuous at $0 \leq x \leq 1$.

Fourth case: If $k = 1$,
LHL = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(0) = 0$
RHL = $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(4x) = 4$,
Here, at $x = 1$, LHL ≠ RHL.
Hence, the function f is discontinuous at x = 1.

Fifth case: If, $k > 1$,

$f(k) = 4k$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(4x) = 4k$.

Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers greater than 1.

Therefore, the function f is discontinuous only at x = 1.

Question 16:

$$f(x) = \begin{cases} -2, & \text{If } x \le -1 \\ 2x, & \text{If } -1 < x \le 1 \\ 2, & \text{If } x > 1 \end{cases}$$

Answer 16:

Let, k be any real number.

According to question, $k < -1$ or $k = -1$ or $-1 < x \le 1$ or $k = 1$ or $k > 1$

First case: If, $k < -1$,

$f(k) = -2$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(-2) = -2$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers less than -1.

Second case: If, $k = -1$, $f(-1) = -2$

LHL = $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-}(-2) = -2$

RHL = $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+}(2x) = -2$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous at x = -1.

Third case: If, $-1 < x \le 1$,

$f(k) = 2k$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(2x) = 2k$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous at $-1 < x \le 1$.

Fourth case: If, $k = 1$,

LHL = $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(2x) = 2$

RHL = $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+}(2) = 2$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous at x = 1.

Fifth case: If, $k > 1$,

$f(k) = 2$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(2) = 2$.

Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers greater than 1.

Therefore, the function f is continuous for all real numbers.

Question 17:

Find the relationship between a and b so that the function f defined by

$$f(x) = \begin{cases} ax + 1, & \text{If } x \le 3 \\ bx + 3, & \text{If } x > 3 \end{cases}$$

is continuous at $x = 3$.

Answer 17:

Given that the function is continuous at $x = 3$. Therefore, LHL = RHL = $f(3)$

$\Rightarrow \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$

$\Rightarrow \lim_{x \to 3^-} ax + 1 = \lim_{x \to 3^+} bx + 3 = 3a + 1$

$\Rightarrow 3a + 1 = 3b + 3 = 3a + 1 \Rightarrow 3a = 3b + 2 \quad \Rightarrow a = b + \frac{2}{3}$

Question 18:

For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$$

continuous at $x = 0$? What about continuity at $x = 1$?

Answer 18:

Given that the function is continuous at $x = 0$. Therefore, LHL = RHL = $f(0)$

$\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$\Rightarrow \lim_{x\to 0^-} \lambda(x^2 - 2x) = \lim_{x\to 0^+} 4x + 1 = \lambda[(0)^2 - 2(0)]$

$\Rightarrow \lambda[(0)^2 - 2(0)] = 4(0) + 1 = \lambda(0)$

$\Rightarrow 0.\lambda = 1 \quad \Rightarrow \lambda = \frac{1}{0}$

Hence, there is no real value of λ for which the given function be continuous.

If, $x = 1$,

$f(1) = 4(1) + 1 = 5$ and $\lim_{x\to 1} f(x) = \lim_{x\to 1} 4(1) + 1 = 5$, Here, $\lim_{x\to 1} f(x) = f(1)$

Hence, the function f is continuous for all real values of λ.

Question 19:

Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x.

Answer 19:

Let, k be any integer.

LHL = $\lim_{x\to k^-} f(x) = \lim_{x\to k^-} x - [x] = k - (\text{k} - 1) = 1$

RHL = $\lim_{x\to k^+} f(x) = \lim_{x\to k^+} x - [x] = k - (\text{k}) = 0$,

Here, at $x = k$, LHL ≠ RHL. Hence, the function f is discontinuous for all integers.

Question 20:

Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$.

Answer 20:

Given function: $f(x) = x^2 - \sin x + 5$,

At $x = \pi$, $f(\pi) = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$

$\lim_{x\to n} f(x) = \lim_{x\to n} x^2 - \sin x + 5 = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$

Here, at $x = \pi$, $\lim_{x\to n} f(x) = f(\pi) = \pi^2 + 5$

Hence, the function f is continuous at $x = \pi$.

Question 21:

Discuss the continuity of the following functions:

(a) $f(x) = \sin x + \cos x$ **(b)** $f(x) = \sin x - \cos x$

(c) $f(x) = \sin x \,.\cos x$

Answer 21:

Let, $g(x) = \sin x$

Let, k be any real number. At $x = k$, $g(k) = \sin k$

LHL = $\lim_{x\to k^-} g(x) = \lim_{x\to k^-} \sin x = \lim_{h\to 0} \sin(k - h) = \lim_{h\to 0} \sin k \cos h - \cos k \sin h = \sin k$

RHL = $\lim_{x\to k^+} g(x) = \lim_{x\to k^+} \sin x = \lim_{h\to 0} \sin(k + h) = \lim_{h\to 0} \sin k \cos h + \cos k \sin h = \sin k$

Here, at $x = k$, LHL = RHL = $g(k)$.

Hence, the function g is continuous for all real numbers.

Let, $h(x) = \cos x$

Let, k be any real number. At $x = k$, $h(k) = \cos k$

LHL = $\lim\limits_{x \to k^-} h(x) = \lim\limits_{x \to k^-} \cos x = \lim\limits_{h \to 0} \cos(k - h) = \lim\limits_{h \to 0} \cos k \cos h + \sin k \sin h = \cos k$

RHL = $\lim\limits_{x \to k^+} h(x) = \lim\limits_{x \to k^+} \cos x = \lim\limits_{h \to 0} \cos(k + h) = \lim\limits_{h \to 0} \cos k \cos h - \sin k \sin h = \cos k$

Here, at $x = k$, LHL = RHL = $h(k)$.

Hence, the function h is continuous for all real numbers.

We know that if g and h are two continuous functions, then the functions $g + h, g - h$ and gh also be a continuous functions.

Hence, **(a)** $f(x) = \sin x + \cos x$ **(b)** $f(x) = \sin x - \cos x$ and **(c)** $f(x) = \sin x \, . \cos x$ are continuous functions.

Question 22:

Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

Answer 22:

Let $g(x) = \sin x$

Let, k be any real number. At $x = k$, $g(k) = \sin k$

LHL = $\lim\limits_{x \to k^-} g(x) = \lim\limits_{x \to k^-} \sin x = \lim\limits_{h \to 0} \sin(k - h) = \lim\limits_{h \to 0} \sin k \cos h - \cos k \sin h = \sin k$

RHL = $\lim\limits_{x \to k^+} g(x) = \lim\limits_{x \to k^+} \sin x = \lim\limits_{h \to 0} \sin(k + h) = \lim\limits_{h \to 0} \sin k \cos h + \cos k \sin h = \sin k$

Here, at $x = k$, LHL = RHL = $g(k)$.

Hence, the function g is continuous for all real numbers.

Let $h(x) = \cos x$

Let, k be any real number. At $x = k$, $h(k) = \cos k$

LHL = $\lim\limits_{x \to k^-} h(x) = \lim\limits_{x \to k^-} \cos x = \lim\limits_{h \to 0} \cos(k - h) = \lim\limits_{h \to 0} \cos k \cos h + \sin k \sin h = \cos k$

RHL = $\lim\limits_{x \to k^+} h(x) = \lim\limits_{x \to k^+} \cos x = \lim\limits_{h \to 0} \cos(k + h) = \lim\limits_{h \to 0} \cos k \cos h - \sin k \sin h = \cos k$

Here, at $x = k$, LHL = RHL = $h(k)$.

Hence, the function h is continuous for all real numbers.

We know that if g and h are two continuous functions, then the functions $\frac{g}{h}$, $h \neq 0$, $\frac{1}{h}$, $h \neq 0$ and $\frac{1}{g}$, $g \neq 0$ be continuous functions.

Therefore, $cosec\ x = \frac{1}{\sin x}$, $\sin x \neq 0$ is continuous $\Rightarrow x \neq n\pi$ $(n \in Z)$ is continuous.

Hence, $cosec\ x$ is continuous except $x = n\pi$ $(n \in Z)$.

$sec\ x = \frac{1}{\cos x}$, $\cos x \neq 0$ is continuous. $\Rightarrow x \neq \frac{(2n+1)\pi}{2}$ $(n \in Z)$ is continuous.

Hence, $sec\ x$ is continuous except $x = \frac{(2n+1)\pi}{2}$ $(n \in Z)$.

$cot\ x = \frac{\cos x}{\sin x}$, $\sin x \neq 0$ is continuous. $\Rightarrow x \neq n\pi$ $(n \in Z)$ is continuous.

Hence, $cot\ x$ is continuous except $x = n\pi$ $(n \in Z)$.

Question 23:

Find all points of discontinuity of f, where

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{If } x < 0 \\ x + 1, & \text{If } x \geq 0 \end{cases}$$

Answer 23:

Let, k be any real number. According to question, $k < 0$ or $k = 0$ or $k > 0$

First case: If, $k < 0$,

$f(k) = \frac{\sin k}{k}$ and $\lim\limits_{x \to k} f(x) = \lim\limits_{x \to k} \left(\frac{\sin x}{x}\right) = \frac{\sin k}{k}$, Here, $\lim\limits_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers less than 0.

Second case: If, $k = 0$, $f(0) = 0 + 1 = 1$

LHL = $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x + 1) = 0 + 1 = 1$

RHL = $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x + 1) = 0 + 1 = 1,$

Here, at $x = 0$, LHL = RHL = $f(0)$.

Hence, the function f is continuous at x = 0.

Third case: If, $k > 0$,

$f(k) = k + 1$ and $\lim_{x \to k} f(x) = \lim_{x \to k} (\mathrm{x} + 1\ ) = \mathrm{k} + 1$, Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers greater than 0.

Therefore, the function f is continuous for all real numbers.

Question 24:

Determine if f defined by

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{If } x \neq 0 \\ 0, & \text{If } x = 0 \end{cases}$$

is a continuous function?

Answer 24:

Let, k be any real number. According to question, $k \neq 0$ or $k = 0$

First case: If, $k \neq 0$,

$f(k) = k^2 \sin\frac{1}{k}$ and $\lim_{x \to k} f(x) = \lim_{x \to k} \left(x^2 \sin\frac{1}{x}\right) = k^2 \sin\frac{1}{k}$, Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for $k \neq 0$.

Second case: If, $k = 0$, $f(0) = 0$

LHL = $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(x^2 \sin\frac{1}{x}\right) = \lim_{x \to 0} \left(x^2 \sin\frac{1}{x}\right)$

We know that, $-1 \leq \sin\frac{1}{x} \leq 1,\ \ x \neq 0 \qquad \Rightarrow -x^2 \leq \sin\frac{1}{x} \leq x^2$

$\Rightarrow\ \lim_{x \to 0}(-x^2) \leq \lim_{x \to 0} \sin\frac{1}{x} \leq \lim_{x \to 0} x^2$

$\Rightarrow 0 \leq \lim_{x \to 0} \sin\frac{1}{x} \leq 0 \qquad \Rightarrow \lim_{x \to 0} \sin\frac{1}{x} = 0 \qquad \Rightarrow \lim_{x \to 0^-} x^2 \sin\frac{1}{x} = 0 \qquad \Rightarrow \lim_{x \to 0^-} f(x) = 0$

Similarly, RHL = $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(x^2 \sin\frac{1}{x}\right) = \lim_{x \to 0} \left(x^2 \sin\frac{1}{x}\right) = 0,$

Here, at $x = 0$, LHL = RHL = $f(0)$

Hence, at$x = 0$, f is continuous.

Hence, the function$f$ is continuous for all real numbers.

Question 25:

Examine the continuity of f, where f is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{If } x \neq 0 \\ -1, & \text{If } x = 0 \end{cases}$$

Answer 25:

Let, k be any real number. According to question, $k \neq 0$ or $k = 0$

First case: If, $k \neq 0$, $f(0) = 0 - 1 = -1$

LHL = $\lim_{k \to 0^-} f(x) = \lim_{k \to 0^-} (\sin x - \cos x) = 0 - 1 = -1$

RHL = $\lim_{k \to 0^+} f(x) = \lim_{k \to 0^+} (\sin x - \cos x) = 0 - 1 = -1,$

Hence, at $x \neq 0$, LHL = RHL = $f(x)$

Hence, the function f is continuous at $x \neq 0$.

Second case: If, $k = 0, f(k) = -1$
and $\lim_{x\to k} f(x) = \lim_{x\to k}(-1) = -1$, Here, $\lim_{x\to k} f(x) = f(k)$
Hence, the function f is continuous at x = 0.
Therefore, the function f is continuous for all real numbers.

Find the values of k so that the function f is continuous at the indicated point in exercises 26 to 29.

Question 26:

$$f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{If } x \neq \dfrac{\pi}{2} \\ 3, & \text{If } x = \dfrac{\pi}{2} \end{cases} \quad \text{at } x = \frac{\pi}{2}$$

Answer 26:

Given that the function is continuous at $x = \frac{\pi}{2}$. Therefore, LHL = RHL = $f\left(\frac{\pi}{2}\right)$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^+} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}^-} \frac{k\cos x}{\pi - 2x} = \lim_{x\to\frac{\pi}{2}^+} \frac{k\cos x}{\pi - 2x} = 3 \Rightarrow \lim_{h\to 0} \frac{k\cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)} = \lim_{h\to 0} \frac{k\cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} = 3$$

$$\Rightarrow \lim_{h\to 0} \frac{k\sin h}{2h} = \lim_{h\to 0} \frac{-k\sin h}{-2h} = 3$$

$$\Rightarrow \frac{k}{2} = \frac{k}{2} = 3 \qquad \left[\because \lim_{h\to 0} \frac{\sin h}{h} = 1\right]$$

$$\Rightarrow k = 6$$

Question 27:

$$f(x) = \begin{cases} kx^2, & \text{If } x \leq 2 \\ 3, & \text{If } x > 2 \end{cases} \quad \text{at } x = 2$$

Answer 27:

Given that the function is continuous at $x = 2$.
Therefore, LHL = RHL = $f(2)$

$$\Rightarrow \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x\to 2^-} kx^2 = \lim_{x\to 2^+} 3 = k(2)^2$$

$$\Rightarrow 4k = 3 = 4k \quad \Rightarrow k = \frac{3}{4}$$

Question 28:

$$f(x) = \begin{cases} kx + 1, & \text{If } x \leq \pi \\ \cos x, & \text{If } x > \pi \end{cases} \quad \text{at } x = \pi$$

Answer 28:

Given that the function is continuous at $x = \pi$.
Therefore, LHL = RHL = $f(\pi)$

$$\Rightarrow \lim_{x\to \pi^-} f(x) = \lim_{x\to \pi^+} f(x) = f(\pi)$$

$$\Rightarrow \lim_{x\to \pi^-} kx + 1 = \lim_{x\to \pi^+} \cos x = k(\pi) + 1$$

$$\Rightarrow k(\pi) + 1 = \cos\pi = k\pi + 1 \Rightarrow k\pi + 1 = -1 = k\pi + 1$$

$$\Rightarrow \pi k = -2 \qquad \Rightarrow k = -\frac{2}{\pi}$$

Question 29:

$$f(x) = \begin{cases} kx + 1, & \text{If } x \leq 5 \\ 3x - 5, & \text{If } x > 5 \end{cases} \text{ at } x = 5$$

Answer 29:

Given that the function is continuous at $x = 5$.
Therefore, LHL = RHL = $f(5)$

$\Rightarrow \lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = f(5)$

$\Rightarrow \lim_{x \to 5^-} kx + 1 = \lim_{x \to 5^+} 3x - 5 = 5k + 1$

$\Rightarrow 5k + 1 = 15 - 5 = 5k + 1$

$\Rightarrow 5k = 9 \quad \Rightarrow k = \frac{9}{5}$

Question 30:

Find the values of a and b such that the function defined by

$$f(x) = \begin{cases} 5, & \text{If } x \leq 2 \\ ax + b, & \text{If } 2 < x < 10 \\ 21, & \text{If } x \geq 10 \end{cases}$$

is a continuous function.

Answer 30:

Given that the function is continuous at $x = 2$. Therefore, LHL = RHL = $f(2)$

$\Rightarrow \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$

$\Rightarrow \lim_{x \to 2^-} 5 = \lim_{x \to 2^+} ax + b = 5$

$\Rightarrow 2a + b = 5 \qquad \text{... (1)}$

Given that the function is continuous at $x = 10$. Therefore, LHL = RHL = $f(10)$

$\Rightarrow \lim_{x \to 10^-} f(x) = \lim_{x \to 10^+} f(x) = f(10)$

$\Rightarrow \lim_{x \to 10^-} ax + b = \lim_{x \to 10^+} 21 = 21$

$\Rightarrow 10a + b = 21 \qquad \text{... (2)}$

Solving the equation (1) and (2), we get

$a = 2 \qquad b = 1$

Question 31:

Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

Answer 31:

Assuming that the functions are well defined for all real numbers, we can write the given function f in the combination of g and h $(f = goh)$. Where, $g(x) = \cos x$ and $h(x) = x^2$. If g and h both are continuous function then f also be continuous.

$[\because goh(x) = g(h(x)) = g(x^2) = \cos(x^2)]$

Function $g(x) = \cos x$

Let, k be any real number. At $x = k$, $g(k) = \cos k$

$\lim_{x \to k} g(x) = \lim_{x \to k} \cos x = \lim_{h \to 0} \cos(k + h) = \lim_{h \to 0} \cos k \cos h - \sin k \sin h = \cos k$

Here, $\lim_{x \to k} g(x) = g(k)$, Hence, the function g is continuous for all real numbers.

Function $h(x) = x^2$

Let, k be any real number. At $x = k$, $h(k) = k^2$

$\lim_{x \to k} h(x) = \lim_{x \to k} x^2 = k^2$

Here, $\lim_{x \to k} h(x) = h(k)$, Hence, the function h is continuous for all real numbers.

Therefore, g and h both are continuous function. Hence, f is continuous.

Question 32:

Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

Answer 32:

Assuming that the functions are well defined for all real numbers, we can write the given function f in the combination of g and h $(f = goh)$. Where, $g(x) = |x|$ and $h(x) = \cos x$. If g and h both are continuous function then f also be continuous.

$[\because goh(x) = g(h(x)) = g(\cos x) = |\cos x|]$

Function $g(x) = |x|$

Rearranging the function g, we get

$$g(x) = \begin{cases} -x, & \text{If } x < 0 \\ \text{x}, & \text{If } x \geq 0 \end{cases}$$

Let, k be any real number. According to question, $k < 0$ or $k = 0$ or $k > 0$

First case: If, $k < 0$,

$g(k) = 0$ and $\lim_{x \to k} g(x) = \lim_{x \to k}(-\text{x}) = 0$, here, $\lim_{x \to k} g(x) = g(k)$

Hence, the function g is continuous for all real numbers less than 0.

Second case: If, $k = 0$, $g(0) = 0 + 1 = 1$

LHL= $\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$

RHL = $\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$,

Here, at $x = 0$, LHL = RHL = $g(0)$

Hence, the function g is continuous at x = 0.

Third case: If, $k > 0$,

$g(k) = 0$ and $\lim_{x \to k} g(x) = \lim_{x \to k}(x) = 0$, Here, $\lim_{x \to k} g(x) = g(k)$

Hence, the function g is continuous for all real numbers greater than 0.

Hence, the function g is continuous for all real numbers.

Function $h(x) = \cos x$

Let, k be any real number. At $x = k$, $h(k) = \cos k$

$\lim_{x \to k} h(x) = \lim_{x \to k} \cos x = \cos k$

Here, $\lim_{x \to k} h(x) = h(k)$, Hence, the function h is continuous for all real numbers.

Therefore, g and h both are continuous function. Hence, f is continuous.

Question 33:

Examine that $\sin |x|$ is a continuous function.

Answer 33:

Assuming that the functions are well defined for all real numbers, we can write the given function f in the combination of g and h $(f = hog)$. Where, $h(x) = \sin x$ and $g(x) = |x|$. If g and h both are continuous function then f also be continuous.

$[\because hog(x) = h(g(x)) = h(|x|) = \sin|x|]$

Function $h(x) = \sin x$

Let, k be any real number. At $x = k$, $h(k) = \sin k$

$\lim_{x \to k} h(x) = \lim_{x \to k} \sin x = \sin k$

Here, $\lim_{x \to k} h(x) = h(k)$, Hence, the function h is continuous for all real numbers.

Function $g(x) = |x|$

Redefining the function g, we get

$$g(x) = \begin{cases} -x, & \text{If } x < 0 \\ \text{x}, & \text{If } x \geq 0 \end{cases}$$

Let, k be any real number. According to question, $k < 0$ or $k = 0$ or $k > 0$
First case: If, $k < 0$,
$g(k) = 0$ and $\lim_{x \to k} g(x) = \lim_{x \to k}(-x) = 0$, Here, $\lim_{x \to k} g(x) = g(k)$
Hence, the function g is continuous for all real numbers less than 0.

Second case: If, $k = 0$, $g(0) = 0 + 1 = 1$
LHL = $\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$
RHL = $\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$,
Here, at $x = 0$, LHL = RHL = $g(0)$
Hence, at $x = 0$, the function g is continuous.

Third case: If, $k > 0$,
$g(k) = 0$ and $\lim_{x \to k} g(x) = \lim_{x \to k}(x) = 0$, Here, $\lim_{x \to k} g(x) = g(k)$
Hence, the function g is continuous for all real numbers greater than 0.
Hence, the function g is continuous for all real numbers.

Therefore, g and h both are continuous function. Hence, f is continuous.

Question 34:

Find all the points of discontinuity of f defined by $f(x) = |x| - |x + 1|$.

Answer 34:

Assuming that the functions are well defined for all real numbers, we can write the given function f in the combination of g and h ($f = g - h$), where, $g(x) = |x|$ and $h(x) = |x + 1|$. If g and h both are continuous function then f also be continuous.
Function $g(x) = |x|$
Redefining the function g, we get,

$$g(x) = \begin{cases} -x, & \text{If } x < 0 \\ x, & \text{If } x \geq 0 \end{cases}$$

Let, k be any real number. According to question, $k < 0$ or $k = 0$ or $k > 0$
First case: If, $k < 0$,
$g(k) = 0$ and $\lim_{x \to k} g(x) = \lim_{x \to k}(-x) = 0$, Here, $\lim_{x \to k} g(x) = g(k)$
Hence, the function g is continuous for all real numbers less than 0.

Second case: If, $k = 0$, $g(0) = 0 + 1 = 1$
LHL = $\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$ and RHL = $\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$,
Here, at $x = 0$, LHL = RHL = $g(0)$
Hence, the function g is continuous at $x = 0$.

Third case: If, $k > 0$,
$g(k) = 0$ and $\lim_{x \to k} g(x) = \lim_{x \to k}(x) = 0$, Here, $\lim_{x \to k} g(x) = g(k)$
Hence, the function g is continuous for all real numbers more than 0.
Hence, the function g is continuous for all real numbers.
Function $h(x) = |x + 1|$
Redefining the function h, we get

$$h(x) = \begin{cases} -(x + 1), & \text{If } x < -1 \\ x + 1, & \text{If } x \geq -1 \end{cases}$$

Let, k be any real number. According to question, $k < -1$ or $k = -1$ or $k > -1$
First case: If, $k < -1$,
$h(k) = -(k + 1)$ and $\lim_{x \to k} h(x) = \lim_{x \to k} -(k + 1) = -(k + 1)$, Here, $\lim_{x \to k} h(x) = h(k)$
Hence, the function g is continuous for all real numbers less than – 1.

Second case: If, $k = -1$, $h(-1) = -1 + 1 = 0$

LHL = $\lim_{x \to -1^-} h(x) = \lim_{x \to -1^-} -(-1+1) = 0$

RHL = $\lim_{x \to -1^+} h(x) = \lim_{x \to -1^+} (x+1) = -1 + 1 = 0,$

Here, at $x = -1$, LHL = RHL = $h(-1)$

Hence, the function h is continuous at $x = -1$.

Third case: If, $k > -1$,

$h(k) = k + 1$ and $\lim_{x \to k} h(x) = \lim_{x \to k} (k+1) = \text{k} + 1$, Here, $\lim_{x \to k} h(x) = h(k)$

Hence, the function g is continuous for all real numbers greater than -1.

Hence, the function h is continuous for all real numbers.

Therefore, g and h both are continuous function. Hence, f is continuous.

Mathematics

(Chapter – 5) (Continuity and Differentiability) (Exercise 5.2)
(Class – XII)

Differentiate the functions with respect to x in Exercises 1 to 8.

Question 1:

$$\sin(x^2+5)$$

Answer 1:

Let $y=\sin(x^2+5)$
Therefore,

$$\frac{dy}{dx}=\cos(x^2+5).\frac{d}{dx}(x^2+5)=cos(x^2+5).2x$$

Question 2:

$$\cos(\sin x)$$

Answer 2:

Let $y=\cos(\sin x)$

Therefore, $\frac{dy}{dx}=-\sin(\sin x).\frac{d}{dx}(\sin x)=-sin(\sin x).\cos x$

Question 3:

$\sin(ax+b)$

Answer 3:

Let $y=\sin(ax+b)$
Therefore,

$$\frac{dy}{dx}=\cos(ax+b).\frac{d}{dx}(ax+b)=cos(ax+b).a$$

Question 4:

$$\sec(\tan(\sqrt{x}))$$

Answer 4:

Let $y=\sec(\tan(\sqrt{x}))$

Therefore, $\frac{dy}{dx}=\sec(\tan\sqrt{x})\tan(\tan\sqrt{x}).\frac{d}{dx}(\tan\sqrt{x})=\sec(\tan\sqrt{x})\tan(\tan\sqrt{x}).\sec^2\sqrt{x}\frac{d}{dx}(\sqrt{x})$

$$=\sec(\tan\sqrt{x})\tan(\tan\sqrt{x}).\sec^2\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)$$

Question 5:

$$\frac{\sin(ax+b)}{\cos(cx+d)}$$

Answer 5:

Let, $y=\frac{sin(ax+b)}{cos(cx+d)}$

Therefore, $\frac{dy}{dx}=\frac{cos(cx+d).\frac{d}{dx}sin(ax+b)-sin(ax+b).\frac{d}{dx}cos(cx+d)}{[cos(cx+d)]^2}$

$$=\frac{cos(cx+d).\sin(ax+b).\frac{d}{dx}(ax+b)-sin(ax+b).[-sin(cx+d).\frac{d}{dx}(cx+d)]}{cos^2(cx+d)}$$

$$=\frac{cos(cx+d).\sin(ax+b).a+sin(ax+b).sin(cx+d)\,c}{cos^2(cx+d)}$$

Question 6:

$$\cos x^3 . \sin^2(x^5)$$

Answer 6:

Let $y = \cos x^3 . \sin^2(x^5)$

Therefore,

$$\frac{dy}{dx} = \cos x^3 . \frac{d}{dx}\sin^2(x^5) + \sin^2(x^5). \frac{d}{dx}\cos x^3$$

$$= \cos x^3 . 2\sin x^5 \cos x^5 . \frac{d}{dx}x^5 + \sin^2(x^5)\,[-\sin x^3]. \frac{d}{dx}x^3$$

$$= \cos x^3 . 2\sin x^5 \cos x^5 . 5x^4 - \sin^2(x^5)\sin x^3 . 3x^2$$

Question 7:

$$2\sqrt{\cot(x^2)}$$

Answer 7:

Let $y = 2\sqrt{\cot(x^2)}$

Therefore,

$$\frac{dy}{dx} = 2. \frac{1}{2\sqrt{\cot(x^2)}}. \frac{d}{dx}[\cot(x^2)]$$

$$= \frac{1}{\sqrt{\cot(x^2)}}. [-cossec\ x^2]. \frac{d}{dx}x^2$$

$$= \frac{1}{\sqrt{\cot(x^2)}}. [-cossec\ x^2]. 2x$$

Question 8:

$$\cos(\sqrt{x})$$

Answer 8:

Let $y = \cos(\sqrt{x})$

Therefore,

$$\frac{dy}{dx} = -\sin(\sqrt{x}). \frac{d}{dx}\sqrt{x}$$

$$= -\sin(\sqrt{x}). \frac{1}{2\sqrt{x}}$$

Question 9:

Prove that the function f given by $f(x) = |x-1|, x \in \boldsymbol{R}$, is not differentiable at $x = 1$.

Answer 9:

At $x = 1$,

$$LHD = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{|1-h-1|-|1-1|}{-h} = \lim_{h\to 0}\frac{h}{-h} = -1$$

$$RHD = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{|1+h-1|-|1-1|}{h} = \lim_{h\to 0}\frac{h}{h} = 1$$

Here, $LHD \neq RHD$, therefore, the function $f(x) = |x-1|, x \in \boldsymbol{R}$, is not differentiable at $x = 1$.

Question 10:

Prove that the greatest integer function defined by $f(x) = [x], 0 < x < 3$, is not differentiable at x = 1 and x = 2.

Answer 10:

At $x = 1$,

$$LHD = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{[1-h]-|1|}{-h}$$

$$= \lim_{h\to 0}\frac{0-1}{-h} = \infty$$

$$RHD = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{[1+h]-[1]}{h}$$

$$= \lim_{h\to 0}\frac{1-1}{h} = 0$$

Here, $LHD \neq RHD$, therefore, the function $f(x) = [x], 0 < x < 3,$ is not differentiable at x = 1.

At $x = 2$,

$$LHD = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{[2-h]-[2]}{-h}$$

$$= \lim_{h\to 0}\frac{1-2}{-h} = \infty$$

$$RHD = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{[2+h]-[2]}{h}$$

$$= \lim_{h\to 0}\frac{2-2}{h} = 0$$

Here, $LHD \neq RHD$, therefore, the function $f(x) = [x], 0 < x < 3$, is not differentiable at x = 2.

Mathematics

(Chapter – 5) (Continuity and Differentiability) (Exercise 5.3)
(Class – XII)

Find $\frac{dy}{dx}$ in the following:

Question 1:

$2x + 3y = \sin x$

Answer 1:

$$2x + 3y = \sin x$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}\sin x$$

$$\Rightarrow 2 + 3\frac{dy}{dx} = \cos x \qquad \Rightarrow \frac{dy}{dx} = \frac{\cos x - 2}{3}$$

Question 2:

$2x + 3y = \sin y$

Answer 2:

$$2x + 3y = \sin y$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}\sin y \qquad \Rightarrow 2 + 3\frac{dy}{dx} = \cos y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(\cos y - 3) = 2 \qquad \Rightarrow \frac{dy}{dx} = \frac{2}{\cos y - 3}$$

Question 3:

$ax + by^2 = \cos y$

Answer 3:

$$ax + by^2 = \cos y$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(ax) + \frac{d}{dx}(by^2) = \frac{d}{dx}\cos y$$

$$\Rightarrow a + 2by\frac{dy}{dx} = -\sin y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(2by + \sin y) = -a \qquad \Rightarrow \frac{dy}{dx} = -\frac{a}{2by + \sin y}$$

Question 4:

$xy + y^2 = \tan x + y$

Answer 4:

$$xy + y^2 = \tan x + y$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}\tan x + \frac{dy}{dx}$$

$$\Rightarrow x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(x + 2y - 1) = \sec^2 x - y$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 x - y}{x + 2y - 1}$$

Question 5:
$x^2 + xy + y^2 = 100$

Answer 5:

$$x^2 + xy + y^2 = 100$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}x^2 + \frac{d}{dx}(xy) + \frac{d}{dx}y^2 = \frac{d}{dx}(100)$$

$$\Rightarrow 2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(x + 2y) = 2x + y \qquad \Rightarrow \frac{dy}{dx} = \frac{2x + y}{x + 2y}$$

Question 6:
$x^3 + x^2y + xy^2 + y^3 = 81$

Answer 6:

$$x^3 + x^2y + xy^2 + y^3 = 81$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}x^3 + \frac{d}{dx}(x^2y) + \frac{d}{dx}(xy^2) + \frac{d}{dx}y^3 = \frac{d}{dx}81$$

$$\Rightarrow 3x^2 + x^2\frac{dy}{dx} + y.2x + x.2y\frac{dy}{dx} + y^2.1 + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(x^2 + 2xy + 3y^2) = -(3x^2 + 2xy + y^2) \qquad \Rightarrow \frac{dy}{dx} = -\frac{3x^2 + 2xy + y^2}{x^2 + 2xy + 3y^2}$$

Question 7:
$\sin^2 y + \cos xy = k$

Answer 7:

$$\sin^2 y + \cos xy = k$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}\sin^2 y + \frac{d}{dx}\cos xy = \frac{d}{dx}k$$

$$\Rightarrow 2\sin y\cos y\frac{dy}{dx} - \sin xy\left(x\frac{dy}{dx} + y\right) = 0 \Rightarrow \sin 2y\frac{dy}{dx} - x\sin xy\frac{dy}{dx} - y\sin xy = 0$$

$$\Rightarrow (\sin 2y - x\sin xy)\frac{dy}{dx} = y\sin xy \qquad \Rightarrow \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

Question 8:
$\sin^2 x + \cos^2 y = 1$

Answer 8:

$$\sin^2 x + \cos^2 y = 1$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}\sin^2 x + \frac{d}{dx}\cos^2 y = \frac{d}{dx}1$$

$$\Rightarrow 2\sin x\cos x + 2\cos y\,(-\sin y)\frac{dy}{dx} = 0$$

$$\Rightarrow \sin 2x - \sin 2y\frac{dy}{dx} = 0 \qquad \Rightarrow \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$$

Question 9:
$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Answer 9:

$$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Let, $x = \tan\theta$

Therefore, $y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}x \quad \Rightarrow y = 2\tan^{-1}x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

Question 10:
$y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Answer 10:

$$y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

Let, $x = \tan\theta$

Therefore, $y = \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right) = \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1}x \quad \Rightarrow y = 3\tan^{-1}x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{3}{1+x^2}$$

Question 11:
$y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

Answer 11:

$$y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Let, $x = \tan\theta$

Therefore, $y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta) = 2\theta = 2\tan^{-1}x \quad \Rightarrow y = 2\tan^{-1}x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

Question 12:
$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

Answer 12:

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Let, $x = \tan\theta$, Therefore,

$y = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\cos 2\theta) = \sin^{-1}\left\{\sin(\frac{\pi}{2} - 2\theta)\right\} = \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2\tan^{-1}x$

$\Rightarrow y = \frac{\pi}{2} - 2\tan^{-1}x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = 0 - \frac{2}{1+x^2} = -\frac{2}{1+x^2}$$

Question 13:

$y = \cos^{-1}\left(\frac{2x}{1+x^2}\right), -1 < x < 1$

Answer 13:

$$y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$$

Let, $x = \tan\theta$

Therefore, $y = \cos^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$= \cos^{-1}(\sin 2\theta) = \cos^{-1}\left\{\cos(\frac{\pi}{2} - 2\theta)\right\} = \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2\tan^{-1}x$

$\Rightarrow y = \frac{\pi}{2} - 2\tan^{-1}x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = 0 - \frac{2}{1+x^2} = -\frac{2}{1+x^2}$$

Question 14:

$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

Answer 14:

$$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$$

Let, $x = \sin\theta$

Therefore, $y = \sin^{-1}\left(2\sin\theta\sqrt{1-sin^2\theta}\right)$

$= \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\sin^{-1}x$

$\Rightarrow y = 2\sin^{-1}x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

Question 15:

$y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

Answer 15:

$$y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$$

Let, $x = \cos\theta$

Therefore, $y = \sec^{-1}\left(\frac{1}{2cos^2\theta-1}\right) = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right) = \sec^{-1}(\sec 2\theta) = 2\theta = 2\cos^{-1}x$

$\Rightarrow y = 2\cos^{-1}x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}$$

Mathematics

(Chapter – 5) (Continuity and Differentiability) (Exercise 5.4)
(Class – XII)

Differentiate the following w.r.t. x:

Question 1:

$\frac{e^x}{\sin x}$

Answer 1:

Let $y = \frac{e^x}{\sin x}$, Therefore,

$$\frac{dy}{dx} = \frac{e^x.\frac{d}{dx}\sin x - \sin x\frac{d}{dx}e^x}{\sin^2 x} = \frac{e^x.\cos x - \sin x.e^x}{\sin^2 x} = \frac{e^x(\cos x - \sin x)}{\sin^2 x}$$

Question 2:

$e^{\sin^{-1} x}$

Answer 2:

Let $y = e^{\sin^{-1} x}$, Therefore,

$$\frac{dy}{dx} = e^{\sin^{-1} x}.\frac{d}{dx}\sin^{-1} x = e^{\sin^{-1} x}.\frac{1}{\sqrt{1-x^2}} = \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$$

Question 3:

e^{x^3}

Answer 3:

Let $y = e^{x^3}$, Therefore,

$$\frac{dy}{dx} = e^{x^3}.\frac{d}{dx}x^3 = e^{x^3}.3x^2 = 3x^2e^{x^3}$$

Question 4:

$\sin(\tan^{-1} e^{-x})$

Answer 4:

Let $y = \sin(\tan^{-1} e^{-x})$, Therefore,

$$\frac{dy}{dx} = \cos(\tan^{-1} e^{-x}).\frac{d}{dx}\tan^{-1} e^{-x} = \cos(\tan^{-1} e^{-x}).\frac{1}{1+(e^{-x})^2}.\frac{d}{dx}e^{-x}$$

$$= \cos(\tan^{-1} e^{-x}).\frac{1}{1+e^{-2x}}.(-e^{-x}) = -\frac{e^{-x}\cos(\tan^{-1} e^{-x})}{1+e^{-2x}}$$

Question 5:

$\log(\cos e^x)$

Answer 5:

Let $y = \log(\cos e^x)$, therefore,

$$\frac{dy}{dx} = \frac{1}{\cos e^x}.\frac{d}{dx}\cos e^x = \frac{1}{\cos e^x}(-\sin e^x)\frac{d}{dx}e^x = -\tan e^x.e^x$$

Question 6:

$e^x + e^{x^2} + \cdots + e^{x^5}$

Answer 6:

Let $y = e^x + e^{x^2} + e^{x^3} + e^{x^4} + e^{x^5}$, Therefore,

$$\frac{dy}{dx} = e^x + e^{x^2}\frac{d}{dx}x^2 + e^{x^3}\frac{d}{dx}x^3 + e^{x^4}\frac{d}{dx}x^4 + e^{x^5}\frac{d}{dx}x^5$$

$$= e^x + e^{x^2}.2x + e^{x^3}.3x^2 + e^{x^4}.4x^3 + e^{x^5}.5x^4$$

$$= e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + 5x^4e^{x^5}$$

Question 7:

$\sqrt{e^{\sqrt{x}}}, x > 0$

Answer 7:

Let $y = \sqrt{e^{\sqrt{x}}}$

Therefore,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{e^{\sqrt{x}}}}\frac{d}{dx}e^{\sqrt{x}}$$

$$= \frac{1}{2\sqrt{e^{\sqrt{x}}}}.e^{\sqrt{x}}.\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{e^{\sqrt{x}}}}.e^{\sqrt{x}}.\frac{1}{2\sqrt{x}} = \frac{\sqrt{e^{\sqrt{x}}}}{4\sqrt{x}}$$

Question 8:

$\log(\log x), x > 1$

Answer 8:

Let $y = \frac{e^x}{\sin x}$

Therefore,

$$\frac{dy}{dx} = \frac{1}{\log x}.\frac{d}{dx}\log x$$

$$= \frac{1}{\log x}.\frac{1}{x} = \frac{1}{x\log x}$$

Question 9:

$\frac{\cos x}{\log x}, x > 0$

Answer 9:

Let $y = \frac{\cos x}{\log x}$

Therefore,

$$\frac{dy}{dx} = \frac{\log x\frac{d}{dx}\cos x - \cos x\frac{d}{dx}\log x}{(\log x)^2}$$

$$= \frac{\log x.(-\sin x) - \cos x.\frac{1}{x}}{(\log x)^2} = \frac{-(x\sin x\log x + \cos x)}{x(\log x)^2}$$

Question 10:

$\cos(\log x + e^x), x > 0$

Answer 10:

Let $y = \cos(\log x + e^x)$

Therefore,

$$\frac{dy}{dx} = -\sin(\log x + e^x).\frac{d}{dx}(\log x + e^x)$$

$$= -\sin(\log x + e^x).\left(\frac{1}{x} + e^x\right)$$

Mathematics

(Chapter – 5) (Continuity and Differentiability) (Exercise 5.5)
(Class – XII)

Differentiate the functions given in Exercises 1 to 11 w.r.t. x.

Question 1:

$\cos x\ .\cos 2x\ .\cos 3x$

Answer 1:

Let $y = \cos x\ .\cos 2x\ .\cos 3x$, taking log on both the sides

$$\log y = \log\cos x + \log\cos 2x + \log\cos 3x$$

Therefore,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos x}.\frac{d}{dx}\cos x + \frac{1}{\cos 2x}.\frac{d}{dx}\cos 2x + \frac{1}{\cos 3x}.\frac{d}{dx}\cos 3x$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{\cos x}.(-\sin x) + \frac{1}{\cos 2x}.(-\sin 2x).2 + \frac{1}{\cos 3x}.(-\sin 3x).3\right]$$

$$\Rightarrow \frac{dy}{dx} = \cos x\ .\cos 2x\ .\cos 3x\,[-\tan x - 2\tan 2x - 3\tan 3x]$$

Question 2:

$\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Answer 2:

Let $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$, taking log on both the sides

$$\log y = \frac{1}{2}[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)]$$

Therefore,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{(x-1)} + \frac{1}{(x-2)} - \frac{1}{(x-3)} - \frac{1}{(x-4)} - \frac{1}{(x-5)}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{(x-1)} + \frac{1}{(x-2)} - \frac{1}{(x-3)} - \frac{1}{(x-4)} - \frac{1}{(x-5)}\right]$$

Question 3:

$(\log x)^{\cos x}$

Answer 3:

Let $y = (\log x)^{\cos x}$, taking log on both the sides

$$\log y = \log(\log x)^{\cos x} = \cos x\,.\log\log x$$

Therefore,

$$\frac{1}{y}\frac{dy}{dx} = \cos x\,.\frac{d}{dx}\log\log x + \log\log x\,.\frac{d}{dx}\cos x$$

$$\Rightarrow \frac{dy}{dx} = y\left[\cos x\,.\frac{1}{\log x}.\frac{1}{x} + \log\log x\,.(-\sin x)\right]$$

$$\Rightarrow \frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x - \sin x\log\log x}{x\log x}\right]$$

Question 4:
$x^x - 2^{\sin x}$

Answer 4:
Let $u = x^x$ and $v = 2^{\sin x}$ therefore, $y = u - v$

Differentiating with respect to x on both sides

$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$ ... (1)

Here, $u = x^x$, taking log on both the sides

$\log u = x\log x$, therefore,

$$\frac{1}{u}\frac{du}{dx} = x.\frac{d}{dx}\log x + \log x\,.\frac{d}{dx}x = x.\frac{1}{x} + \log x\,.1 = 1 + \log x$$

$$\frac{du}{dx} = u[1 + \log x] = x^x[1 + \log x] \quad ...(2)$$

and , $v = 2^{\sin x}$, taking log on both the sides

$\log v = \sin x \log 2$, therefore,

$$\frac{1}{v}\frac{dv}{dx} = \log 2\,.\frac{d}{dx}\sin x = \log 2\,.\cos x$$

$$\frac{dv}{dx} = v[\cos x \log 2] = 2^{\sin x}[\cos x \log 2] \quad ...(3)$$

Putting the value of $\frac{du}{dx}$ from (2) and $\frac{dv}{dx}$ from (3) in equation (1), we get

$$\frac{dy}{dx} = x^x[1 + \log x] - 2^{\sin x}[\cos x \log 2]$$

Question 5:
$(x+3)^2\,.\,(x+4)^3\,.\,(x+5)^4$

Answer 5:
Let $y = (x+3)^2\,.\,(x+4)^3\,.\,(x+5)^4$, taking log on both the sides

$$\log y = 2\log(x+3) + 3\log(x+4) + 4\log(x+5)$$

Therefore,

$$\frac{1}{y}\frac{dy}{dx} = 2.\frac{1}{(x+3)} + 3.\frac{1}{(x+4)} + 4.\frac{1}{(x+5)}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{2(x+4)(x+5) + 3(x+3)(x+5) + 4(x+3)(x+4)}{(x+3)(x+4)(x+5)}\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{2(x^2+9x+20) + 3(x^2+8x+15) + 4(x^2+7x+12)}{(x+3)(x+4)(x+5)}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2\,.\,(x+4)^3\,.\,(x+5)^4\left[\frac{9x^2+70x+133}{(x+3)(x+4)(x+5)}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)\,.\,(x+4)^2\,.\,(x+5)^3(9x^2+70x+133)$$

Question 6:

$\left(x+\frac{1}{x}\right)^{x}+x^{\left(1+\frac{1}{x}\right)}$

Answer 6:

Let $u=\left(x+\frac{1}{x}\right)^{x}$ and $v=x^{\left(1+\frac{1}{x}\right)}$, therefore, $y=u+v$

Differentiating with respect to x, we get

$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ ... (1)

Here, $u=\left(x+\frac{1}{x}\right)^{x}$, taking log on both the sides

$\log u=x\log\left(x+\frac{1}{x}\right)$, therefore,

$$\frac{1}{u}\frac{du}{dx}=x.\frac{d}{dx}\log\left(x+\frac{1}{x}\right)+\log\left(x+\frac{1}{x}\right).\frac{d}{dx}x$$

$$=x.\frac{1}{\left(x+\frac{1}{x}\right)}.\left(1-\frac{1}{x^2}\right)+\log\left(x+\frac{1}{x}\right).1=\frac{x^2}{x^2+1}.\frac{x^2-1}{x^2}+\log\left(x+\frac{1}{x}\right)$$

$$\frac{du}{dx}=\left(x+\frac{1}{x}\right)^{x}\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right] \quad ...(2)$$

and , $v=x^{\left(1+\frac{1}{x}\right)}$, taking log on both the sides

$\log v=\left(1+\frac{1}{x}\right)\log x$, therefore,

$$\frac{1}{v}\frac{dv}{dx}=\left(1+\frac{1}{x}\right).\frac{d}{dx}\log x+\log x.\frac{d}{dx}\left(1+\frac{1}{x}\right)=\left(1+\frac{1}{x}\right).\frac{1}{x}+\log x.\left(-\frac{1}{x^2}\right)$$

$$\frac{dv}{dx}=v\left[\left(\frac{x^2+1}{x}\right).\frac{1}{x}-\frac{\log x}{x^2}\right]=x^{\left(1+\frac{1}{x}\right)}\left[\frac{x^2+1-\log x}{x^2}\right] \quad ...(3)$$

Putting the value of $\frac{du}{dx}$ from (2) and $\frac{dv}{dx}$ from (3) in equation (1), we get

$$\frac{dy}{dx}=\left(x+\frac{1}{x}\right)^{x}\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right]+x^{\left(1+\frac{1}{x}\right)}\left[\frac{x^2+1-\log x}{x^2}\right]$$

Question 7:

$(\log x)^{x}+x^{\log x}$

Answer 7:

Let $u=(\log x)^{x}$ and $v=x^{\log x}$, therefore, $y=u+v$

Differentiating with respect to x, we get

$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ ... (1)

Here, $u=(\log x)^{x}$, taking log on both the sides

$\log u=x\log\log x$, therefore,

$$\frac{1}{u}\frac{du}{dx}=x.\frac{d}{dx}\log\log x+\log\log x.\frac{d}{dx}x$$

$= x.\frac{1}{\log x}.\frac{1}{x} + \log\log x\,.\,1 = \frac{1}{\log x} + \log\log x$

$\frac{du}{dx} = (\log x)^x \left[\frac{1 + \log x\,.\log\log x}{\log x}\right]$

$= (\log x)^{x-1}(1 + \log x\,.\log\log x) \quad ...(2)$

and , $v = x^{\log x}$, taking log on both the sides

$\log v = \log x \log x$, therefore,

$\frac{1}{v}\frac{dv}{dx} = \log x\,.\frac{d}{dx}\log x + \log x\,.\frac{d}{dx}\log x$

$= \log x\,.\frac{1}{x} + \log x\,.\frac{1}{x}$

$\frac{dv}{dx} = v\left[\frac{2\log x}{x}\right] = x^{\log x}\left[\frac{2\log x}{x}\right] = x^{\log x-1}(2\log x) \quad ...(3)$

Putting the value of $\frac{du}{dx}$ from (2) and $\frac{dv}{dx}$ from (3) in equation (1), we get

$\frac{dy}{dx} = (\log x)^{x-1}(1 + \log x\,.\log\log x) + x^{\log x-1}(2\log x)$

Question 8:

$(\sin x)^x + \sin^{-1}\sqrt{x}$

Answer 8:

Let $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$, therefore, $y = u + v$

Differentiating with respect to x, we get

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(1)$

Here, $u = (\sin x)^x$, taking log on both the sides

$\log u = x\log\sin x$, therefore,

$\frac{1}{u}\frac{du}{dx} = x.\frac{d}{dx}\log\sin x + \log\sin x\,.\frac{d}{dx}x$

$= x.\frac{1}{\sin x}.\cos x + \log\sin\ x\,.\,1 = x\cot x + \log\sin x$

$\frac{du}{dx} = (\sin x)^x(x\cot x + \log\sin x) \quad ...(2)$

and , $v = \sin^{-1}\sqrt{x}$, therefore,

$\frac{1}{v}\frac{dv}{dx} = \log x\,.\frac{d}{dx}\log x + \log x\,.\frac{d}{dx}\log x\ = \log x\,.\frac{1}{x} + \log x\,.\frac{1}{x}$

$\frac{dv}{dx} = \frac{1}{\sqrt{1-x}}.\frac{d}{dx}\sqrt{x} = \frac{1}{\sqrt{1-x}}.\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x-x^2}} \quad ...(3)$

Putting the value of $\frac{du}{dx}$ from (2) and $\frac{dv}{dx}$ from (3) in equation (1), we get

$\frac{dy}{dx} = (\sin x)^x(x\cot x + \log\sin x) + \frac{1}{2\sqrt{x-x^2}}$

Question 9:

$x^{\sin x} + (\sin x)^{\cos x}$

Answer 9:

Let $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$ therefore, $y = u + v$

Differentiating with respect to x, we get, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ... (1)

Here, $u = x^{\sin x}$, taking log on both the sides

$\log u = \sin x \log x$, therefore,

$$\frac{1}{u}\frac{du}{dx} = \sin x \,.\, \frac{d}{dx}\log x + \log x \,.\, \frac{d}{dx}\sin x = \sin x \,.\, \frac{1}{x} + \log x \,.\, \cos x = \frac{\sin x}{x} + \log x \cos x$$

$$\frac{du}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \log x \cos x\right] = x^{\sin x - 1}(\sin x + x \log x \cos x) \quad ... (2)$$

and , $v = (\sin x)^{\cos x}$, taking log on both the sides, we get $\log v = \cos x \log \sin x$, therefore,

$$\frac{1}{v}\frac{dv}{dx} = \cos x \,.\, \frac{d}{dx}\log \sin x + \log \sin x \,.\, \frac{d}{dx}\cos x = \cos x \,.\, \frac{1}{\sin x}\cos x + \log \sin x\,(-\sin x)$$

$$\frac{dv}{dx} = v[\cos x \cot x - \sin x \log \sin x] = (\sin x)^{\cos x}(\cos x \cot x - \sin x \log \sin x) \quad ... (3)$$

Putting the value of $\frac{du}{dx}$ from (2) and $\frac{dv}{dx}$ from (3) in equation (1), we get

$$\frac{dy}{dx} = x^{\sin x - 1}(\sin x + x \log x \cos x) + (\sin x)^{\cos x}(\cos x \cot x - \sin x \log \sin x)$$

Question 10:

$x^{x \cos x} + \frac{x^2+1}{x^2-1}$

Answer 10:

Let $u = x^{x \cos x}$ and $v = \frac{x^2+1}{x^2-1}$ therefore, $y = u + v$

Differentiating with respect to x, we get $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ... (1)

Here, $u = x^{x \cos x}$, taking log on both the sides, we get $\log u = x \log x$, therefore,

$$\frac{1}{u}\frac{du}{dx} = x \cos x \,.\, \frac{d}{dx}\log x + \log x \,.\, \frac{d}{dx}x \cos x = x \cos x \,.\, \frac{1}{x} + \log x \,.\, (-x.\sin x + \cos x)$$

$$= \cos x - x \sin x \log x + \cos x \log x$$

$$\frac{du}{dx} = u[\cos x - x \sin x \log x + \cos x \log x]$$

$$= x^{x \cos x}[\cos x - x \sin x \log x + \cos x \log x] \quad ... (2)$$

and , $v = \frac{x^2+1}{x^2-1}$, Taking log on both the sides

$\log v = \log(x^2+1) - \log(x^2-1)$, therefore,

$$\frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2+1}.2x - \frac{1}{x^2-1}.2x = \frac{2x(x^2-1) - 2x(x^2+1)}{(x^2+1)(x^2-1)} = \frac{-4x}{(x^2+1)(x^2-1)}$$

$$\frac{dv}{dx} = v\left[\frac{-4x}{(x^2+1)(x^2-1)}\right] = \frac{x^2+1}{x^2-1}\left[\frac{-4x}{(x^2+1)(x^2-1)}\right] = -\frac{4x}{(x^2-1)^2} \quad ... (3)$$

Putting the value of $\frac{du}{dx}$ from (2) and $\frac{dv}{dx}$ from (3) in equation (1), we get

$$\frac{dy}{dx} = x^{x \cos x}[\cos x - x \sin x \log x + \cos x \log x] - \frac{4x}{(x^2-1)^2}$$

Question 11:

$(x\cos x)^x + (x\sin x)^{\frac{1}{x}}$

Answer 11:

Let $u = (x\cos x)^x$ and $v = (x\sin x)^{\frac{1}{x}}$, therefore, $y = u + v$

Differentiating with respect to x, we get

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ... (1)

Here, $u = (x\cos x)^x$, Taking log on both the sides

$\log u = x\log(x\cos x)$, therefore,

$$\frac{1}{u}\frac{du}{dx} = x.\frac{d}{dx}\log(x\cos x) + \log(x\cos x).\frac{d}{dx}x$$

$$= x.\frac{1}{(x\cos x)}(-x\sin x + \cos x) + \log(x\cos x).1 = -x\tan x + 1 + \log(x\cos x)$$

$$\frac{du}{dx} = (x\cos x)^x[1 - x\tan x + \log(x\cos x)]$$

$$= (x\cos x)^x[1 - x\tan x + \log(x\cos x)] \quad ... (2)$$

and , $v = (x\sin x)^{\frac{1}{x}}$, Taking log on both the sides

$\log v = \frac{1}{x}\log(x\sin x)$, therefore,

$$\frac{1}{v}\frac{dv}{dx} = \frac{1}{x}.\frac{d}{dx}\log(x\sin x) + \log(x\sin x).\frac{d}{dx}\frac{1}{x}$$

$$= \frac{1}{x}.\frac{1}{x\sin x}(x\cos x + \sin x) + \log(x\sin x)\left(-\frac{1}{x^2}\right)$$

$$\frac{dv}{dx} = v\left[\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right]$$

$$= (x\sin x)^{\frac{1}{x}}\left[\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right] \quad ... (3)$$

Putting the value of $\frac{du}{dx}$ from (2) and $\frac{dv}{dx}$ from (3) in equation (1), we get

$$\frac{dy}{dx} = (x\cos x)^x[1 - x\tan x + \log(x\cos x)] + (x\sin x)^{\frac{1}{x}}\left[\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right]$$

Find $\frac{dy}{dx}$ of the functions given in Exercises 12 to 15:

Question 12:

$x^y + y^x = 1$

Answer 12:

Let $u = x^y$ and $v = y^x$, therefore, $u + v = 1$

Differentiating with respect to x, we get

$\frac{du}{dx} + \frac{dv}{dx} = 0$ ... (1)

Here, $u = x^y$, Taking log on both the sides, $\log u = y\log x$, therefore,

$$\frac{1}{u}\frac{du}{dx} = y.\frac{d}{dx}\log x + \log x.\frac{d}{dx}y = y.\frac{1}{x} + \log x.\frac{dy}{dx}$$

$$\frac{du}{dx} = x^y \left[\frac{y}{x} + \log x \cdot \frac{dy}{dx}\right] \quad \dots (2)$$

and , $v = y^x$, Taking log on both the sides

$\log v = x \log y$, therefore,

$$\frac{1}{v}\frac{dv}{dx} = x.\frac{d}{dx}\log y + \log y \cdot \frac{d}{dx}x = x.\frac{1}{y}\frac{dy}{dx} + \log y \cdot 1$$

$$\frac{dv}{dx} = v\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] = y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] \quad \dots (3)$$

Putting the value of $\frac{du}{dx}$ from (2) and $\frac{dv}{dx}$ from (3) in equation (1), we get

$$x^y\left[\frac{y}{x} + \log x \cdot \frac{dy}{dx}\right] + y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] = 0$$

$$\Rightarrow yx^{y-1} + x^y \log x \frac{dy}{dx} + xy^{x-1}\frac{dy}{dx} + y^x \log y = 0$$

$$\Rightarrow \frac{dy}{dx}(x^y \log x + xy^{x-1}) = -(y^x \log y + yx^{y-1})$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^x \log y + yx^{y-1}}{x^y \log x + xy^{x-1}}$$

Question 13:

$y^x = x^y$

Answer 13:

$$y^x = x^y$$

Taking log on both the sides, $x \log y = y \log x$, therefore,

$$x.\frac{d}{dx}\log y + \log y \cdot \frac{d}{dx}x = y.\frac{d}{dx}\log x + \log x \cdot \frac{d}{dx}y$$

$$\Rightarrow x.\frac{1}{y}\frac{dy}{dx} + \log y \cdot 1 = y.\frac{1}{x} + \log x \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{x}{y} - \log x\right) = \frac{y}{x} - \log y$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{x - y\log x}{y}\right) = \frac{y - x\log y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(y - x\log y)}{x(x - y\log x)}$$

Question 14:

$(\cos x)^y = (\cos y)^x$

Answer 14:

$$(\cos x)^y = (\cos y)^x$$

Taking log on both the sides, $y \log[\cos x] = x\, log[\cos y]$, therefore,

$$y.\frac{d}{dx}\log\cos x + \log\cos x \cdot \frac{d}{dx}y = x.\frac{d}{dx}\log\cos y + \log\cos y \cdot \frac{d}{dx}x$$

$$\Rightarrow y\left(\frac{-\sin x}{\cos x}\right) + \log\cos x \cdot \frac{dy}{dx} = x.\left(\frac{-\sin y}{\cos y}\right)\frac{dy}{dx} + \log\cos y \cdot 1$$

$$\Rightarrow \frac{dy}{dx}(\log\cos x + x\tan y) = \log\cos y + y\tan x \qquad \Rightarrow \frac{dy}{dx} = \frac{\log\cos y + y\tan x}{\log\cos x + x\tan y}$$

Question 15:
$xy = e^{(x-y)}$

Answer 15:

$$xy = e^{(x-y)}$$

Taking log on both the sides,

$\log x + \log y = (x-y)\log e \quad \Rightarrow \log x + \log y = (x-y)$, therefore,

$$\frac{1}{x} + \frac{1}{y}.\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y}+1\right) = 1 - \frac{1}{x} \qquad \Rightarrow \frac{dy}{dx}\left(\frac{1+y}{y}\right) = \frac{x-1}{x} \qquad \Rightarrow \frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$$

Question 16:
Find the derivative of the function given by $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$.

Answer 16:
$f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$

Taking log on both the sides,

$\log f(x) = \log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^8)$, therefore,

$$\frac{1}{f(x)}.\frac{d}{dx}f(x) = \frac{1}{1+x} + \frac{1}{1+x^2}.\frac{d}{dx}x^2 + \frac{1}{1+x^4}.\frac{d}{dx}x^4 + \frac{1}{1+x^8}.\frac{d}{dx}x^8$$

$$\Rightarrow \frac{1}{f(x)}.f'(x) = \frac{1}{1+x} + \frac{1}{1+x^2}.2x + \frac{1}{1+x^4}.4x^3 + \frac{1}{1+x^8}.8x^7$$

$$\Rightarrow f'(x) = f(x)\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$$

$$\Rightarrow f'(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$$

$$\Rightarrow f'(1) = (1+1)(1+1)(1+1)(1+1)\left[\frac{1}{1+1} + \frac{2}{1+1} + \frac{4}{1+1} + \frac{8}{1+1}\right]$$

$$\Rightarrow f'(1) = (2)(2)(2)(2)\left[\frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2}\right] = 16\left(\frac{15}{2}\right) = 120$$

Question 17:
Differentiate $(x^2-5x+8)(x^3+7x+9)$ in three ways mentioned below:

(i) by using product rule

(ii) by expanding the product to obtain a single polynomial.

(iii) by logarithmic differentiation.

Do they all give the same answer?

Answer 17:
Let $y = (x^2-5x+8)(x^3+7x+9$

(i) Differentiating using product rule

$$\frac{dy}{dx} = (x^2-5x+8)\frac{\mathrm{d}}{\mathrm{dx}}(x^3+7x+9) + (x^3+7x+9)\frac{\mathrm{d}}{\mathrm{dx}}(x^2-5x+8)$$

$$= (x^2-5x+8)(3x^2+7) + (x^3+7x+9)(2x-5)$$

$$= (3x^4+7x^2-15x^3-35x+24x^2+56) + 2x^4-5x^3+14x^2-35x+18x-45$$

$$= 5x^4-20x^3+45x^2-52x+11$$

(ii) Differentiating by expanding the product to obtain a single polynomial

$y = (x^2 - 5x + 8)(x^3 + 7x + 9) = x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72$

$= x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$

$\frac{dy}{dx} = \frac{d}{dx}x^5 - 5\frac{d}{dx}x^4 + 15\frac{d}{dx}x^3 - 26\frac{d}{dx}x^2 + 11\frac{d}{dx}x + \frac{d}{dx}72 = 5x^4 - 20x^3 + 45x^2 - 52x + 11$

(iii) Logarithmic differentiation

$y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

Taking log on both sides, $\log y = \log(x^2 - 5x + 8) + \log(x^3 + 7x + 9)$

$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{(x^2 - 5x + 8)}.\frac{d}{dx}(x^2 - 5x + 8) + \frac{1}{(x^3 + 7x + 9)}.\frac{d}{dx}(x^3 + 7x + 9)$

$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{x^2 - 5x + 8}.(2x - 5) + \frac{1}{x^3 + 7x + 9}.(3x^2 + 7)$

$\frac{dy}{dx} = y\left[\frac{(2x - 5)(x^3 + 7x + 9) + (3x^2 + 7)(x^2 - 5x + 8)}{(x^2 - 5x + 8)(x^3 + 7x + 9)}\right]$

$= y\left[\frac{2x^4 + 14x^2 + 18x - 5x^3 - 35x - 45 + 3x^4 - 15x^3 + 24x^2 + 7x^2 - 35x + 56}{(x^2 - 5x + 8)(x^3 + 7x + 9)}\right]$

$\Rightarrow \frac{dy}{dx} = (x^2 - 5x + 8)(x^3 + 7x + 9)\left[\frac{5x^5 - 20x^3 + 45x^2 - 52x + 11}{(x^2 - 5x + 8)(x^3 + 7x + 9)}\right]$

$\Rightarrow \frac{dy}{dx} == 5x^4 - 20x^3 + 45x^2 - 52x + 11$

Hence, all the three answers are same.

Question 18:

If u, v and w are functions of x, then show that

$$\frac{d}{dx}(u.v.w) = \frac{du}{dx}v.w + u.\frac{dv}{dx}.w + u.v\frac{dw}{dx}$$

in two ways - first by repeated application of product rule, second by logarithmic differentiation.

Answer 18:

Let $y = u.v.w = u.(v.w)$

Differentiation by repeated application of product rule

$\frac{dy}{dx} = u.\frac{d}{dx}(v.w) + (v.w).\frac{d}{dx}u = u\left[v\frac{d}{dx}w + w\frac{d}{dx}v\right] + v.w.\frac{du}{dx}$

$\Rightarrow \frac{dy}{dx} = u.v.\frac{dw}{dx} + u.w.\frac{dv}{dx} + v.w.\frac{du}{dx}$

Differentiation using logarithmic

Let $y = u.v.w$

Taking log on both the sides, $\log y = \log u + \log v + \log w$

$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{u}.\frac{du}{dx} + \frac{1}{v}.\frac{dv}{dx} + \frac{1}{w}.\frac{dw}{dx} \Rightarrow \frac{dy}{dx} = y\left[\frac{1}{u}.\frac{du}{dx} + \frac{1}{v}.\frac{dv}{dx} + \frac{1}{w}.\frac{dw}{dx}\right]$

$\Rightarrow \frac{dy}{dx} = u.v.w\left[\frac{1}{u}.\frac{du}{dx} + \frac{1}{v}.\frac{dv}{dx} + \frac{1}{w}.\frac{dw}{dx}\right] \Rightarrow \frac{dy}{dx} = \frac{u.v.w}{u}.\frac{du}{dx} + \frac{u.v.w}{v}.\frac{dv}{dx} + \frac{u.v.w}{w}.\frac{dw}{dx}$

$\Rightarrow \frac{dy}{dx} = v.w.\frac{du}{dx} + u.w.\frac{dv}{dx} + u.v.\frac{dw}{dx}$

Mathematics

(Chapter – 5) (Continuity and Differentiability) (Exercise 5.6)
(Class – XII)

If x and y are connected parametrically by the equations given in Exercises 1 to 10, without eliminating the parameter, Find $\frac{dy}{dx}$.

Question 1:

$x = 2at^2 , y = at^4$

Answer 1:

Here, $x = 2at^2 , y = at^4$

Therefore, $\frac{dx}{dt} = 2a(2t)$ and $\frac{dy}{dt} = a(4t^3)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4at^3}{4at} = t^2$$

Question 2:

$x = a\cos\theta , y = b\cos\theta$

Answer 2:

Here, $x = a\cos\theta , y = b\cos\theta$, therefore, $\frac{dx}{d\theta} = a(-\sin\theta)$ and $\frac{dy}{d\theta} = b(-\sin\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$$

Question 3:

$x = \sin t , y = \cos 2t$

Answer 3:

Here, $x = \sin t , y = \cos 2t$, therefore, $\frac{dx}{dt} = \cos t$ and $\frac{dy}{dt} = -\sin 2t . 2$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\sin 2t}{\cot t} = -\frac{2(2\sin t\cos t)}{\cos t} = -4\sin t$$

Question 4:

$x = 4t, y = \frac{4}{t}$

Answer 4:

Here, $x = 4t , y = \frac{4}{t}$, therefore, $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = -\frac{4}{t^2}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{4}{t^2}}{4} = -\frac{1}{t^2}$$

Question 5:

$x = \cos\theta - \cos 2\theta , y = \sin\theta - \sin 2\theta$

Answer 5:

Here, $x = \cos\theta - \cos 2\theta , y = \sin\theta - \sin 2\theta$

Therefore, $\frac{dx}{d\theta} = -\sin\theta + 2\sin 2\theta$ and $\frac{dy}{d\theta} = \cos\theta - 2\cos 2\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta - 2\cos 2\theta}{-\sin\theta + 2\sin 2\theta}$$

Question 6:

$x = a(\theta - \sin\theta), y = a(1 + \cos\theta)$

Answer 6:

Here, $x = a(\theta - \sin\theta), y = a(1 + \cos\theta)$

Therefore, $\frac{dx}{d\theta} = a(1 - \cos\theta)$ and $\frac{dy}{d\theta} = a(0 - \sin\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a\sin\theta)}{a(1 - \cos\theta)} = -\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

Question 7:

$x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

Answer 7:

Here, $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$, therefore, $\frac{dx}{dt} = \frac{\sin^3 t\frac{d}{dt}\sqrt{\cos 2t} - \sqrt{\cos 2t}\frac{d}{dt}\sin^3 t}{\left(\sqrt{\cos 2t}\right)^2}$

$$= \frac{\sin^3 t.\frac{1}{2\sqrt{\cos 2t}}.(-\sin 2t).2 - \sqrt{\cos 2t}.3\sin^2 t\cos t}{\cos 2t} = \frac{-\sin^3 t.\sin 2t - 3\cos 2t.\sin^2 t\cos t}{\cos 2t\sqrt{\cos 2t}}$$

and $\frac{dy}{dt} = \frac{\cos^3 t\frac{d}{dt}\sqrt{\cos 2t} - \sqrt{\cos 2t}\frac{d}{dt}\cos^3 t}{\left(\sqrt{\cos 2t}\right)^2}$

$$= \frac{\cos^3 t.\frac{1}{2\sqrt{\cos 2t}}.(-\sin 2t).2 - \sqrt{\cos 2t}.3\cos^2 t(-\sin t)}{\cos 2t} = \frac{-\cos^3 t.\sin 2t + 3\cos 2t.\cos^2 t\sin t}{\cos 2t\sqrt{\cos 2t}}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-cos^3 t\,.\,sin\,2t + 3cos\,2t\,.\,cos^2 t\,sin\,t}{-sin^3 t\,.\,sin\,2t - 3cos\,2t\,.\,sin^2 t\,cos\,t}$$

$$= \frac{-cos^3 t\,.\,(2\,sin\,t\cos t) + 3cos\,2t\,.\,cos^2 t\,sin\,t}{-sin^3 t\,.\,(2\,sin\,t\cos t) - 3cos\,2t\,.\,sin^2 t\,cos\,t} = \frac{cos^2 t\,sin\,t\,(-2cos^2 t + 3\cos 2t)}{sin^2 t\,cos\,t\,(-2\sin^2 t - 3\cos 2t)}$$

$$= \frac{\cos t\,[-2cos^2 t + 3(2cos^2 t - 1)]}{\sin t\,[-2\sin^2 t - 3(1 - 2\sin^2 t)]} = \frac{\cos t\,[-2cos^2 t + 6cos^2 t - 3]}{\sin t\,[-2\sin^2 t - 3 + 6\sin^2 t]}$$

$$= \frac{\cos t\,[4cos^2 t - 3]}{\sin t\,[-3 + 4\sin^2 t]} = -\frac{4cos^3 t - 3\cos t}{3\sin t - 4\sin^3 t} = -\frac{\cos 3t}{\sin 3t} = -\cot 3t$$

Question 8:

$x = a\left(\cos t + \log\tan\frac{t}{2}\right)\ y = a\sin t$

Answer 8:

Here, $x = a\left(\cos t + \log\tan\frac{t}{2}\right)\ y = a\sin t$,

Therefore, $\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}}.\sec^2\frac{t}{2}.\frac{1}{2}\right) = a\left(-\sin t + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}.\frac{1}{\cos^2\frac{t}{2}}.\frac{1}{2}\right)$

$= a\left(-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right) = a\left(-\sin t + \frac{1}{\sin t}\right) = a\left(\frac{-\sin^2 t + 1}{\sin t}\right) = a\left(\frac{\cos^2 t}{\sin t}\right)$

and $\frac{dy}{dt} = a\cos t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{a\left(\frac{\cos^2 t}{\sin t}\right)} = \frac{\sin t}{\cos t} = \tan t$$

Question 9:

$x = a \sec\theta\,, y = b\tan\theta$

Answer 9:

Here, $x = a \sec\theta\,, y = b\tan\theta$

Therefore, $\frac{dx}{d\theta} = a\sec\theta\tan\theta$

and $\frac{dy}{d\theta} = b\sec^2\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta} = \frac{b\left(\frac{1}{\cos\theta}\right)}{a\left(\frac{\sin\theta}{\cos\theta}\right)} = \frac{b}{a}\operatorname{cosec}\theta$$

Question 10:

$x = a\,(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$

Answer 10:

Here, $x = a\,(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$

Therefore, $\frac{dx}{d\theta} = a\,[-\sin\theta + (\theta\cos\theta + \sin\theta)] = a\theta\cos\theta$

and $\frac{dy}{d\theta} = a\,[\cos\theta - (-\theta\sin\theta + \cos\theta)] = a\theta\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

Question 11:

If $x = \sqrt{a^{\sin^{-1}t}}, y = \sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

Answer 11:

Here, $x = \sqrt{a^{\sin^{-1}t}},\ \ y = \sqrt{a^{\cos^{-1}t}}$

Therefore,

$$\frac{dx}{dt} = \frac{1}{2\sqrt{a^{\sin^{-1}t}}}.\frac{d}{dx}a^{\sin^{-1}t} = \frac{1}{2\sqrt{a^{\sin^{-1}t}}}.a^{\sin^{-1}t}.\log a\frac{1}{\sqrt{1-t^2}}$$

$$= \frac{1}{2x}.x^2.\log a\frac{1}{\sqrt{1-t^2}} = \frac{x\log a}{\sqrt{1-t^2}}$$

and

$$\frac{dy}{dt} = \frac{1}{2\sqrt{a^{\cos^{-1}t}}}.\frac{d}{dx}a^{\cos^{-1}t} = \frac{1}{2\sqrt{a^{\cos^{-1}t}}}.a^{\cos^{-1}t}.\log a\frac{-1}{\sqrt{1-t^2}}$$

$$= \frac{1}{2y}.y^2.\log a\frac{1}{\sqrt{1-t^2}} = -\frac{y\log a}{\sqrt{1-t^2}}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{y\log a}{\sqrt{1-t^2}}}{\frac{x\log a}{\sqrt{1-t^2}}} = -\frac{y}{x}$$

Find the second order derivatives of the functions given in Exercises 1 to 10.

Question 1:

$x^2 + 3x + 2$

Answer 1:

Let $y = x^2 + 3x + 2$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 3x + 2) = 2x + 3$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(2x + 3) = 2$$

Question 2:

x^{20}

Answer 2:

Let $y = x^{20}$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(x^{20}) = 20x^{19}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(20x^{19}) = 380x^{18}$$

Question 3:

$x.\cos x$

Answer 3:

Let $y = x.\cos x$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(x.\cos x) = x.\frac{d}{dx}\cos x + \cos x.\frac{d}{dx}x = -x\sin x + \cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(-x\sin x + \cos x) = -\left(x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}x\right) - \sin x$$

$$= -x\cos x - \sin x - \sin x = -(x\cos x + 2\sin x)$$

Question 4:

$\log x$

Answer 4:

Let $y = \log x$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

Question 5:

$x^3 \log x$

Answer 5:

Let $y = x^3 \log x$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(x^3\log x) = x^3.\frac{d}{dx}\log x + \log x.\frac{d}{dx}x^3 = x^3.\frac{1}{x} + \log x.3x^2 = x^2 + 3x^2\log x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(x^2 + 3x^2\log x) = 2x + 3\left(x^2\frac{d}{dx}\log x + \log x\frac{d}{dx}x^2\right)$$

$$= 2x + 3\left(x^2.\frac{1}{x} + \log x.2x\right) = 2x + 3x + 6x\log x = 5x + 6x\log x = x(5 + 6\log x)$$

Question 6:

$e^x \sin 5x$

Answer 6:

Let $y = e^x \sin 5x$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(e^x \sin 5x) = e^x.\frac{d}{dx}\sin 5x + \sin 5x\,.\frac{d}{dx}e^x$$

$$= e^x.\cos 5x\,.5\ + \sin 5x\,.e^x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(5e^x \cos 5x + e^x \sin 5x)$$

$$= 5\left(e^x.\frac{d}{dx}\cos 5x + \cos 5x\,.\frac{d}{dx}e^x\right) + \left(e^x.\frac{d}{dx}\sin 5x + \sin 5x\,.\frac{d}{dx}e^x\right)$$

$$= 5[e^x.(-\sin 5x).5 + \cos 5x\,.e^x] + [e^x.\cos 5x\,.5\ + \sin 5x\,.e^x]$$

$$= e^x(-25\sin 5x + 5\cos 5x + 5\cos 5x + \sin 5x) = e^x(10\cos 5x - 24\sin 5x)$$

Question 7:

$e^{6x}\cos 3x$

Answer 7:

Let $y = e^{6x}\cos 3x$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(e^{6x}\cos 3x) = e^{6x}.\frac{d}{dx}\cos 3x + \cos 3x\,.\frac{d}{dx}e^{6x}$$

$$= e^{6x}.(-\sin 3x).3\ + \cos 3x\,.e^{6x}.6$$

$$= 3e^{6x}(-\sin 3x + 2\cos 3x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}[3e^{6x}(-\sin 3x + 2\cos 3x)]$$

$$= 3e^{6x}.\frac{d}{dx}(-\sin 3x + 2\cos 3x) + (-\sin 3x + 2\cos 3x).\frac{d}{dx}3e^{6x}$$

$$= 3e^{6x}.(-3\cos 3x - 6\sin 3x) + (-\sin 3x + 2\cos 3x).18e^{6x}$$

$$= e^{6x}(-9\,\text{cso}\,3x - 18\sin 3x - 18\sin 3x + 36\cos 3x)$$

$$= e^{6x}(27\cos 3x - 36\text{sin}\,3x)$$

$$= 9e^{6x}(3\cos 3x - 4\text{sin}\,3x)$$

Question 8:

$\tan^{-1} x$

Answer 8:

Let $y = \tan^{-1} x$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{(1+x^2)\frac{d}{dx}1 - 1.\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{0-2x}{(1+x^2)^2} = -\frac{2x}{(1+x^2)^2}$$

Question 9:
$\log(\log x)$

Answer 9:
Let $y = \log(\log x)$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(\log(\log x)) = \frac{1}{\log x}.\frac{1}{x} = \frac{1}{x\log x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x\log x}\right) = \frac{(x\log x)\frac{d}{dx}1 - 1.\frac{d}{dx}(x\log x)}{(x\log x)^2}$$

$$= \frac{0 - \left(x.\frac{1}{x} + \log x\right)}{(x\log x)^2} = -\frac{1+\log x}{(x\log x)^2}$$

Question 10:
$\sin(\log x)$

Answer 10:
Let $y = \sin(\log x)$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(\sin(\log x)) = \cos(\log x).\frac{1}{x} = \frac{\cos(\log x)}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right] = \frac{x\frac{d}{dx}\cos(\log x) - \cos(\log x).\frac{d}{dx}x}{(x)^2}$$

$$= \frac{x\left\{-\sin(\log x).\frac{1}{x}\right\} - \cos(\log x).1}{(x)^2} = \frac{-\sin(\log x) - \cos(\log x)}{(x)^2}$$

Question 11:
If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$

Answer 11:
Given that: $y = 5\cos x - 3\sin x$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(5\cos x - 3\sin x) = -5\sin x - 3\cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(-5\sin x - 3\cos x) = -5\cos x + 3\sin x = -(5\cos x - 3\sin x) = -y$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = 0$$

Question 12:
If $y = \cos^{-1}x$, find $\frac{d^2y}{dx^2}$ in terms of y alone.

Answer 12:
Given that: $y = \cos^{-1}x$

$\Rightarrow \cos y = x$, therefore,

$$-\sin y\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y} = -\operatorname{cosec} y$$

$$\Rightarrow \frac{d^2y}{dx^2} = -(-\operatorname{cosec} y\cot y).\frac{dy}{dx} = (\operatorname{cosec} y\cot y).(-\operatorname{cosec} y) = -\operatorname{cosec}^2 y\cot y$$

Question 13:

If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2y_2 + xy_1 + y = 0$

Answer 13:

Given that: $y = 3\cos(\log x) + 4\sin(\log x)$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(3\cos(\log x) + 4\sin(\log x)) = -3\sin(\log x).\frac{1}{x} + 4\cos(\log x).\frac{1}{x}$$

$$\Rightarrow x\frac{dy}{dx} = -3\sin(\log x) + 4\cos(\log x)$$

$$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx}.\frac{d}{dx}x = \frac{d}{dx}[-3\sin(\log x) + 4\cos(\log x)]$$

$$= -3\cos(\log x).\frac{1}{x} - 4\sin(\log x).\frac{1}{x} = -\frac{1}{x}[3\cos(\log x) + 4\sin(\log x)] = -\frac{1}{x}.y$$

$$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{1}{x}y \qquad \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y \qquad \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

$$\Rightarrow x^2y_2 + xy_1 + y = 0$$

Question 14:

If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

Answer 14:

Given that: $y = Ae^{mx} + Be^{nx}$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(Ae^{mx} + Be^{nx}) = mAe^{mx} + nBe^{nx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(mAe^{mx} + nBe^{nx}) = m^2Ae^{mx} + n^2Be^{nx}$$

Putting the value of $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ in $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$, we get

$$\text{LHS} = (m^2Ae^{mx} + n^2Be^{nx}) - (m+n)(mAe^{mx} + nBe^{nx}) + mny$$

$$= m^2Ae^{mx} + n^2Be^{nx} - (m^2Ae^{mx} + mnBe^{nx} + mnAe^{mx} + n^2Be^{nx}) + mny$$

$$= -(mnAe^{mx} + mnBe^{nx}) + mny$$

$$= -mn(Ae^{mx} + Be^{nx}) + mny$$

$$= -mny + mny = 0 = \text{RHS}$$

Question 15:

If $y = 500e^{7x} + 600e^{-7x}$, Show that $\frac{d^2y}{dx^2} = 49y$.

Answer 15:

Given that: $y = 500e^{7x} + 600e^{-7x}$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(500e^{7x} + 600e^{-7x}) = 500e^{7x}.7 + 600e^{-7x}.(-7) = 7(500e^{7x} - 600e^{-7x})$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}7(500e^{7x} - 600e^{-7x}) = 7[500e^{7x}.7 + 600e^{-7x}.(-7)]$$

$$= 49(500e^{7x} - 600e^{-7x}) = 49y$$

$$\Rightarrow \frac{d^2y}{dx^2} = 49y$$

Question 16:

If $e^y(x+1)=1$, show that $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$.

Answer 16:

Given that: $e^y(x+1)=1$, therefore,

$$e^y\frac{dy}{dx}(x+1)+(x+1)\frac{d}{dx}e^y=\frac{d}{dx}1$$

$$\Rightarrow e^y+(x+1)e^y\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{1}{x+1}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{d}{dx}\left(-\frac{1}{x+1}\right)$$

$$=-\left[\frac{(x+1).\frac{d}{dx}1-1.\frac{d}{dx}(x+1)}{(x+1)^2}\right]=-\left[\frac{0-1}{(x+1)^2}\right]=\frac{1}{(x+1)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(-\frac{1}{x+1}\right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$$

Question 17:

If $y=(\tan^{-1}x)^2$, show that $(x^2+1)^2y_2+2x(x^2+1)y_1=2$.

Answer 17:

Given that: $y=(\tan^{-1}x)^2$, therefore,

$$\frac{dy}{dx}=\frac{d}{dx}[(\tan^{-1}x)^2]=2\tan^{-1}x.\frac{1}{1+x^2}=\frac{2\tan^{-1}x}{1+x^2}$$

$$\Rightarrow (1+x^2)\frac{dy}{dx}=2\tan^{-1}x$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}+\frac{dy}{dx}.\frac{d}{dx}(1+x^2)=\frac{d}{dx}(2\tan^{-1}x)$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}+\frac{dy}{dx}.2x=\frac{2}{1+x^2}$$

$$\Rightarrow (1+x^2)^2\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx}=2$$

$$\Rightarrow (x^2+1)^2y_2+2x(x^2+1)y_1=2$$

Mathematics

(Chapter – 5) (Continuity and Differentiability) (Miscellaneous Exercise)
(Class – XII)

Differentiate w.r.t. x the function in Exercises 1 to 11.

Question 1:

$(3x^2 - 9x + 5)^9$

Answer 1:

Let $y = (3x^2 - 9x + 5)^9$, therefore,

$$\frac{dy}{dx} = 9(3x^2 - 9x + 5)^8 . \frac{d}{dx}(3x^2 - 9x + 5) = 9(3x^2 - 9x + 5)^8 . (6x - 9)$$

$$= 27(3x^2 - 9x + 5)^8 . (2x - 3)$$

Question 2:

$\sin^3 x + \cos^6 x$

Answer 2:

Let $y = \sin^3 x + \cos^6 x$, therefore,

$$\frac{dy}{dx} = 3\sin^2 x . \frac{d}{dx}\sin x + 6\cos^5 x . \frac{d}{dx}\cos x = 3\sin^2 x . \cos x + 6\cos^5 x . (-\sin x)$$

$$= 3\sin x \cos x\,(\sin x - 2\cos^4 x)$$

Question 3:

$(5x)^{3\cos 2x}$

Answer 3:

Let $y = (5x)^{3\cos 2x}$, taking log on both sides

$\log y = \log(5x)^{3\cos 2x} = 3\cos 2x . \log 5x$

Therefore,

$$\frac{1}{y}\frac{dy}{dx} = 3\cos 2x . \frac{d}{dx}\log 5x + \log 5x . \frac{d}{dx} 3\cos 2x$$

$$\Rightarrow \frac{dy}{dx} = y\left[3\cos 2x . \frac{1}{5x} . 5 + \log 5x . 3(-\sin 2x) . 2\right]$$

$$\Rightarrow \frac{dy}{dx} = 3(5x)^{3\cos 2x}\left[\frac{\cos 2x - 2\sin 2x \log 5x}{x}\right]$$

Question 4:

$\sin^{-1}(x\sqrt{x}), 0 \le x \le 1$

Answer 4:

Let $y = \sin^{-1}(x\sqrt{x})$, therefore,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (x\sqrt{x})^2}} . \frac{d}{dx}(x\sqrt{x}) = \frac{1}{\sqrt{1 - x^3}} . \left[x\frac{d}{dx}\sqrt{x} + \sqrt{x} . \frac{d}{dx}x\right]$$

$$= \frac{1}{\sqrt{1 - x^3}} . \left[x\frac{1}{2\sqrt{x}} + \sqrt{x} . 1\right]$$

$$= \frac{1}{\sqrt{1 - x^3}} . \left[\frac{x + 2x}{2\sqrt{x}}\right]$$

$$= \frac{3x}{2\sqrt{x}\sqrt{1 - x^3}} = \frac{3}{2}\sqrt{\frac{x}{1 - x^3}}$$

Question 5:

$\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}, -2 < x < 2.$

Answer 5:

Let $y = \frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$, therefore,

$$\frac{dy}{dx} = \frac{\cos^{-1}\frac{x}{2}.\frac{d}{dx}\sqrt{2x+7} - \sqrt{2x+7}\frac{d}{dx}\cos^{-1}\frac{x}{2}}{\left(\sqrt{2x+7}\right)^2}$$

$$= \frac{\left[\cos^{-1}\frac{x}{2}.\frac{1}{2\sqrt{2x+7}}.2\right] - \sqrt{2x+7}\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}.\frac{1}{2}}{2x+7}$$

$$= \frac{\cos^{-1}\frac{x}{2}.\frac{1}{\sqrt{2x+7}} + \sqrt{2x+7}\frac{1}{\sqrt{4-(x)^2}}}{2x+7} = \frac{\cos^{-1}\frac{x}{2}.\sqrt{4-x^2} + 2x + 7}{(2x+7)\sqrt{2x+7}\sqrt{4-x^2}}$$

Question 6:

$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right], 0 < x < \frac{\pi}{2}$

Answer 6:

Let $y = \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right]$, therefore,

$$y = \cot^{-1}\left[\frac{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}} + \sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}} - \sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}\right]$$

$$= \cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2} + \sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2} - \sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right]$$

$$= \cot^{-1}\left[\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right] = \cot^{-1}\left[\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right] = \cot^{-1}\left[\cot\frac{x}{2}\right] = \frac{x}{2}$$

Therefore, $\frac{dy}{dx} = \frac{1}{2}$

Question 7:

$(\log x)^{\log x}, x > 1$

Answer 7:

Let $y = (\log x)^{\log x}$, taking log on both sides

$\log y = \log(\log x)^{\log x} = \log x . \log(\log x)$

Therefore,

$$\frac{1}{y}\frac{dy}{dx} = \log x . \frac{d}{dx}\log(\log x) + \log(\log x) . \frac{d}{dx}\log x$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log x . \frac{1}{\log x}.\frac{1}{x} + \log(\log x).\frac{1}{x}\right] \Rightarrow \frac{dy}{dx} = (\log x)^{\log x}\left[\frac{1 + \log(\log x)}{x}\right]$$

Question 8:
$\cos(a\cos x + b\sin x)$, for some constant a and b.

Answer 8:
Let $y = \cos(a\cos x + b\sin x)$, therefore,

$$\frac{dy}{dx} = -\sin(a\cos x + b\sin x)\,.\frac{d}{dx}(a\cos x + b\sin x)$$

$$= -\sin(a\cos x + b\sin x)\,(-a\sin x + b\cos x)$$

$$= \sin(a\cos x + b\sin x)\,(a\sin x - b\cos x)$$

Question 9:
$(\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$

Answer 9:
Let $y = (\sin x - \cos x)^{(\sin x - \cos x)}$, taking log on both sides

$\log y = \log(\sin x - \cos x)^{(\sin x - \cos x)} = (\sin x - \cos x).\log(\sin x - \cos x)$

Therefore,

$$\frac{1}{y}\frac{dy}{dx} = (\sin x - \cos x).\frac{d}{dx}\log(\sin x - \cos x) + \log(\sin x - \cos x)\,.\frac{d}{dx}(\sin x - \cos x)$$

$$\Rightarrow \frac{dy}{dx} = y\left[(\sin x - \cos x).\frac{(\cos x + \sin x)}{(\sin x - \cos x)} + \log(\sin x - \cos x)\,(\cos x + \sin x)\right]$$

$$\Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}(\cos x + \sin x)[1 + \log(\cos x - \sin x)]$$

Question 10:
$x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$.

Answer 10:
Let $u = x^x$ and $y = u + x^a + a^x + a^a$ therefore,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{\mathrm{d}}{\mathrm{dx}}x^a + \frac{\mathrm{d}}{\mathrm{dx}}a^x + \frac{\mathrm{d}}{\mathrm{dx}}a^a$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + ax^{a-1} + a^x\log a + 0 \qquad \dots(1)$$

Here, $u = x^x$, taking log on both sides

$\log u = \log x^x = x.\log x$

Therefore,

$$\frac{1}{u}\frac{du}{dx} = x.\frac{d}{dx}\log x + \log x\,.\frac{d}{dx}x = x.\frac{1}{x} + \log x\,.\,1 \;\Rightarrow \frac{du}{dx} = u(1 + \log x) = x^x(1 + \log x)$$

Putting the value of $\frac{du}{dx}$ in equation (1), we get

$$\frac{dy}{dx} = x^x(1 + \log x) + ax^{a-1} + a^x\log a$$

Question 11:
$x^{x^2-3} + (x-3)^{x^2}$, for $x > 3$.

Answer 11:
Let $u = x^{x^2-3}$ and $v = (x-3)^{x^2}$ therefore, $y = u + v$

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \dots(1)$$

Here, $u = x^{x^2-3}$, taking log on both sides
$\log u = (x^2 - 3)\log x$, therefore,

$$\frac{1}{u}\frac{du}{dx} = (x^2-3).\frac{d}{dx}\log x + \log x\,.\frac{d}{dx}(x^2-3) = (x^2-3).\frac{1}{x} + \log x\,.\,2x$$

$$\frac{du}{dx} = u\left[\frac{x^2-3+2x^2\log x}{x}\right]$$

$$\frac{du}{dx} = x^{x^2-3}\left[\frac{x^2-3+2x^2\log x}{x}\right] = x^{x^2-4}(x^2-3+2x^2\log x) \quad \dots(2)$$

and, $v = (x-3)^{x^2}$, taking log on both sides
$\log v = x^2\log(x-3)$, therefore,

$$\frac{1}{v}\frac{dv}{dx} = x^2.\frac{d}{dx}\log(x-3) + \log(x-3)\,.\frac{d}{dx}x^2 = x^2.\frac{1}{x-3} + \log(x-3)\,.\,2x = \frac{x^2}{x-3} + 2x.\log(x-3)$$

$$\frac{dv}{dx} = v\left[\frac{x^2}{x-3} + 2x.\log(x-3)\right] = (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x.\log(x-3)\right] \quad \dots(3)$$

Putting the value of $\frac{du}{dx}$ from (2) and value of $\frac{dv}{dx}$ from (3) in equation (1), we have

$$\frac{dy}{dx} = x^{x^2-4}(x^2-3+2x^2\log x) + (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x.\log(x-3)\right]$$

Question 12:

Find $\frac{dy}{dx}$, if $y = 12(1-\cos t),\ x = 10(t-\sin t), -\frac{\pi}{2} < t < \frac{\pi}{2}$.

Answer 12:

Here, $x = 10(t-\sin t)\,, y = 12(1-\cos t)$

Therefore, $\frac{dx}{dt} = 10(1-\cos t)$ and $\frac{dy}{dt} = 12(0+\sin t)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12\sin t}{10(1-\cos t)} = \frac{6\left(2\sin\frac{t}{2}\cos\frac{t}{2}\right)}{5\left(2\sin^2\frac{t}{2}\right)} = \frac{6}{5}\cot\frac{t}{2}$$

Question 13:

Find $\frac{dy}{dx}$, if $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}, 0 < x < 1$.

Answer 13:

Here, $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}$, therefore

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}x + \frac{d}{dx}\sin^{-1}\sqrt{1-x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\frac{d}{dx}\sqrt{1-x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{x}.\frac{1}{2\sqrt{1-x^2}}\frac{d}{dx}(1-x^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{x}.\frac{1}{2\sqrt{1-x^2}}(-2x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

Question 14:

If $x\sqrt{1+y}+y\sqrt{1+x}=0$, for $-1<x<1$, Prove that $\frac{dy}{dx}=-\frac{1}{(1+x)^2}$

Answer 14:

Given that: $x\sqrt{1+y}+y\sqrt{1+x}=0 \quad \Rightarrow x\sqrt{1+y}=-y\sqrt{1+x}$

Squaring both sides

$x^2(1+y)=y^2(1+x) \quad \Rightarrow x^2+x^2y=y^2+y^2x$

$\Rightarrow x^2-y^2+x^2y-y^2x=0$

$\Rightarrow (x+y)(x-y)+xy(x-y)=0 \quad \Rightarrow (x-y)(x+y+xy)=0$

$\Rightarrow (x+y+xy)=0 \quad [\because x\neq y \quad \Rightarrow x-y\neq 0]$

$\Rightarrow y(1+x)=-x \Rightarrow y=-\dfrac{x}{1+x}$

Therefore,

$$\frac{dy}{dx}=-\left[\frac{(1+x)\frac{d}{dx}x-x\frac{d}{dx}(1+x)}{(1+x)^2}\right]=-\frac{1+x-x}{(1+x)^2}=-\frac{1}{(1+x)^2}$$

Question 15:

If $(x-a)^2+(y-b)^2=c^2$ for some $c>0$, prove that

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

is a constant independent of a and b.

Answer 15:

Given that: $(x-a)^2+(y-b)^2=c^2$. Differentiating with respect to x, we have

$\frac{d}{dx}(x-a)^2+\frac{d}{dx}(y-b)^2=\frac{d}{dx}c^2 \quad \Rightarrow 2(x-a)+2(y-b)\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}=-\frac{x-a}{y-b}$

Differentiating again, we have

$$\frac{d^2y}{dx^2}=-\frac{(y-b)\frac{d}{dx}(x-a)-(x-a)\frac{d}{dx}(y-b)}{(y-b)^2}=-\frac{(y-b)1-(x-a)\frac{dy}{dx}}{(y-b)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2}=-\frac{(y-b)1-(x-a)\left(-\frac{x-a}{y-b}\right)}{(y-b)^2}=-\frac{(y-b)^2+(x-a)^2}{(y-b)^3}=-\frac{c^2}{(y-b)^3}$$

Putting the values in $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$, we have

$$\frac{\left[1+\left(-\frac{x-a}{y-b}\right)^2\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}}=\frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}}=\frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}}=\frac{\left[\frac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}}$$

$$=\frac{\frac{c^3}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}}=-\frac{c^3}{c^2}=-c,$$ which is a constant independent of a and b.

Question 16:

If $\cos y = x\cos(a+y)$, with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

Answer 16:

Given that: $\cos y = x\cos(a+y)$ $\Rightarrow x = \frac{\cos y}{\cos(a+y)}$, therefore,

Differentiating with respect to y, we have

$$\frac{dx}{dy} = \frac{\cos(a+y)\frac{d}{dy}\cos y - \cos y\frac{d}{dy}\cos(a+y)}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\cos(a+y)\,(-\sin y) - \cos y\,(-\sin(a+y))}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy} = \frac{-\sin y\cos(a+y) + \cos y\sin(a+y)}{\cos^2(a+y)}$$

$$= \frac{\sin(a+y-y)}{\cos^2(a+y)} = \frac{\sin a}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

Question 17:

If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\frac{d^2y}{dx^2}$.

Answer 17:

Here, $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$

Therefore,

$$\frac{dx}{dt} = a[-\sin t + (t\cos t + \sin t)] = at\cos t$$

and

$$\frac{dy}{dt} = a[(\cos t - (-t\sin t + \cos t))] = at\sin t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{at\sin t}{at\cos t} = \tan t$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec^2 t\,.\frac{dt}{dx} = \sec^2 t\,.\frac{1}{at\cos t} = \frac{\sec^3 t}{at}$$

Question 18:

If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it.

Answer 18:

Rewriting the function $f(x) = |x|^3$ in the following form:

$$f(x) = \begin{cases} x^3 & \text{if } x \geq 0 \\ -x^3 & \text{if } x < 0 \end{cases}$$

If $x \geq 0$, $f(x) = x^3$ $\Rightarrow f'(x) = 3x^2$ $\Rightarrow f''(x) = 6x$

If $x < 0$, $f(x) = -x^3$ $\Rightarrow f'(x) = -3x^2$ $\Rightarrow f''(x) = -6x$

Hence, $f''(x)$ exists for all real x and it can be represented as follows:

$$f''(x) = \begin{cases} 6x & \text{if } x \geq 0 \\ -6x & \text{if } x < 0 \end{cases}$$

Question 19:
Using the fact that sin (A + B) = sin A cos B + cos A sin B and the differentiation, obtain the sum formula for cosines.

Answer 19:
Given that: $\sin(A+B) = \sin A \cos B + \cos A \sin B$,

Differentiating with respect to x, we have

$$\frac{d}{dx}\sin(A+B) = \left(\sin A \frac{d}{dx}\cos B + \cos B \frac{d}{dx}\sin A\right) + \left(\cos A \frac{d}{dx}\sin B + \sin B \frac{d}{dx}\cos A\right)$$

$$\Rightarrow \cos(A+B).\left(\frac{dA}{dx} + \frac{dB}{dx}\right)$$

$$= \left(\sin A\,(-\sin B)\frac{dB}{dx} + \cos B \cos A \frac{dA}{dx}\right) + \left(\cos A \cos B \frac{dB}{dx} + \sin B\,(-\sin A)\frac{dA}{dx}\right)$$

$$\Rightarrow \cos(A+B).\left(\frac{dA}{dx} + \frac{dB}{dx}\right)$$

$$= (\cos A \cos B - \sin A \sin B)\frac{dB}{dx} + (\cos A \cos B - \sin A \sin B)\frac{dA}{dx}$$

$$\Rightarrow \cos(A+B).\left(\frac{dA}{dx} + \frac{dB}{dx}\right) = (\cos A \cos B - \sin A \sin B)\left(\frac{dA}{dx} + \frac{dB}{dx}\right)$$

$$\Rightarrow \cos(A+B) = \cos A \cos B - \sin A \sin B$$

Question 20:
Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer.

Answer 20:
Function $f(x) = |x-1| + |x-3|$ is continuous for all real points but not differentiable at two points ($x = 1$ and $x = 3$).

Question 21:
If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$ prove that $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Answer 21:
Given that: $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, therefore,

$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ \frac{dl}{dx} & \frac{dm}{dx} & \frac{dn}{dx} \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ \frac{da}{dx} & \frac{db}{dx} & \frac{dc}{dx} \end{vmatrix}$$

$$\Rightarrow \frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ 0 & 0 & 0 \end{vmatrix}$$

$$\Rightarrow \frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix} + 0 + 0 \quad \Rightarrow \frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Question 22:

If $y = e^{a\cos^{-1}x}, -1 \le x \le 1$, show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

Answer 22:

Given that: $y = e^{a\cos^{-1}x}$, therefore,

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}e^{a\cos^{-1}x} = e^{a\cos^{-1}x}\frac{d}{dx}a\cos^{-1}x$$

$$\Rightarrow \frac{dy}{dx} = e^{a\cos^{-1}x}a.\frac{-1}{\sqrt{1-x^2}} = -\frac{ay}{\sqrt{1-x^2}}$$

Squaring both the sides, we have

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$$

$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

Differentiating again with respect to x, we have

$$(1-x^2).2\frac{dy}{dx}.\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\frac{d}{dx}(1-x^2) = a^2 2y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left[2(1-x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}(-2x)\right] = 2a^2y\frac{dy}{dx}$$

$$\Rightarrow 2\frac{dy}{dx}\left[(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}\right] = 2a^2y\frac{dy}{dx}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = a^2y$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$