5

Complex Numbers

5.01 Introduction

Jain Mathematician Shree Mahaveerachara in 850 B.C. gave a hint that negative numbers do not have square roots. In 1637 BC. Rene De'cartes described the numbers as Real and Imaginary. Euler in 1748 B.C. denoted the value of $\sqrt{-1}$ as i. Gauss in 1832 called $a+b\sqrt{-1}$ as *Complex Number.*

Real solution of quadratic equation $ax^2+bx+c=0$ can be obtained by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$ (Shri-Dhar Aacharya formula)

If discriminant $b^2-4ac\geq 0$ then roots are real but in the case of $b^2-4ac<0$ so we are not able to find the roots of above equation because in this case $\sqrt{b^2-4ac}$ (square root of negative number) is not real number. To solve these kind of problems, expanding the real number system a new number system is developed in which the square root of negative number is also included. These number are called as complex numbers.

We know that product of two real numbers having same sign is positive but if $\sqrt{-1}$ is multiplied by $\sqrt{-1}$ then the product is negative. Mathematicians gave the name as imaginary quantities to all these numbers.

Defiition "Every number whose square is negative is termed as Complex Number"

Example : $\sqrt{-2}, \sqrt{-13}, \sqrt{-15}$ etc. are imaginary numbers.

IOTA : $\left(i=\sqrt{-1}\right)$ is an imaginary number with the following properties

(i) $i=\sqrt{-1}, i^2=-1, i^3=i^2\cdot i=-i, i^4=\left(i^2\right)\left(i^2\right)=(-1)(-1)=1$

(ii) $1/i=-i$

(iii) $i^{4m}=\left(i^4\right)^m=(1)^m=1$

i.e. if the power of i is multiple of 4, then its value is always 1.

5.02 Complex Numbers

If $a, b\in R$ then $a+ib$ or $a-ib$ are called as complex numbers. Generally it is indicated as z, as $z=a+ib$, where $i=\sqrt{-1}$ (measure form), where a is called as real part and b as imaginary part.

Complex Number as an Ordered Pair- Hamilton defined complex number in the form of an ordered pair according to which $z=a+ib$ is represented as (a,b) where $a,b\in R$.

Real and imaginary parts of complex number z are expressed as $Re(z)$ and $Im(z)$ respectively.

Special case: If $Im(z)=0$ then complex quantity is called as real quantity and if $Re(z)=0$ then it is called as imaginary quantity.

For Example : $-2, 0, -2i, 1-\sqrt{2}\,i$ are all complex numbers of the form $a+ib$ which can be written as

$$-2 = 2 + i\cdot 0 = (-2, 0),\quad -2i = 0-2i = (0,-2),\quad 0 = 0 + i\cdot 0 = (0,0),\quad 1-\sqrt{2}i = \left(1, -\sqrt{2}\right)$$

5.03 Set of Complex Numbers

The set of Complex number is denoted by C and is defined as

$$C = \{a+ib : a,\, b \in R\}$$

5.04 Theorems on Complex Numbers

Theorem 5-1 If any Complex Number is equal to zero then its real and imaginary parts are also equal to zeo.

Proof - Let $z = a + ib$ and given $a+ib=0$

$\therefore \quad a = -ib$

squaring both sides

$$(a)^2 = (-ib)^2 = (-1)^2\left(i^2\right)b^2 = -b^2$$

or $\quad a^2 + b^2 = 0$

but a and b are real, therefore equating a and b individually equal to zero we have

$a = 0$ and $b = 0$.

Theorem 5.2 If two Complex numbers are equal then their real and Imginary parts are equal,

Proof - Let $z_1 = a+ib$ and $z_2 = c+id$ are equal

now $\quad z_1 = z_2$

$\Rightarrow \quad a+ib = c+id$

$\Rightarrow \quad (a-c) + i(b-d) = 0$

$\Rightarrow \quad a-c=0$ and $b-d=0$

$\Rightarrow \quad a=c$ and $b=d$

Two complex number $Z_1 = a+ib$ and $Z_2 = c+ib$ are equal if $a=c$ and $b=d$.

5.05 Addition Operation in the Set of Complex Numbers C

Let $z_1 = a+ib$ and $Z_2 = c+ib$ be any two complex number. Then, the sum Z_1+Z_2 is defined as follows:

$$\begin{aligned} z_1 + z_2 &= (a+ib)+(c+id) \\ &= (a+c) + i(b+d) \end{aligned}$$

Example : If $z_1 = 2+5i,\ z_2 = 3+7i$ then

$$z_1 + z_2 = (2+5i)+(3+7i) = 5+12i\ .$$

5.06 Properties of Addition on the Set of Complex Numbers C

1. Closure Property

The sum of two complex numbers is a complex number, i.e.

$\forall \; z_1, z_2 \in C \;\Rightarrow\; z_1 + z_2 \in C$

Proof : If $z_1 = a + ib$ and $z_2 = c + id$ are two complex numbers

$$z_1 + z_2 = (a+ib) + (c+id) = (a+c) + i(b+d)$$

$\Rightarrow \quad c \in R \;\Rightarrow\; a + c \in R$ .and $b \in R,\; d \in R \;\Rightarrow\; b + d \in R$

$\therefore \quad z_1 + z_2 \in C$

2. Associative property:

For any three complex numbers the associative property holds i.e.

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \quad \forall \; z_1, z_2, z_3 \in C$$

Prof : Let $z_1 = a_1 + ib_1, \quad z_2 = a_2 + ib_2, \quad z_3 = a_3 + ib_3$

then $z_1 + (z_2 + z_3) = (a_1 + ib_1) + [(a_2 + ib_2) + (a_3 + ib_3)]$

$$= (a_1 + ib_1) + [(a_2 + a_3) + i(b_2 + b_3)]$$

$$= [a_1 + (a_2 + a_3) + i\{b_1 + (b_2 + b_3)\}]$$

$$= [(a_1 + a_2) + a_3 + i\{(b_1 + b_2) + b_3\}]$$

$$= [a_1 + a_2 + i(b_1 + b_2)] + (a_3 + ib_3)$$

$$= [(a_1 + ib_1) + (a_2 + ib_2)] + (a_3 + ib_3)$$

$$= (z_1 + z_2) + z_3$$

3. Additive Identity

There exists the complex number $0 = 0 + i0$ (denoted as 0), called the *additive identity* or the *zero complex number*, such that, for every complex number

Proof : Let $z = a + ib$ then $\because \; 0 + i\,0 = 0$

$\therefore \quad z + 0 = (a+ib) + (0+i0) = (a+0) + i(b+0) = a + ib = z$

$$0 + z = (0+i0) + (a+ib) = (0+a) + i(0+b) = a + ib = z$$

$\therefore \quad z + 0 = z = 0 + z$

4. Additive Inverse)

To every complex number $z = a + ib$, we have the complex number $-a + i(-b)$ (denoted as $-z$), called the *addit*ve inverse or negative of z. We observe that $z + (-z) = 0$ (the additive identity).

$\therefore \quad \begin{aligned} z + (-z) &= (a+ib) + (-a-ib) = (a-a) + i(b-b) = 0 + i0 = 0 \\ (-z) + z &= (-a-ib) + (a+ib) = (-a+a) + i(-b+b) = 0 + i0 = 0 \end{aligned}$

$\therefore \quad z + (-z) = 0 = (-z) + z$

5. Commutative Property

For any two complex numbers

$$z_1 + z_2 = z_2 + z_1, \quad \forall \; z_1, z_2 \in C$$

Proof : Let $z_1 = a + ib, \quad z_2 = c + id$

Then $z_1 + z_2 = (a+ib)+(c+id)$

$$= (a+c)+i(b+d)$$

$$= (c+a)+i(d+b)$$

$$= (c+id)+(a+ib)$$

$$= z_2 + z_1$$

$\therefore \quad z_1 + z_2 = z_2 + z_1$

6. Cancellation Law

The set of complex numbers follows the cancellation law

$z_1 + z_3 = z_2 + z_3 \Rightarrow z_1 = z_2, \ \forall z_1 z_2, z_3 \in C$ [right Cancellation law]

$z_3 + z_1 = z_3 + z_2 \Rightarrow z_1 = z_2, \ \forall \ z_1, z_2, z_3 \in C$ [left Cancellation law]

Proof : Let $z_1 = a_1 + ib_1, \quad z_2 = a_2 + ib_2, \quad z_3 = a_3 + ib_3$

then, $z_1 + z_3 = (a_1 + ib_1)+(a_3 + ib_3) = (a_1 + a_3) + i(b_1 + b_3)$

$z_2 + z_3 = (a_2 + ib_2)+(a_3 + ib_3) = (a_2 + a_3) + i(b_2 + b_3)$

$\therefore \quad z_1 + z_3 = z_2 + z_3 \quad \Rightarrow \quad (a_1 + a_3) + i(b_1 + b_3) = (a_2 + a_3) + i(b_2 + b_3)$

$\Rightarrow \quad a_1 + a_3 = a_2 + a_3, \ b_1 + b_3 = b_2 + b_3$ [comparing the real and imaginary parts]

$\Rightarrow \quad a_1 = a_2, \ b_1 = b_2$

$\Rightarrow \quad a_1 + ib_1 = a_2 + ib_2$

$\Rightarrow \quad z_1 = z_2$

Similarly we can prove $z_3 + z_1 = z_3 + z_2 \quad \Rightarrow \quad z_1 = z_2$

5.07 Subtraction Operation in the Set of Complex Numbers C

Given any two complex numbers $z_1 = a + ib$ and $z_2 = c + id$, the product $z_1 - z_2$ is defined as follows.

$$z_1 - z_2 = (a+ib)-(c+id)$$

$$= (a-c)+i(b-d)$$

Thus, the difference of two complex numbers is again a complex number where the real parts are subtracted together and the imaginary are subtracted together.

Example - If $z_1 = 4 + 3i$ and $z_2 = 2 + i$

then $z_1 - z_2 = (4-2)+(3-1)i$

$= 2 + 2i$ $\left[i^2 = -1\right]$

5.08 Multiplication Operation in the Set of Complex Numbers C

Given any two complex numbers $z_1 = a + ib$ and $z_2 = c + ib$, the product of is defined as follows:

$$z_1 \cdot z_2 = (a+ib)\cdot(c+id)$$

$$= ac + ibc + iad + i^2bd$$

$$= ac + i(bc + ad) - bd \qquad (\because \; i^2 = -1)$$

$$= (ac - bd) + i(bc + ad)$$

i.e. the product of two Compex Numbers is again a complex number.

Example : If $z_1 = 2 + 3i$ and $z_2 = 2 + 4i$

then $z_1 \cdot z_2 = (2 + 3i) \cdot (2 + 4i)$

$$= 4 + 6i + 8i + 12i^2 = 4 + 14i - 12 = -8 + 14i$$

5.09 Properties of Multiplication on the Set of Complex Numbers C

1. Closure property:

The product of two complex numbers is a complex number. The product $z_1 z_2$ is a complex number for all complex numbers z_1 and z_2. i.e. $\forall \; z_1, \, z_2 \in C \Rightarrow z_1 z_2 \in C$

Proof : If $z_1 = a + ib$ and $z_2 = c + id$ are two complex numbers, where $a, b, c, \in R$,

then $z_1 \cdot z_2 = (a + ib) \cdot (c + id)$

$$= ac + ibc + iad + i^2bd$$

$$= (ac - bd) + i(bc + ad)$$

$\because \quad a, b, c, d \in R \qquad \therefore \qquad (ac - bd) \in R$ and $(bc + ad) \in R$

$\therefore \quad z_1 z_2 \in C$

2. Associative Property:

For any three complex numbers z_1, z_2, z_3,

$$z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3, \qquad \forall \; z_1, z_2, z_3 \in C$$

Proof : If $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$ and $z_3 = a_3 + ib_3$.

$$z_1 \cdot (z_2 \cdot z_3) = (a_1 + ib_1) \cdot [(a_2 + ib_2) \cdot (a_3 + ib_3)]$$

$$= (a_1 + ib_1) \cdot \left[\left\{(a_2a_3 - b_2b_3) + i(a_2b_3 + b_2a_3)\right\}\right]$$

$$= \left[\left\{a_1(a_2a_3 - b_2b_3) - b_1(a_2b_3 + b_2a_3)\right\} + i\left\{b_1(a_2a_3 - b_2b_3) + a_1(a_2b_3 + b_2a_3)\right\}\right]$$

$$= \left[(a_1a_2a_3 - a_1b_2b_3 - b_1a_2b_3 - b_1b_2a_3) + i(b_1a_2a_3 - b_1b_2b_3 + a_1a_2b_3 + a_1b_2a_3)\right] \qquad (1)$$

$$(z_1 \cdot z_2) \cdot z_3 = [(a_1 + ib_1)(a_2 + ib_2)] \cdot (a_3 + ib_3)$$

$$= \left[\left\{(a_1a_2 - b_1b_2) + i(b_1a_2 + a_1b_2)\right\}\right] \cdot (a_3 + ib_3)$$

$$= \left[\left\{(a_1a_2 - b_1b_2)a_3 - (b_1a_2 + a_1b_2)b_3\right\} + i\left\{a_3(b_1a_2 + a_1b_2) + b_3(a_1a_2 - b_1b_2)\right\}\right]$$

$$= \left[(a_1a_2a_3 - b_1b_2a_3 - b_1a_2b_3 - a_1b_2b_3) + i(a_1a_2b_3 - b_1b_2b_3 + b_1a_2a_3 + a_1b_2a_3)\right] \qquad (2)$$

from eq. (1) and (2),

$$z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$$

3. Multiplicative Identity

There exists the complex number $1 + i\,0 = (1, 0)$ (denoted as 1), called the *multiplicative identity* such that $z.\ 1 = z$, for every complex number z.

Proof : let $z = a + ib$ then $1 = 1 + i0$

$$z \cdot 1 = (a + ib) \cdot (1 + i0) = \{(a - 0) + i(b + 0) = a + ib = z\}$$

$$1 \cdot z = (1 + i0) \cdot (a + ib) = \{(a - 0) + i(b + 0) = a + ib = z\}$$

$\therefore \quad z \cdot 1 = z = 1 \cdot z$

4. Multiplicative Inverse

For every non-zero complex number $z = a + ib$ there lies a multiplicative inverse in set C

Proof : let $z = a + ib \neq 0$ is a complex number where a and b are non-zero and $a, b \in R$

let $x + iy$ be the multiplicative inverse of $a + ib$

$\Rightarrow \quad (a + ib) \cdot (x + iy) = 1 + i \cdot 0$

$\Rightarrow \quad (ax - by) + i(bx + ay) = 1 + i \cdot 0$

comparing the real and imaginary parts we have

$\Rightarrow \quad ax - by = 1$ and $bx + ay = 0$

$\Rightarrow \quad x = \dfrac{a}{a^2 + b^2}$ and $y = \dfrac{-b}{a^2 + b^2}$; $a^2 + b^2 \neq 0$

$\because \quad a, b \in R \Rightarrow \dfrac{a}{a^2 + b^2} = x \in R$ and $\dfrac{-b}{a^2 + b^2} = y \in R$

$\Rightarrow \quad \dfrac{a}{a^2 + b^2} + i\left(\dfrac{-b}{a^2 + b^2}\right) \in C \quad \Rightarrow \quad x + iy \in C$

we have the complex number $z = a + ib$ whose multiplicative inverse is $\dfrac{a}{a^2 + b^2} + i\left(\dfrac{-b}{a^2 + b^2}\right)$ denoted by $1/z$ or z^{-1}

$$\therefore \quad z = a + ib \Rightarrow z^{-1} = \frac{1}{z} = \frac{a}{a^2 + b^2} + i\frac{-b}{a^2 + b^2}.$$

5. Commutative Property

For any two complex numbers z_1 and z_2

Proof : If $z_1 = a + ib$ and $z_2 = c + id$

$$\therefore \quad z_1 \cdot z_2 = (a + ib) \cdot (c + id) = (ac - bd) + i(bc + ad)$$

$$= (ca - db) + i(cb + da) = (c + id) \cdot (a + ib) = z_2 \cdot z_1$$

6. Cancellation Law

The set of complex numbers follows the multiplicative law of cancellation.

Thus if $z_3 \neq 0$ and $z_1, z_2, z_3 \in C$ then

$z_1 z_3 = z_2 z_3 \Rightarrow z_1 = z_2$ [right cancellation law]

$z_3 z_1 = z_3 z_2 \Rightarrow z_1 = z_2$ [left cancellation law]

Proof : If $z_1 = a_1 + ib_1,\quad z_2 = a_2 + ib_2,\quad z_3 = a_3 + ib_3$

$\therefore \quad z_1 \cdot z_3 = (a_1 + ib_1)\cdot(a_3 + ib_3) = (a_1a_3 - b_1b_3) + i(a_1b_3 + a_3b_1)$

and $\quad z_2 \cdot z_3 = (a_2 + ib_2)\cdot(a_3 + ib_3) = (a_2a_3 - b_2b_3) + i(a_2b_3 + a_3b_2)$

$\therefore \quad z_1z_3 = z_2z_3 \Rightarrow (a_1a_3 - b_1b_3) + i(a_1b_3 + b_1a_3) = (a_2a_3 - b_2b_3) + i(a_2b_3 + b_2a_3)$

comparing the real and imaginary parts

$a_1a_3 - b_1b_3 = a_2a_3 - b_2b_3$ and $a_1b_3 + a_3b_1 = a_2b_3 + b_2a_3$

$(a_1 - a_2)a_3 - (b_1 - b_2)b_3 = 0$ and $(a_1 - a_2)b_3 + a_3(b_1 - b_2) = 0$

$\Rightarrow \quad a_1 = a_2,\ b_1 = b_2 \qquad (\because\ z_3 \neq 0)$

$\Rightarrow \quad z_1 = z_2$

similarly we can prove that

$$z_3z_1 = z_3z_2 \Rightarrow z_1 = z_2$$

7. Distributive Law

For any three complex numbers z_1, z_2, z_3

$z_1(z_2 + z_3) = z_1z_2 + z_1z_3$ and $(z_2 + z_3)z_1 = z_2z_1 + z_3z_1 \qquad \forall\ z_1, z_2, z_3 \in C$

Proof : If $z_1 = a_1 + ib_1,\quad z_2 = a_2 + ib_2,\quad z_3 = a_3 + ib_3$

$$z_1(z_2 + z_3) = (a_1 + ib_1)[(a_2 + ib_2) + (a_3 + ib_3)]$$
$$= (a_1 + ib_1)[(a_2 + a_3) + i(b_2 + b_3)]$$
$$= (a_1a_2 + a_1a_3 - b_1b_2 - b_1b_3) + i(a_1b_2 + a_1b_3 + b_1a_2 + b_1a_3) \qquad (1)$$
$$z_1z_2 + z_1z_3 = (a_1 + ib_1)(a_2 + ib_2) + (a_1 + ib_1)(a_3 + ib_3)$$
$$= (a_1a_2 + a_1a_3 - b_1b_2 - b_1b_3) + i(a_1b_2 + b_1a_2 + b_1a_3 + b_3a_1) \qquad (2)$$

by (1) and (2) $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$.

similarly we can prove that $(z_1 + z_2)z_3 = z_1z_3 + z_2z_3$

5.10 Division Operation in the Set of Complex Numbers *C*

Let $z_1 = a + ib$ and $z_2 = c + id\ (z_2 \neq 0)$, where $a, b, c, d \in R$ are two complex numbers the quotient of which is denoted by $z_1 \div z_2$ therefore

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{(a+ib)}{(c+id)} \times \frac{(c-id)}{(c-id)}$$
$$= \frac{ac + ibc - iad - i^2bd}{c^2 - i^2d^2}$$

$$= \frac{(ac+bd)+i(bc-ad)}{c^2+d^2} \qquad (\because\ i^2=-1)$$

$$= \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2} \qquad (c^2+d^2 \neq 0 \quad \because\ z_2 \neq 0)$$

Example : If $z_1 = 3+4i$ and $z_2 = 1+3i$ then

$$z_1 \div z_2 = \frac{3+4i}{1+3i}$$

$$= \frac{3+4i}{1+3i} \times \frac{1-3i}{1-3i} = \frac{3+4i-9i-12i^2}{(1)^2-9i^2}$$

$$= \frac{3-5i+12}{1+9} = \frac{3}{2} - \frac{1}{2}i$$

5.11 Conjugate Complex Numbers

Two complex numbers are called conjugate of each other if their real parts are equal and the imaginary parts are equal but with of opposite sign.

$\therefore$ If $z = a+ib$ then conjugate is represented by $\bar{z}$ where $\bar{z} = a-ib$

Example :

(i) If $z = (3, -5) = 3-5i \Rightarrow \bar{z} = (3, 5) = (3+5i)$

(ii) If $z = (-2, \sqrt{11}) = -2+\sqrt{11}\,i \Rightarrow \bar{z} = (-2, -\sqrt{11}) = -2-\sqrt{11}\,i$

5.12 Some Properties of Conjugate Complex Numbers

If $z, z_1, z_2 \in C$, then

(i) $z+\bar{z} = 2\,\mathrm{Re}(z)$ **(ii)** $z-\bar{z} = 2i\,\mathrm{Im}(z)$ **(iii)** $\overline{(\bar{z})} = z$

(iv) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$ **(v)** $\overline{z_1-z_2} = \bar{z}_1 - \bar{z}_2$ **(vi)** $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

(vii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2},\ z_2 \neq 0$ **(viii)** $z\bar{z} = [\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2$

Proof : If $z = a+ib;\ a, b \in R,$ then

(i) $z+\bar{z} = (a+ib)+(a-ib) = 2a = 2\,\mathbf{Re}(z)$

(ii) $z-\bar{z} = (a+ib)-(a-ib) = 2ib = 2i\,\mathbf{Im}(z)$

(iii) $\overline{(\bar{z})} = \overline{(a-ib)} = a-(-ib) = a+ib = z$

(iv) let $z_1 = a+ib$ and $z_2 = c+id$ then

$$z_1+z_2 = (a+ib)+(c+id) = (a+c)+i(b+d)$$

$$\Rightarrow \quad \overline{z_1+z_2} = (a+c)-i(b+d) = (a-ib)+(c-id) = \bar{z}_1+\bar{z}_2$$

(v) $z_1 - z_2 = (a+ib)-(c+id) = (a-c)+i(b-d)$

$\Rightarrow \quad \overline{z_1 - z_2} = (a-c)-i(b-d) = (a-ib)-(c-id) = \overline{z}_1 - \overline{z}_2$

(vi) $z_1 \cdot z_2 = (a+ib)\cdot(c+id) = (ac-bd)+i(bc+ad)$

$\Rightarrow \quad \overline{z_1 \cdot z_2} = (ac-bd)-i(bc+ad)$ (1)

and $\overline{z}_1 \cdot \overline{z}_2 = (a-ib)(c-id) = (ac-bd)-i(bc+ad)$ (2)

from equation (1) and (2) $\overline{z_1 \cdot z_2} = \overline{z}_1 \cdot \overline{z}_2$

(vii) $\dfrac{z_1}{z_2} = \dfrac{a+ib}{c+id} = \dfrac{(a+ib)}{(c+id)} \times \dfrac{(c-id)}{(c-id)} = \left(\dfrac{ac+bd}{c^2+d^2}\right) + i\left(\dfrac{bc-ad}{c^2+d^2}\right)$

$\Rightarrow \quad \overline{\left(\dfrac{z_1}{z_2}\right)} = \left(\dfrac{ac+bd}{c^2+d^2}\right) - i\left(\dfrac{bc-ad}{c^2+d^2}\right)$ (1)

$$\frac{\overline{z}_1}{\overline{z}_2} = \frac{a-ib}{c-id} = \left(\frac{a-ib}{c-id}\right) \times \left(\frac{c+id}{c+id}\right)$$

$$= \left(\frac{ac+bd}{c^2+d^2}\right) - i\left(\frac{bc-ad}{c^2+d^2}\right) \quad (2)$$

from equation (1) and (2) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z}_1}{\overline{z}_2}$

(viii) $z\overline{z} = (a+ib)(a-ib) = a^2 - i^2b^2 = a^2 + b^2 = [\operatorname{Re}(z)]^2 + [\operatorname{Im}(z)]^2$

5.13 Modulus of a Complex Number

If $z = a+ib$ be a complex number, then the modulus of z is denoted by $|z|$ is defined to be the nonnegative real number $\sqrt{a^2+b^2}$

$$z = a+ib$$

$$\Rightarrow \quad |z| = \sqrt{a^2+b^2} = \sqrt{\left[\{\operatorname{Re}(z)\}^2 + \{\operatorname{Im}(z)\}^2\right]}$$

$$\therefore \quad |z| \geq 0, \quad \forall \; z \in C$$

(i) $z_1 = 2+3i \Rightarrow |z_1| = \sqrt{4+9} = \sqrt{13}$ (ii) $z_2 = -2i \Rightarrow |z_2| = \sqrt{0+4} = 2$

Note: In complex numbers $z_1 > z_2$ or $z_1 < z_2$ is meaningless but $|z|$ is positive therefore $|z_1| > |z_2|$, $|z_1| < |z_2|$ are meaningful.

5.14 Properties Related to Moduli of Complex Numbers

If $z, z_1, z_2 \in C$ then

(i) $|z| \geq |\mathbf{Re}(z)| \geq \mathbf{Re}(z)$; $|z| \geq |\mathbf{Im}(z)| \geq \mathbf{Im}(z)$ **(ii)** $|z| = |\overline{z}| = |-z|$

(iii) $z\overline{z} = |z|^2$ **(iv)** $|z_1 z_2| = |z_1||z_2|$ **(v)** $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $|z_2| \neq 0$

Proof : (i) Let $z = a + ib$ then $|z| = \sqrt{a^2 + b^2}$ where $\mathbf{Re}(z) = a$, $\mathbf{Im}(z) = b$

$\therefore$ $|z| \geq \mathbf{Re}(z)$ and $|z| \geq \mathbf{Im}(z)$

(ii) Let $z = a + ib$ then $\overline{z} = a - ib$ and $-z = -a - ib$

$\therefore$ $|z| = \sqrt{a^2 + b^2}$, $|\overline{z}| = \sqrt{a^2 + b^2}$ and $|-z| = \sqrt{a^2 + b^2} \Rightarrow |z| = |\overline{z}| = |-z|$

(iii) Let $z = a + ib$ then $\overline{z} = a - ib$

$\therefore$ $$z\overline{z} = (a + ib)(a - ib) = a^2 - i^2 b^2 = a^2 + b^2 \quad (1)$$

and $$|z|^2 = |a + ib|^2 = \left(\sqrt{a^2 + b^2}\right)^2 = a^2 + b^2 \quad (2)$$

from equation (1) and (2) $z\overline{z} = |z|^2$

Note: If $z \neq 0$, then $z^{-1} = \overline{z}/|z|^2$ formula is used to find the multiplicative inverse.

(iv) Let $z_1 = a + ib$ and $z_2 = c + id$

then $$z_1 z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

$\therefore$ $$|z_1 z_2| = \sqrt{(ac - bd)^2 + (ad + bc)^2}$$

$$= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}$$

$$= \sqrt{(a^2 + b^2)}\sqrt{(c^2 + d^2)} = |z_1||z_2|$$

(v) Let $z_1 = a + ib$ and $z_2 = c + id \neq 0$ then

$$\frac{z_1}{z_2} = \frac{a + ib}{c + id} = \frac{(a + ib)}{(c + id)} \times \frac{(c - id)}{(c - id)} = \left(\frac{ac + bd}{c^2 + d^2}\right) + i\left(\frac{bc - ad}{c^2 + d^2}\right)$$

$\therefore$ $$\left|\frac{z_1}{z_2}\right| = \sqrt{\left(\frac{ac + bd}{c^2 + d^2}\right)^2 + \left(\frac{bc - ad}{c^2 + d^2}\right)^2} = \sqrt{\frac{(a^2 + b^2)(c^2 + d^2)}{(c^2 + d^2)^2}} = \frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}} = \frac{|z_1|}{|z_2|}$$

Triangular Inequalities

If $z_1, z_2 \in C$ then

(i) $|z_1 + z_2| \leq |z_1| + |z_2|$ (ii) $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

Proof : (i) $$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} \qquad \left[\because |z|^2 = z\overline{z}\right]$$

$$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \qquad \left[\because \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2\right]$$

$$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$$

$$= |z_1|^2 + z_1\bar{z}_2 + z_2\bar{z}_1 + |z_2|^2$$

$$= |z_1|^2 + z_1\bar{z}_2 + \overline{(z_1\bar{z}_2)} + |z_2|^2 \qquad \left[\because \overline{(\bar{z})} = z\right]$$

$$= |z_1|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2) + |z_2|^2 \qquad [\because z + \bar{z} = 2\,\mathrm{Re}(z)]$$

$$\le |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2 \qquad [\because 2\,\mathrm{Re}(z) \le |z|]$$

$$\le |z_1|^2 + 2|z_1||\bar{z}_2| + |z_2|^2 \qquad [\because |z_1 z_2| = |z_1||z_2|]$$

$$\le |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \qquad [\because |\bar{z}| = |z|]$$

$$\le [|z_1| + |z_2|]^2$$

$\therefore \qquad |z_1 + z_2| \le |z_1| + |z_2|$

(ii) $\qquad |z_1 - z_2|^2 = (z_1 - z_2)\overline{(z_1 - z_2)} \qquad \left[\because |z|^2 = z\bar{z}\right]$

$$= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \qquad \left[\because \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2\right]$$

$$= z_1\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_1 + z_2\bar{z}_2$$

$$= |z_1|^2 - (z_1\bar{z}_2 + z_2\bar{z}_1) + |z_2|^2$$

$$= |z_1|^2 - \left[z_1\bar{z}_2 + \overline{(z_1\bar{z}_2)}\right] + |z_2|^2 \qquad \left[\overline{(\bar{z})} = z\right]$$

$$= |z_1|^2 - [2\,\mathrm{Re}(z_1\bar{z}_2)] + |z_2|^2 \qquad [\because z + \bar{z} = \mathrm{Re}(z)]$$

$$\ge |z_1|^2 - 2|z_1\bar{z}_2| + |z_2|^2 \qquad [\because 2\,\mathrm{Re}(z) \le |z|]$$

$$\ge |z_1|^2 - 2|z_1||\bar{z}_2| + |z_2|^2$$

$$\ge |z_1|^2 - 2|z_1||z_2| + |z_2|^2 \qquad [|\bar{z}| = |z|]$$

$$\ge [|z_1| - |z_2|]^2 \qquad \text{(i)}$$

$\therefore \qquad |z_1 - z_2| \ge |z_1| - |z_2|$

similarly $| z_2 - z_1 | \ge | z_2 | - | z_1 |$

but $\quad | z_1 - z_2 | = | z_2 - z_1 |$

from (i) and (ii)

$$| z_1 - z_2 | \ge || z_1 | - | z_2 ||$$

Illustrative Examples

Example 1. Find the value of the following

(i) $\dfrac{1}{i}$ $\qquad$ (ii) i^{17} $\qquad$ (iii) $1 - i^{200}$

Solution : (i) $\frac{1}{i}=\frac{1}{i}\times\frac{i}{i}=\frac{i}{i^2}=\frac{i}{-1}=-i$ (ii) $i^{17}=\left(i^2\right)^8 i=\left(-1\right)^8 i=i$

(iii) $1-i^{200}=1-\left(i^2\right)^{100}1-\left(-1\right)^{100}=1-1=0$

Example 2. Solve and write the real and imaginary part of the following:

(i) $20i$ (iii) $-1-5\sqrt{-1}$ (iii) $\left(1+\sqrt{-25}\right)-\left(-2+\sqrt{-9}\right)+\left(3-\sqrt{-4}\right)$

Solution : (i) $\because\ 20i=0+20i$

$\therefore$ Real part = 0 and Imaginary part = 0

(ii) $\because\ -1-5\sqrt{-1}=-1-5i$

$\therefore$ Real part = –1 and Imaginary part = –5

(iii) $\because\ \left(1+\sqrt{-25}\right)-\left(-2+\sqrt{-9}\right)+\left(3-\sqrt{-4}\right)$

$=\left(1+5i\right)-\left(-2+3i\right)+\left(3-2i\right)$

$=6+0i$

$\therefore$ Real part = 6 and Imaginary part = 0 .

Example 3. Find the additive and multiplicative inverse of the complex number $\left(2-3i\right)$.

Solution : Let $\left(a+ib\right)$ is the additive inverse of $\left(2-3i\right)$

$\Rightarrow$ $\left(2-3i\right)+\left(a+ib\right)=0+i0$

$\Rightarrow$ $\left(2+a\right)+\left(-3+b\right)i=0+i0$

$\Rightarrow$ $2+a=0$ and $-3+b=0$

$\Rightarrow$ $a=-2$ and $b=3$

$\therefore$ the additive inverse of $\left(2-3i\right)$ is $\left(-2+3i\right)$

multiplicative inverse of $\left(2-3i\right)=\frac{1}{2-3i}$

$$=\frac{1}{2-3i}\times\frac{2+3i}{2+3i}=\frac{2+3i}{4-9i^2}=\frac{2+3i}{4+9}=\frac{2}{13}+\frac{3}{13}i$$

Example 4. Express the following in the form $A+iB$.

(i) $\frac{2+\sqrt{-16}}{1-\sqrt{-25}}$ (ii) $1+i^7-15i^{16}+4i^{23}$

Solution : (i) $\frac{2+\sqrt{-16}}{1-\sqrt{-25}}=\frac{2+\sqrt{16}\sqrt{-1}}{1-\sqrt{25}\sqrt{-1}}=\frac{2+4i}{1-5i}=\frac{2+4i}{1-5i}\times\frac{1+5i}{1+5i}$

$$= \frac{2+10i+4i+20i^2}{1-25i^2} = \frac{2+14i+20(-1)}{1-25(-1)}$$

$$= \frac{-18+14i}{1+25} = \frac{-18}{26} + \frac{14}{26}i = \frac{-9}{13} + \frac{7}{13}i.$$

(ii) $1+i^7-15i^{16}+4i^{23} = 1+(i^2)^3 i - 15(i^2)^8 + 4(i^2)^{11} i$

$= 1+(-1)^3 i - 15(-1)^8 + 4(-1)^{11} i$ $\quad [\because i^2 = -1]$

$= 1 - i - 15 - 4i$

$= -14 - 5i$.

Example 5. Find the value of x and y if

$$(x+iy)(2+3i) = 1+8i.$$

Solution : $\because \quad (x+iy)(2+3i) = 1+8i$

$\Rightarrow \quad 2x+3xi+2yi+3yi^2 = 1+8i$

$\Rightarrow \quad (2x-3y)+(3x+2y)i = 1+8i$ $\quad [\because i^2 = -1]$

$\Rightarrow \quad 2x-3y=1$ एवं $3x+2y=8$

on solving $x=2$ and $y=1$.

Example 6. If $z_1 = 2+3i$ and $z_2 = 1-i$ then verify the following:

(i) $|z_1||z_2| = |z_1 z_2|$.

(ii) $|z_1+z_2| \le |z_1| + |z_2|$.

Solution : $z_1 = 2+3i$ and $z_2 = 1-i$

$\therefore \quad z_1 + z_2 = (2+1)+i(3-1) = 3+2i$

and $\quad z_1 \cdot z_2 = (2+3i)\cdot(1-i) = 2-2i+3i-3i^2 = 5+i$

now $\quad |z_1| = \sqrt{4+9} = \sqrt{13}$; $|z_2| = \sqrt{1+1} = \sqrt{2}$; $|z_1 z_2| = \sqrt{25+1} = \sqrt{26}$;

and $\quad |z_1+z_2| = \sqrt{9+4} = \sqrt{13}$

$\therefore$ (i) $|z_1|\cdot|z_2| = \sqrt{13}\cdot\sqrt{2} = \sqrt{26} = |z_1 z_2|$

(ii) $|z_1|+|z_2| = \sqrt{13}+\sqrt{2} > \sqrt{13} = |z_1+z_2|$

Example 7. If $|z|=1$ then Prove that $\frac{z-1}{z+1}$, $z \neq -1$ is purely imaginary

Solution : Let $z = x+iy$ then as per question $|z| = 1 \Rightarrow |z|^2 = 1 \Rightarrow x^2+y^2 = 1$

$$\frac{z-1}{z+1}=\frac{x+iy-1}{x+iy+1}=\frac{x-1+iy}{x+1+iy}\times\frac{x+1-iy}{x+1-iy}$$

$$=\frac{(x-1)(x+1)-iy(x-1)+iy(x+1)-i^2y^2}{(x+1)^2-i^2y^2}$$

$$=\frac{x^2-1-iy(x-1-x-1)-(-1)y^2}{(x+1)^2-(-1)y^2}$$

$$=\frac{x^2+y^2-1+2yi}{x^2+1+2x+y^2}=\frac{2yi}{2(1+x)} \qquad \left[\because\ x^2+y^2=1\right]$$

$$=\frac{y}{x+1}i \qquad \text{(purely imaginary)}$$

Exercise 5.1

1. Simplify the following

   (i) i^{52} (ii) $\sqrt{-2}\sqrt{-3}$ (iii) $(1+i)^5(1-i)^5$

2. Find the Additive and Multiplicative Inverse of the following

   (i) $1+2i$ (ii) $1/(3+4i)$ (iii) $(3+i)^2$

3. Find the complex conjugate of the complex number $\frac{(2+i)^3}{3+i}$

4. Find the Modulus of the following

   (i) $4+i$ (ii) $-2-3i$ (iii) $1/(3-2i)$

5. If $a^2+b^2=1$ then find the value of $\frac{1+b+ia}{1+b-ia}$

6. If $a=\cos\theta+i\sin\theta$ then find the value of $(1+a)/(1-a)$

7. Find the values of x, y satisfying the equation $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$

8. If z_1 and z_2 are two complex numbers then prove that

   $$|z_1+z_2|^2+|z_1-z_2|^2=2|z_1|^2+2|z_2|^2$$

9. If $a+ib=\frac{c+i}{c-i}$ where c is real, the prove that

   $$a^2+b^2=1 \text{ and } \frac{b}{a}=\frac{2c}{c^2-1}$$

10. If $(x+iy)^{1/3}=a+ib$ then prove that $\frac{x}{a}+\frac{y}{b}=4(a^2-b^2)$.

11. If $\frac{(x+i)^2}{3x+2}=a+ib$ then prove that $\frac{(x^2+1)^2}{(3x+2)^2}=a^2+b^2$.

5.15 Geometrical Reprsentation of Complex Numbers

(A) Cartesian Representation

We already know that corresponding to each ordered pair of real numbers (*x*, *y*), we get a unique point in the XY plane and vice-versa with reference to a set of mutually perpendicular lines known as the *x*-axis and the *y*-axis. The complex numbers *x* + *iy* which corresponds to the ordered pair (*x*, *y*) can be represented geometrically as the unique point P(*x*, *y*) in the XY plane and vice-versa. The plane having a complex number assigned to each of its point is called the complex plane or the Argand plane.

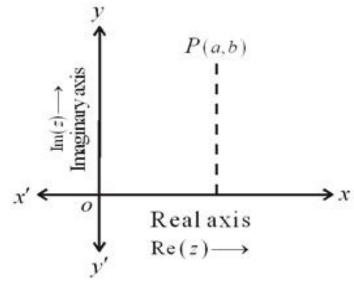

fig. 5.01

Some complex numbers such as $2+4i$, $-2+3i$, $0+1i$, $2+0i$, $-5-2i$ and $1-2i$ which correspond to the ordered pairs (2, 4), (–2, 3), (0, 1), (2, 0), (–5, –2) and (1, –2), respectively, have been represented geometrically by the points A, B, C, D, E and F, respectively.

(B) Polar Representation

Let the point P represent the nonzero complex number $z=a+ib\neq 0$. Let the directed line segment OP be of length *r* and θ be the angle which OP makes with the positive direction of *x*-axis. We may note that the point P is uniquely determined by the ordered pair of real numbers (r,θ), called the *polar coordinates of the point* P. We consider the origin as the pole and the positive direction of the *x*-axis as the initial line.

We have $a/r=\cos\theta$, $b/r=\sin\theta$ and therefore, $z=r\cos\theta+ir\sin\theta$. The letter is said to be the polar form of the complex number. Here $r=\sqrt{a^2+b^2}=|z|$, is the modulus of *z* and θ is called the argument (or amplitude) of *z* which is denoted by arg *z*.

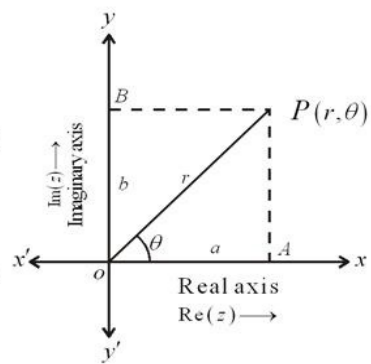

fig. 5.02

Note:

(i) For any complex number $z\neq 0$, therefore corresponds only one value of θ in $0\leq\theta<2\pi$. However, any other interval of length 2π, for example $-\pi<\theta\leq\pi$ S, can be such an interval. We shall take the value of, such that $-\pi<\theta\leq\pi$, called **principle argument** of *z* and is denoted by arg *z*, unless specified otherwise.

(ii) Formula to find the amplitude(arg) $\theta=\tan^{-1}(b/a)$.

(iii) The amplitude(arg) of *z* depends on the quadrant in which *z* lies

(a) If $a>0, b>0$ (first quadrant) then amplitude(arg) $z=\tan^{-1}\frac{b}{a}$

(b) If $a<0, b>0$ (second quadrant) then amplitude(arg) $z=\pi-\tan^{-1}|b/a|$

(c) If $a<0, b<0$ (third quadrant) then amplitude(arg) $z=-\pi+\tan^{-1}|b/a|$

(d) If $a>0, b<0$ (fourth quadrant) then amplitude(arg) $z=-\tan^{-1}|b/a|$

(iv) The argument of complex number 0 is not defined.

(v) Conditions of Complex number in the polar form $r(\cos\theta+i\sin\theta)$

(a) $1=\cos 0°+i\sin 0°$ (b) $-1=\cos\pi+i\sin\pi$

(c) $i=\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)$ (d) $-i=\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)$

Example : Represent the complex number $\frac{1+2i}{1-3i}$ in polar form and also find the modulus and argument.

Solution : Let $z=\frac{1+2i}{1-3i}=\frac{(1+2i)}{(1-3i)}\times\frac{(1+3i)}{(1+3i)}=\frac{-5+5i}{10}=-\frac{1}{2}+\frac{1}{2}i$

Let $z=-\frac{1}{2}+\frac{1}{2}i=r(\cos\theta+i\sin\theta)$ (Polar form)

then $r\cos\theta=-1/2$ (1)

$r\sin\theta=1/2$ (2)

squaring and adding (1) and (2)

$r^2=1/2 \quad\Rightarrow\quad r=1/\sqrt{2}$

dividing (2) and (1)

$\frac{r\sin\theta}{r\cos\theta}=\frac{(1/2)}{(-1/2)} \quad\Rightarrow\quad \tan\theta=-1$

$\because$ $(-1/2, 1/2)$ lies in II quadrant hence the agruement $z=\pi-\tan^{-1}|b/a|$ will be $z=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

$\therefore$ Polar form of $\frac{1+2i}{1-3i}$ $=\frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

modulus $z=r=\frac{1}{\sqrt{2}}$ and argument $z=\frac{3\pi}{4}$

5.16 Geometrical Representation of Sum of Two Complex Numbers

Let us denote the complex numbers $z_1=a_1+ib_1$ and $z_2=a_2+ib_2$ by the points P_1 and P_2 respectively constructing a parallelogram $OP_1\,PP_2$. The coordinates of P_1 and P_2 are (a_1, b_1) and (a_2, b_2). We know that the diagonals of parallelogram bisect each other. Taking M as the mid-point of P_1P_2, the coordinates will be $\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right)$.

As M is the mid-point of OP and O is the origin with coordinates (0, 0). Therefore the coordinates of P will be (a_1+a_2, b_1+b_2) . Point P will represent the sum z_1+z_2 of complex number.

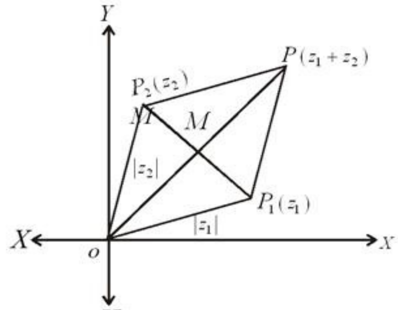

Fig. 5.03

Note: $\because\ OP \le OP_1 + P_1P \quad \Rightarrow \quad |z_1+z_2| \le |z_1| + |z_2|$

5.17 Geometrical Representation of Subtraction of Two Complex Numbers

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are two complex numbers. They are represented by the points P_1 and P_2 whose coordinates are (a_1, b_1) and (a_2, b_2) respectively P_2O is extended till P_2' so that $|OP_2| = |OP_2'|$, then P_2' denotes the complex number $(-z_2)$ with coordinates $(-a_2, -b_2)$, constructing a parallelogram OP_1RP_2' , the sum (z_1) and $(-z_2)$ is represented by R, therefore the coordinates of R will be (a_1-a_2, b_1-b_2) which will denote the *Difference* of z_1 and z_2 .

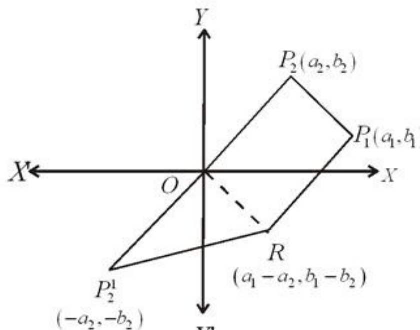

Fig. 5.04

5.18 Geometrical Representation of the Product of Two Complex Numbers

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. are two complex numbers, represented by the points P_1 and P_2

Let $\quad z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad$ and $\qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore \quad OP_1 = r_1, OP_2 = r_2$

$\angle XOP_1 = \theta_1$ and $\angle XOP_2 = \theta_2$

Let A be any point on x-axis such that $OA = 1$.

$\therefore \quad A$ point denotes $1 + i0$. Triangle OAP_1 and OP_2Q are similar, by property of similarity

$$\frac{OQ}{OP_2} = \frac{OP_1}{OA} \quad \Rightarrow \quad OQ = OP_1 \times OP_2$$
$$= r_1 r_2 \ (\because\ OA = 1)$$

$$\angle XOQ = \angle XOP_2 + \angle P_2OQ$$
$$= \angle XOP_2 + \angle XOP_1$$
$$= \angle XOP_1 + \angle XOP_2$$
$$= \theta_1 + \theta_2$$

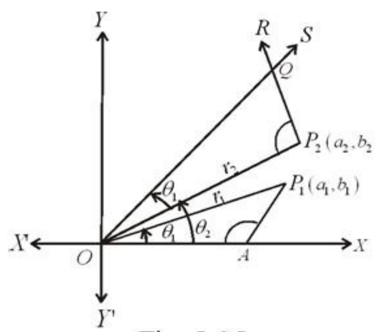

Fig. 5.05

Thus point Q denotes the product $r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$ i.e. $z_1 z_2$ of the two complex numbers.

Illustrative Examples

Example 8. Find the argument of the following complex numbers:-

(i) $1+20i$ (ii) $-7+i$ (iii) $1-i$ (iv) $-1-4i$

Solution : (i) $1 + 20\, i$ lies in first quadrant.

$\therefore$ argument $\theta = \tan^{-1}(b/a) = \tan^{-1}(20/1)$

(ii) $-7+i$ lies in second quadrant

$\therefore$ argument $\theta = \pi - \tan^{-1}|b/a| = \pi - \tan^{-1}(1/7)$

(iii) $1-i$ lies in fourth quadrant

$\therefore$ argument $\theta = -\tan^{-1}|b/a| = -\tan^{-1}|-1/1| = -\tan^{-1}1 = -\pi/4$

(iv) $-1-4i$ lies in third quadrant

$\therefore$ argument $\theta = -\pi + \tan^{-1}|b/a| = -\pi + \tan^{-1}|-4/-1| = -\pi + \tan^{-1}4$

Example 9. Express the following complex numbers in polar form.

(i) $\sqrt{3}+i$ (ii) $\frac{i}{3}-\frac{3}{i}$

(iii) $-\frac{3}{2}+i\frac{3\sqrt{3}}{2}$ (iv) $\sin\frac{\pi}{5}+i\left(1-\cos\frac{\pi}{5}\right)$

Solution : (i) Let $\sqrt{3}+i = r(\cos\theta + i\sin\theta)$

$\therefore$ $r\cos\theta = \sqrt{3}$ (1)

$\therefore$ $r\sin\theta = 1$ (2)

squaring and adding (1) and (2)

$$r^2(\cos^2\theta + \sin^2\theta) = 3+1 \Rightarrow r^2 = 4 \Rightarrow r = 2$$

dividing (1) and (2)

$\tan\theta = \frac{1}{\sqrt{3}}$ therefore $\theta = \frac{\pi}{6}$ ($\because$ $\sqrt{3}+i$ lies in first quadrant)

$\therefore$ Polar form of $\sqrt{3}+i$ will be $= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

(ii) $\frac{i}{3}-\frac{3}{i} = \frac{i}{3}-\frac{3i}{i^2} = \frac{i}{3}+3i = \frac{10}{3}i$

Let $\frac{10}{3}i = r(\cos\theta + i\sin\theta)$

$\therefore$ $r\cos\theta = 0$ and $r\sin\theta = \frac{10}{3}$

$\Rightarrow \quad r^2 = 0 + \frac{100}{9}$ i.e. $r = \frac{10}{3}$ and $\cos\theta = 0,\ \sin\theta = 1$ therefore $\theta = \frac{\pi}{2}$

$\therefore$ Polar form $\frac{i}{3} - \frac{3}{i} = \frac{10}{3}i$ will be $= \frac{10}{3}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

(iii) Let $-\frac{3}{2} + i\frac{3\sqrt{3}}{2} = r(\cos\theta + i\sin\theta)$

$\therefore \quad r\cos\theta = -\frac{3}{2}$ and $r\sin\theta = \frac{3\sqrt{3}}{2}$

squaring and adding $r^2 = \frac{9}{4} + \frac{27}{4} = \frac{36}{4} \quad \Rightarrow \quad r = 3$

$\because \quad -\frac{3}{2} + i\frac{3\sqrt{3}}{2}$ lies in the second quadrant therefore

argument $\theta = \pi - \tan^{-1}|b/a|$

$$= \pi - \tan^{-1}\left|\frac{3\sqrt{3}}{2} \Big/ -\frac{3}{2}\right| = \pi - \tan^{-1}\sqrt{3} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$\therefore$ Polar form of $-\frac{3}{2} + i\frac{3\sqrt{3}}{2}$ will be $= 3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

(iv) $\because \quad \sin\frac{\pi}{5} + i\left(1 - \cos\frac{\pi}{5}\right)$

$$= 2\sin\frac{\pi}{10}\cos\frac{\pi}{10} + i \cdot 2\sin^2\frac{\pi}{10}$$

$$= 2\sin\frac{\pi}{10}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$$

which is the required polar form whose modulus is $r = 2\sin\frac{\pi}{10}$ and argument $\theta = \frac{\pi}{10}$.

Example 10. If z_1 and z_2 are two non-zero complex numbers then prove that

(i) $\arg\ z_1 z_2 = \arg\ z_1 + \arg\ z_2$

(ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg\ z_1 - \arg\ z_2$

Solution : Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ here θ_1 and θ_2 are the arguments of z_1 and z_2

(i) $\quad z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$

$$= r_1 r_2 \left[(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2) \right]$$

$$= r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right]$$

$\therefore$ argument of $z_1 z_2 = \theta_1 + \theta_2 = \arg\ z_1 + \arg\ z_2$

(ii) $\left(\dfrac{z_1}{z_2}\right) = \dfrac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$

$$= \frac{r_1}{r_2} \frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2} \times \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$

$$= \frac{r_1}{r_2} \frac{\{(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2)\}}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2} \left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2) \right]$$

$\therefore$ argument of $z_1 / z_2 = \theta_1 - \theta_2 = \arg\ z_1 - \arg\ z_2$

Exercise 5.2

1. Find the argument of the following complex numbers

(i) $\dfrac{1+i}{1-i}$ (ii) $-1+\sqrt{3}\,i$ (iii) $\dfrac{5+i\sqrt{3}}{4-i2\sqrt{3}}$

2. Convert the following complex numbers into polar form.

(i) $\dfrac{1+i}{\sqrt{2}}$ (ii) $\dfrac{1+7i}{(2-i)^2}$ (iii) $\sin\dfrac{\pi}{3} + i\cos\dfrac{\pi}{3}$

3. If z_1 and z_2 are two non-zero complex numbers then prove that

are $z_1 \overline{z}_2 = \arg\ z_1 - \arg\ z_2$

5.19 To find the square root of a complex number $a+ib$

Let $\sqrt{a+ib} = x+iy$ then $(a+ib) = (x+iy)^2$

$\Rightarrow$ $(a+ib) = (x^2 - y^2) + (2xy)\,i$

equating the real and imaginary parts

$$x^2 - y^2 = a \quad (1)$$

and $$2xy = b \quad (2)$$

$\therefore$ $$x^2 + y^2 = \sqrt{(x^2 - y^2)^2 + 4x^2y^2} = \sqrt{a^2 + b^2} \quad (3)$$

adding (1) and (3)

$$2x^2 = a + \sqrt{a^2 + b^2}$$

$$\therefore \qquad x = \pm\sqrt{\left(\frac{a+\sqrt{a^2+b^2}}{2}\right)}$$

subtracting (3) from (1)

$$-2y^2 = a - \sqrt{a^2+b^2}$$

$$\therefore \qquad y = \pm\sqrt{\left(\frac{-a+\sqrt{a^2+b^2}}{2}\right)}$$

$$\therefore \qquad \sqrt{a+ib} = \pm\left[\sqrt{\left(\frac{a+\sqrt{a^2+b^2}}{2}\right)} + i\sqrt{\left(\frac{-a+\sqrt{a^2+b^2}}{2}\right)}\right]$$

5.20 Cube roots of unity:

Let $\sqrt[3]{1} = z,$ then $z^3 = 1$

$\Rightarrow \quad z^3 - 1 = 0$

$\Rightarrow \quad (z-1)(z^2+z+1) = 0$

$\Rightarrow \quad (z-1) = 0$ or $z^2+z+1=0$

$\Rightarrow \quad z = 1$ or $z = \dfrac{-1\pm\sqrt{(1-4)}}{2}$

$\Rightarrow \quad z = 1$ or $z = \dfrac{-1\pm\sqrt{3}i}{2}$

$\therefore$ Cube root of unity are $1, \dfrac{-1+i\sqrt{3}}{2}, \dfrac{-1-i\sqrt{3}}{2}$

thus there are three cube roots of unity out of which two are complex numbers

THEOREM - Prove that in the cube root of unity the complex roots are square of one other.

Proof : Let $\alpha = \dfrac{-1+i\sqrt{3}}{2}$ and $\beta = \dfrac{-1-i\sqrt{3}}{2}$ be the two complex roots

$$\therefore \qquad \alpha^2 = \left(\frac{-1+i\sqrt{3}}{2}\right)^2 = \frac{1-3-i2\sqrt{3}}{4} = \frac{-1-i\sqrt{3}}{2} = \beta$$

$$\beta^2 = \left(\frac{-1-i\sqrt{3}}{2}\right)^2 = \frac{1-3+i2\sqrt{3}}{4} = \frac{-1+i\sqrt{3}}{2} = \alpha$$

Hence proved

Note :

1. The cube roots of unity are represented by $1, \omega, \omega^2$
2. Properties - $(a)\omega^3 = 1$ and $(b) 1 + \omega + \omega^2 = 0$.

Illustrative Examples

Example 11: Find the cube root of $3 - 4i$.

Solution : Let $\sqrt{3-4i} = x + iy \Rightarrow 3 - 4i = x^2 - y^2 + 2ixy$

Comparing the real and imaginary parts

$$x^2 - y^2 = 3 \qquad (1)$$

and $$2xy = -4 \qquad (2)$$

$\therefore \quad x^2 + y^2 = \sqrt{\left(x^2 - y^2\right) + 4x^2y^2} = \sqrt{9+16} = 5$

adding (1) and (3) subtracting (3) from (1)

$2x^2 = 8 \Rightarrow x = \pm 2 \qquad 2y^2 = 2 \Rightarrow y = \pm 1$

$\therefore$ $2xy$ is negative thus taking x and y will be of opposite sign

$\therefore$ $x = 2$ and $y = -1$ and $x = -2$ then $y = 1$

$\therefore$ $\sqrt{3-4i} = \pm(2 - i)$

Example 12: Find the cube root of –27.

Solution : $-27 = (-3)(1)^{1/3} = -3, \ -3w, \ -3w^2$. $\quad \left[\because \ (1)^{1/3} = 1, \ \omega, \ \omega^2\right]$

Example 13: If $1, \ w, \ w^2$ are the cube root of unity then prove that $\left(1 - w + w^2\right)\left(1 + w - w^2\right) = 4$

Solution : $\left(1 - w + w^2\right)\left(1 + w - w^2\right) = \left(-w - w\right)\left(-w^2 - w^2\right) \quad \left[\because \ 1 + w + w^2 = 0\right]$

$$= \left(-2w\right)\left(-2w^2\right) = 4w^3 = 4(1) = 4.$$

Example 14: If $x = a + b, \ y = aw + bw^2$ and $z = aw^2 + bw$ then prove that $x^2 + y^2 + z^2 = 6ab$

Solution : $\because \quad x^2 = a^2 + 2ab + b^2, \qquad y^2 = a^2w^2 + 2abw^3 + b^2w^4 = a^2w^2 + 2ab + b^2w \qquad \left(\because \ w^3 = 1\right)$

and $z^2 = a^2w^4 + 2abw^3 + b^2w^2 = a^2w + 2ab + b^2w^2$

$\therefore \quad x^2 + y^2 + z^2 = a^2\left(1 + w^2 + w\right) + 6ab + b^2\left(1 + w + w^2\right)$

$= a^2(0) + 6ab + b^2(0) = 6ab \qquad \left(\because \ 1 + w + w^2 = 0\right)$

Example 15: Prove that

$\left(1 - w + w^2\right)\left(1 - w^2 + w^4\right)\left(1 - w^4 + w^8\right)...2n$ Factors $= 2^{2n}$

Solution : $\because \quad w^3 = 1$ and $1 + w + w^2 = 0$

$\therefore \quad \left(1 - w + w^2\right)\left(1 - w^2 + w^4\right)\left(1 - w^4 + w^8\right)...2n$ factors

$= \left(-w - w\right)\left(-w^2 - w^2\right)\left(-w - w\right)\left(-w^2 - w^2\right)...2n$ factors

$$= \{(-2w)(-2w^2)\}\{(-2w)(-2w^2)\}...n \text{ factors}$$

$$= (4w^3)(4w^3)...n \text{ factors}$$

$$= 4^n = 2^{2n}$$

Example 16: Solve $(x-1)^3 + 8 = 0$.

Solution : Let $x - 1 = y$ thus from the given equation

$$y^3 + 8 = 0 \quad \Rightarrow \quad y = (-8)^{1/3} = -2(1)^{1/3} = -2, -2w, -2w^2$$

$$\therefore \quad x - 1 = -2, -2\omega, -2\omega^2$$

$$\Rightarrow \quad x = 1-2, \ 1-2w, \ 1-2w^2$$

$$= -1, \quad 1-2w, \ 1-2w^2$$

Exercise 5.3

1. Find the square root of the following complex numbers

   (i) $-5+12i$ (ii) $8-6i$ (iii) $-i$

2. Find the value of $\sqrt{4+3\sqrt{-20}} + \sqrt{4-3\sqrt{-20}}$.

3. Find the cube root of

   (i) -216 (ii) -512.

4. Prove that

   (i) $1 + w^n + w^{2n} = 0$, where $n = 2, 4$

   (ii) $1 + w^n + w^{2n} = 3$, where n is a multiple of 3.

5. Prove that

   (i) $$\left(\frac{-1+\sqrt{-3}}{2}\right)^{29} + \left(\frac{-1-\sqrt{-3}}{2}\right)^{29} = -1$$

   (ii) $$(1+5w^2+w)(1+5w+w^2)(5+w+w^2) = 64.$$

6. If $1, w, w^2$ are the cube root of unity then prove that

   $$(1+w)(1+w^2)(1+w^4)(1+w^8)....2n \text{ factors} = 1$$

5.21 Quadratic Equation

We are already familiar with the quadratic equations and have solved them in the set of real numbers in the cases where discriminant is non-negative, i.e., ≥ 0.

Let us consider the following quadratic equation : $ax^2 + bx + c = 0$. We shall now study the various methods of finding the roots of quadratic equations.

(A) Vedic method of solving the equation $ax^2 + bx + c = 0$

To solve the equation by this method firstly the knowledge of differential is essential.

(i) First find the derivative of $ax^2+bx+c=0$, read it as D_1

Example : The derivative of x^2-5x+6 is $2x-5$, say $D_1=2x-5$

(ii) Find the discriminant i.e., $\pm\sqrt{(-5)^2-4\times1\times6}=\pm1$

(iii) Now put $D_1=\pm1$ and solve

$\therefore \quad 2x-5=\pm1 \quad \Rightarrow \quad x=2$ or 3

Example : (i) $7x^2-5x-2=0$

$$14x-5=\pm\sqrt{25+4\times7\times2}=\pm9 \quad \Rightarrow 14x=5\pm9 \Rightarrow x=1 \text{ or } x=-\frac{4}{14}$$

(ii) $ax^2+bx+c=0$

$2ax+b=\pm\sqrt{b^2-4ac}$ can be found

(B) Properties of quadratic equation $ax^2+bx+c=0$

(i) If $c=0$, then one root will be zero.
(ii) If $b=0$, then roots are equal but opposite sign.
(iii) If $b=c=0$, both the roots will be zero.
(iv) If $a=c$, roots will be reciprocal of each other.
(v) If $a=0$, one root will be infinity.
(vi) If $a=b=0$ both the roots are infinity.

Example : If the roots are equal then find the value of m : $x^2-10x+21=m$

Solution : For equal roots $\left(b^2=4ac\right) \Rightarrow (-10)^2=4\times1\times(21-m)\Rightarrow m=-4$

(C) Relation between the roots and Coefficients of quadratic equation $ax^2+bx+c=0$

Let the roots of $ax^2+bx+c=0$ be α and β then

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

sum $\alpha+\beta=-\dfrac{b}{a}$ i.e. $=-\dfrac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

product $\alpha\beta=\dfrac{c}{a}$ i.e. $==\dfrac{\text{Constant term}}{\text{Coefficient of } x^2}$

Note: If the coefficient of x^2 is 1 then the form of equation will be $x^2+\frac{b}{a}x+\frac{c}{a}=0$ i.e.

sum of the roots = – coefficient of x
product of the roots = constant term

(D) Formation of quadratic equation

(i) The required equation when $(x-\alpha)$ and $(x-\beta)$ are the factors then $(x-\alpha)(x-\beta)=0$ i.e.

$$x^2-(\alpha+\beta)x+\alpha\beta=0$$

$$x^2-(\text{sum of the roots})x+\text{product of the roots}=0.$$

Illustrative Examples

Example 17 : Find the solution using vedic method : $x^2+x+3=0$.

Solution : Here $D_1=2x+1$, Discriminant $=\pm\sqrt{1-12}=\pm\sqrt{11}\,i$

$\therefore \quad 2x+1=\pm\sqrt{11}\,i \quad$ therefore $x=\dfrac{-1\pm\sqrt{11}\,i}{2}$

Example 18 : Find the solution using Vedic Method : $2x^2-9ix-9=0$.

Solution : Here $D_1=4x-9i$ Discriminant $=\pm\sqrt{-81+4\times2\times9}=\pm3i$

$\therefore \quad 4x-9i=\pm3i \Rightarrow \quad x=3i$ or $\dfrac{3}{2}i$

Example 19 : Solve $x^2-2x+(-2+4i)=0$

Solution : Here $D_1=2x-2$ Discriminant $=\pm\sqrt{4-4\times1\times(-2+4i)}=\pm2\sqrt{3-4i}$

finding the square root of $\sqrt{3-4i}$

let the square root be $u+iv$

$\therefore \quad \sqrt{3-4i}=u+iv$

$\Rightarrow \quad 3-4i=u^2-v^2+2uvi$

$\therefore \quad u^2-v^2=3$ and $2uv=-4$

$\therefore \quad u^2+v^2=\sqrt{(u^2-v^2)^2+4u^2v^2} \Rightarrow u^2+v^2=\sqrt{9+16}=5$

$\therefore \quad u^2-v^2=3$ and $u^2+v^2=5$ we have $2u^2=8$ and $2v^2=2$

$\Rightarrow \quad u^2=4$ and $v^2=1$

$\Rightarrow \quad u=\pm2$ and $v=\pm1$

but $2uv=-4$ taking u and v with opposite sign

$\sqrt{3-4i}=\pm(2-i) \quad \therefore \quad D_1=2x-2=\pm2(2-i)$ [discriminant]

$\Rightarrow \quad x-1=\pm(2-i) \quad \Rightarrow \quad x=3-i$ या $-1+i$

Example 20 : Find the equation whose roots are reciprocal of the roots of the equation ax^2+bx+c .

Solution : Let the roots of $ax^2+bx+c=0$ be α and β

the roots of required equation will be $1/\alpha$ and $1/\beta$

$\therefore$ the required equation will be $x^2-\left(\dfrac{1}{\alpha}+\dfrac{1}{\beta}\right)x+\dfrac{1}{\alpha}\dfrac{1}{\beta}=0$

$$\Rightarrow \quad x^2 - \left(\frac{\alpha+\beta}{\alpha\beta}\right)x + \frac{1}{\alpha\beta} = 0$$

$$\Rightarrow \quad x^2 - \left(\frac{-b/a}{c/a}\right)x + \frac{1}{c/a} = 0 \qquad \text{[using (1) and (2)]}$$

$$\Rightarrow \quad \frac{c}{a}x^2 + \frac{b}{a}x + 1 = 0$$

$\Rightarrow \quad cx^2 + bx + a = 0$

Exercise 5.4

1. Find the solution using vedic method.

   (i) $x^2 + 4x + 13 = 0$ (ii) $2x^2 + 5x + 4 = 0$ (iii) $ix^2 + 4x - 15/2 = 0$.

2. Find the quadratic equation if the roots are

   (i) 5 and –2 (ii) $1 + 2i$.

3. If one roots of $x^2 - px + q = 0$ is double the other then prove that $2p^2 = 9q$.

4. Find the condition if the roots of $ax^2 + bx + c = 0$ are in the ratio $m : n$

Miscellaneous Exercise 5

1. The real and imaginary parts of complex number $\frac{1+i}{1-i}$ are

(A) 1, 1 (B) 0, 0 (C) 0, 1 (D) 1, 0

2. If $2 + (2a + 5ib) = 8 + 10i$ then

(A) $a = 2,\ b = 3$ (B) $a = 2,\ b = -3$ (C) $a = 3, b = 2$ (D) $a = 3,\ b = -2$

3. The multiplicative inverse of $3 - i$ is

(A) $\frac{3+i}{10}$ (B) $\frac{-3+i}{10}$ (C) $\frac{3-i}{10}$ (D) $\frac{-3-i}{10}$

4. The conjugate of $\frac{2-3i}{4+i}$

(A) $\frac{-5+14i}{17}$ (B) $\frac{5+14i}{17}$ (C) $\frac{14+5i}{17}$ (D) $\frac{14-5i}{17}$

5. If $z_1, z_2 \in C$ then the correct statement is

(A) $|z_1 - z_2| \geq |z_1| + |z_2|$ (B) $|z_1 + z_2| \leq |z_1 - z_2|$

(C) $|z_1 + z_2| \geq |z_1 - z_2|$ (D) $|z_1 - z_2| \leq |z_1| + |z_2|$

6. If $|z - 3| = |z + 3|$ then z lies at

(A) $x-$ axis (B) $y-$ axis (C) on line $x=y$ (D) on line $x=-y$

7. Find the argument (θ) of –2.

8. Convert $\frac{5\sqrt{3}}{2}+\frac{5}{2}i$ into polar form.

9. Find the value of $4+5w^4+3w^5$.

10. Write $\frac{1}{1-\cos\theta+i\sin\theta}$ in the form of $a+ib$

11. Find the number of non-zero integral solution of $|1-i|^x=2^x$

12. If $z_1, z_2 \in C$ then prove that

(i) $|z_1-z_2|\le|z_1|+|z_2|$ (ii) $|z_1+z_2|\ge|z_1|-|z_2|$

(iii) $|z_1+z_2+\ldots+z_n|\le|z_1|+|z_2|+\ldots+|z_n|$

13. If $|z_1|=1=|z_2|$ then prove that $|z_1+z_2|=\left|\frac{1}{z_1}+\frac{1}{z_2}\right|$

14. If $\frac{(a+i)^2}{2a-i}=p+iq$ then prove that $p^2+q^2=\frac{(a^2+1)^2}{4a^2+1}$.

15. If $|z_1|=|z_2|$ and arg $z_1+\arg z_2=0$ then prove that $z_1=\overline{z}_2$

16. If θ_1, θ_2 are the arguments of z_1, z_2 then prove that

$$z_1\overline{z}_2+\overline{z}_1z_2=2|z_1||z_2|\cos(\theta_1-\theta_2)$$

17. Prove that:-

(i) $(a+bw+cw^2)(a+bw^2+cw)=a^2+b^2+c^2-ab-bc-ca$

(ii) $(a+b+c)(a+bw+cw^2)(a+bw^2+cw)=a^3+b^3+c^3-3abc$

18. If α, β are the two complex numbers and $|\beta|=1$ then prove that $\left|\frac{\beta-\alpha}{1-\overline{\alpha}\beta}\right|=1$.

19. If α and β are the roots fo equation $px^2-qx-r=0$ then find the equation whose roots are $1/\alpha$ and $1/\beta$

20. Find the relation when one roots of $\ell x^2-2mx+n=0$ is P times the other.

Important Point

1. $i=\sqrt{-1}$

2. $z=a+ib$ is complex number where $a,b\in R$. also written as (a,b)

Real z $=\text{Re}(z)=a$ and Imaginary $z=\text{Im}(z)=b$.

3. $z = a + ib = 0 \Rightarrow a = 0$ and $b = 0$

4. Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ then $z_1 = z_2 \Rightarrow a_1 = a_2$ and $b_1 = b_2$

5. The set of complex number is denoted by C i.e.

   $C = \{a + ib;\ a, b \in R\}$ or $C = \{(a, b);\ a, b \in R\}$

6. Following are the properties related to complex number

   (i) addition (ii) Difference (iii) product (iv) division

7. Properties of Complex numbers:

   (i) Closure law $z_1, z_2 \in C \Rightarrow z_1 + z_2 \in C$ and $z_1 \cdot z_2 \in C$

   (ii) Associative law $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

   and $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3,\ \forall\ z_1, z_2, z_3 \in C$

   (iii) Additive identity $= 0 + i\,0$,

   Multiplicative identity $= 1 + i\,0 = 1$

   (iv) Additive inverse $z\ = -z$ and multiplicative inverse of $z = a + ib$ is

   $$\frac{a}{a^2 + b^2} + \frac{i(-b)}{a^2 + b^2} \quad [z \neq 0]$$

   (v) Commutative law $z_1 + z_2 = z_2 + z_1$ and $z_1 \cdot z_2 = z_2 \cdot z_1\ \forall\ z_1, z_2 \in C$

   (vi) Cancellation law $z_1 + z_3 = z_2 + z_3 \Rightarrow z_1 = z_2;\quad z_3 + z_1 = z_3 + z_2 \Rightarrow z_1 = z_2$

   $z_1 \cdot z_3 = z_2 \cdot z_3 \Rightarrow z_1 = z_2;\quad z_3 \cdot z_1 = z_3 \cdot z_2 \Rightarrow z_1 = z_2\ [z_3 \neq 0]$

8. The conjugate of $z = x + iy$ is given by $\bar{z} = x - iy$.

9. Properties of conjugate

   (i) $z + \bar{z} = 2\,\text{Re}(z)$ and $z - \bar{z} = 2\,\text{Im}(z)$

   (ii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ and $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

   (iii) $\overline{(\bar{z})} = z,\ \overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2,\ \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2},\ z_2 \neq 0$

10. If $z = a + ib$ then $|z| = |\bar{z}| = \sqrt{a^2 + b^2}$

11. $z\bar{z} = |z|^2 \quad \therefore \quad z^{-1} = \frac{\bar{z}}{|z|^2}$

12. A polynomial equation of n degree has n roots.

13. The Polar form of complex number is $z = a + ib = r(\cos\theta + i\sin\theta)$ where $r = |z|$ and argument $z = \theta$.

14. The principle argument θ takes teh condition $-\pi < \theta \leq \pi$

15. The square root of $a + ib$ is

   $$\sqrt{a + ib} = \pm\left[\left(\sqrt{a^2 + b^2} + a\right)/2\right]^{1/2} \pm i\left[\left(\sqrt{a^2 + b^2} - a\right)/2\right]^{1/2}$$

16. For the quadratic equation $ax^2 + bx + c = 0$. Sum of the roots $= -\frac{b}{a}$ and product of the roots $= \frac{c}{a}$

17. If the roots are α and β, then the equation will be $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

Answers

Exercise 5.1

1. (i) 1, (ii) $-\sqrt{6}$, (iii) 32 **2.** (i) $-1-2i,\ \frac{1}{5}-\frac{2}{5}i$, (ii) $-\frac{3}{25}+\frac{4}{25}i,\ 3+4i$, (iii) $-8-6i,\ \frac{2}{25}-\frac{3}{50}i$

3. $\frac{17}{10}-\frac{31}{10}i$ **4.** (i) $\sqrt{17}$, (ii) $\sqrt{13}$, (iii) $\frac{1}{\sqrt{13}}$ **5.** $b+ia$ **6.** $i\cot\theta/2$ **7.** $x=3,\ y=-1$

Exercise 5.2

1. (i) $\frac{\pi}{2}$ (ii) $\frac{2\pi}{3}$ (iii) $\frac{\pi}{3}$ **2.** (i) $\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}$ (ii) $\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ (iii) $\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$

Exercise 5.3

1. (i) $\pm(2+3i)$ (ii) $\pm(3-i)$ (iii) $\pm\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)$

2. $\pm 6,\ \pm 2\sqrt{5}\,i$ **3.** (i) $-6,\ -6w,\ -6w^2$ (ii) $-8,\ -8w,\ -8w^2$

Exercise 5.4

1. (i) $-2\pm 3i$ (ii) $\frac{-5\pm\sqrt{7}\,i}{4}$ (iii) $\frac{3-i}{2},\frac{-3+9i}{2}$

2. (i) $x^2-3x-10=0$ (ii) $x^2-2x+5=0$ **4.** $mnb^2=ac(m+n)^2$

Miscellaneous Exercise 5

1. C **2.** C **3.** A **4.** B **5.** D **6.** B

7. π **8.** $5\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ **9.** $\sqrt{3}\,i$ **10.** $\frac{1}{2}-\frac{i}{2}\cot\frac{\theta}{2}$

11. zero **19.** $rx^2+qx-p=0$ **20.** $4m^2p=\ell n(1+p)^2$