Chapter 2

Laplace Transform

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- The region of convergence
- Poles and zeroes
- Properties of ROC
- Laplace transforms of some elementary signals
- Inverse laplace transform
- Unilateral laplace transform
- The system function
- Stability
- Causal and stable system
- Invertibility
- Systems interconnections

Laplace Transform

A continuous time LTI system with impulse response $h(t)$, the output $y(t)$ of the system to the complex exponential input of the form e^{st} is

$$y(t) = h(t) * e^{st}$$

$$= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau$$

$$= e^{st}\int_{-\infty}^{\infty} h(\tau)\cdot e^{-s\tau}\cdot d\tau$$

$$y(t) = T\{e^{st}\} = H(s)e^{st}$$

where, $H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$

The function $H(s)$ in above equation is referred to as the Laplace transform of $h(t)$. For a general continuous-time signal $x(t)$, the Laplace transform $X(s)$ is defined as,

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \text{ (bilateral Laplace transform)}$$

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt \text{ (unilateral Laplace transform)}$$

Both unilateral and bilateral transform are same if $x(t) = 0$, for $t < 0$.

Symbolically, the transform will be represented by

$$X(s) = L\{x(t)\}.$$

The Region of Convergence

The range of values of the complex variable 's' for which the Laplace transform converges is called the region of convergence (ROC).

Consider the signal $x(t) = e^{-at}u(t)$ then Laplace transform

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt$$

$$= \int_{0}^{\infty} e^{-(s+a)t}dt$$

$$= \left|\frac{-1}{s+a}e^{-(s+a)t}\right|_0^{\infty} = \frac{1}{s+a}, \text{ Re}(s) > -a,$$

because $\underset{t\to\infty}{\text{Lt}}\, e^{-(s+a)t} = 0$, when $\text{Re}(s + a) > 0$ or $\text{Re}(s) > -a$ is the ROC.

The convenient way to display the ROC is shown in the Figure 1, the variable 's' is a complex number, and in the figure, we display the complex plane, generally referred to as the s-plane, the horizontal and vertical axes are sometimes referred to as $\text{Re}\{s\}$ or σ-axis and the $\text{Im}\{s\}$- or $j\omega$-axis, respectively.

The shaded region in the figure represents the set of points in the s-plane corresponding to the ROC.

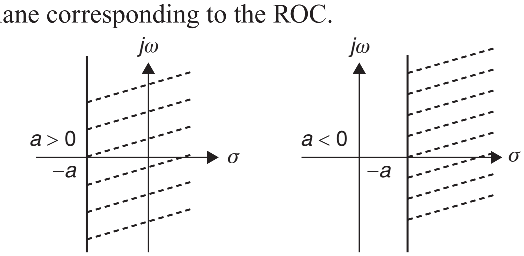


Figure 1 ROC for $X(s)$

The ROC of $X(s)$ is $\text{Re}(s) > -a$. This fact means that the integral defining $X(s)$, exists, only for the values of s in the shaded region, for other values of s, the integral does not converge. For this reason, the shaded region is called ROC or region of existence of $X(s)$.

Poles and Zeroes of $X(s)$

Usually, $X(s)$ will be a rational function in s, that is $X(s)=\frac{a_0}{b_0}\frac{(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}$, where m, n are positive integers, $X(s)$ is proper rational function if $n > m$, the roots of numerator polynomial z_k are called the zeroes of $X(s)$, as $X(s) = 0$ for those values. Similarly, the roots of denominator p_k are called the poles of $X(s)$ as for those values $X(s)$ is infinite.

Therefore, the poles of $X(s)$ lie outside of ROC, since $X(s)$ does not converge at the poles, but zeroes may lie inside or outside of ROC.

Properties of ROC

1. The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the s-plane and ROC does not contain any poles.
2. If $x(t)$ is finite-duration signal and absolutely integrable, i.e.,

$$x(t) \neq 0,\ t_1 \leq t \leq t_2 = 0$$

otherwise, ROC is the entire s-plane except possibly $s = 0$ or $s = \infty$.

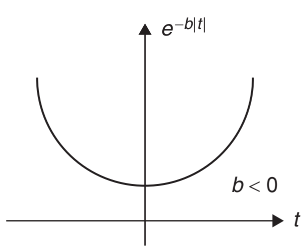
3. If $x(t)$ is a right-sided signal, that is $x(t) = 0$ for $t < t_1 < \infty$, then ROC is of the form $\text{Re}(s) > \sigma_{max}$, where σ_{max} equals the maximum real part of any poles of $X(s)$, thus ROC is to the right of all the poles of $X(s)$.
4. If $x(t)$ is a left-sided signal, that is $x(t) = 0$ for $t > t_2 > -\infty$, then ROC is of the form $\text{Re}(s) < \sigma_{min}$, where σ_{min} is the minimum real part of any of the poles of $X(s)$, thus ROC is to the left of all the poles of $X(s)$.
5. If $x(t)$ is a two-sided signal, that is $x(t)$ is an infinite-duration signal, then ROC is of the form $\sigma_1 < \text{Re}(s) < \sigma_2$ where σ_1, σ_2 are real parts of the two poles of $X(s)$, thus ROC is a vertical strip in the s-plane between σ_1 and σ_2.

Example 1: Find the Laplace transform of $x(t) = e^{-b|t|}$.

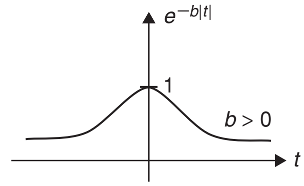
Solution: $x(t)$ be expressed as

$$X(t) = e^{-bt}u(t) + e^{+bt}u(-t)$$

So, the Laplace transform of $e^{-bt}u(t) \leftrightarrow \frac{1}{s+b}$, $\text{Re}\{s\} > -b$ and for $e^{+bt}u(-t) \leftrightarrow \frac{-1}{s-b}$, $\text{Re}\{s\} < +b$. If $b \leq 0$, then there is no common region of ROC. Thus, for those values of b, $x(t)$ has no Laplace transform.



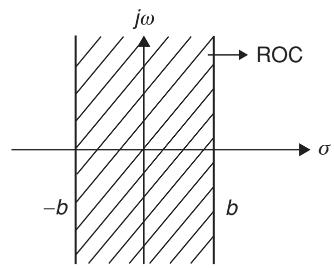
For $b < 0$, the signal is also not finite integrable, so no Laplace transform exists.



For $b > 0$, the signal is integrable, and there is common ROC is $-b < \text{Re}\{s\} < +b$

$$e^{-b|t|} \xleftrightarrow{L} \frac{1}{s+b} - \frac{1}{s-b} = \frac{-2b}{s^2-b^2},\ -b < \text{Re}\{s\} < +b$$

The corresponding pole zero plot is



Laplace Transforms of Some Elementary Signals

1. Unit impulse function $\delta(t)$

$$L(\delta(t)] = \int_{-\infty}^{\infty} \delta(t)dt = 1, \text{ all } s$$

$$L\{\delta(t-T)\} = e^{-sT}, \text{ all } s$$

2. Unit step function $u(t)$

$$L[u(t)] = \int_{-\infty}^{\infty} u(t)e^{-st}dt = \int_{0}^{\infty} e^{-st}dt$$

$$= \frac{1}{s}e^{-st}\Big|_0^{\infty} = \frac{1}{s} \text{ Re}(s) > 0$$

3. Some other Laplace transform pairs

$$L\{-u\ (-t)\} = \frac{1}{s},\ \text{Re}(s) < 0$$

$$L\{tu\ (t)\} = \frac{1}{s^2},\ \text{Re}(s) > 0$$

$$L\{t^k u\ (t)\} = \frac{k!}{s^{k+1}},\ \text{Re}(s) > 0$$

x(t)	x(s)	ROC
$e^{-at}u(t)$	$\frac{1}{s+a}$	$Re(s) > -a$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$Re(s) < -a$

(Continued)

$x(t)$	$x(s)$	ROC
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$Re(s) > -a$
$-te^{-at}u(-t)$	$\frac{1}{(s+a)^2}$	$Re(s) < -a$
$\cos\omega_0 t \cdot u(t)$	$\frac{s}{s^2+\omega_0^2}$	$Re(s) > 0$
$\sin\omega_0 t \cdot u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$Re(s) > 0$
$e^{-at}\cos\omega_0 t \cdot u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$Re(s) > -a$
$e^{-at}\sin\omega_0 t \cdot u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$Re(s) > -a$
$t^k e^{-at}u(t)$	$\frac{k!}{(s+a)^{k+1}}$	$Re(s) > -a$
$\frac{d^n}{dt^n}\delta(n)$	s^n	All s
$u(t) * u(t) * \ldots$ n times	$1/s^n$	$Re\{s\} > 0$

Properties of Laplace Transform

1. **Linearity:** Laplace transform is a linear operator, It holds the principle of super position.

If $\quad L\{x_1(t)\} = X_1(s)$, ROC $= R_1$

$\quad L\{x_2(t)\} = X_2(s)$, ROC $= R_2$

$$L\{a_1x_1(t) + a_2x_2(t)\} = a_1X_1(s) + a_2X_2(s),\ \text{ROC} = R_1 \cap R_2$$

2. **Time shifting:** The time shifting property states that delaying a signal by t_0 seconds, amounts to multiplying its transform with e^{-st_0}.

$$L\{x(t)\} = X(s),\ \text{ROC} = R$$

$$L\{x(t-t_0)\} = e^{-st_0}X(s),\ \text{ROC} = R$$

3. **Shifting in s-domain:** Shifting in s-domain (or) frequency shifting property states that frequency shift by s_0 is same as multiplying its inverse transform with e^{sot}.

$$e^{s_0 t}x(t) \leftrightarrow x(s-s_0),\ \text{ROC} = R + \text{Re}(s_0)$$

If the Laplace transform of $x(t)$ has a pole or zero at $s = a$, then the Laplace transform of $e^{j\omega_0 t}x(t)$ has a pole or zero at $s = a + j\omega_0$.

4. **Time scaling**

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{s}{a}\right),\ \text{ROC} = R/\text{a}$$

x(at) is the signal $x(t)$, time compressed by the factor a, and $X\left(\frac{s}{a}\right)$ is $X(s)$ expanded along the s-scale by the same factor a.

The scaling property states that time compression of a signal by a factor 'a' causes expansion of its Laplace transform in the s-scale by the same factor.

Similarly, time expansion $x(t)$ causes compression of $X(s)$ in the s-scale by the same factor.

5. **Time reversal**

$$x(-t) \leftrightarrow X(-s),\ \text{ROC} = -R$$

6. **Differentiation in time-domain**

$$\frac{d}{dt}[x(t)] \leftrightarrow sX(s) - x(0^-),\ \text{ROC} \supset R$$

Repeated application of this property yields

$$\frac{d^2x(t)}{dt^2} \leftrightarrow s^2X(s) - sx(0^-) - \frac{d}{dt}x(0^-)$$

$$\frac{d^nx(t)}{dt^n} \leftrightarrow s^nX(s) - \sum_{k=1}^{n} s^{n-k}\frac{d^{k-1}x(0^-)}{dt^{k-1}}$$

7. **Differentiation in the s-domain:** The dual of the time-differentiation property is the frequency-differentiation property.

$$-tx(t) \leftrightarrow \frac{d}{ds}X(s),\ R^1 = R$$

8. **Integration in the time domain**

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(t)dt,$$

ROC $= R \cap \{\text{Re}(s) > 0\}$

9. **Time convolution**

$$x_1(t) * x_2(t) \leftrightarrow X_1(s) \cdot X_2(s),\ \text{ROC} \supset R_1 \cap R_2.$$

We can apply the time-convolution property to the LTI system input–output relationship $y(t) = x(t) * h(t)$ to obtain $Y(s) = X(s)\,H(s)$.

$$H(s) = \frac{Y(s)}{X(s)} = \text{transfer function } H(s).$$

10. **Time differentiation**

$$\frac{d}{dt}x(t) \leftrightarrow sX(s) - x(0^-)$$

$$\frac{d^2}{dt^2}x(t) \leftrightarrow s^2X(s) - sx(0^-) - x'(0^-)$$

$$\frac{d^n}{dt^n}x(t) \leftrightarrow s^nX(s) - s^{n-1}x(0^-) - s^{n-2}x'(0^-) - \cdots - SX^{(n-2)}(0) - X^{n-1}(0^-)$$

11. **Frequency integration:** The dual of the time-integration property is the frequency-integration property.

$$\frac{x(t)}{t} \leftrightarrow \int_{s}^{\infty} X(p)dp$$

12. Frequency convolution

$$x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi j}[X_1(s) * X_2(s)]$$

13. Initial value theorem

$$x(0^+) = \underset{s\to\infty}{\text{Lt}}\ sX(s)$$

14. Final value theorem

$$\underset{s\to 0}{\text{Lt}}\ sX(s) = x(\infty)$$

It is desirable to know the values of $x(t)$ as $t \to 0$ and $t \to \infty$ (initial and final values of $x(t)$) from the knowledge of its Laplace transform $X(s)$, then initial and final value theorems provide such information.

15. Conjugation property: $x(t) \leftrightarrow X(s)$ with ROC $= R$, then conjugate of $x(t)$, $x^*(t) \leftrightarrow X^*(s^*)$ with ROC $= R$. When $x(t)$ is real, $x(t) = x^*(t)$, so

$$X(s) = X^*(s^*).$$

Consequently if $x(t)$ is real, and if $X(s)$ has a pole/zero at $s = s_0$, then $X(s)$ also has pole or zero at the complex point $s = s^*_0$.

Inverse Laplace Transform

$$x(t) \leftrightarrow X(s)$$

$$x(t) = L^{-1}\{X(s)\}$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$$

In this integral, the real 'σ' is to be selected such that if the ROC of $X(s)$ is $\sigma_1 < \text{Re}(s) < \sigma_2$, then $\sigma_1 < \sigma < \sigma_2$.

Unilateral Laplace Transform

The Laplace transforms for the signals $e^{-at}u(t)$ and $-e^{-at}u(-t)$ are identical except for their regions of convergence. Therefore, for a given $X(s)$, there may be more than one inverse transform, depending on the ROC. If we restrict all our signals to the causal type, such an ambiguity does not arise. There is only one inverse transform of $X(s) = \dfrac{1}{(s+a)}$, namely $e^{-at}u(t)$. To find $x(t)$ from $X(s)$, we need not even specify ROC. If all signals are restricted to the causal type, then for a given $X(s)$, there is only one inverse Laplace transform $x(t)$.

The unilateral Laplace transform is a special case of the bilateral Laplace transform in which all signals are restricted to being causal.

The unilateral Laplace transform $X(s)$ of $x(t)$ is $X(s) = \int_0^\infty x(t)e^{-st}\,dt.$

Example 1: What is the Laplace transform of the signal $x(t) = e^{-3t}\,u(t) + e^{-t}(\cos 2t)\,u(t)$?

Solution: $x(t) = \{e^{-3t} + \frac{1}{2}(e^{2jt} + e^{-2jt})\cdot e^{-t}\}u(t)$

$$= e^{-3t}u(t) + \frac{1}{2}e^{(-1+2j)t}u(t) + \frac{1}{2}\cdot e^{(-1-2j)t}u(t)$$

By taking Laplace transform

$$X(s) = \frac{1}{s+3} + \frac{1}{2}\cdot\frac{1}{s+(1+2j)} + \frac{1}{2}\cdot\frac{1}{s+(1-2j)}$$ and the ROC is $\text{Re}\{s\} > -3$ for first term, for second and third terms $\text{Re}\{s\} > -1$.

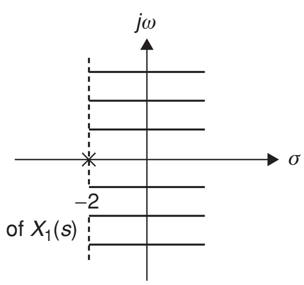
For all three Laplace transforms to converge simultaneously, we must have $\text{Re}\{s\} > -1$

$$\text{Consequently, } X(s) = \frac{1}{s+3} + \frac{1}{2}\cdot\frac{2s+2}{s^2+2s+5}$$

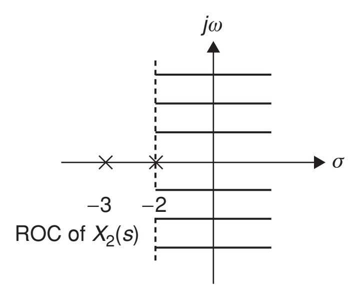
$$= \frac{1}{s+3} + \frac{s+1}{s^2+2s+5},\ \text{Re}\{s\} > -1$$

Example 2: Consider $X_1(s) = \dfrac{1}{s+2}$ $\text{Re}\{s\} > -2$ and $X_2(s) = \dfrac{1}{(s+2)(s+3)}$, $\text{Re}\{s\} > -2$. Then what is the ROC of $X(s) = X_1(s) - X_2(s)$?

Solution: $X_1(s) = \dfrac{1}{s+2}, \text{Re}\{s\} > -2$



$$X_2(s) = \frac{1}{(s+2)(s+3)},\quad \text{Re}\{s\} > -2$$



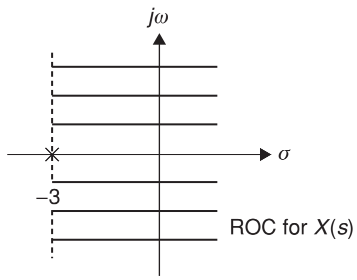
$$X(s) = X_1(s) - X_2(s)$$

$$= \frac{1}{s+2} - \frac{1}{(s+2)(s+3)} = \frac{s+3-1}{(s+2)(s+3)}$$

$$= \frac{(s+2)}{(s+2)(s+3)} = \frac{1}{s+3},\quad \text{Re}\{s\} > -3$$

Thus, in the linear combination of $X_1(s)$ and $X_2(s)$, the pole at $s = -2$ is cancelled by a zero at $s = -2$. The intersection of the ROC for $X_1(s)$ and $X_2(s)$ is $\text{Re}\{s\} > -2$.

However, since the ROC is always bounded by a pole or infinity, for this example, the ROC of $X(s)$ can be extended to the left to be bounded by the pole at $s = -3$, as a result of the pole-zero cancellation at $s = -2$.



Example 3: What is the inverse Laplace transform of $X(s) = \dfrac{2s^2+5}{s^2+3s+2}$?

Solution: $X(s) = \dfrac{2s^2+5}{s^2+3s+2} = 2 + \dfrac{A}{(s+1)} + \dfrac{B}{(s+2)}$

where, $A = (s+1)X(s)\big|_{s=-1} = \dfrac{2+5}{-1+2} = 7$

$$B = (s+2)X(s)\big|_{s=-2} = \frac{8+5}{-2+1} = -13$$

So, $X(s) = 2 + \dfrac{7}{s+1} - \dfrac{13}{s+2}$

By inverse Laplace transform

$$x(t) = 2\delta(t) + 7e^{-t}u(t) - 13e^{-2t}u(t)$$

Example 4: Find the inverse Laplace transform of $X(s) = \dfrac{s+2+4e^{-3s}}{(s+3)(s+1)}$.

Solution: Observe the exponential term e^{-3s} in numerator of $X(s)$, indicating time delay.

$$X(s) = \frac{s+2}{(s+3)(s+1)} + \frac{4e^{-3s}}{(s+3)(s+1)}$$
$$= X_1(s) + X_2(s)\, e^{-3s}$$

So, $X_1(s) = \dfrac{s+2}{(s+1)(s+3)} = \dfrac{1}{2}\dfrac{1}{s+1} + \dfrac{1}{2}\dfrac{1}{s+3}$

By inverse Laplace transform

$$x_1(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

Similarly, $X_2(s) = \dfrac{4}{(s+1)(s+3)} = \dfrac{2}{s+1} - \dfrac{2}{s+3}$

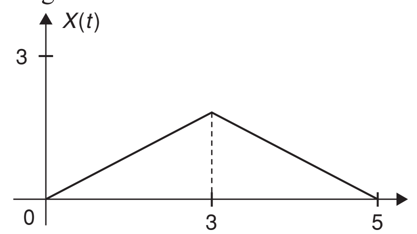
By inverse Laplace transform

$$x_2(t) = 2e^{-t}u(t) - 2e^{-3t}u(t)$$
$$X(s) = X_1(s) + X_2(s)\, e^{-3s}$$

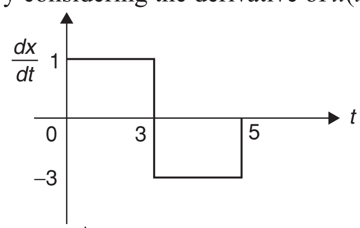
So, $x(t) = x_1(t) + x_2(t-3)$

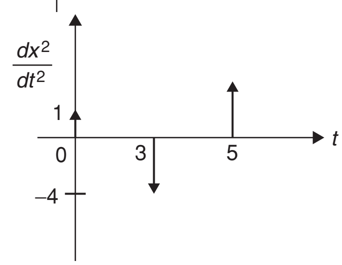
$$= \frac{1}{2}[e^{-t} + e^{-3t}]u(t) + 2[e^{-(t-3)} - e^{-3(t-3)}]u(t-3)$$

Example 5: Find the Laplace transform of the signal $x(t)$ depicted in figure.



Solution: By considering the derivative of $x(t)$





$$\Rightarrow \frac{d^2x}{dt^2} = \delta(t) - 4\delta(t-3) + 3\delta(t-5)$$

By applying Laplace transform

$$L\left\{\frac{d^2x}{dt^2}\right\} = s^2 X(s) = 1 - 4e^{-3s} + 3e^{-5s}$$

$$X(s) = \frac{1}{s^2}(1 - 4e^{-3s} + 3e^{-5s})$$

Analysis of Continuous Time LTI System

The System Function

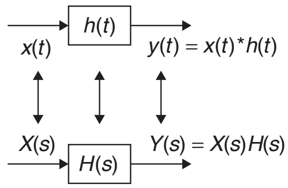
The output of continuous time LTI system equals to the convolution of the input $x(t)$ with the impulse response $h(t)$ that is

$$y(t) = x(t) * h(t)$$

By applying Laplace transform

$$Y(s) = X(s) \cdot H(s)$$

and $$H(s) = \frac{Y(s)}{X(s)}$$



The Laplace transform $H(s)$ of $h(t)$ is referred to as the system function (or the transfer function) of the system.

Characteristics of Continuous Time LTI System

Many properties of continuous time LTI systems can be closely associated with the characteristics of $H(s)$ in s-plane and ROC.

Causality

For causal continuous time LTI system, we have $h(t) = 0$, $t < 0$ Since h(t) is right handed signal, the corresponding $H(s)$, will have ROC $\text{Re}(s) > \sigma_{\max}$.

In other words, for a system with a rational system function, causality of the system is equivalent to the ROC being the right-half plane, to the right of the right most pole.

Similarly, for Anti causal $h(t) = 0$, $t > 0$ and $H(s)$ ROC will be $\text{Re}(s) < \sigma_{\min}$ (left-half plane).

Stability

A continuous time LTI system is BIBO stable if $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty.$

The corresponding $H(s)$, ROC contains the $j\omega$-axis or an LTI system is stable if and only if the ROC of its system function $H(s)$ includes the $j\omega$-axis [i.e., $\text{Re}\{s\} = 0$].

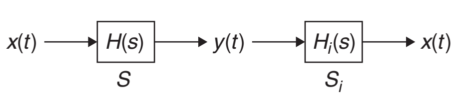
Causal and Stable System

If the system is both causal and stable, then the poles of $H(s)$ must lie in the left half of s-plane.

Since ROC is of form $\text{Re}(s) > \sigma_{\max,}$ and $j\omega$-axis is included in ROC, we must have $\sigma_{\max} < 0$. In other terms, a causal system with rational system function $H(s)$ is stable if and only if all of the poles of $H(s)$ lie in the left-half of the s-plane, i.e., all of the poles have negative real parts.

Invertibility

If $H(s)$ is the transfer function of a system S, then S_i, its inverse system has a transfer function. $H_i(s)$ given by $H_i(s) = \dfrac{1}{H(s)}$



The cascade of systems S and S_i is an identity system with impulse response $\delta(t)$.

Implying $H(s) \cdot H_i(s) = 1$

Consider an ideal differentiator system $y(t) = \dfrac{d}{dt}x(t)$ its Laplace transform $Y(s) = s\,X(s)$ (for causal system $x(0^-) = 0$).

So, $H(s) = \dfrac{Y(s)}{X(s)} = s$ is transfer function.

For an ideal integrator system with zero initial state

$$y(t) = \int_0^t x(\tau)d\tau \to Y(s) = \frac{1}{s}X(s)$$

Transfer function $H(s) = \dfrac{Y(s)}{X(s)} = \dfrac{1}{s}$

From the above discussion, an ideal integrator and its inverse, an ideal differentiator have transfer functions $1/s$ and s respectively, leading to $H(s) \cdot H_i(s) = 1$.

Examples 6: Check the causality of the system function

$$H(s) = \frac{e^s}{s+1}, \ \text{Re}\{s\} > -1.$$

Solution: The ROC is to the right of the right most pole. Therefore, impulse response must be one sided.

$$e^{-t}u(t) \leftrightarrow \frac{1}{s+1}\text{Re}\{s\} > -1$$

By using time-shift property

$$x(t-t_0) \leftrightarrow e^{-t_0 s}X(s)$$

$$e^{-(t+1)}u(t+1) \leftrightarrow \frac{e^s}{s+1}\text{Re}\{s\} > -1$$

So, impulse response $h(t) = e^{-(t+1)}\,u(t+1)$ which is non zero for $-1 < t < 0$, hence the system is not causal.

Note: The causality implies that the ROC is to the right of the right most pole, but the converse is not in general true, unless the system function is rational.

Example 7: Consider an LTI system with system function $H(s) = \dfrac{s-1}{(s+2)(s-3)}$. Check the stability and causality for different ROCs.

Solution: $H(s) = \dfrac{s-1}{(s+2)(s-3)}$

$$= \frac{3}{5}\cdot\frac{1}{s+2} + \frac{2}{5}\frac{1}{s-3}$$

Possible ROCs are $\text{Re}\{s\} > 3$, $\text{Re}\{s\} < -2$ and $-2 < \text{Re}\{s\} < 3$.

When ROC is $\text{Re}\{s\} > 3$,

$$h(t) = \frac{3}{5}e^{-2t}u(t) + \frac{2}{5}e^{+3t}u(t)$$

$$= \frac{1}{5}\{3e^{-2t} + 2e^{3t}\}u(t)$$

The system is anti-causal, and unstable.

When ROC is $-2 < \text{Re}\{s\} < 3$,

$$\text{Re}\{s\} < 3 \rightarrow -e^{3t}u(-t) \leftrightarrow \frac{+1}{s-3}$$

$$\text{Re}\{s\} > -2 \rightarrow e^{-2t}u(t) \leftrightarrow \frac{1}{s+2}$$

So, $h(t) = \frac{3}{5} \cdot e^{-2t}u(t) - \frac{2}{5}e^{3t}u(-t)$

The system is non-causal and stable.

When ROC is $\text{Re}\{s\} < -2$,

$$h(t) = -\frac{3}{5} \cdot e^{-2t}u(-t) - \frac{2}{5}e^{3t}u(-t)$$

The system is anti-causal and unstable.

System Function for LTI Systems Described by Linear Constant-coefficient Differential Equation

An LTI system with input $x(t)$, and output of $y(t)$ in differential equation form

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

Applying Laplace transform and using differentiation property

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s) \Big|_{\text{Zero initial condition}}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Examples 8: Consider an LTI system for which the input $x(t)$ and output $y(t)$ satisfy the linear constant co-efficient differential equation $\frac{dy(t)}{dt} + 2y(t) = x(t)$ with zero initial conditions. Find the impulse response of the system (1) when it is causal system and (2) when it is anti causal system.

Solution: $\frac{dy(t)}{dx} + 2y(t) = x(t)$

By applying Laplace transform $sY(s) + 2Y(s) = X(s)$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+2}$$

The above expression is system function, but we don't know the ROC of the system. When the system is causal, the ROC can be inferred to be to the right of the right most pole, which is this case corresponds to $\text{Re}\{s\} > -2$.

Then $h(t) = e^{-2t}\, u(t)$

When the system is anti causal, the ROC associate with $H(s)$ would be $\text{Re}\{s\} < -2$. Then $h(t) = -e^{-2t}\, u(-t)$.

Example 9: If input to an LTI system is $x(t) = e^{-2t}u(t)$ then the output is $y(t) = [e^{-t} - e^{-3t}]u(t)$. What is the transfer function of the system, and the differential equation of the system with zero initial conditions?

Solution: By Laplace transform for $x(t)$, $y(t)$ we get $X(s) = \frac{1}{s+2}$, $\text{Re}\{s\} > -2$ and $Y(s) = \frac{1}{s+1} - \frac{1}{s+3}$, $\text{Re}\{s\} > -1$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2/(s+1)(s+3)}{1/s+2} = \frac{2(s+2)}{(s+1)(s+3)}$$

According to convolution property, the ROC of $Y(s)$ must include at least the inter sections of the ROCs of $X(s)$ and $H(s)$.

So, $H(s)$ ROC should be $\text{Re}\{s\} > -1$ and $H(s)$ is causal.

Since both poles of $H(s)$ have negative real parts it follows that the system is stable.

We can write $\frac{Y(s)}{X(s)} = \frac{2(s+2)}{(s+1)(s+3)} = \frac{2s+4}{s^2+4s+3}$

$$s^2Y(s) + 4sY(s) + 3 \cdot Y(s) = 2s\, X(s) + 4X(s)$$

$\frac{d^2}{dt^2} y(t) + 4\frac{dy(t)}{dt} + 3y(t) = 2\frac{dx(t)}{dt} + 4x(t)$ is the differential equation of the system specified.

Example 10: From the following information about an LTI system, find the system function of that LTI system:

1. The system is causal.
2. The system function is rational and has only one pole at $s = -1$ and $s = 3$
3. if $x(t) = 1$, then output $y(t) = 0$
4. The value of the impulse response at $t = 0^+$ is 2.

Solution: Consider the system function $H(s) = \frac{P(s)}{Q(s)}$, and $Q(s) = (s + 1)\,(s - 3)$.

$$H(s) = \frac{P(s)}{(s+1)(s-3)}$$

$y(t)$ is the output of the system for input $x(t) = e^{st}$, then $y(t) = H(s)\, e^{st}$, here for $x(t) = 1 = e^{0t}$.

$$y(t) = H(0)e^{0t} = 0,\ H(0) = 0$$

Which means that $H(s)$ must have a zero at $s = 0$. So, $P(S) = s \cdot P^1(s)$.

The value of impulse response at

$$t = 0 + h(t)\Big|_{t=0^+} = \underset{s\to\infty}{\text{Lt}}\ sH(s)$$

$$= \underset{s\to\infty}{\text{Lt}}\ \frac{s^2P^1(s)}{s^2 - 2s - 3} = 2$$

So, $P^1(s) = 2$, i.e., it is a constant

So, $H(s) = \dfrac{2s}{(s+1)(s-3)}$

And the system is unstable as it has a pole at $s = 3$ with positive real part.

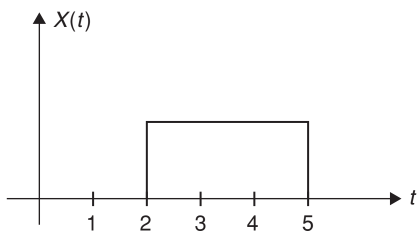
Example 11: A system is described by the differential equation $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 3y(t) = x(t)$.

Let $x(t)$ be a rectangular pulse given by

$$x(t) = \begin{cases} 1 & 2 < t < 5 \\ 0 & \text{otherwise} \end{cases}$$

Assuming zero initial conditions, the Laplace transform of $y(t)$ is?

Solution: $x(t) = \begin{cases} 1 & 2 < t < 5 \\ 0 & \text{otherwise} \end{cases}$



$$x(t) = u(t-2) - u(t-5)$$

By taking Laplace transform

$$X(s) = \frac{e^{-2s}}{s} - \frac{e^{-5s}}{s}$$

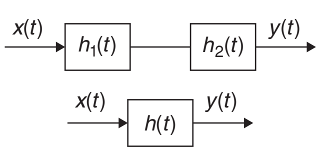
For the given differential equation, by considering Laplace transform $s^2Y(s) + 4sY(s) + 3 = X(s)$

$$Y(s) = \frac{X(s)}{s^2+4s+3} = \frac{e^{-2s} - e^{-5s}}{s(s+1)(s+3)}.$$

Systems Interconnections

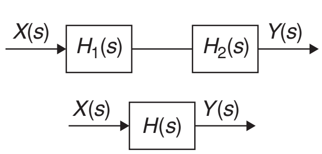
Two systems in cascade,

(a)



$h(t) = h_1(t) * h_2(t)$

(b)

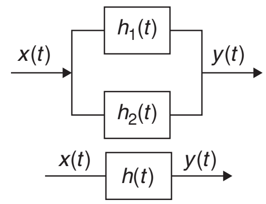


$H(s) = H_1(s) \cdot H_2(s)$

(i) Time-domain, (ii) s-domain

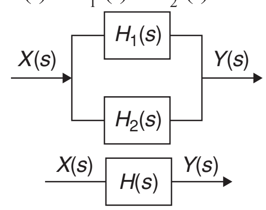
Two systems in parallel

(a)



$h(t) = h_1(t) + h_2(t)$

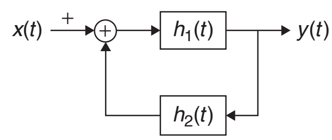
(b)

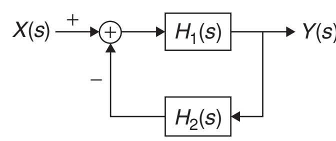


$H(s) = H_1(s) + H_2(s)$

(i) Time-domain, (ii) s-domain

Feedback interconnection of two LTI systems.





$$\frac{Y(s)}{X(s)} = H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

Solved Examples

Example 1: $X(s) = e^{-s}\left[\dfrac{-2}{s(s+2)}\right]$. What are the initial and final values of $x(t)$?

(A) -1 and 0 (B) 0 and 1
(C) 0 and -1 (D) 1 and 1

Solution: (C)

Initial value theorem

$$x(0^+) = \underset{s\to\infty}{\text{Lt}}\, sX(s) = \underset{s\to\infty}{\text{Lt}}\, se^{-s}\left[\frac{-2}{s(s+2)}\right] = 0$$

Final value theorem $x(\infty) = \underset{s\to 0}{\text{Lt}}\, sX(s)$

$$= \underset{s\to 0}{\text{Lt}}\, se^{-s}\frac{-2}{s(s+2)} = -1$$

Example 2: Using Laplace transform find out output signal. when $h(t) = e^{-2t}u(t), t > 0; x(t) = e^{-6t}u(t), t > 0; y(t) = ?$

(A) $\left(\frac{1}{4}e^{-2t} - \frac{1}{4}e^{-6t}\right)u(t) = y(t)$, Re$(s) > -2$

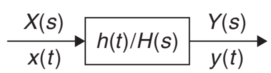
(B) $\left(\frac{1}{4}e^{-2t}+\frac{1}{4}e^{-6t}\right)u(t)=y(t)$, Re$(s)>2$

(C) $\left(-\frac{1}{4}e^{-2t}-\frac{1}{4}e^{6t}\right)u(t)=y(t)$, Re$(s)>-2$

(D) $\left(\frac{1}{4}e^{-2t}-\frac{1}{4}e^{-6t}\right)u(t)=y(t)$, Re$(s)>-2$

Solution: (A)

$$h(t)=e^{-2t}\,u(t)$$
$$x(t)=e^{-6t}\,u(t)$$



Time response

$$\text{output } y(t)=h(t)*x(t)$$

Frequency response

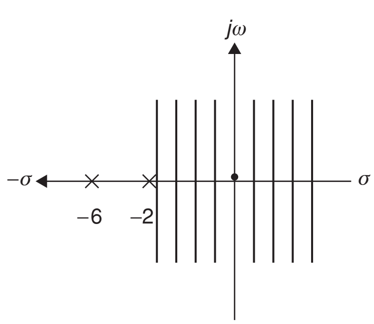
$$Y(s)=X(s)\,H(s)$$

$$Y(s)=\frac{1}{s+6}\cdot\frac{1}{s+2},\ \text{Re}(s)>-2 \text{ (or) } \sigma>-2$$

$$Y(s)=\frac{A}{s+6}+\frac{B}{s+2}$$

By using factorial method

$$A=\underset{s\to-6}{\text{Lt}}\,(s+6)Y(s)$$
$$=\underset{s\to-6}{\text{Lt}}\,(s+6)\frac{1}{(s+2)(s+6)}=\frac{-1}{4}$$
$$B=\underset{s\to-2}{\text{Lt}}\,(s+2)\frac{1}{(s+2)(s+6)}=\frac{1}{4}$$
$$Y(s)=\frac{\frac{-1}{4}}{s+6}+\frac{\frac{1}{4}}{s+2},\ \sigma>-2$$
$$y(t)=\left(\frac{1}{4}e^{-2t}-\frac{1}{4}e^{-6t}\right)u(t),$$



Right most signal stable and causal.

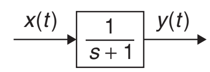
Example 3: If Laplace transform of a signal $Y(s)=\frac{1}{s(s-1)}$, then its final value is

(A) 1 (B) −1
(C) 0 (D) ∞

Solution: (B)
Final value theorem

$$y(\infty)=\underset{s\to0}{\text{Lt}}\,s\cdot Y(s)=\underset{s\to0}{\text{Lt}}\,\frac{s}{s(s-1)}=-1$$

Example 4: For the system shown below, $x(t)=(\sin t)\,u(t)$. In steady state, the response $y(t)$ will be



(A) $\frac{1}{\sqrt{2}}\sin t$ (B) $\frac{1}{\sqrt{2}}\sin\left(t-\frac{\pi}{4}\right)$

(C) $\frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}\right)$ (D) $\frac{1}{\sqrt{2}}\cos t$

Solution: (B)

$$H(s)=\frac{1}{s+1}$$
$$x(t)=(\sin t)u(t)\leftrightarrow X(s)=\frac{1}{s^2+1}$$
$$Y(s)=H(s)\cdot X(s)=\frac{1}{(s+1)(s^2+1)}$$
$$=\frac{1}{2}\left[\frac{1}{s+1}-\frac{s-1}{s^2+1}\right]$$
$$=\frac{1}{2}\left[\frac{1}{s+1}-\frac{s}{s^2+1}+\frac{1}{s^2+1}\right]$$
$$y(t)=\frac{1}{2}[e^{-t}-\cos t+\sin t]u(t)$$

at steady state, $y(t)=\frac{1}{2}[\sin t-\cos t]u(t)$

$$=\frac{1}{\sqrt{2}}\sin\left(t-\frac{\pi}{4}\right)$$

Example 5: Unit step response of a system starting from rest is given by $c(t)=1-e^{-2t}$, for $t\geq0$. The transfer function of the system is

(A) $\frac{s}{s+2}$ (B) $\frac{1}{s+2}$

(C) $\frac{s}{s-2}$ (D) $\frac{2}{s+2}$

Solution: (D)
Unit step response $c(t)=1-e^{-2t}$, for $t\geq0$

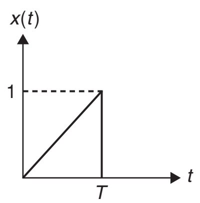
Impulse response $h(t)=\frac{d}{dt}c(t)=2e^{-2t}$, for $t\geq0$. So, transfer function $H(s)=\frac{2}{s+2}$.

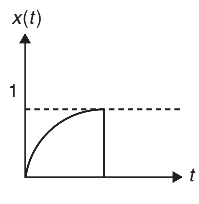
Exercises

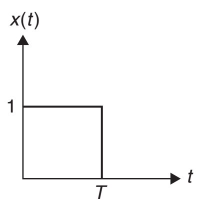
Practice Problems I

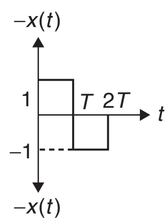
Directions for questions 1 to 22: Select the correct alternative from the given choices.

1. Inverse Laplace transform of $X(s)$ is $x(t)$. When $X(s) = \dfrac{1-e^{-sT}}{s}$, shape of $x(t)$ is?

(A) 

(B) 

(C) 

(D) 

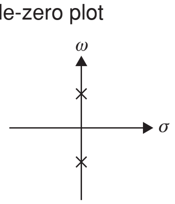
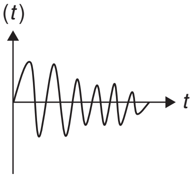
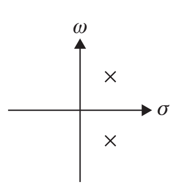
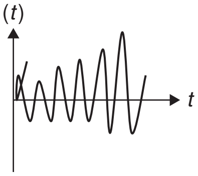
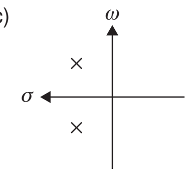
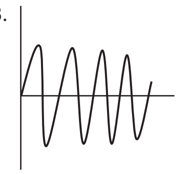
2. Match the following:

List I	List II
(a) $\dfrac{10}{s(s+10)}$	1. $10\,\delta(t)$
(b) $\dfrac{10}{s^2+100}$	2. $(e^{-10t}\cos 10t)u(t)$
(c) $\dfrac{(s+10)}{(s+10)^2+100}$	3. $(\sin 10t)u(t)$
(d) 10	4. $(1-e^{-10t})u(t)$

Codes:

	a	b	c	d		a	b	c	d
(A)	3	4	1	2	(B)	4	3	2	1
(C)	3	4	1	2	(D)	4	3	1	2

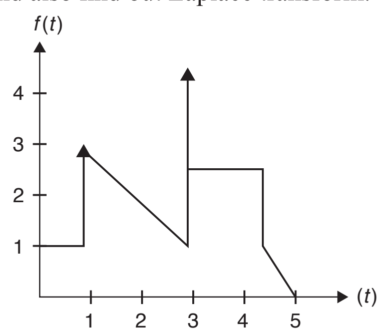
3. Match the following:

List I	List II
Pole-zero plot	Response
(a) 	1. (t) 
(b) 	2. (t) 
(c) 	3. 

	a	b	c		a	b	c
(A)	1	2	3	(B)	3	2	1
(C)	2	3	1	(D)	3	1	2

Common Data for Questions 4 and 5:

4. Resolve $f(t)$ in terms of step, impulse and ramp functions and also find out Laplace transform.



(A) $\dfrac{1}{s}[1+2e^{-s}+e^{-3s}-e^{-4s}]$
$+\dfrac{1}{s^2}[e^{-s}-e^{-3s}+4e^{-4s}-e^{-5s}]+2e^{-3s}$

(B) $\dfrac{1}{s}[1+2e^{-s}+e^{-3s}-e^{-4s}]$
$+\dfrac{1}{s^2}[e^{-s}-e^{-3s}+e^{-4s}-e^{-5s}]+2e^{-3s}$

(C) $\dfrac{1}{s}[1+2e^{-s}+3e^{-3s}-e^{-4s}]$
$-\dfrac{1}{s^2}[e^{-s}-e^{-3s}+e^{-4s}-e^{-5s}]+2e^{-3s}$

(D) $\dfrac{1}{s}[1+2e^{-s}+e^{-3s}-e^{-4s}]$
$-\dfrac{1}{s^2}[e^{-s}-e^{-3s}+e^{-4s}-e^{-5s}]-2e^{-3s}$

5. From above figure find out the initial value.

(A) 0 (B) 4
(C) 2 (D) 1

Common Data for Questions 6 and 7: The impulse response $h(t)$ of a LTI continuous time system is given by $h(t) = \exp(-2t)\,u(t)$ where $u(t)$ denotes the unit step function.

6. The frequency response $H(\omega)$ of this system in terms of angular frequency 'ω' is given by $H(\omega)$

(A) $\frac{1}{1+\omega}$ (B) $\frac{\sin\omega}{\omega}$

(C) $\frac{1}{2+j\omega}$ (D) $\frac{3\omega}{2+\omega}$

7. The output of this system to the sinusoidal input $x(t) = 2\cos(2t)$ for all time 't' is

(A) 0 (B) $2^{-0.25}\cos(2t-0.125\pi)$

(C) $2^{-0.5}\cos(2t-0.125\pi)$ (D) $2^{-0.5}\cos(2t-0.25\pi)$

8. The impulse response of LTI system is $h(t) = e^{-a|t|}$, the step respons e is

(A) $\frac{1}{a}[e^{at} - e^{-at}u(t) - e^{at}u(t) + 2u(t)]$

(B) $\frac{1}{a}[e^{-at} + e^{at}u(t) - e^{-at}u(t) - 2u(t)]$

(C) $\frac{1}{a}[e^{at} + e^{-at}u(t) - e^{at}u(t) + 2u(t)]$

(D) $\frac{1}{a}[e^{at} + e^{-at}u(t) - e^{at}u(t) - 2u(t)]$

9. Laplace transform of $(t^2 - 2t)u(t-1)$ is

(A) $\frac{2e^{-s}(1-s)}{s^3}$ (B) $\frac{2e^{s}(1-s)}{s^3}$

(C) $\frac{2e^{-s}(1+s)}{s^3}$ (D) $\frac{2e^{s}(1+s)}{s^3}$

10. The convolution of $x_1(t) = u(t+3)$ and $x_2(t) = \delta(t-5)$ is

(A) $u(t)$ (B) $u(t-2)$

(C) $u(t+2)$ (D) $u(2)$

11. Laplace transform of $x(t) = \sin^2 t \cdot u(t)$ is

(A) $\frac{4}{s(s^2+2)}$ (B) $\frac{2}{s(s^2+2)}$

(C) $\frac{2}{(s^2+4)}$ (D) $\frac{2}{s(s^2+4)}$

12. The inverse Laplace transform of $\frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$ is

(A) $\cos^2 t$ (B) $\sin^2 t$

(C) $\cos^2 2t$ (D) $\sin^2 2t$

Linked answer questions 13 and 14:

13. The inverse Laplace transform of $\ln\left(\frac{s-a}{s-b}\right)$ is

(A) $(e^{bt} - e^{at})u(t)$ (B) $\frac{1}{t}(e^{-bt} - e^{-at})u(t)$

(C) $\frac{1}{t}(e^{bt} - e^{at})u(t)$ (D) $\frac{1}{t}(e^{-bt} + e^{-at})u(t)$

14. The inverse Laplace transform of $s\ln\left(\frac{s-a}{s-b}\right)$ is

(A) $\frac{1}{t^2}[(bt-1)e^{bt} + (at-1)e^{at}]\cdot u(t)$

(B) $\frac{1}{t^2}[(bt-1)e^{bt} - (at-1)e^{at}]\cdot u(t)$

(C) $\frac{1}{t^2}[(bt+1)e^{bt} - (at+1)e^{at}]\cdot u(t)$

(D) $\frac{1}{t^2}[(bt+1)e^{-bt} - (at-1)e^{-at}]\cdot u(t)$

15. Laplace transform of $e^{-3t}u(t) * tu(t)$

(A) $\frac{-1}{s^2(s+3)}$ (B) $\frac{1}{s^2(s-3)}$

(C) $\frac{1}{s^2(s+3)}$ (D) $\frac{-1}{s^2(s-3)}$

16. Poles and zeros of $e^{2t}u(t) + e^{-3t}u(-t)$

(A) Zero at $\frac{-5}{2}$, poles at 2, 3

(B) Zero at $\frac{5}{2}$, poles at –2, –3

(C) No common ROC

(D) Zero at $\frac{-5}{2}$, poles at –2, –3

17. The initial and final values of $X(s) = \frac{2s}{s^2+2s+2}$ are

(A) 2 and 1 (B) 2 and 0

(C) 1 and 2 (D) 2 and 2

18. A system is formed by connecting two systems in cascade. The impulse response of these systems are $e^{-2t}u(t)$, $2e^{-t}u(t)$. The impulse response of over all system is

(A) $2(e^{-t} - e^{-2t})u(t)$ (B) $2(e^{-t} + e^{-2t})u(t)$

(C) $(e^{-t} - e^{-2t})u(t)$ (D) $2(e^{-t}\cdot e^{-2t})u(t)$

19. $H(s)$ is the transfer function.

$H(s) = \frac{s}{s^2 - s - 2}(\sigma < -1)$. Then function is

(A) Causal and stable

(B) Causal and unstable

(C) Non-causal and stable

(D) Non-causal and unstable

20. A continuous time LTI system is described by

$$\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 3y(t) = 2\frac{d}{dt}x(t) + 4x(t).$$

Assuming zero initial conditions, the response $y(t)$ of the above system for the input $x(t) = e^{-2t}u(t)$ is

(A) $(e^{-t} - e^{-3t})u(t)$ (B) $(e^{t} + e^{3t})u(t)$

(C) $(e^{-t} + e^{-3t})u(t)$ (D) $(e^{t} - e^{3t})u(t)$

21. A linear time invariant, causal, continuous time system has a natural transfer function with sample poles at $s = -2$ and $s = -4$, and one sample zero at $s = -1$. A unit step $u(t)$ is applied at the input of the system. At steady state the output has constant value of 1. The impulse response of this system is

(A) $-4e^{-2t}\,u(t)$ (B) $12e^{-4t}\,u(t)$
(C) $(-4e^{-2t} + 12e^{-4t})u(t)$ (D) None

22. Unit impulse response of a system is $f(t) = e^{-t}$, for $t \geq 0$, for this system, the steady state value of the output for unit step input is equal to

(A) 1 (B) –1
(C) 0 (D) ∞

Practice Problems 2

Directions for questions 1 to 17: Select the correct alternative from the given choices.

1. Match the function with ROC.

P.	$e^{-at}u(t)$	**1.**	$\sigma > 0$
Q.	$-e^{-at}u(-t)$	**2.**	$\sigma > a$
R.	$te^{at}\,u(t)$	**3.**	$\sigma < -a$
S.	$t^n\,u(t)$	**4.**	$\sigma > -a$

(A) P – 3, Q – 4, R – 2, S – 1
(B) P – 4, Q – 3, R – 2, S – 1
(C) P – 3, Q – 4, R – 1, S – 2
(D) P – 4, Q – 3, R – 1, S – 2

2. The initial $x(0)$, and final value $x(\infty)$ of $X(s) = \dfrac{1}{s(s-1)}$ is

(A) 1, 0 (B) 0, –1
(C) 2, 1 (D) 1, 2

3. If $x(t)$ and $X(s)$ are Laplace transfom pairs then Laplace transform of $e^{at}\,x(t)$ is

(A) $e^{as}\,X(s)$ (B) $X(s)$
(C) $X(s - a)$ (D) $X(s + a)$

4. If $x(t)$ and $X(s)$ are Laplace transform pairs then Laplace transform of $t \times (t)$ is

(A) $\int_0^\infty X(s)ds$ (B) $\dfrac{-d}{ds}X(s)$
(C) $\dfrac{\partial}{\partial s}X(s)$ (D) $X(s)\,e^{-s}$

5. Laplace transform of $\left[\dfrac{\sin \pi t}{t}\right]u(t)$ is

(A) $\tan^{-1}\left(\dfrac{s}{\pi}\right)$ (B) $\cos^{-1}\left(\dfrac{s}{\pi}\right)$
(C) $\cot^{-1}\left(\dfrac{s}{\pi}\right)$ (D) $\sin^{-1}\left(\dfrac{s}{\pi}\right)$

6. Laplace transform of a rectified half sine wave with period 2π, amplitude 1 is

(A) $\dfrac{(1+e^{-\pi s})}{s^2+1}$ (B) $\dfrac{(1-e^{-\pi s})}{s^2+1}$
(C) $\dfrac{1}{(s^2+1)(1-e^{-\pi s})}$ (D) $\dfrac{1}{(s^2+1)(1+e^{-\pi s})}$

7. Laplace transforms of $x(t) = e^{at}u(t)$, $y(t) = -e^{at}\,u(-t)$ have

(A) same transforms but different ROCs
(B) different transforms but same ROCs
(C) same transforms and same ROCs
(D) different transforms and different ROCs

8. Laplace transform and ROC of $e^{-5t}u(-t+3)$ is

(A) $-\dfrac{e^{-3(s+5)}}{s+5}$, Re$(s) < 5$ (B) $\dfrac{e^{-3(s+5)}}{s+5}$, Re$(s) > 5$
(C) $-\dfrac{e^{-3(s+5)}}{s+5}$, Re$(s) > 5$ (D) $\dfrac{e^{-3(s+5)}}{s+5}$, Re$(s) < 5$

9. Match the Laplace transform pairs

P.	$x(at)$	**1.**	$asX(s)$
Q.	$x(t - a)$	**2.**	$X(s - a)$
R.	$e^{at} \cdot x(t)$	**3.**	$e^{-sa}\,X(s)$
S.	$\dfrac{d}{dt}[ax(t)]$	**4.**	$\dfrac{1}{a}X\left(\dfrac{s}{a}\right) a > 0$

(A) P – 3, Q – 2, R – 1, S – 4
(B) P – 4, Q – 2, R – 1, S – 3
(C) P – 2, Q – 4, R – 3, S – 1
(D) P – 4, Q – 3, R – 2, S – 1

10. Poles and zeros of $x(t) = e^{-2t}\,u(t) + e^{-3t}u(t)$ are

(A) Zero at $\dfrac{-5}{2}$, poles at 2, 3
(B) Zero at $\dfrac{5}{2}$, poles at –2, –3
(C) No common ROC
(D) Zero at $\dfrac{-5}{2}$, poles at –2, –3

11. Inverse Laplace transform of $\dfrac{2s}{s+2}$ is

(A) $2\delta(t) - 4e^{-2t}u(t)$ (B) $2\delta(t) + 4e^{-2t}u(t)$
(C) $2\delta(t) - 4e^{2t}u(t)$ (D) $2\delta(t) + 4e^{2t}u(t)$

12. Given $H(s) = e^{-5s}$, what is the impulse response of the system?

(A) $u(t - 5)$ (B) $\delta(t - 5)$
(C) $e^{-5t}\,u(t)$ (D) $t \cdot e^{-5t}u(t)$

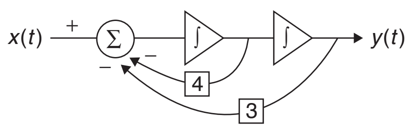
13. Convolution of $a^n u[n]$ and $b^n u[n]$ is

(A) $\dfrac{b^{n+1}+a^{n+1}}{a+b}u[n]$ (B) $\dfrac{b^{n+1}-a^{n+1}}{b-a}u[n]$

(C) $\dfrac{b^{n-1}-a^{n-1}}{b-a}u[n]$ (D) $\dfrac{b^{n-1}+a^{n-1}}{b-a}u[n]$

Linked answer questions 14 and 15:

14. For a linear time invariant system whose block diagram is shown here with input $x(t)$ and output $y(t)$



The transfer function is

(A) $\dfrac{1}{s^2+4s+3}$ (B) $\dfrac{1}{s+4}$

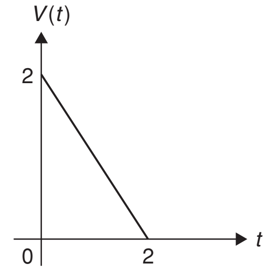
(C) $\dfrac{1}{s-3}$ (D) $\dfrac{1}{s^2-4s-3}$

15. The step response of the system is

(A) $\dfrac{-1}{2}e^{-t}+\dfrac{1}{6}e^{-3t}$ (B) $\dfrac{1}{3}+\dfrac{1}{2}e^{-t}-\dfrac{1}{6}e^{-3t}$

(C) $\dfrac{1}{3}-\dfrac{1}{2}e^{-t}+\dfrac{1}{6}e^{-3t}$ (D) $\dfrac{1}{3}-\dfrac{1}{2}e^{t}+\dfrac{1}{6}e^{-3t}$

16. Express Laplace transform of



(A) $\dfrac{-2}{s}+\dfrac{1}{s^2[1-e^{-2s}]}$ (B) $\dfrac{2}{s}-\dfrac{1}{s^2}[1+e^{-2s}]$

(C) $\dfrac{-2}{s}-\dfrac{1}{s^2[1-e^{-2s}]}$ (D) $\dfrac{2}{s}-\dfrac{1}{s^2}[1-e^{-2s}]$

17. Given $L\{f(t))\}=\dfrac{s+2}{s^2+1}$, $L\{g(t)\}=\dfrac{s^2+1}{(s+3)(s+2)}$,

$h(t)=\int_0^t f(\tau)g(t-\tau)d\tau$, then $L\ \{h\ (t)\}$ is

(A) $\dfrac{1}{s+2}$ (B) $\dfrac{1}{s+1}$

(C) $\dfrac{1}{s+3}$ (D) $\dfrac{s}{s+3}$

Previous Years' Questions

1. The running integrator, given by $y(t)=\int_0^H \pi R^2 \times (t)dt$

or $Y(S)=\int_{-\alpha}^{t} \times(\alpha)d\alpha$ **[2006]**

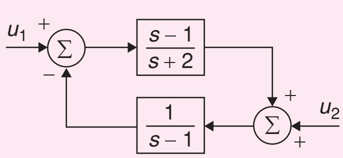
(A) has no finite singularities in its double sided Laplace transform $Y(s)$.

(B) produces a bounded output for every causal bounded input.

(C) produces a bounded output for every anticausal bounded input.

(D) has no finite zeroes in its double sided Laplace transform $Y(s)$.

2. The system shown in the figure is **[2007]**

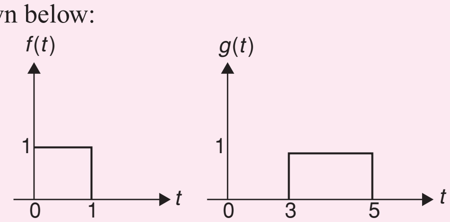


(A) stable
(B) unstable
(C) conditionally stable
(D) stable for input u_1, but unstable for input u_2

3. For the system $\dfrac{2}{(s+1)}$ the approximate time taken for a step response to reach 98% of its final value is **[2010]**

(A) 1 s (B) 2 s
(C) 4 s (D) 8 s

Common Data for Questions 4 and 5: Given $f(t)$ and $g(t)$ as shown below:



4. $g(t)$ can be expressed as **[2010]**

(A) $g(t)=f(2t-3)$ (B) $g(t)=f\left(\dfrac{t}{2}-3\right)$

(C) $g(t)=f\left(2t-\dfrac{3}{2}\right)$ (D) $g(t)=f\left(\dfrac{t}{2}-\dfrac{3}{2}\right)$

5. The Laplace transform of $g(t)$ is **[2010]**

(A) $\dfrac{1}{S}(e^{3s}-e^{5s})$ (B) $\dfrac{1}{S}(e^{-5s}-e^{-3s})$

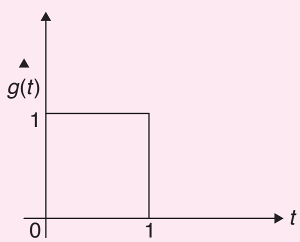
(C) $\dfrac{e^{-3s}}{S}(1-e^{-2s})$ (D) $\dfrac{1}{S}(e^{5s}-e^{3s})$

6. Let the Laplace transform of a function $f(t)$ which exists for $t > 0$ be $F_1'(s)$ and the Laplace transform of its delayed version $f(t - \tau)$ be $F_2(s)$. Let $f_1''(s)$ be the complex conjugate of $F_1(s)$ with the Laplace variable set as $s = \sigma + j\omega$.

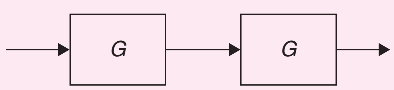
If $G(s) = \dfrac{F_2(s) \cdot F_1'(s)}{|F_1(s)|^2}$, then the inverse Laplace transform of $G(s)$ is **[2011]**

(A) an ideal impulse $\delta(t)$
(B) an ideal delayed impulse $\delta(t - \tau)$
(C) an ideal step function $u(t)$
(D) an ideal delayed step function $u(t - \tau)$

7. The impulse response $g(t)$ of a system G, is as shown in Figure (a). What is the maximum value attained by the impulse response of two cascaded blocks of G as shown in Figure (b)? **[2015]**



(a)



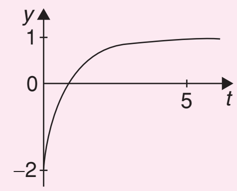
(b)

(A) $\frac{2}{3}$ (B) $\frac{3}{4}$

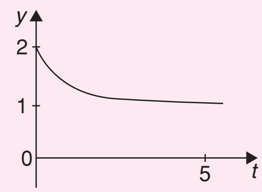
(C) $\frac{4}{5}$ (D) 1

8. The unit step response of a system with the transfer function $G(s) = \dfrac{1-2s}{1+s}$ is given by which one of the following waveforms? **[2015]**

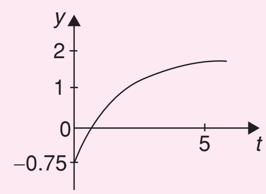
(a)



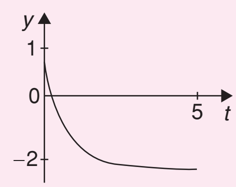
(b)



(c)



(d)



Answer Keys

Exercises

Practice Problems 1

1. C	**2.** B	**3.** B	**4.** C	**5.** D	**6.** C	**7.** D	**8.** A	**9.** A	**10.** B
11. D	**12.** A	**13.** C	**14.** B	**15.** C	**16.** C	**17.** B	**18.** A	**19.** D	**20.** A
21. C	**22.** A								

Practice Problems 2

1. B	**2.** B	**3.** C	**4.** B	**5.** C	**6.** C	**7.** A	**8.** A	**9.** D	**10.** D
11. A	**12.** B	**13.** B	**14.** A	**15.** C	**16.** D	**17.** C			

Previous Years' Questions

1. C	**2.** D	**3.** C	**4.** D	**5.** C	**6.** B	**7.** D	**8.** A