CHAPTER

9

# Area

➤ Different Cases of Bounded Area

## DIFFERENT CASES OF BOUNDED AREA

**a.** The area bounded by the continuous curve $y = f(x)$, the axis of $x$ and the ordinates $x = a$ and $x = b$ (where $b > a$) is given by

$$A = \int_a^b f(x)\,dx = \int_a^b y\,dx$$

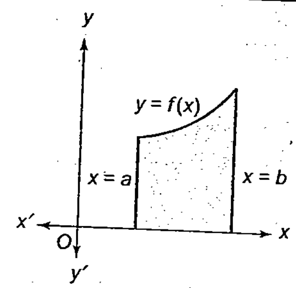



Fig. 9.1

**b.** The area bounded by the straight lines $x = a$, $x = b (a < b)$ and the curves $y = f(x)$ and $y = g(x)$, provided $f(x) \le g(x)$ (where $a \le x \le b$), is given by $A = \int_a^b [g(x) - f(x)]dx$

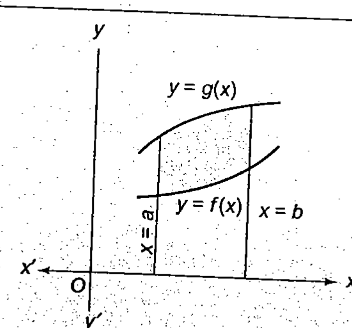



Fig. 9.2

**c.** When two curves $y = f(x)$ and $y = g(x)$ intersect, the bounded area is $A = \int_a^b [g(x) - f(x)]dx$, where $a$ and $b$ are the roots of the equation $f(x) = g(x)$.

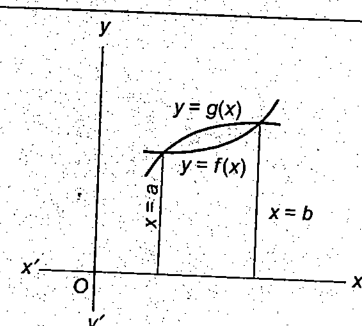



Fig. 9.3

**d.** If the curve crosses the $x$-axis at $c$, then the area bounded by the curve $y = f(x)$ and the ordinates $x = a$ and $x = b$

(where $b > a$) is given by $A = \left|\int_a^c f(x)dx\right| + \left|\int_c^b f(x)dx\right|$

$$= A = \int_a^c f(x)dx - \int_c^b f(x)dx$$

$\left(\because \int_a^c f(x)\,dx > 0 \text{ and } \int_c^b f(x)dx < 0\right)$

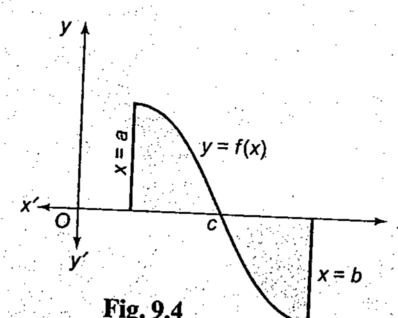



Fig. 9.4

**e.** The area bounded by $y = f(x)$ and $y = g(x)$ (where $a \le x \le b$), when they intersect at $x = c \in (a, b)$ is given by

$$A = \int_a^b |f(x) - g(x)|\,dx$$

or $\int_a^c (f(x) - g(x))dx + \int_c^b (g(x) - f(x))dx$

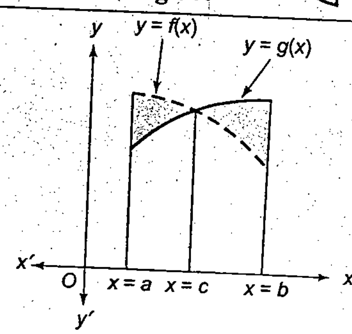



Fig. 9.5

## Curve Tracing

To find the approximate shape of a curve, the following procedure is adopted in order:

a. Symmetry

- Symmetry about the $x$-axis
  If all the powers of '$y$' in the equation are even, then the curve is symmetrical about the $x$-axis.
  e.g., $y^2 = 4ax$.
- Symmetry about the $y$-axis
  If all the powers of '$x$' in the equation are even, then the curve is symmetrical about the $y$-axis.
  e.g., $x^2 = 4ay$.
- Symmetry about both axes
  If all the powers of '$x$' and '$y$' in the equation are even, the curve is symmetrical about the axis of '$x$' as well as of '$y$'.
  e.g., $x^2 + y^2 = a^2$.
- Symmetry about the line $y = x$
  If the equation of the curve remains unchanged on interchanging '$x$' and '$y$', then the curve is symmetrical about the line $y = x$
  e.g., $x^3 + y^3 = 3xy$.

b. Find the points where the curve crosses the $x$-axis and the $y$-axis.

c. Find $\frac{dy}{dx}$ and examine, if possible, the intervals when $f(x)$ is increasing or decreasing and also its stationary points.

d. Examine $y$ when $x \to \infty$ or $x \to -\infty$.

**Example 9.1** Find the area bounded by the parabola $y = x^2 + 1$ and the straight line $x + y = 3$.

**Sol.**

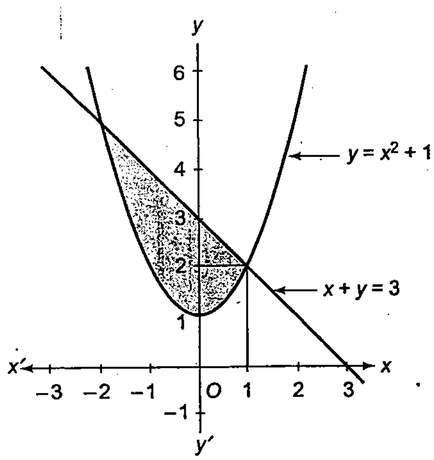



**Fig. 9.6**

The two curves meet at points where $3 - x = x^2 + 1$, i.e., $x^2 + x - 2 = 0$

$\Rightarrow (x+2)(x-1) = 0 \Rightarrow x = -2, 1$

$\therefore$ required area $= \int_{-2}^{1} \left[(3-x) - (x^2+1)\right] dx$

$$= \int_{-2}^{1} (2 - x - x^2)\, dx$$

$$= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1}$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - \frac{4}{2} + \frac{8}{3}\right)$$

$$= \frac{9}{2} \text{ sq. units}$$

**Example 9.2** Find the smaller area enclosed by the circle $x^2 + y^2 = 9$ and the line $x = 1$.

**Sol.**

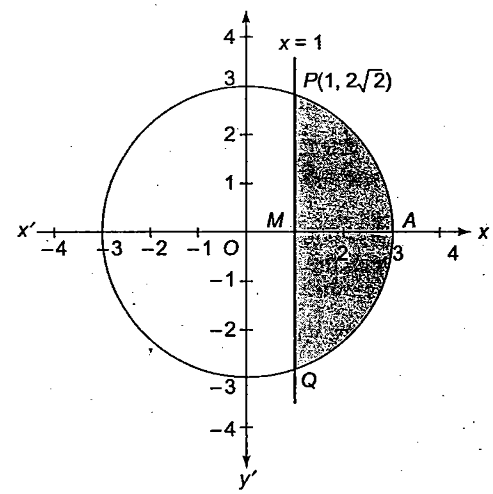



**Fig. 9.7**

Required area = 2 area $MAPM$

$$= 2\int_{1}^{3} \sqrt{9 - x^2}\, dx$$

$$= 2\left|\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right|_{1}^{3}$$

$$= 2\left[\frac{9}{2}\sin^{-1} 1 - \frac{2\sqrt{2}}{2} - \frac{9}{2}\sin^{-1}\frac{1}{3}\right]$$

$$= 9 \times \frac{\pi}{2} - 2\sqrt{2} - 9\sin^{-1}\frac{1}{3}$$

$$= 9\left[\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right] - 2\sqrt{2}$$

$$= 9\cos^{-1}\frac{1}{3} - 2\sqrt{2} = 9\sec^{-1} 3 - \sqrt{8} \text{ sq. units}$$

**Example 9.3** Find the area of the closed figure bounded by the curves $y = \sqrt{x}$, $y = \sqrt{4 - 3x}$ and $y = 0$.

**Sol.**

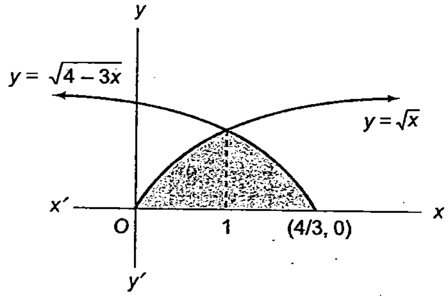



**Fig. 9.8**

$$A=\int_0^1\left(\sqrt{x}dx\right)+\int_1^{4/3}\sqrt{4-3x}dx$$

$$=\left(\frac{x^{3/2}}{3/2}\right)_0^1+\left(\frac{(4-3x)^{3/2}}{-3(3/2)}\right)_1^{4/3}$$

$$=\frac{2}{3}+\frac{2}{3}\left[\frac{1}{3}\right]=\frac{2}{3}+\frac{2}{9}=\frac{8}{9}\text{ sq. units}$$

**Example 9.4** Find the area, lying above the $x$-axis and included between the circle $x^2+y^2=8x$ and the parabola $y^2=4x$.

**Sol.** Solving the curves, we get $x^2+4x=8x \Rightarrow x=0, 4$

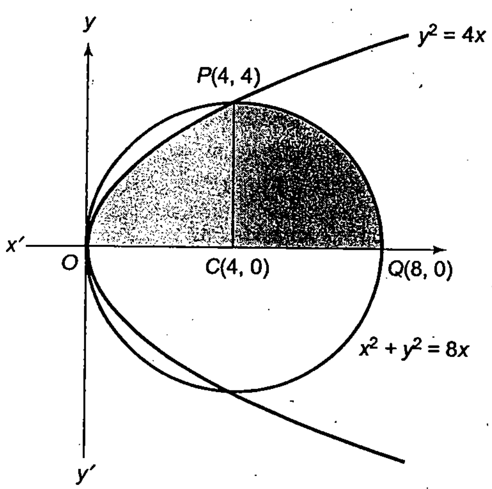



**Fig. 9.9**

Required area

$$=\int_0^4 y_{\text{parabola}}\,dx+\int_4^8 y_{\text{circle}}\,dx$$

Circle is $(x-4)^2+y^2=4^2$,

Area of circle in 1st quadrant $=\frac{1}{4}\pi 4^2=4\pi$

$$A=\int_0^4 2\sqrt{x}\,dx+4\pi$$

$$=\frac{4}{3}\left[x^{3/2}\right]_0^4+4\pi$$

$$=\frac{4}{3}\times 4\sqrt{4}+4\pi\text{ sq. units}$$

**Example 9.5** Find the area bounded by the curve $y=(x-1)(x-2)(x-3)$ lying between the ordinates $x=0$ and $x=3$.

**Sol.** $y=(x-1)(x-2)(x-3)$

The curves will intersect the $x$-axis, when $y=0$

$\Rightarrow (x-1)(x-2)(x-3)=0$

$\Rightarrow x=1,2,3$

And the curve intersects the $y$-axis,

when $x=0 \Rightarrow y=-6$

Thus, the graph of the given function for $0\le x\le 3$ is as shown in Fig. 9.10.

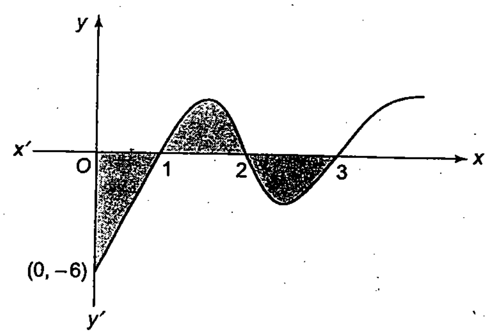



**Fig. 9.10**

Hence, the required area $A$ = shaded area

$$=\left|\int_0^1 ydx\right|+\left|\int_1^2 ydx\right|+\left|\int_2^3 ydx\right| \quad (1)$$

Since $\int ydx=\int(x-1)(x-2)(x-3)dx$

$$=\int(x^3-6x^2+11x-6)dx$$

$$=\frac{x^4}{4}-2x^3+\frac{11x^2}{2}-6x$$

$\therefore$ from (1)

$$A=\left|\left[\frac{x^4}{4}-2x^3+\frac{11x^2}{2}-6x\right]_0^1\right|+\left|\left[\frac{x^4}{4}-2x^3+\frac{11x^2}{2}-6x\right]_1^2\right|$$

$$+\left|\left[\frac{x^4}{4}-2x^3+\frac{11x^2}{2}-6x\right]_2^3\right|$$

$=|-9/4|+(1/4)+|-1/4|$

$=11/4$ sq. units

**Example 9.6** Consider the region formed by the lines $x=0$, $y=0$, $x=2$, $y=2$. If the area enclosed by the curves $y=e^x$ and $y=\ln x$, within this region, is being removed, then find the area of the remaining region.

**Sol.** Required area = shaded region

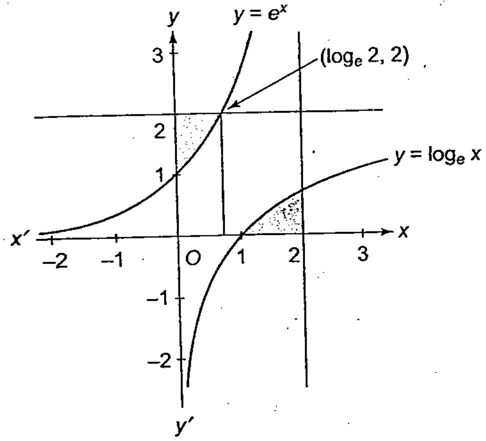



Fig. 9.11

$= 2\int_0^{\ln 2} (2 - e^x)\,dx$

$= 2[2x - e^x]_0^{\ln 2}$

$= 2(2\ln 2 - 1)$ sq. units

**Example 9.7** Find the area bounded by the curves $y = \sin x$ and $y = \cos x$ between two consecutive points of the intersection.

**Sol.**

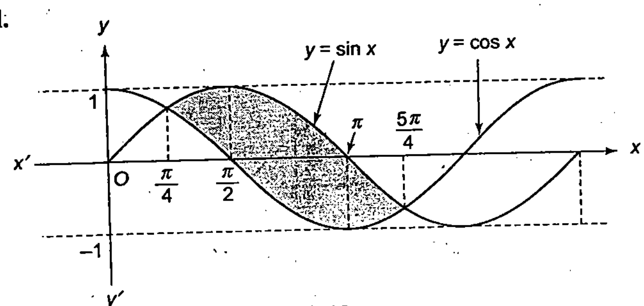



Fig. 9.12

Two consecutive points of intersection of $y = \sin x$ and $y = \cos x$ can be taken as $x = \pi/4$ and $x = 5\pi/4$

$\therefore$ required area $= \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)dx$

$= [-\cos x - \sin x]_{\pi/4}^{5\pi/4}$

$= \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} = 2\sqrt{2}$ sq. units

## Some Standard Areas

1. Area bounded by $y = \sin x$, where $0 \le x \le \pi$, and the $x$-axis is 2 sq. units. In fact, area of one loop of $y = \sin x$ and $y = \cos x$ is 2 sq. units.
2. Area bounded by $y = \log_e x$, $y = 0$, and $x = 0$ is 1 sq. units.
3. Area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\pi ab$ sq. units.
4. Area bounded by $y^2 = 4ax$ and $x^2 = 4by$, where $a > 0$, $b > 0$ is $A = \int_0^k \left(2\sqrt{a}\sqrt{x} - \frac{x^2}{4b}\right)dx = \frac{16ab}{3}$ sq. units, where $k = (64ab^2)^{\frac{1}{3}}$.

**Example 9.8** $AOB$ is the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in which $OA = a$, $OB = b$. Then find the area between the arc $AB$ and the chord $AB$ of the ellipse.

**Sol.**

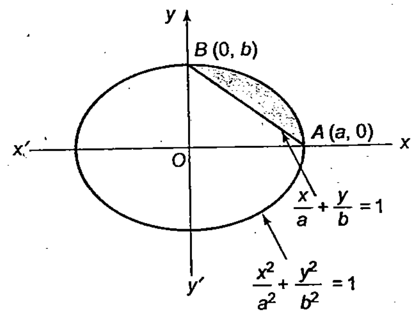



Fig. 9.13

Area of ellipse is $\pi ab$. Then the area of ellipse in the first quadrant is $\frac{1}{4}\pi ab$ sq. units.

Now area of triangle $OAB = \frac{1}{2}ab$ sq. units

Hence, the required area is $\frac{1}{4}\pi ab - \frac{1}{2}ab = \frac{ab}{4}(\pi - 2)$ sq. units.

**Example 9.9** Find the ratio in which the area bounded by the curves $y^2 = 12x$ and $x^2 = 12y$ is divided by the line $x = 3$.

**Sol.**

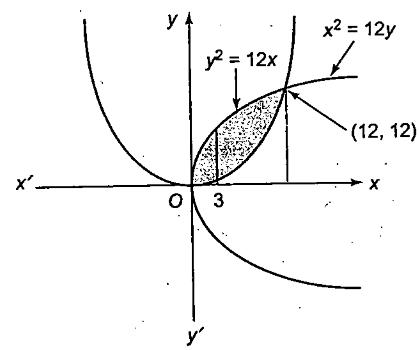



Fig. 9.14

$A_1$ = area bounded by $y^2 = 12x$, $x^2 = 12y$ and line $x = 3 = \int_0^3 \sqrt{12x}\,dx - \int_0^3 \frac{x^2}{12}dx$

$= \sqrt{12}\left|\frac{2x^{3/2}}{3}\right|_0^3 - \left|\frac{x^3}{36}\right|_0^3 = \frac{45}{4}$ sq. units.

$A_2$ = area bounded by $y^2 = 12x$ and $x^2 = 12y$

$= \frac{16(3)(3)}{3} = 48$ sq. units

$\therefore$ required ratio $= \frac{\frac{45}{4}}{48 - \frac{45}{4}} = \frac{45}{147}$

**Example 9.10** Find the area bounded by
**a.** $y = \log_e|x|$ and $y = 0$
**b.** $y = |\log_e|x||$ and $y = 0$

**Sol.** **a.** $y = \log_e|x|$ and $y = 0$

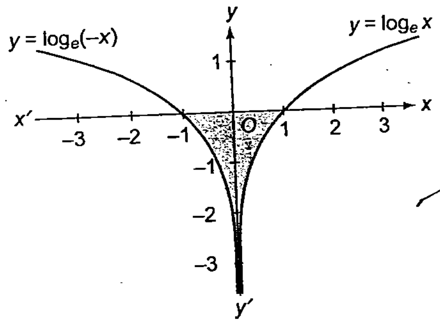



**Fig. 9.15**

From the figure, required area = area of the shaded region = 1 + 1 = 2 sq. units (as we know that area bounded by $y = \log_e x$, $x = 0$ and $y = 0$, is 1 sq. units)

**b.** $y = |\log_e|x||$ and $y = 0$

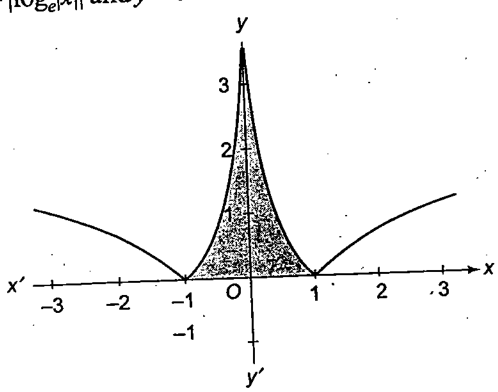


**Fig. 9.16**

From the figure, required area = area of the shaded region = 1 + 1 = 2 sq. units.

## Area Bounded by Curves While Integrating Along y-axis

Sometimes integration w.r.t. $y$ is very useful, i.e., along the horizontal strip. Area bounded by the curve, $y$-axis and the two abscissas at $y = a$ and $y = b$ is written as $A = \int_a^b x\, dy$.

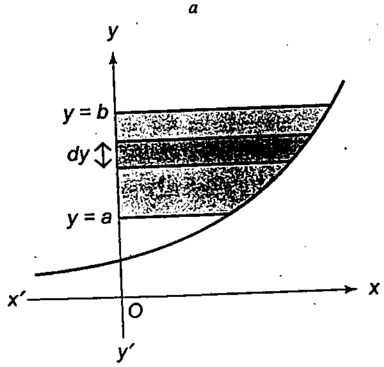



**Fig. 9.17**

**Example 9.11** Find the area bounded by $x = 2y - y^2$ and the $y$-axis.

**Sol.**

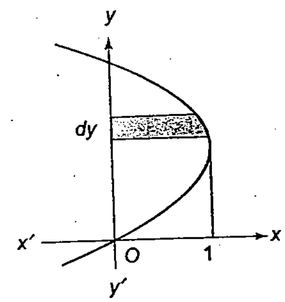



**Fig. 9.18**

$$A = \int_0^2 x\, dy = \int_0^2 (2y - y^2)\, dy = \left[y^2 - \frac{y^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$$

**Example 9.12** Find the area bounded by $y = \sin^{-1} x$, $y = \cos^{-1} x$ and the $x$-axis.

**Sol.**

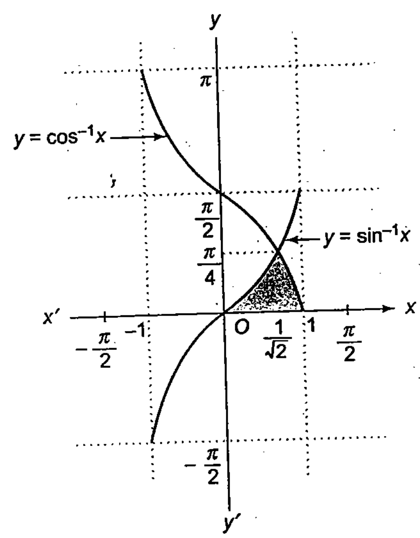



**Fig. 9.19**

$y = \sin^{-1} x$, $y = \cos^{-1} x$ and the $x$-axis if vertical strip is used, we get

$$A = \int_0^{1/\sqrt{2}} \sin^{-1} x\, dx + \int_{1/\sqrt{2}}^{1} \cos^{-1} x\, dx$$

If horizontal strip is used, then

$$A = \int_0^{\pi/4} (\cos y - \sin y)\, dy$$

$$= [\sin y + \cos y]_0^{\pi/4}$$

$$= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right]$$

$$= \sqrt{2} - 1$$

**Example 9.13** Find the area of the figure bounded by the parabolas $x = -2y^2, x = 1 - 3y^2$.

**Sol.**

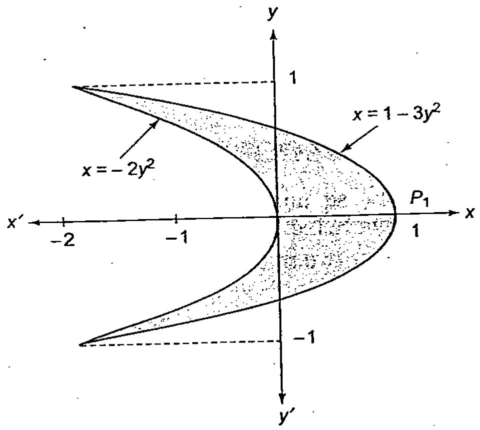



**Fig. 9.20**

Solving the equation $x = -2y^2, x = 1 - 3y^2$, we find that the ordinates of the point of intersection of the two curves are $y_1 = -1, y_2 = 1$. The points are (–2, – 1) and (–2, 1).

The required area is given by

$$A = 2\int_0^1 (\underset{R}{x_1} - \underset{L}{x_2})dy = 2\int\left[\left(1-3y^2\right)-\left(-2y^2\right)\right]dy$$

$$= 2\int_0^1 \left(1 - y^2\right) dy = 2\left[y - \frac{y^3}{3}\right]_0^1 = \frac{4}{3}$$

## Concept Application Exercise 9.1

1. Find the area lying in the first quadrant and bounded by the curve $y = x^3$ and the line $y = 4x$.
2. Find the area enclosed by the curves $x^2 = y, y = x + 2$ and $x$-axis.
3. A curve is given by $y = \begin{cases}\sqrt{(4-x^2)}, & 0 \le x < 1\\ \sqrt{(3x)}, & 1 \le x \le 3.\end{cases}$ Find the area lying between the curve and $x$-axis.
4. Find the area of the region bounded by the limits $x = 0$, $x = \frac{\pi}{2}$ and $f(x) = \sin x, g(x) = \cos x$.
5. Find the area bounded by the curve $y = \sin^{-1} x$ and the line $x = 0, |y| = \frac{\pi}{2}$.
6. Find the area bounded by $y = \tan^{-1} x$, $y = \cot^{-1} x$ and $y$-axis in first quadrant.
7. Prove that area common to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and its auxiliary circle $x^2 + y^2 = a^2$ is equal to area of another ellipse of semi-axis $a$ and $a - b$.
8. Find the area bounded by $y = \log_e x$, $y = -\log_e x$, $y = \log_e(-x)$ and $y = -\log_e(-x)$.

# EXERCISES

## Subjective Type

*Solutions on page 9.15*

1. Draw a rough sketch of the curve $y = \frac{x^2+3x+2}{x^2-3x+2}$ and find the area of the bounded region between the curve and the $x$-axis.
2. $f(x)$ is a continuous and bijective function on $R$. If $\forall t \in R$, then the area bounded by $y = f(x), x = a - t, x = a$ and the $x$-axis is equal to the area bounded by $y = f(x), x = a + t$, $x = a$ and the $x$-axis. Then prove that $\int_{-\lambda}^{\lambda} f^{-1}(x)dx = 2a\lambda$ (given that $f(a) = 0$).
3. Find a continuous function '$f$', where $(x^4 - 4x^2) \le f(x) \le (2x^2 - x^3)$ such that the area bounded by $y = f(x), y = x^4 - 4x^2$, the $y$-axis and the line $x = t$, where $(0 \le t \le 2)$ is $k$ times the area bounded by $y = f(x)$, $y = 2x^2 - x^3$, $y$-axis and line $x = t$(where $0 \le t \le 2$).
4. Find the area bounded by the curves $y = -x^2 + 6x - 5$, $y = -x^2 + 4x - 3$ and the straight line $y = 3x - 15$ and lying right to $x = 1$.
5. Find the value of $a$ where $(a > 2)$ for which the reciprocal of the area enclosed between $y = \frac{1}{x^2}; y = \frac{1}{4(x-1)}$; $x = 2$ and $x = a$ is $a$ itself and for what values of $b \in (1, 2)$, the area of the figure bounded by the lines $x = b$ and $x = 2$ is $1 - \frac{1}{b}$.
6. If the area bounded by $y = x^2 + 2x - 3$ and the line $y = kx + 1$ is the least, find $k$ and also the least area.
7. Find the area of the figure enclosed by the curve $5x^2 + 6xy + 2y^2 + 7x + 6y + 6 = 0$.
8. **a.** Sketch and find the area bounded by the curve $\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}$ and $x^2 + y^2 = a^2$ (where $a > 0$).
   **b.** If curve $|x| + |y| = a$ divides the area in two parts, then find their ratio in the first quadrant only.

## Objective Type

*Solutions on page 9.17*

**Each question has four choices a, b, c, and d, out of which *only one* is correct.**

1. Area enclosed by the curve $y = f(x)$ defined parametrically as $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$ is equal to
   **a.** $\pi$ sq. units  **b.** $\pi/2$ sq. units
   **c.** $\frac{3\pi}{4}$ sq. units  **d.** $\frac{3\pi}{2}$ sq. units

2. The area inside the parabola $5x^2 - y = 0$ but outside the parabola $2x^2 - y + 9 = 0$ is

   a. $12\sqrt{3}$ sq. units  b. $6\sqrt{3}$ sq. units
   c. $8\sqrt{3}$ sq. units  d. $4\sqrt{3}$ sq. units

3. Let $f(x) =$ minimum $(x+1, \sqrt{1-x})$ for all $x \le 1$. Then the area bounded by $y = f(x)$ and the $x$-axis is

   a. $\frac{7}{3}$ sq. units  b. $\frac{1}{6}$ sq. units
   c. $\frac{11}{6}$ sq. units  d. $\frac{7}{6}$ sq. units

4. Area enclosed between the curves $|y| = 1 - x^2$ and $x^2 + y^2 = 1$ is

   a. $\frac{3\pi - 8}{3}$ sq. units  b. $\frac{\pi - 8}{3}$ sq. units
   c. $\frac{2\pi - 8}{3}$ sq. units  d. None of these

5. The area of the region enclosed by the curves $y = x \log x$ and $y = 2x - 2x^2$ is

   a. $\frac{7}{12}$ sq. units  b. $\frac{1}{2}$ sq. units
   c. $\frac{5}{12}$ sq. units  d. None of these

6. The area enclosed between the curves $y = \log_e(x + e)$, $x = \log_e\left(\frac{1}{y}\right)$ and the $x$-axis is

   a. 2 sq. units  b. 1 sq. units
   c. 4 sq. units  d. None of these

7. Area bounded by $y = \frac{1}{x^2 - 2x + 2}$ and $x$-axis is

   a. $2\pi$ sq. units  b. $\frac{\pi}{2}$ sq. units
   c. 2 sq. units  d. $\pi$ sq. units

8. Area bounded by the curve $xy^2 = a^2(a - x)$ and the $y$-axis is

   a. $\pi a^2/2$ sq. units  b. $\pi a^2$ sq. units
   c. $3\pi a^2$ sq. units  d. None of these

9. The area of the closed figure bounded by $x = -1$, $y = 0$, $y = x^2 + x + 1$ and the tangent to the curve $y = x^2 + x + 1$ at $A(1, 3)$ is

   a. 4/3 sq. units  b. 7/3 sq. units
   c. 7/6 sq. units  d. None of these

10. The area bounded by $y = \sec^{-1} x$, $y = \operatorname{cosec}^{-1} x$ and line $x - 1 = 0$ is

    a. $\log(3 + 2\sqrt{2}) - \frac{\pi}{2}$ sq. units
    b. $\frac{\pi}{2} - \log(3 + 2\sqrt{2})$ sq. units
    c. $\pi - \log_e 3$ sq. units
    d. None of these

11. The area of the region whose boundaries are defined by the curves $y = 2\cos x$, $y = 3\tan x$ and the $y$-axis is

    a. $1 + 3\ln\left(\frac{2}{\sqrt{3}}\right)$ sq. units
    b. $1 + \frac{3}{2}\ln 3 - 3\ln 2$ sq. units
    c. $1 + \frac{3}{2}\ln 3 - \ln 2$ sq. units
    d. $\ln 3 - \ln 2$ sq. units

12. The area between the curve $y = 2x^4 - x^2$, the $x$-axis and the ordinates of the two minima of the curve is

    a. 11/60 sq. units  b. 7/120 sq. units
    c. 1/30 sq. units  d. 7/90 sq. units

13. The area bounded by the curve $a^2y = x^2(x + a)$ and the $x$-axis is

    a. $a^2/3$ sq. units  b. $a^2/4$ sq. units
    c. $3a^2/4$ sq. units  d. $a^2/12$ sq. units

14. The area of the region in 1st quadrant bounded by the $y$-axis, $y = \frac{x}{4}$, $y = 1 + \sqrt{x}$ and $y = \frac{2}{\sqrt{x}}$ is

    a. 2/3 sq. units  b. 8/3 sq. units
    c. 11/3 sq. units  d. 13/6 sq. units

15. The area of the closed figure bounded by $y = \frac{x^2}{2} - 2x + 2$ and the tangents to it at (1, 1/2) and (4, 2) is

    a. 9/8 sq. units  b. 3/8 sq. units
    c. 3/2 sq. units  d. 9/4 sq. units

16. The area of the closed figure bounded by $x = -1$, $x = 2$ and $y = \begin{cases} -x^2 + 2, & x \le 1 \\ 2x - 1, & x > 1 \end{cases}$ and the abscissa axis is

    a. 16/3 sq. units  b. 10/3 sq. units
    c. 13/3 sq. units  d. 7/3 sq. units

17. The area of the region bounded by $x^2 + y^2 - 2x - 3 = 0$ and $y = |x| + 1$ is

    a. $\frac{\pi}{2} - 1$ sq. units  b. $2\pi$ sq. units
    c. $4\pi$ sq. units  d. $\pi/2$ sq. units

18. The value of the parameter $a$ such that the area bounded by $y = a^2x^2 + ax + 1$, coordinate axes and the line $x = 1$ attains its least value, is equal to

    a. $-\frac{1}{4}$ sq. units  b. $-\frac{1}{2}$ sq. units
    c. $-\frac{3}{4}$ sq. units  d. $-1$ sq. units

19. The area enclosed by the curve $y = \sqrt{4 - x^2}$, $y \ge \sqrt{2}\sin\left(\frac{x\pi}{2\sqrt{2}}\right)$ and the $x$-axis is divided by the $y$-axis in the ratio

    a. $\frac{\pi^2 - 8}{\pi^2 + 8}$  b. $\frac{\pi^2 - 4}{\pi^2 + 4}$
    c. $\frac{\pi - 4}{\pi - 4}$  d. $\frac{2\pi^2}{2\pi + \pi^2 - 8}$

20. If $f(x) = \sin x$, $\forall\, x \in \left[0, \frac{\pi}{2}\right]$, $f(x) + f(\pi - x) = 2$, $\forall\, x \in \left(\frac{\pi}{2}, \pi\right]$ and $f(x) = f(2\pi - x)$, $\forall\, x \in (\pi, 2\pi)$, then the area enclosed by $y = f(x)$ and the $x$-axis is
   a. $\pi$ sq. units  b. $2\pi$ sq. units
   c. 2 sq. units  d. 4 sq. units

21. The area of the region bounded by $x = 0$, $y = 0$, $x = 2$, $y = 2$, $y \le e^x$ and $y \ge \ln x$ is
   a. $6 - 4\ln 2$ sq. units  b. $4\ln 2 - 2$ sq. units
   c. $2\ln 2 - 4$ sq. units  d. $6 - 2\ln 2$ sq. units

22. The area of the loop of the curve, $ay^2 = x^2(a - x)$ is
   a. $4a^2$ sq. units  b. $\frac{8a^2}{15}$ sq. units
   c. $\frac{16a^2}{9}$ sq. units  d. None of these

23. The area of the region enclosed between the curves $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$ is
   a. 1 sq. units  b. 4/3 sq. units
   c. 2/3 sq. units  d. 2 sq. units

24. The area bounded by the loop of the curve $4y^2 = x^2(4 - x^2)$ is
   a. 7/3 sq. units  b. 8/3 sq. units
   c. 11/3 sq. units  d. 16/3 sq. units

25. The area enclosed by the curves $xy^2 = a^2(a - x)$ and $(a - x)y^2 = a^2x$ is
   a. $(\pi - 2)a^2$ sq. units  b. $(4 - \pi)a^2$ sq. units
   c. $\pi a^2/3$ sq. units  d. None of these

26. The area bounded by the curves $y = xe^x$, $y = xe^{-x}$ and the line $x = 1$ is
   a. $\frac{2}{e}$ sq. units  b. $1 - \frac{2}{e}$ sq. units
   c. $\frac{1}{e}$ sq. units  d. $1 - \frac{1}{e}$ sq. units

27. The area of the figure bounded by the parabola $(y - 2)^2 = x - 1$, the tangent to it at the point with the ordinate $x = 3$ and the $x$-axis is
   a. 7 sq. units  b. 6 sq. units
   c. 9 sq. units  d. None of these

28. The area bounded by $y = 3 - |3 - x|$ and $y = \frac{6}{|x + 1|}$ is
   a. $\frac{15}{2} - 6\ln 2$ sq. units  b. $\frac{13}{2} - 3\ln 2$ sq. units
   c. $\frac{13}{2} - 6\ln 2$ sq. units  d. None of these

29. The area of the region of the plane bounded by $\max(|x|, |y|) \le 1$ and $xy \le \frac{1}{2}$ is
   a. $1/2 + \ln 2$ sq. units  b. $3 + \ln 2$ sq. units
   c. 31/4 sq. units  d. $1 + 2\ln 2$ sq. units

30. The area bounded by the two branches of curve $(y - x)^2 = x^3$ and the straight line $x = 1$ is
   a. 1/5 sq. units  b. 3/5 sq. units
   c. 4/5 sq. units  d. 8/4 sq. units

31. Area bounded by the curves $y = \log_e x$ and $y = (\log_e x)^2$ is
   a. $e - 2$ sq. units  b. $3 - e$ sq. units
   c. $e$ sq. units  d. $e - 1$ sq. units

32. The area of the region containing the points $(x, y)$ satisfying $4 \le x^2 + y^2 \le 2(|x| + |y|)$ is
   a. 8 sq. units  b. 2 sq. units
   c. $4\pi$ sq. units  d. $2\pi$ sq. units

33. Let $f(x) = x^3 + 3x + 2$ and $g(x)$ is the inverse of it. Then the area bounded by $g(x)$, the $x$-axis and the ordinate at $x = -2$ and $x = 6$ is
   a. 1/4 sq. units  b. 4/3 sq. units
   c. 5/4 sq. units  d. 7/3 sq. units

34. The area bounded by the curve $f(x) = x + \sin x$ and its inverse function between the ordinates $x = 0$ and $x = 2\pi$ is
   a. $4\pi$ sq. units  b. $8\pi$ sq. units
   c. 4 sq. units  d. 8 sq. units

35. The area bounded by the $x$-axis, the curve $y = f(x)$ and the lines $x = 1$, $x = b$ is equal to $\sqrt{b^2 + 1} - \sqrt{2}$ for all $b > 1$, then $f(x)$ is
   a. $\sqrt{x - 1}$  b. $\sqrt{x + 1}$
   c. $\sqrt{x^2 + 1}$  d. $\frac{x}{\sqrt{1 + x^2}}$

36. Let $f(x)$ be a non-negative continuous function such that the area bounded by the curve $y = f(x)$, $x$-axis and the ordinates $x = \frac{\pi}{4}$ and $x = \beta > \frac{\pi}{4}$ is $\beta \sin\beta + \frac{\pi}{4}\cos\beta + \sqrt{2}\beta$. Then $f'\left(\frac{\pi}{2}\right)$ is
   a. $\left(\frac{\pi}{2} - \sqrt{2} - 1\right)$  b. $\left(\frac{\pi}{4} + \sqrt{2} - 1\right)$
   c. $-\frac{\pi}{2}$  d. $\left(1 - \frac{\pi}{4} - \sqrt{2}\right)$

37. The area bounded by the curves $y = \sin^{-1}|\sin x|$ and $y = (\sin^{-1}|\sin x|)^2$, where $0 \le x \le 2\pi$, is
   a. $\frac{1}{3} + \frac{\pi^2}{4}$ sq. units  b. $\frac{1}{6} + \frac{\pi^3}{8}$ sq. units
   c. 2 sq. units  d. None of these

38. Consider two curves $C_1$: $y^2 = 4[\sqrt{y}]x$ and $C_2$: $x^2 = 4[\sqrt{x}]y$, where [.] denotes the greatest integer function. Then the area of region enclosed by these two curves within the square formed by the lines $x = 1$, $y = 1$, $x = 4$, $y = 4$ is
   a. 8/3 sq. units  b. 10/3 sq. units
   c. 11/3 sq. units  d. 11/4 sq. units

39. The area enclosed between the curve $y^2(2a - x) = x^3$ and the line $x = 2$ above the $x$-axis is
   a. $\pi a^2$ sq. units  b. $\frac{3\pi a^2}{2}$ sq. units
   c. $2\pi a^2$ sq. units  d. $3\pi a^2$ sq. units

40. The area bounded by the curve $y^2 = 1 - x$ and the lines $y = \frac{|x|}{x}$, $x = -1$ and $x = \frac{1}{2}$ is

a. $\frac{3}{\sqrt{2}} - \frac{11}{6}$ sq. units
b. $3\sqrt{2} - \frac{11}{4}$ sq. units
c. $\frac{6}{\sqrt{2}} - \frac{11}{5}$ sq. units
d. None of these

## Multiple Correct Answers Type

Solutions on page 9.28

**Each question has four choices a, b, c, and d, out of which *one or more* answers are correct.**

1. Let $A(k)$ be the area bounded by the curves $y = x^2 - 3$ and $y = kx + 2$.
 a. The range of $A(k)$ is $\left[\frac{10\sqrt{5}}{3}, \infty\right)$
 b. The range of $A(k)$ is $\left[\frac{20\sqrt{5}}{3}, \infty\right)$
 c. If function $k \to A(k)$ is defined for $k \in [-2, \infty)$, then $A(k)$ is many-one function
 d. The value of $k$ for which area is minimum is 1
2. The parabolas $y^2 = 4x$ and $x^2 = 4y$ divide the square region bounded by the lines $x = 4, y = 4$ and the co-ordinate axes. If $S_1, S_2, S_3$ are the areas of these parts numbered from top to bottom, respectively, then
 a. $S_1{:}S_2 \equiv 1{:}1$
 b. $S_2{:}S_3 \equiv 1{:}2$
 c. $S_1{:}S_3 \equiv 1{:}1$
 d. $S_1{:}(S_1 + S_2) = 1{:}2$
3. Which of the following have the same bounded area
 a. $f(x) = \sin x, g(x) = \sin^2 x$, where $0 \le x \le 10\pi$
 b. $f(x) = \sin x, g(x) = |\sin x|$, where $0 \le x \le 20\pi$
 c. $f(x) = |\sin x|, g(x) = \sin^3 x$, where $0 \le x \le 10\pi$
 d. $f(x) = \sin x, g(x) = \sin^4 x$, where $0 \le x \le 10\pi$
4. If the curve $y = ax^{1/2} + bx$ passes through the point (1, 2) and lies above the $x$-axis for $0 \le x \le 9$ and the area enclosed by the curve, the $x$-axis and the line $x = 4$ is 8 sq. units. Then
 a. $a = 1$ b. $b = 1$ c. $a = 3$ d. $b = -1$
5. The area enclosed by the curves $x = a\sin^3 t$ and $y = a\cos^3 t$ is equal to
 a. $12a^2 \int_0^{\pi/2} \cos^4 t \sin^2 t\, dt$
 b. $12a \int_0^{\pi/2} \cos^2 t \sin^4 t\, dt$
 c. $2\int_{-a}^{a} \left(a^{2/3} - x^{2/3}\right)^{3/2} dx$
 d. $4\int_0^{a} \left(a^{2/3} - x^{2/3}\right)^{3/2} dx$
6. If $A_i$ is the area bounded by $|x - a_i| + |y| = b_i$, $i \in N$, where $a_{i+1} = a_i + \frac{3}{2}b_i$ and $b_{i+1} = \frac{b_i}{2}$, $a_1 = 0, b_1 = 32$, then
 a. $A_3 = 128$
 b. $A_3 = 256$
 c. $\lim_{n\to\infty} \sum_{i=1}^{n} A_i = \frac{8}{3}(32)^2$
 d. $\lim_{n\to\infty} \sum_{i=1}^{n} A_i = \frac{4}{3}(16)^2$

## Reasoning Type

Solutions on page 9.29

**Each question has four choices a, b, c, and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.**

a. if both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1
b. if both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1
c. if STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.
d. if STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. **Statement 1:** Area bounded by $y = e^x$, $y = 0$ and $x = 0$ is 1 sq. units.
 **Statement 2:** Area bounded by $y = \log_e x$, $x = 0$ and $y = 0$ is 1 sq. units.
2. $f(x)$ is a polynomial of degree 3 passing through origin having local extrema at $x = \pm 2$.
 **Statement 1:** Ratio of areas in which $f(x)$ cuts the circle $x^2 + y^2 = 36$ is 1:1.
 **Statement 2:** Both $y = f(x)$ and the circle are symmetric about origin.
3. **Statement 1:** The area bounded by parabola $y = x^2 - 4x + 3$ and $y = 0$ is 4/3 sq. units
 **Statement 2:** The area bounded by curve $y = f(x) \ge 0$ and $y = 0$ between ordinates $x = a$ and $x = b$ (where $b > a$) is $\int_a^b f(x)\,dx$.
4. **Statement 1:** The area enclosed between the parabolas $y^2 - 2y + 4x + 5 = 0$ and $x^2 + 2x - y + 2 = 0$ is same as that of bounded by curves $y^2 = -4x$ and $x^2 = y$.
 **Statement 2:** Shifting of origin to point $(h, k)$ does not change the bounded area.
5. **Statement 1:** The area of the region bounded by the curve $2y = \log_e x$, $y = e^{2x}$ and the pair of lines $(x + y - 1) \times (x + y - 3) = 0$ is $2k$ sq. units.
 **Statement 2:** The area of the region bounded by the curves $y = e^{2x}$, $y = x$ and the pair of lines $x^2 + y^2 + 2xy - 4x - 4y + 3 = 0$ is $k$ units.
6. **Consider two regions**
 $R_1$ : Point $P$ is nearer to (1, 0) than to $x = -1$.
 $R_2$ : Point $P$ is nearer to (0, 0) than to (8, 0).
 **Statement 1:** Area of the region common to $R_1$ and $R_2$ is $\frac{128}{3}$ sq. units.
 **Statement 2:** Area bounded by $x = 4\sqrt{y}$ and $y = 4$ is $\frac{32}{3}$ sq. units.
7. **Statement 1:** Area bounded by $2 \ge \max.\{|x - y|, |x + y|\}$ is 8 sq. units.
 **Statement 2:** Area of the square of side length 4 is 16 sq. units.

## Linked Comprehension Type

Solutions on page 9.30

**Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c, and d, out of which *only one* is correct.**

**For Problems 1–2**

Let $A_r$ be the area of the region bounded between the curves $y^2 = (e^{-kr})x$ (where $k > 0, r \in N$) and the line $y = mx$ (where $m \neq 0$), $k$ and $m$ are some constants.

1. $A_1, A_2, A_3, \ldots$ are in G.P. with common ratio
   a. $e^{-k}$ b. $e^{-2k}$
   c. $e^{-4k}$ d. None of these

2. $\lim_{n\to\infty} \sum_{i=1}^{n} A_i = \frac{1}{48(e^{2k}-1)}$, then the value of $m$ is
   a. 3 b. 1 c. 2 d. 4

**For Problems 3–5**

If $y = f(x)$ is a monotonic function in $(a, b)$, then the area bounded by the ordinates at $x = a, x = b, y = f(x)$ and $y = f(c)$ (where $c \in (a, b)$) is minimum when $c = \frac{a+b}{2}$.

**Proof:** $A = \int_a^c (f(c) - f(x))\,dx + \int_c^b (f(x) - f(c))\,dx$

$= f(c)(c-a) - \int_a^c (f(x))\,dx + \int_c^b (f(x))\,dx - f(c)(b-c)$

$\Rightarrow A = [2c - (a+b)]f(c) + \int_c^b (f(x))\,dx - \int_a^c (f(x))\,dx$

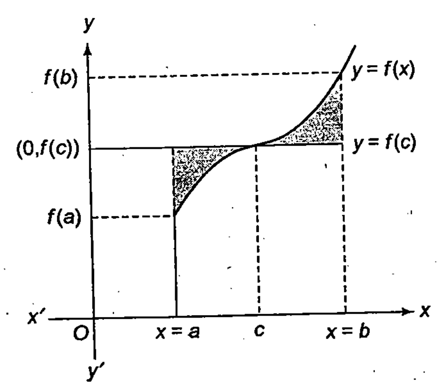



**Fig. 9.21**

Differentiating w.r.t. $c$,

$\frac{dA}{dc} = [2c - (a+b)]f'(c) + 2f(c) + 0 - f(c) - (f(c) - 0)$

for maxima and minima $\frac{dA}{dc} = 0$

$\Rightarrow f'(c)[2c - (a+b)] = 0$ (as $f'(c) \neq 0$)

hence $c = \frac{a+b}{2}$

also for $c < \frac{a+b}{2}, \frac{dA}{dc} < 0$ and for $c > \frac{a+b}{2}, \frac{dA}{dc} > 0$

Hence $A$ is minimum when $c = \frac{a+b}{2}$.

3. If the area bounded by $f(x) = \frac{x^3}{3} - x^2 + a$ and the straight lines $x = 0, x = 2$ and the $x$-axis is minimum, then the value of $a$ is
   a. 1/3 b. 2 c. 1 d. 2/3

4. The value of the parameter $a$ for which the area of the figure bounded by the abscissa axis, the graph of the function $y = x^3 + 3x^2 + x + a$ and the straight lines, which are parallel to the axis of ordinates and cut the abscissa axis at the point of extremum of the function, which is the least, is
   a. 2 b. 0 c. −1 d. 1

5. If the area enclosed by $f(x) = \sin x + \cos x$, $y = a$ between two consecutive points of extremum is minimum, then the value of $a$ is
   a. 0 b. −1 c. 1 d. 2

**For Problems 6–8**

Consider the areas $S_0, S_1, S_2 \ldots$ bounded by the $x$-axis and half-waves of the curve $y = e^{-x}\sin x$, where $x \geq 0$.

6. The value of $S_0$ is
   a. $\frac{1}{2}(1+e^{\pi})$ sq. units b. $\frac{1}{2}(1+e^{-\pi})$ sq. units
   c. $\frac{1}{2}(1-e^{-\pi})$ sq. units d. $\frac{1}{2}(e^{\pi}-1)$ sq. units

7. The sequence $S_0, S_1, S_2, \ldots$, forms a G.P. with common ratio
   a. $\frac{e^{\pi}}{2}$ b. $e^{-\pi}$ c. $e^{\pi}$ d. $\frac{e^{-\pi}}{2}$

8. $\sum_{n=0}^{\infty} S_n$ is equal to
   a. $\frac{1+e^{\pi}}{1-e^{-\pi}}$ b. $\frac{\frac{1}{2}(1+e^{\pi})}{1-e^{-\pi}}$
   c. $\frac{1}{2(1-e^{-\pi})}$ d. None of these

**For Problems 9–11**

Two curves $C_1 \equiv [f(y)]^{2/3} + [f(x)]^{1/3} = 0$ and $C_2 \equiv [f(y)]^{2/3} + [f(x)]^{2/3} = 12$, satisfying the relation $(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2-y^2)$.

9. The area bounded by $C_1$ and $C_2$ is
   a. $2\pi - \sqrt{3}$ sq. units b. $2\pi + \sqrt{3}$ sq. units
   c. $\pi + \sqrt{6}$ sq. units d. $2\sqrt{3} - \pi$ sq. units

10. The area bounded by the curve $C_2$ and $|x| + |y| = \sqrt{12}$ is
   a. $12\pi - 24$ sq. units b. $6 - \sqrt{12}$ sq. units
   c. $2\sqrt{12} - 6$ sq. units d. None of these

**11.** The area bounded by $C_1$ and $x+y+2=0$ is

a. 5/2 sq. units  b. 7/2 sq. units
c. 9/2 sq. units  d. None of these

**For Problems 12–13**

Consider the two curves $C_1: y = 1 + \cos x$ and $C_2: y = 1 + \cos(x-\alpha)$ for $\alpha \in \left(0, \frac{\pi}{2}\right)$, where $x \in [0, \pi]$. Also the area of the figure bounded by the curves $C_1$, $C_2$ and $x = 0$ is same as that of the figure bounded by $C_2$, $y = 1$ and $x = \pi$.

**12.** The value of $\alpha$ is

a. $\frac{\pi}{4}$  b. $\frac{\pi}{3}$  c. $\frac{\pi}{6}$  d. $\frac{\pi}{8}$

**13.** For the values of $\alpha$, area bounded by $C_1$, $C_2$, $x = 0$ and $x = \pi$ is

a. 1 sq. units  b. 2 sq. units
c. $2 + \sqrt{3}$ sq. units  d. None of these

**For Problems 14–16**

Consider the function defined implicitly by the equation $y^2 - 2ye^{\sin^{-1}x} + x^2 - 1 + [x] + e^{2\sin^{-1}x} = 0$ (where $[x]$ denotes the greatest integer function).

**14.** The area of the region bounded by the curve and the line $x = -1$ is

a. $\pi + 1$ sq. units  b. $\pi - 1$ sq. units
c. $\frac{\pi}{2} + 1$ sq. units  d. $\frac{\pi}{2} - 1$ sq. units

**15.** Line $x = 0$ divides the region mentioned above in two parts. The ratio of area of left-hand side of line to that of right-hand side of line is

a. $1+\pi:\pi$  b. $2-\pi:\pi$
c. 1:1  d. $\pi+2:\pi$

**16.** The area of the region of curve and line $x = 0$ and $x = \frac{1}{2}$ is

a. $\frac{\sqrt{3}}{4} + \frac{\pi}{6}$ sq. units  b. $\frac{\sqrt{3}}{2} + \frac{\pi}{6}$ sq. units
c. $\frac{\sqrt{3}}{4} - \frac{\pi}{6}$ sq. units  d. $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$ sq. units

**For Problems 17 – 19**

Computing area with parametrically represented boundaries:

If the boundary of a figure is represented by parametric equation, i.e., $x = x(t)$, $y = y(t)$, then the area of the figure is evaluated by one of the three formulas

$$S = -\int_\alpha^\beta y(t)x'(t)\,dt,\quad S = \int_\alpha^\beta x(t)y'(t)\,dt,$$

$$S = \frac{1}{2}\int_\alpha^\beta (xy' - yx')\,dt,$$

where $\alpha$ and $\beta$ are the values of the parameter $t$ corresponding respectively to the beginning and the end of the traversal of the curve corresponding to increasing $t$.

**17.** The area of the region bounded by an arc of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ and the $x$-axis is

a. $6\pi a^2$ sq. units  b. $3\pi a^2$ sq. units
c. $4\pi a^2$ sq. units  d. None of these

**18.** The area of the loop described as $x = \frac{t}{3}(6-t)$, $y = \frac{t^2}{8}(6-t)$ is

a. $\frac{27}{5}$ sq. units  b. $\frac{24}{5}$ sq. units
c. $\frac{27}{6}$ sq. units  d. $\frac{21}{5}$ sq. units

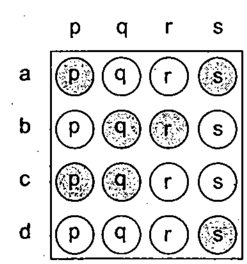
**19.** If the curve given by parametric equation $x = t - t^3$, $y = 1 - t^4$ forms a loop for all values of $t \in [-1, 1]$, then the area of the loop is

a. $\frac{1}{7}$ sq. units  b. $\frac{3}{5}$ sq. units
c. $\frac{16}{35}$ sq. units  d. $\frac{8}{35}$ sq. units

## Matrix-Match Type

*Solutions on page 9.34*

**Each question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct match are a-p, a-s, b-r, c-p, c-q and d-s, then the correctly bubbled 4 × 4 matrix should be as follows:**



**1.**

| Column 1 | Column 2 |
|---|---|
| a. The area bounded by the curve $y = x\|x\|$, $x$-axis and the ordinates $x = 1, x = -1$ | p. 10/3 sq. units |
| b. The area of the region lying between the lines $x - y + 2 = 0$, $x = 0$ and the curve $x = \sqrt{y}$ | q. 64/3 sq. units |
| c. The area enclosed between the curves $y^2 = x$ and $y = \|x\|$ | r. 2/3 sq. units |
| d. The area bounded by parabola $y^2 = x$, straight line $y = 4$ and $y$-axis | s. 1/6 sq. units |

2.

| Column 1 | Column 2 |
|---|---|
| a. Area enclosed by $y = [x]$ and $y = \{x\}$, where $[\cdot]$ and $\{\cdot\}$ represent greatest integer and fractional part functions, respectively | p. 32/5 sq. units |
| b. The area bounded by the curves $y^2 = x^3$ and $\|y\| = 2x$. | q. 1 sq. units |
| c. The smaller area included between the curves $\sqrt{x} + \sqrt{\|y\|} = 1$ and $\|x\| + \|y\| = 1$. | r. 4 sq. units |
| d. Area bounded by the curves $y = \left[\frac{x^2}{64} + 2\right]$ (where [.] denotes the greatest integer function), $y = x - 1$ and $x = 0$ above the $x$-axis. | s. 2/3 sq. units |

3.

| Column 1: [·] represents greatest integer function. | Column 2 |
|---|---|
| a. Area enclosed by $[x]^2 = [y]^2$ for $1 \le x \le 4$ | p. 8 sq. units |
| b. Area enclosed by $[\|x\|] + [\|y\|] = 2$ | q. 6 sq. units |
| c. Area enclosed by $[\|x\|]\,[\|y\|] = 2$ | r. 4 sq. units |
| d. Area enclosed by $\frac{[\|x\|]}{[\|y\|]} = 2, -5 \le x \le 5$ | s. 12 sq. units |

## Integer Type

Solutions on page 9.36

1. The area enclosed by the curve $C: y = x\sqrt{9 - x^2}$ $(x \ge 0)$ and the $x$-axis is
2. Let $S$ be the area bounded by the curve $y = \sin x\ (0 \le x \le \pi)$ and the $x$-axis and $T$ be the area bounded by the curves $y = \sin x\left(0 \le x \le \frac{\pi}{2}\right)$, $y = a\cos x\left(0 \le x \le \frac{\pi}{2}\right)$ and the $x$-axis (where $a \in R^+$).

   The value of $(3a)$ such that $S: T = 1 : \frac{1}{3}$ is
3. Let $C$ be a curve passing through $M(2, 2)$ such that the slope of the tangent at any point to the curve is reciprocal of the ordinate of the point. If the area bounded by curve $C$ and line $x = 2$ is $A$, then the value of $\frac{3A}{2}$ is
4. The area enclosed by $f(x) = 12 + ax + -x^2$ coordi-nates axes and the ordinates at $x = 3$ $(f(3) > 0)$ is 45 square units. If $m$ and $n$ are the $x$-axis intercepts of the graph of $y = f(x)$ then the value of $(m + n + a)$ is
5. If the area bounded by the curve $f(x) = x^{1/3}(x - 1)$ and $x$-axis is $A$, then the value of $28A$ is
6. If the area bounded by the curve $y = x^2 + 1$ and the tangents to it drawn from the origin is $A$, then the value of $3A$ is
7. If the area enclosed by the curve $y = \sqrt{x}$ and $x = -\sqrt{y}$, the circle $x^2 + y^2 = 2$ above the $x$-axis, is $A$ then the value of $\frac{16}{\pi}A$ is
8. The value of '$a$' $(a > 0)$ for which the area bounded by the curves $y = \frac{x}{6} + \frac{1}{x^2}$, $y = 0$, $x = a$ and $x = 2a$ has the least value is
9. Area bounded by the relation $[2x] + [y] = 5$, $x, y > 0$, is (where $[\cdot]$ represents greatest integer function)
10. The area bounded by the curves $y = x(x - 3)^2$ and $y = x$ is (in sq. units) :
11. If the area of the region $\{(x, y): 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$ is $A$, then the value of $3A - 17$ is
12. If $S$ is the sum of possible values of $c$ for which the area of the figure bounded by the curves $y = \sin 2x$, the straight lines $x = \pi/6$, $x = c$ and the abscissa axis is equal to 1/2, then the value of $\pi/S$ is
13. If $A$ is the area bounded by the curves $y = \sqrt{1 - x^2}$ and $y = x^3 - x$, then the value of $\pi/A$.
14. Consider two curves $C_1: y = \frac{1}{x}$ and $C_2: y = \ln x$ on the $xy$ plane. Let $D_1$ denotes the region surrounded by $C_1$, $C_2$ and the line $x = 1$ and $D_2$ denotes the region surrounded by $C_1$, $C_2$ and the line $x = a$. If $D_1 = D_2$, then the sum of logarithm of possible values of $a$ is
15. If '$a$' $(a > 0)$ is the value of parameter for each of which the area of the figure bounded by the straight line, $y = \frac{a^2 - ax}{1 + a^4}$ and the parabola $y = \frac{x^2 + 2ax + 3a^2}{1 + a^4}$ is the greatest, then the value of $a^4$ is
16. If $S$ is the sum of cubes of possible value of '$c$' for which the area of the figure bounded by the curve $y = 8x^2 - x^5$, then straight lines $x = 1$ and $x = c$ and the abscissa axis is equal to 16/3, then the value of $[S]$, where $[\cdot]$ denotes the greatest integer function, is

## Archives

Solutions on page 9.40

### Subjective

1. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$. **(IIT-JEE, 1981)**
2. For any real $t$, $x = \frac{1}{2}(e^t + e^{-t})$, $y = \frac{1}{2}(e^t - e^{-t})$ is a point on the hyperbola $x^2 - y^2 = 1$. Show that the area bounded by the hyperbola and the lines joining its centre to the points corresponding to $t_1$ and $-t_1$ is $t_1$. **(IIT-JEE, 1982)**
3. Find the area bounded by the $x$-axis, part of the curve $y = \left(1 + \frac{8}{x^2}\right)$ and the ordinates at $x = 2$ and $x = 4$. If the ordinate at $x = a$ divides the area into two equal parts, then find $a$. **(IIT-JEE, 1983)**

4. Find the area of the region bounded by the $x$-axis and the curves defined by $y = \tan x$, $\left(\text{where } -\frac{\pi}{3} \le x \le \frac{\pi}{3}\right)$ and $y = \cot x$ $\left(\text{where } \frac{\pi}{6} \le x \le \frac{3\pi}{2}\right)$. **(IIT-JEE, 1984)**

5. Sketch the region bounded by the curves $y = \sqrt{5-x^2}$ and $y = |x-1|$ and find its area. **(IIT-JEE, 1985)**

6. Find the area bounded by the curves $x^2 + y^2 = 4$, $x^2 = -\sqrt{2}y$ and $x = y$. **(IIT-JEE, 1986)**

7. Find the area bounded by the curves $x^2 + y^2 = 25$, $4y = |4 - x^2|$ and $x = 0$ above the $x$-axis. **(IIT-JEE, 1987)**

8. Find the area of the region bounded by the curve $C: y = \tan x$, tangent drawn to $C$ at $x = \frac{\pi}{4}$ and the $x$-axis.

9. Compute the area of the region bounded by the curves $y = ex \log_e x$ and $y = \frac{\log x}{ex}$. **(IIT-JEE, 1990)**

10. Sketch the curves and identify the region bounded by $x = \frac{1}{2}$, $x = 2$, $y = \ln x$ and $y = 2^x$. Find the area of this region. **(IIT-JEE, 1991)**

11. Sketch the region bounded by the curves $y = x^2$ and $y = \frac{2}{1+x^2}$. Find the area. **(IIT-JEE, 1992)**

12. In what ratio does the $x$-axis divide the area of the region bounded by the parabolas $y = 4x - x^2$ and $y = x^2 - x$?

13. Consider a square with vertices at (1, 1), (−1, 1), (−1, −1) and (1, −1). Let $S$ be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region $S$ and find its area. **(IIT-JEE, 1995)**

14. Let $A_n$ be the area bounded by the curve $y = (\tan x)^n$ and the lines $x = 0$, $y = 0$ and $x = \frac{\pi}{4}$. Prove that for $n > 2$, $A_n + A_{n-2} = \frac{1}{n-1}$ and deduce $\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$.

15. Find all the possible values of $b > 0$, so that the area of the bounded region enclosed between the parabolas $y = x - bx^2$ and $y = \frac{x^2}{b}$ is maximum.

16. Let $O$ (0, 0), $A$(2, 0) and $B\left(1, \frac{1}{\sqrt{3}}\right)$ be the vertices of a triangle. Let $R$ be the region consisting of all those points $P$ inside $\Delta OAB$ which satisfy $d(P, OA) \le \min[d(P, OB), d(P, AB)]$, where $d$ denotes the distance from the point to the corresponding line. Sketch the region $R$ and find its area.

17. Let $f(x) = \text{Maximum } \{x^2, (1-x)^2, 2x(1-x)\}$, where $0 \le x \le 1$. Determine the area of the region bounded by the curves $y = f(x)$, $x$-axis, $x = 0$ and $x = 1$.

18. Let $C_1$ and $C_2$ be the graphs of the functions $y = x^2$ and $y = 2x$, respectively, where $0 \le x \le 1$. Let $C_3$ be the graph of a function $y = f(x)$, where $0 \le x \le 1$, $f(0) = 0$. For a point $P$ on $C_1$, let the lines through $P$, parallel to the axis, meet $C_2$ and $C_3$ at $Q$ and $R$, respectively (see Fig. 9.22). If for every position of $P$ (on $C_1$), the areas of the shaded regions $OPQ$ and $ORP$ are equal, determine the function $f(x)$.

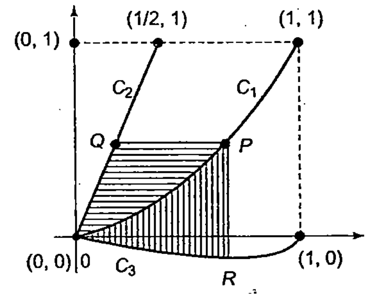



**Fig. 9.22**

19. Let $f(x)$ be a continuous function given by $f(x) = \begin{cases} 2x, & |x| \le 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}$. Find the area of the region in the third quadrant bounded by the curves $x = -2y^2$ and $y = f(x)$ lying on the left of the line $8x + 1 = 0$.

20. Find the area of the region bounded by the curves $y = x^2$, $y = |2 - x^2|$ and $y = 2$, which lies to the right of the line $x = 1$.

21. Find the area bounded by the curve $x^2 = y$, $x^2 = -y$ and $y^2 = 4x - 3$.

22. If $f(x)$ be a differentiable function such that $f'(x) = g(x)$, $g''(x)$ exists, $|f(x)| < 1$ and $(f(0))^2 + (g(0))^2 = 9$. Prove that there is a point $c$ where $c \in (-3, 3)$ such that $g(c).g''(c) < 0$. **(IIT-JEE, 2005)**

23. If $f(x)$ is a quadratic polynomial and $a$, $b$, $c$ are three real and distinct numbers satisfying

$$\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix}.$$

Given $f(x)$ cuts the $x$-axis at $A$ and $V$ is the point of maxima. If $AB$ is any chord which subtends a right angle at $V$, find curve $f(x)$ and area bounded by the chord $AB$ and curve $f(x)$. **(IIT-JEE, 2005)**

## Objective

***Multiple choice questions with one correct answer***

1. The area bounded by the curves $y = f(x)$, the $x$-axis and the ordinates $x = 1$ and $x = b$ is $(b-1)\sin(3b+4)$. Then $f(x)$ is
   a. $(x-1)\cos(3x+4)$
   b. $\sin(3x+4)$
   c. $\sin(3x+4) + 3(x-1)\cos(3x+4)$
   d. None of these **(IIT-JEE, 1982)**

2. The area bounded by the curves $y = |x| - 1$ and $y = -|x| + 1$ is
   a. 1 sq. units
   b. 2 sq. units
   c. $2\sqrt{2}$ sq. units
   d. 4 sq. units **(IIT-JEE, 2002)**

3. The area bounded by the curves $y = \sqrt{x}$, $2y + 3 = x$ and $x$-axis in the 1st quadrant is
   a. 9 sq. units  b. 27/4 sq. units
   c. 36 sq. units  d. 9 sq. units
   **(IIT-JEE, 2002)**

4. The area bounded by the parabolas $y = (x + 1)^2$ and $y = (x - 1)^2$ and the line $y = 1/4$ is
   a. 4 sq. units  b. 1/6 sq. units
   c. 4/3 sq. units  d. 1/3 sq. units
   **(IIT-JEE, 2005)**

5. The area enclosed between the curves $y = ax^2$ and $x = ay^2$ (where $a > 0$) is 1 sq. unit, then the value of $a$ is
   a. $1/\sqrt{3}$  b. 1/2  c. 1  d. 1/3
   **(IIT-JEE, 2004)**

6. Let the straight line $x = b$ divide the area enclosed by $y = (1 - x)^2$, $y = 0$ and $x = 0$ into two parts $R_1$ $(0 \le x \le b)$ and $R_2$ $(b \le x \le 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then $b$ equals
   a. 3/4  b. 1/2  c. 1/3  d. 1/4
   **(IITJEE 2011)**

***Multiple choice questions with one or more than one correct answer***

1. For which of the following values of $m$ is the area of the regions bounded by the curve $y = x - x^2$ and the line $y = mx$ equal 9/2?
   a. −4  b. −2  c. 2  d. 4
   **(IIT-JEE, 1999)**

2. Area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is **(IIT-JEE, 2009)**
   a. $e - 1$  b. $\int_1^e \ln (e + 1 - y)\, dy$
   c. $e - \int_0^1 e^x\, dx$  d. $\int_1^e \ln y\, dy$

# ANSWERS AND SOLUTIONS

## Subjective Type

1. $f(x) = \dfrac{(x+1)(x+2)}{(x-1)(x-2)}$

Graph will cut $x$-axis at $x = -1$ and $x = -2$.
It is discontinuous at $x = 1$ and $x = 2$.

$\lim_{x \to \pm\infty} f(x) \to 1$, $\lim_{x \to 1^-} f(x) \to +\infty$

$\lim_{x \to 1^+} f(x) \to -\infty$

$\lim_{x \to 2^-} f(x) \to -\infty$,

$\lim_{x \to 2^+} f(x) \to +\infty$, $f(0) = 1$.

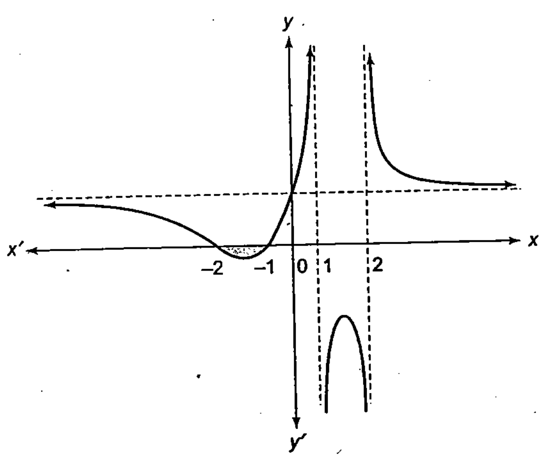



**Fig. 9.23**

Now we have to find the area of the shaded region. The required area

$$= \left|\int_{-2}^{-1} f(x)dx\right| = \left|\int_{-2}^{-1}\left(\frac{x^2+3x+2}{x^2-3x+2}\right)dx\right| = \left|\int_{-2}^{-1} 1 + \frac{6x}{(x-1)(x-2)}dx\right|$$

$$= \left|\left[x\Big|_{-2}^{-1} + 6\int_{-2}^{-1}\left(\frac{2}{x-2} - \frac{1}{x-1}\right)dx\right]\right|$$

$$= |1 + 6[2\ln|x-2| - \ln|x-1|]_{-2}^{-1}|$$

$$= |1 + 6[2(\ln 3 - \ln 4) - (\ln 2 - \ln 3)]|$$

$$= |1 + 6[3\ln 3 - 5\ln 2]|$$

$$= 6 \ln\left(\frac{32}{27}\right) - 1 \text{ sq. units.}$$

2. Given $\left|\int_{a-t}^{a} f(x)dx\right| = \left|\int_{a}^{a+t} f(x)dx\right|, \forall t \in R$

$$\Rightarrow \int_{a-t}^{a} f(x)dx = -\int_{a}^{a+t} f(x)dx$$

[$\because f(a) = 0$ and $f(x)$ is monotonic]

$\Rightarrow f(a - t) = -f(a + t)$

$\Rightarrow (a - t) + f(a + t) = 0$  (1)

$f(a + t) = -f(a - t) = x$ (say)

$\Rightarrow t = f^{-1}(x) - a$  (2)

and $t = a - f^{-1}(-x)$  (3)

From equations (3) and (2), $(a - f^{-1}(x)) + (a - f^{-1}(-x)) = 0$

$$\Rightarrow \int_{-\lambda}^{\lambda} f^{-1}(x)dx = \frac{1}{2}\int_{-\lambda}^{\lambda} (f^{-1}(x) + f^{-1}(-x))\,dx = 2a\lambda.$$

3. According to the given conditions

$$\int_0^t \left[f(x) - (x^4 - 4x^2)\right]dx = k\int_0^t \left[(2x^2 - x^3) - f(x)\right]dx$$

Differentiate both sides w.r.t. '$t$', we get

$f(t) - (t^4 - 4t^2) = k(2t^2 - t^3 - f(t))$ or

$(1 + k) f(t) = k2t^2 - kt^3 + t^4 - 4t^2$

$$\Rightarrow f(t) = \frac{1}{k+1}[t^4 - kt^3 + (2k-4)t^2]$$

Hence, required $f$ is given by $f(x) = \frac{1}{k+1}(x^4 - kx^3 + 2(k-2)x^2)$.

**4.** The given curves are

$y = -x^2 + 6x - 5$ or $(x-3)^2 = -(y-4)$ (1)

which is a parabola with vertex at $A_1(3, 4)$ and axis parallel to the $y$-axis. It intersects the $x$-axis at the points $P(1, 0)$ and $Q(5, 0)$

$y = -x^2 + 4x - 3$ or $(x-2)^2 = -(y-1)$ (2)

which is a parabola with vertex at $A_2(2, 1)$ and axis parallel to the $y$-axis. It intersects the $x$-axis at the points $P(1, 0)$ and $R(3, 0)$.

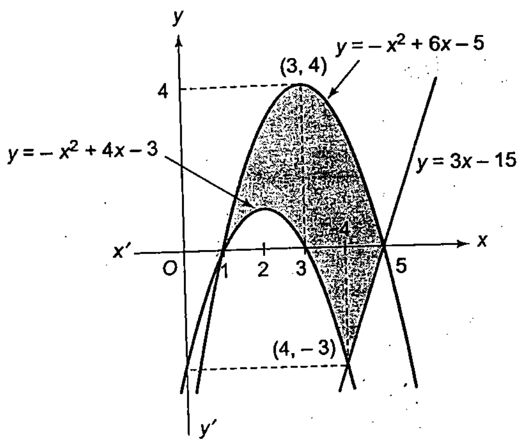



**Fig. 9.24**

and $y = 3x - 15$ (3)

Solving, the points of intersections of (1), (2) is (1,0); (1), (3) are (–2,–21) and (5,0) and (2), (3) are (–3,–24) and (4, –3).

Thus, the required area is the shaded area in the diagram.

Required area

$$= \left|\int_1^4 (y_1 - y_2)dx\right| + \left|\int_4^5 (y_1 - y_3)dx\right|$$

$$= \left|\int_1^4 [(-x^2 + 6x - 5) - (-x^2 + 4x - 3)]dx\right| + \left|\int_4^5 [(-x^2 + 6x - 5) - (3x - 15)]dx\right|$$

$$= \left|\int_1^4 (2x-2)dx\right| + \left|\int_4^5 (-x^2 + 3x + 10)\,dx\right|$$

$= 9 + 19/6 = 73/6$ sq. units.

**5.** Solving the given curves $y = \frac{1}{x^2}; y = \frac{1}{4(x-1)}$

$x^2 = 4(x-1) \Rightarrow (x-2)^2 = 0$

$\Rightarrow$ curves touch other

$$\therefore A = \int_2^a \left(\frac{1}{4(x-1)} - \frac{1}{x^2}\right)dx = \frac{1}{a}$$

$$\Rightarrow \left[\frac{1}{4}\log(x-1) + \frac{1}{x}\right]_2^a = \frac{1}{a}$$

$$\Rightarrow \frac{1}{4}\log(a-1) + \frac{1}{a} - \frac{1}{2} = \frac{1}{a}$$

$\Rightarrow \log(a-1) = 2$

$\Rightarrow a = e^2 + 1$

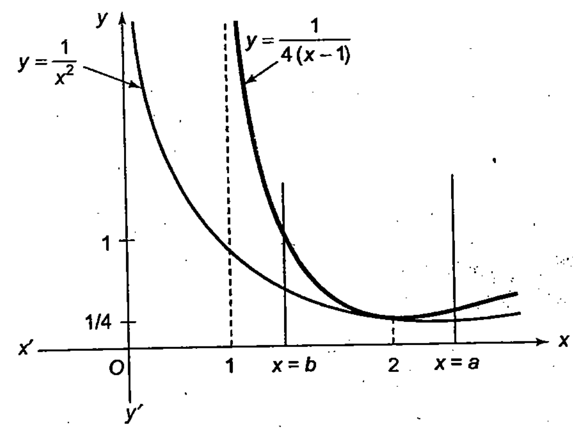



**Fig. 9.25**

$a = e^2 + 1$

Also, $1 - \frac{1}{b} = \int_b^2 \left(\frac{1}{4(x-1)} - \frac{1}{x^2}\right)dx \Rightarrow b = 1 + e^{-2}$

**6.** $x_1$ and $x_2$ are the roots of the equation

$x^2 + 2x - 3 = kx + 1$, or

$x^2 + (2-k)x - 4 = 0$

$$\Rightarrow \left.\begin{array}{l} x_1 + x_2 = k-2 \\ x_1 x_2 = -4 \end{array}\right\}$$

8.

$$A = \int_{x_1}^{x_2} \left[(kx+1) - (x^2 + 2x - 3)\right]dx$$

$$= \left[(k-2)\frac{x^2}{2} - \frac{x^3}{3} + 4x\right]_{x_1}^{x_2}$$

$$= \left[(k-2)\frac{x_2^2 - x_1^2}{2} - \frac{1}{3}(x_2^3 - x_1^3) + 4(x_2 - x_1)\right]$$

$$= (x_2 - x_1)\left[\frac{(k-2)^2}{2} - \frac{1}{3}\left((x_2 + x_1)^2 - x_1x_2\right) + 4\right]$$

$$= \sqrt{(x_2 + x_1)^2 - 4x_1x_2}\left[\frac{(k-2)^2}{2} - \frac{1}{3}\left((k-2)^2 + 4\right) + 4\right]$$

$$= \frac{\sqrt{(k-2)^2+16}}{6}\left[\frac{1}{6}(k-2)^2+\frac{8}{3}\right]$$

$$= \frac{\left[(k-2)^2+16\right]^{3/2}}{6}$$

which is least when $k = 2$ and $A_{\text{least}} = 32/3$ sq. units.

7. Equation of curve can be re-written as

$2y^2 + 6(1+x)y + 5x^2 + 7x + 6 = 0$

$$\Rightarrow y_1 = \frac{-3(1+x) - \sqrt{(3-x)(x-1)}}{2},$$

$$y_2 = \frac{-3(1+x) + \sqrt{(3-x)(x-1)}}{2}$$

Therefore, the curves ($y_1$ and $y_2$) are defined for values of $x$ for which $(3-x)(x-1) \geq 0$, i.e., $1 \leq x \leq 3$.

(Actually the given equation denotes an ellipse, because $\Delta \neq 0$ and $h^2 < ab$.)

Required area will be given by

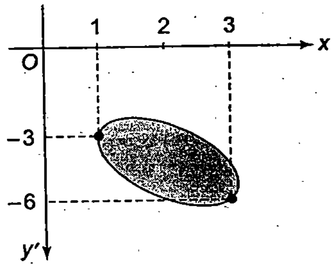



Fig. 9.26

$$A = \int_1^3 (y_1 - y_2)\,dx \Rightarrow A = \int_1^3 \sqrt{(3-x)(x-1)}\,dx$$

Put $x = 3\cos^2\theta + \sin^2\theta$, i.e., $dx = -2\sin 2\theta\,d\theta$

$$A = 2\int_0^{\pi/2} \sin^2 2\theta\,d\theta = \frac{\pi}{2} \text{ sq. units.}$$

8. a. $\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}$

$x = 0 \Rightarrow y = \pm a$

$y = 0 \Rightarrow x = \pm a$

(a) Required area

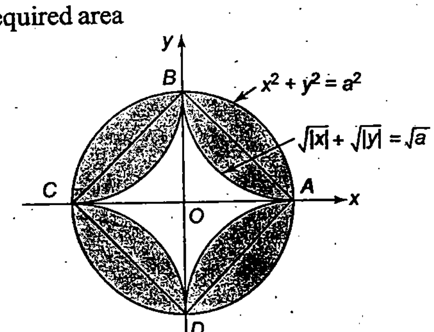



Fig. 9.27

$$\Delta = 4\int_0^a \sqrt{a^2 - x^2}\,dx - 4\int_0^a (\sqrt{a} - \sqrt{x})^2\,dx$$

$$= \pi a^2 - 4\int_0^a (\sqrt{a} - \sqrt{x})^2\,dx$$

$$= \pi a^2 - 4\int_0^a [a + x - 2\sqrt{a}\sqrt{x}]\,dx$$

$$= \pi a^2 - 4\left[a^2 + \frac{a^2}{2} - 2\sqrt{a}\,\frac{2}{3}a^{3/2}\right]$$

$$= \pi a^2 - 4\left[\frac{3a^2}{2} - \frac{4}{3}a^2\right] = \pi a^2 - 4\frac{a^2}{6}$$

$$= \left(\pi - \frac{2}{3}\right)a^2 \text{ sq. units.}$$

b. Area included between curves and circle in Ist quadrant

$$= \frac{1}{4}\pi a^2 - \frac{1}{2}a \times a = \frac{(\pi-2)a^2}{4}$$

Area included between $|x| + |y| = a$ and curve $\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}$ in Ist quadrant

$$= \frac{1}{4}\left(\pi - \frac{2}{3}\right)a^2 - \left(\frac{\pi}{4} - \frac{1}{2}\right)a^2 = \frac{a^2}{3}$$

$$\text{Area ratio} = \frac{4}{3(\pi - 2)}$$

## Objective Type

1. a. Clearly $t$ can be any real number

Let $t = \tan\theta \Rightarrow x = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$

$\Rightarrow x = \cos 2\theta$, and

$$y = \frac{2\tan\theta}{1 + \tan^2\theta} = \sin 2\theta$$

$\Rightarrow x^2 + y^2 = 1$

Thus, required area $= \pi$ sq. units.

2. a. Given $5x^2 - y = 0$, and (1)

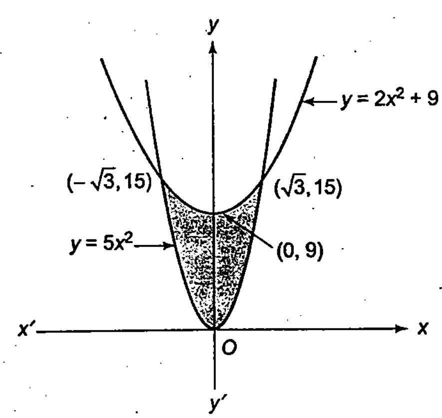



Fig. 9.28

$2x^2 - y + 9 = 0$ (2)

Eliminating $y$, we get

$5x^2 - (2x^2 + 9) = 0$

$\Rightarrow 3x^2 = 9 \Rightarrow x = -\sqrt{3}, \sqrt{3}$

$\therefore$ required area

$= 2\int_0^{\sqrt{3}} \left((2x^2 + 9) - 5x^2\right) dx$

$= 2\int_0^{\sqrt{3}} \left(9 - 3x^2\right) dx$

$= 2\left[9x - x^3\right]_0^{\sqrt{3}}$

$= 2\left[9\sqrt{3} - 3\sqrt{3}\right]$

$= 12\sqrt{3}$ sq. units

**3. d.**

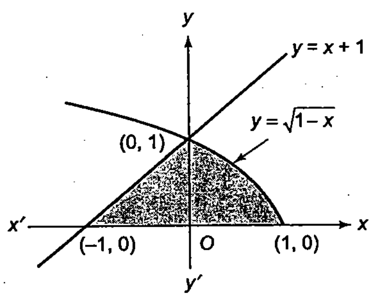



**Fig. 9.29**

Required area = shaded region

$= \int_0^1 (x_2 - x_1)dy$ (integrating along $y$-axis)

$= \int_0^1 [(1 - y^2) - (y - 1)]dy$

$= \frac{7}{6}$ sq. unit

**4. a.**

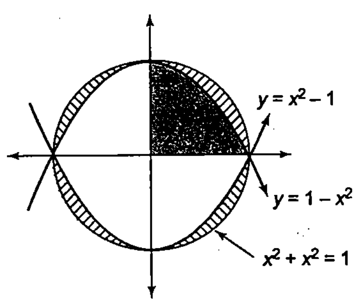



**Fig. 9.30**

The dotted area is

$A = \int_0^1 (1 - x^2)dx = \left(x - \frac{x^3}{3}\right)_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$

Hence, area bounded by circle $x^2 + y^2 = 1$ and

$|y| = 1 - x^2$

= lined area

= Area of circle – area bounded by $|y| = 1 - x^2$

$= \pi - 4 \cdot \left(\frac{2}{3}\right) = \frac{3\pi - 8}{3}$ sq. units.

**5. a.** Curve tracing : $y = x \log_e x$

Clearly, $x > 0$,

For $0 < x < 1$, $x \log_e x < 0$, and for $x > 1$, $x \log_e x > 0$

Also $x \log_e x = 0 \Rightarrow x = 1$.

Further, $\frac{dy}{dx} = 0 \Rightarrow 1 + \log_e x = 0 \Rightarrow x = 1/e$, which is a point of minima.

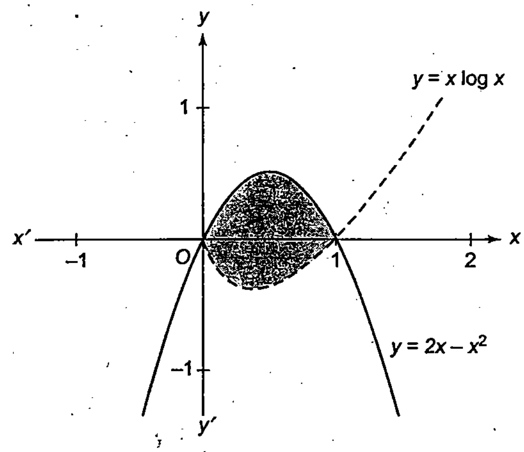



**Fig. 9.31**

Required area

$= \int_0^1 (2x - 2x^2)dx - \int_0^1 x \log x \, dx$

$= \left[x^2 - \frac{2x^3}{3}\right]_0^1 - \left[\frac{x^2}{2} \log x - \frac{x^2}{4}\right]_0^1$

$= \left(1 - \frac{2}{3}\right) - \left[0 - \frac{1}{4} - \frac{1}{2}\lim_{x \to 0} x^2 \log x\right] = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$

**6. a.** $y = \log_e(x + e)$, $x = \log_e\left(\frac{1}{y}\right) \Rightarrow y = e^{-x}$.

for $y = \log_e(x + e)$ shift the graph of $y = \log_e x$, $e$ units left hand side.

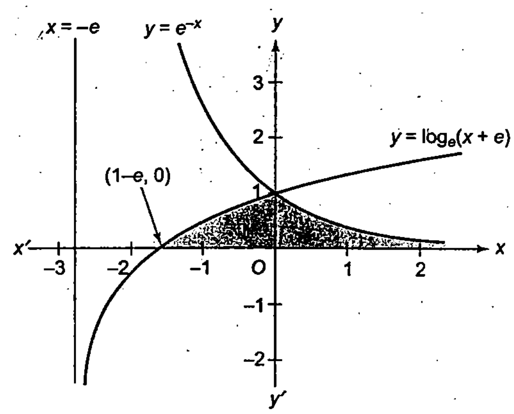



**Fig. 9.32**

$$\text{Required area} = \int_{1-e}^{0} \log_e (x+e)dx + \int_0^{\infty} e^{-x}\,dx$$

$$= |x\log_e(x+e)|_{1-e}^{0} - \int_{1-e}^{0} \frac{x}{x+e}dx - |e^{-x}|_0^{\infty}$$

$$= \int_0^{1-e}\left(1-\frac{e}{x+e}\right)dx - e^{-\infty} + e^0$$

$$= |x - e\log(x+e)|_0^{1-e} - 0 + 1$$

$$= 1 - e + e\log e + 1 = 2 \text{ sq. units.}$$

**7.d** $y = \dfrac{1}{(x-1)^2+1}$

$y$ is maximum when $(x-1)^2 = 0$. Also, graph is symmetrical about line $x = 1$.

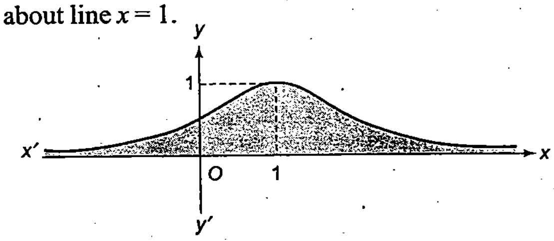



**Fig. 9.33**

$$\text{Area} = 2\int_1^{\infty}\frac{1}{(x-1)^2+1}dx = 2[\tan^{-1}(x-1)]_1^{\infty} = \pi \text{ sq. units.}$$

**8.b** $xy^2 = a^2(a-x)$

$$\Rightarrow x = \frac{a^3}{y^2+a^2}$$

The given curve is symmetrical about $x$-axis, and meets it at $(a, 0)$.

The line $x = 0$, i.e., $y$-axis is an asymptote (tangent at infinity).

$$\text{Area} = \int_0^{\infty} x\,dy = 2\int_0^{\infty}\frac{a^3}{y^2+a^2}dx$$

$$= 2a^3\,\frac{1}{a}\left[\tan^{-1}\frac{y}{a}\right]_0^{\infty} = 2a^2\,\frac{\pi}{2} = \pi a^2 \text{ sq. units.}$$

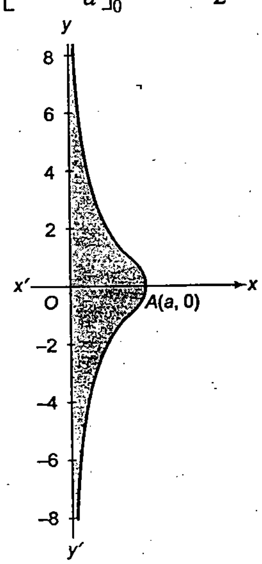



**Fig. 9.34**

**9. c.** Given $y = x^2 + x + 1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} \Rightarrow y - \frac{3}{4} = \left(x+\frac{1}{2}\right)^2$.

This is a parabola with vertex at $\left(-\frac{1}{2}, \frac{3}{4}\right)$ and the curve is concave upwards.

$$y = x^2 + x + 1 \Rightarrow \frac{dy}{dx} = 2x + 1 \Rightarrow \left(\frac{dy}{dx}\right)_{(1,3)} = 3$$

Equation of the tangent at $A$ (1, 3) is $y = 3x$

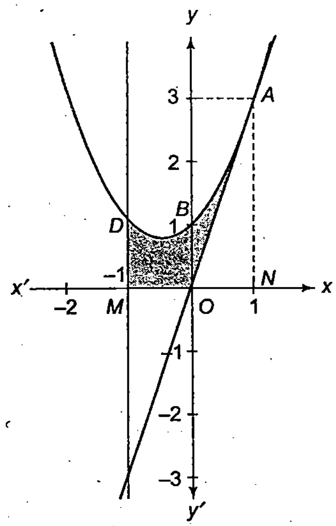



**Fig. 9.35**

Required (shaded) area = area $ABDMN$ – area $ONA$

$$\text{Now, area } ABDMN = \int_{-1}^{1}(x^2+x+1)dx$$

$$= 2\int_0^1 (x^2+1) = \frac{8}{3}$$

$$\text{Area of } ONA = \frac{1}{2}\times 1\times 3 = \frac{3}{2}.$$

$$\therefore \text{ required area} = \frac{8}{3} - \frac{3}{2} = \frac{16-9}{6} = \frac{7}{6} \text{ sq. units.}$$

**10. a.**

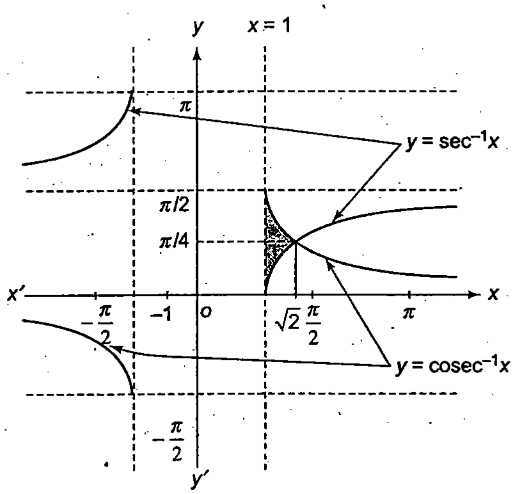



**Fig. 9.36**

Integrating along $x$-axis, we get

$$A = \int_1^{\sqrt{2}} \left(\operatorname{cosec}^{-1} x - \sec^{-1} x\right) dx$$

Integrating along $y$-axis, we get

$$A = 2\int_0^{\pi/4} (\sec y - 1)\, dy$$

$$= 2\left[\log|\sec y + \tan y| - y\right]_0^{\pi/4}$$

$$= 2\left[\log\left|\sqrt{2}+1\right| - \frac{\pi}{4}\right] = \log(3 + 2\sqrt{2}) - \frac{\pi}{2} \text{ sq. units.}$$

**1. b.** Solving $2\cos x = 3\tan x$, we get

$$2 - 2\sin^2 x = 3\sin x \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}.$$

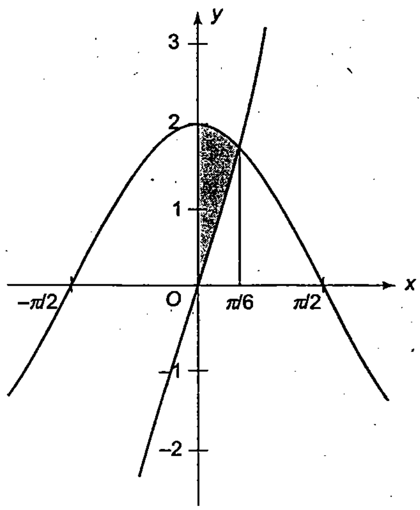



**Fig. 9.37**

$$\text{Required area} = \int_0^{\pi/6} (2\cos x - 3\tan x)\, dx$$

$$= 2\sin x - 3\log\sec x\Big|_0^{\pi/6} = 1 - 3\ln 2 + \frac{3}{2}\ln 3 \text{ sq. units.}$$

**. b.** The curve is $y = 2x^4 - x^2 = x^2(2x^2 - 1)$
The curve is symmetrical about the axis of $y$.
Also, it is a polynomial of 4 degree having roots 0, 0, $\pm\frac{1}{\sqrt{2}}$. $x = 0$ is repeated root. Hence, graph touches at (0, 0).
The curve intersects the axes at $O(0, 0)$, $A(-1/\sqrt{2}, 0)$ and $B(1/\sqrt{2}, 0)$.
Thus, the graph of the curve is shown in Fig. 9.38.

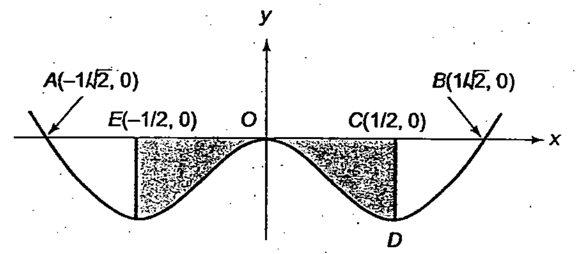



**Fig. 9.38**

Here, $y \le 0$, as $x$ varies from $x = -1/2$ to $x = 1/2$
∴ The required area
$= 2$ Area $OCDO$

$$= 2\left|\int_0^{1/2} y\,dx\right|$$

$$= 2\left|\int_0^{1/2} (2x^4 - x^2)\,dx\right|$$

$= 7/120$ sq. units

**13. d.** The curve is $y = \dfrac{x^2(x+a)}{a^2}$, which is a cubic polynomial.

Since $\dfrac{x^2(x+a)}{a^2} = 0$ has repeated root $x = 0$, it touches $x$-axis at (0, 0) and intersects at $(-a, 0)$.

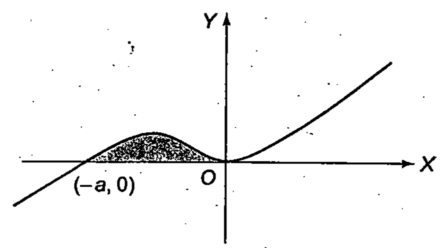



**Fig. 9.39**

Required area $= \int_{-a}^{0} y\,dx = \int_{-a}^{0}\left[\dfrac{x^2(x+a)}{a^2}\right]dx = a^2/12$ sq. units.

**14. c.**

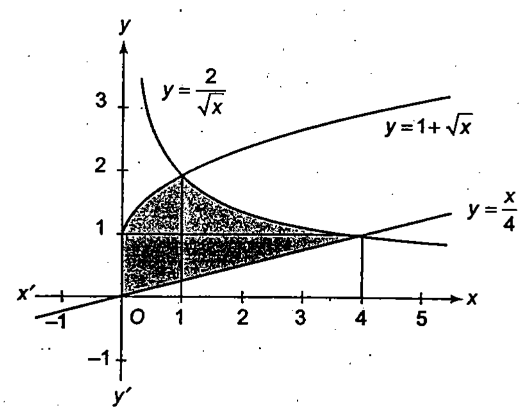



**Fig. 9.40**

15.

$$A_1=\int_0^1\left(1+\sqrt{x}-\frac{x}{4}\right)dx$$

$$=\left[x+\frac{2x^{3/2}}{3}-\frac{x^2}{8}\right]_0^1=1+\frac{2}{3}-\frac{1}{8}=\frac{37}{24}.$$

$$A_2=\int_1^4\left(\frac{2}{\sqrt{x}}-\frac{x}{4}\right)dx$$

$$=\left[4\sqrt{x}-\frac{x^2}{8}\right]_1^4$$

$$=\left[8-2-4+\frac{1}{8}\right]=\frac{17}{8}$$

$$\Rightarrow A=A_1+A_2==\frac{88}{24}=\frac{11}{3}\text{ sq. units}$$

**5. a.** $y=\frac{x^2}{2}-2x+2=\frac{(x-2)^2}{2}$,

$\frac{dy}{dx}=x-2$, $\left(\frac{dy}{dx}\right)_{x=1}=-1$, $\left(\frac{dy}{dx}\right)_{x=4}=2$

$\Rightarrow$ Tangent at (1, 1/2) is $y-1/2=-1(x-1)$ or $2x+2y-3=0$

Tangent at (4, 2) is $y-2=2(x-4)$ or $2x-y-6=0$

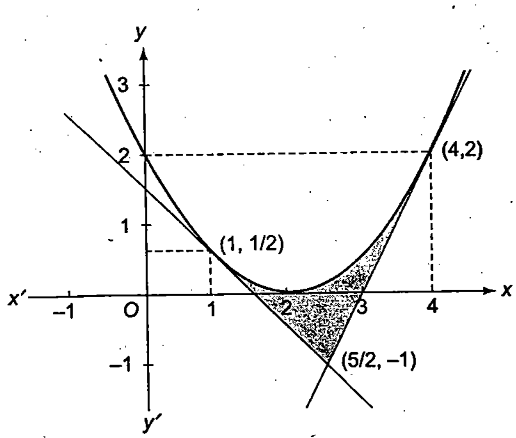



**Fig. 9.41**

$$\text{Hence, }A=\int_1^{5/2}\left(\frac{x^2}{2}-2x+2-\frac{3-2x}{2}\right)dx$$

$$+\int_{5/2}^4\left(\frac{x^2}{2}-2x+2-(2x-6)\right)dx$$

$$=\int_1^4\left(\frac{x^2}{2}-2x+2\right)dx-\int_1^{5/2}\left(\frac{3-2x}{2}\right)dx-\int_{5/2}^4(2x-6)dx$$

$$=\left(\frac{x^3}{6}-x^2+2x\right)_1^4-\frac{1}{2}\left(3x-x^2\right)_1^{5/2}-\left(x^2-6x\right)_{5/2}^4$$

$$=\left(\frac{63}{6}-15+6\right)-\frac{1}{2}\left(3\times\frac{3}{2}-\left(\frac{25}{4}-1\right)\right)$$

$$-\left(\left(16-\frac{25}{4}\right)-6\left(4-\frac{5}{2}\right)\right)$$

$$=\frac{3}{2}-\frac{1}{2}\left(\frac{9}{2}-\frac{21}{4}\right)-\left(\frac{39}{4}-6\left(\frac{3}{2}\right)\right)$$

$$=\frac{9}{8}\text{ sq. units}$$

**16. a.**

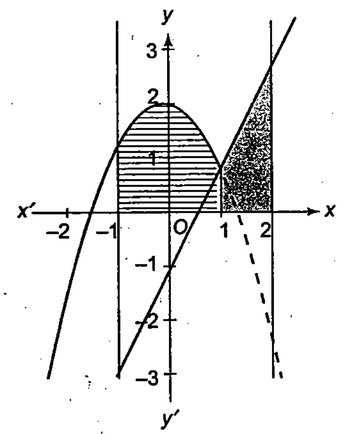



**Fig. 9.42**

$$A=\int_{-1}^{1}(-x^2+2)dx+\int_1^2(2x-1)dx$$

$$=\left(-\frac{x^3}{3}+2x\right)_{-1}^{1}+(x^2-x)_1^2$$

$$=\frac{16}{3}\text{ sq. units}$$

**17. a.**

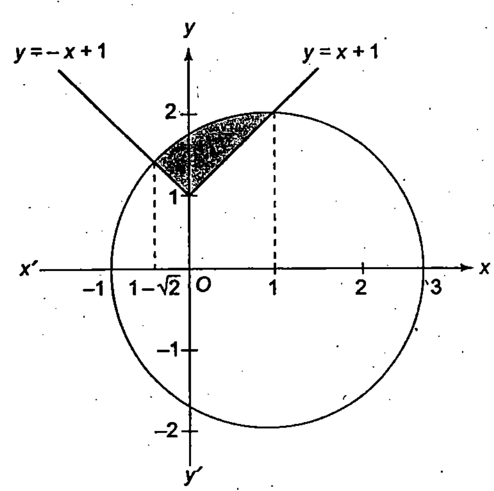



**Fig. 9.43**

$x^2+y^2-2x-3=0$

$\Rightarrow (x-1)^2+y^2=4$

$$A=\int_{1-\sqrt{2}}^{0}\left(\sqrt{4-(x-1)^2}-(-x+1)\right)dx+\int_{0}^{1}\left(\sqrt{4-(x-1)^2}-(x+1)\right)dx$$

$$=\left.\frac{x-1}{2}\sqrt{4-(x-1)^2}+\frac{4}{2}\sin^{-1}\frac{x-1}{2}+\frac{x^2}{2}-x\right|_{1-\sqrt{2}}^{0}+\left.\frac{x-1}{2}\sqrt{4-(x-1)^2}+\frac{4}{2}\sin^{-1}\frac{x-1}{2}-\frac{x^2}{2}-x\right|_{0}^{1}$$

$$=\left(-\frac{\sqrt{3}}{2}-\frac{\pi}{3}\right)-\left(\frac{-\sqrt{2}}{2}\sqrt{2}-\frac{\pi}{2}+\frac{3-2\sqrt{2}}{2}-1+\sqrt{2}\right)+\left(-\frac{1}{2}-1\right)-\left(-\frac{\sqrt{3}}{2}-\frac{\pi}{3}\right)$$

$$=-\left(-1-\frac{\pi}{2}+\frac{3}{2}-\sqrt{2}-1+\sqrt{2}\right)-\frac{3}{2}=\frac{\pi}{2}-1 \text{ sq. units.}$$

**18. c.** $a^2x^2+ax+1$ is clearly positive for all real values of $x$. Area under consideration

$$A=\int_0^1\left(a^2x^2+ax+1\right)dx$$

$$=\frac{a^2}{3}+\frac{a}{2}+1$$

$$=\frac{1}{6}\left(2a^2+3a+6\right)$$

$$=\frac{1}{6}\left(2\left(a^2+\frac{3}{2}a+\frac{9}{16}\right)+6-\frac{18}{16}\right)$$

$$=\frac{1}{6}\left(2\left(a+\frac{3}{4}\right)^2+\frac{39}{8}\right),$$ which is clearly minimum for $a=-\frac{3}{4}$.

**19. d.** $y=\sqrt{4-x^2}$, $y=\sqrt{2}\sin\left(\frac{x\pi}{2\sqrt{2}}\right)$

intersect at $x=\sqrt{2}$

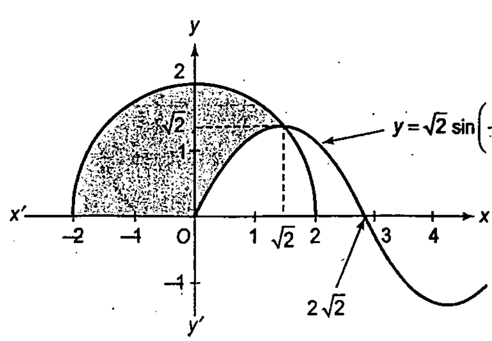



Fig. 9.44

Area to the left of $y$-axis is $\pi$

Area to the right of $y$-axis

$$=\int_0^{\sqrt{2}}\left(\sqrt{4-x^2}-\sqrt{2}\sin\frac{x\pi}{2\sqrt{2}}\right)dx$$

$$=\left(\frac{x\sqrt{4-x^2}}{2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right)_0^{\sqrt{2}}+\left(\frac{4}{\pi}\cos\frac{x\pi}{2\sqrt{2}}\right)_0^{\sqrt{2}}$$

$$=\left(1+2\times\frac{\pi}{4}\right)+\frac{4}{\pi}(0-1)$$

$$=1+\frac{\pi}{2}-\frac{4}{\pi}$$

$$=\frac{2\pi+\pi^2-8}{2\pi} \text{ sq. units.}$$

$$\therefore \text{ ratio}=\frac{2\pi^2}{2\pi+\pi^2-8}.$$

**20. b.** $f(x)=\sin x$

$f(x)+f(\pi-x)=2$

$f(x)=2-f(\pi-x)=2-\sin(\pi-x)=2-\sin x$, where $x\in\left(\frac{\pi}{2},\pi\right]$

$f(x)=f(2\pi-x)=2-\sin(2\pi-x)$, where $x\in\left(\pi,\frac{3\pi}{2}\right]$

$f(x)=f(2\pi-x)=-\sin x$, where $x\in\left(\frac{3\pi}{2},2\pi\right]$

$$f(x)=\begin{cases}\sin x, & x\in\left[0,\frac{\pi}{2}\right]\\ 2-\sin x, & x\in\left(\frac{\pi}{2},\pi\right]\\ 2+\sin x, & x\in\left(\pi,\frac{3\pi}{2}\right]\\ -\sin x, & x\in\left(\frac{3\pi}{2},2\pi\right]\end{cases}$$

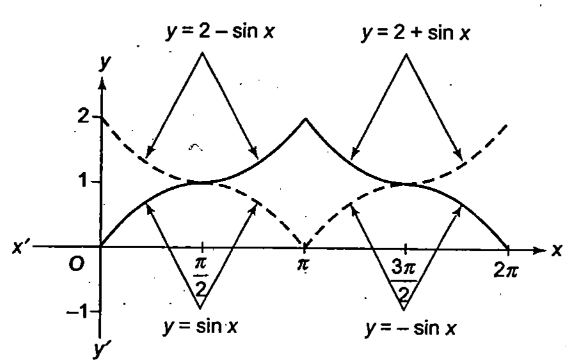



Fig. 9.45

$$\text{Area} = \int_0^{\pi/2} \sin x \sin dx + \int_{\pi/2}^{\pi} (2 - \sin x)\, dx + \int_{\pi}^{3\pi/2} (2 + \sin x) dx + \int_{3\pi/2}^{2\pi} (-\sin x)\, dx$$

$$= 1 + 2 \times \frac{\pi}{2} - 1 + 2.\frac{\pi}{2} - 1 + 1 = 2\pi \text{ sq. units.}$$

**21. a.**

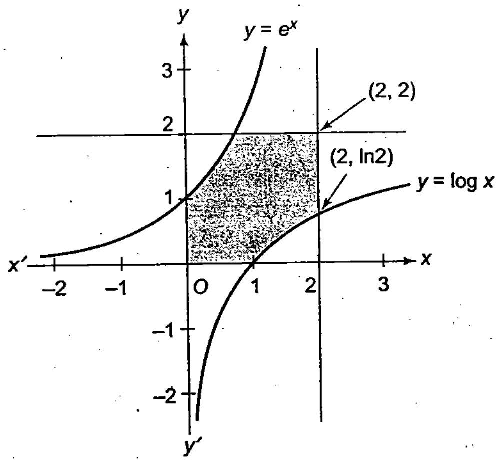



**Fig. 9.46**

$$A = \int_1^2 \ln x\, dx$$

$$= [x \log x - x]_1^2$$

$$= 2 \log 2 - 1$$

$\Rightarrow$ Required area $= 4 - 2(2 \ln 2 - 1) = 6 - 4 \ln 2$ sq. units.

**22. b.** $ay^2 = x^2 (a - x) \Rightarrow y = \pm x \sqrt{\dfrac{a-x}{a}}$

Curve tracing : $y = x \sqrt{\dfrac{a-x}{a}}$

We must have $x \le a$

For $0 < x \le a, y > 0$ and for $x < 0, y < 0$

Also $y = 0 \Rightarrow x = 0, a$

Curve is symmetrical about $x$-axis.

When $x \to -\infty, y \to -\infty$

Also, it can be verified that $y$ has only one point of maxima for $0 < x < a$.

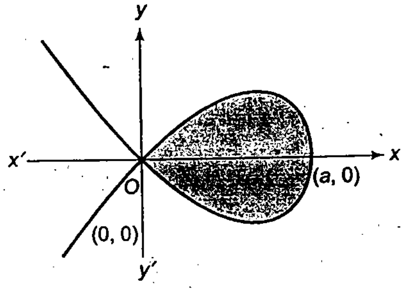



**Fig. 9.47**

$$\text{Area} = 2\int_0^a x \sqrt{\frac{a-x}{a}}\, dx$$

$$\sqrt{\frac{a-x}{a}} = t \Rightarrow 1 - \frac{x}{a} = t^2 \Rightarrow x = a(1 - t^2)$$

$$\Rightarrow A = 2\int_1^0 a(1-t^2)t(-2at)dt$$

$$= 4a^2 \int_0^1 (t^2 - t^4)dt$$

$$= 4a^2 \left[\frac{t^3}{3} - \frac{t^5}{5}\right]_0^1$$

$$= 4a^2 \left[\frac{1}{3} - \frac{1}{5}\right] = \frac{8a^2}{15} \text{ sq. units.}$$

**23. d.**

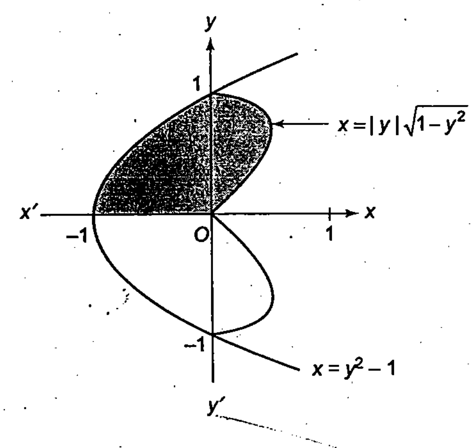



**Fig. 9.48**

$$A = 2\int_0^1 \left[y\sqrt{1-y^2} - (y^2 - 1)\right] dy$$

$= 2$ sq. units

**24. d.** $4y^2 = x^2 (4 - x^2)$ (1)

$$\Rightarrow y = \pm \frac{1}{2}\sqrt{x^2 \left(4 - x^2\right)}$$

$$\Rightarrow y = \pm \frac{x}{2} \sqrt{\left(4 - x^2\right)}$$

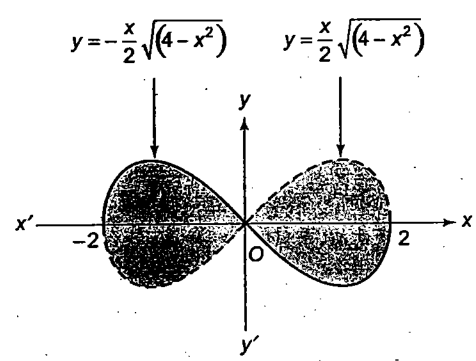



**Fig. 9.49**

$\therefore$ Area $(A) = 4 \times \int_0^2 \frac{x}{2}\sqrt{(4-x^2)}\,dx$

Let $4 - x^2 = t \Rightarrow -2x\,dx = dt$

$$\Rightarrow A = \int_0^4 \sqrt{t}\,dt = \left[\frac{t^{3/2}}{3/2}\right]_0^4 = \frac{2}{3} \times [\sqrt{64} - 0]$$

$$\Rightarrow A = \frac{16}{3} \text{ sq. units}$$

**25. a.** The two curves are

$$xy^2 = a^2(a-x) \Rightarrow x = \frac{a^3}{a^2+y^2} \qquad (1)$$

and $\qquad (a-x)y^2 = a^2x$

$$\Rightarrow x = \frac{ay^2}{a^2+y^2} = \frac{ay^2 + a^3 - a^3}{a^2+y^2} = a - \frac{a^3}{a^2+y^2} \qquad (2)$$

Curve (1) is symmetrical about $x$-axis, and have $y$-axis as the asymptote.

Curve (2) is symmetrical about $x$-axis, tangent at origin as $y$-axis and the asymptote $x = a$.

The two curves intersect at the point $P(a/2, a)$ and $Q(a/2, -a)$.

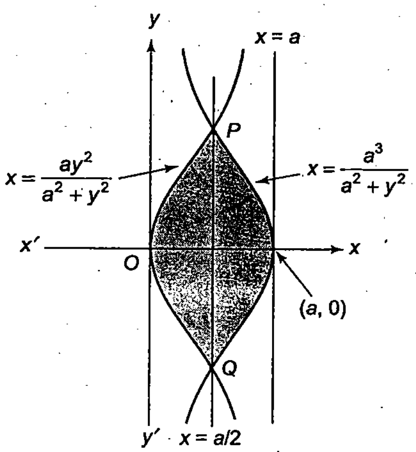



**Fig. 9.50**

Required area

$$= 2\int_0^a \left[-a + \frac{a^3}{a^2+y^2} + \frac{a^3}{a^2+y^2}\right] dy \text{ (integrating along } y\text{-axis)}$$

$$= 2\left[-ay + 2a^2 \tan^{-1}\frac{y}{a}\right]_0^a$$

$$= 2\left[-a^2 + 2a^2\frac{\pi}{4}\right]$$

$= (\pi - 2)a^2$ sq. units.

**26. a.** Curve tracing : $y = x e^x$

Let $\frac{dy}{dx} = 0 \Rightarrow e^x + xe^x = 0 \Rightarrow x = -1$.

Also, at $x = -1$, $\frac{dy}{dx}$ changes sign from –ve to +ve, hence, $x = -1$ is a point of minima.

When $x \to \infty, y \to \infty$

Also $\lim\limits_{x\to-\infty} xe^x = \lim\limits_{x\to-\infty} \frac{x}{e^{-x}} = \lim\limits_{x\to-\infty} \frac{1}{-e^{-x}} = 0$

With similar types of arguments, we can draw the graph of $y = x e^{-x}$.

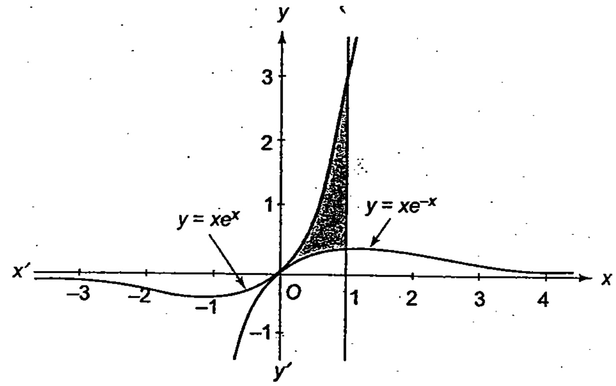



**Fig. 9.51**

Required area

$$= \int_0^1 xe^x dx - \int_0^1 xe^{-x} dx$$

$$= \left[xe^x\right]_0^1 - \int_0^1 e^x dx - \left(\left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x} dx\right)$$

$$= e - (e-1) - \left(-e^{-1} - (e^{-1} - 1)\right) = \frac{2}{e} \text{ sq. units.}$$

**27. c.** Given parabola is $(y-2)^2 = x - 1$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2(y-2)}$$

When $y = 3, x = 2$

$$\therefore \frac{dy}{dx} = \frac{1}{2(3-2)} = \frac{1}{2}.$$

Tangent at (2, 3) is $y - 3 = \frac{1}{2}(x-2) \Rightarrow x - 2y + 4 = 0$

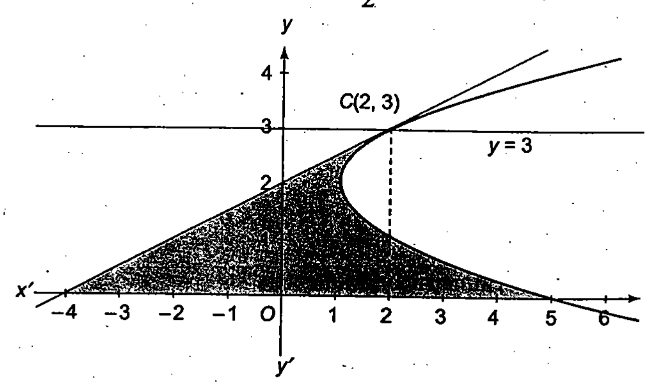



**Fig. 9.52**

$\therefore$ required area

$$= \int_0^3 \left((y-2)^2 + 1\right) dy - \int_0^3 (2y-4)\,dy$$

$$= \left|\frac{(y-2)^3}{3} + y\right|_0^3 - |y^2 - 4y|_0^3$$

$$= \frac{1}{3} + 3 + \frac{8}{3} - (9 - 12) = 9 \text{ sq. units.}$$

**28. c.**

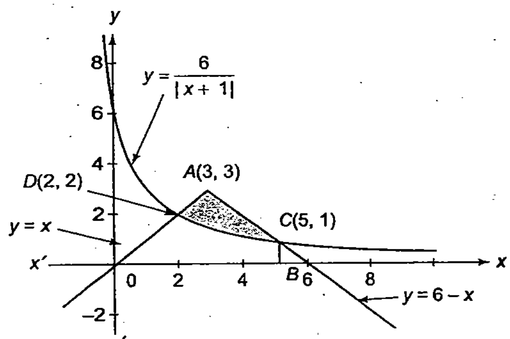



**Fig. 9.53**

First consider $y = 3 - |3 - x|$

For $x < 3$; $y = 3 - (3 - x) = x$

For $x \geq 3$; $y = 3 - (x - 3) = 6 - x$

Consider $y = \frac{6}{|x+1|}$

For $x < -1$; $y = \frac{6}{-1-x}$

$\Rightarrow (1 + x)y = -6$

For $x > -1$; $y = \frac{6}{x+1}$

Required area

$$= \left[\int_2^3 \left(x - \frac{6}{x+1}\right)dx + \int_3^5 \left((6-x) - \frac{6}{x+1}\right)dx\right]$$

$$= \left[\left(\frac{x^2}{2}\right)_2^3 + \left(6x - \frac{x^2}{2}\right)_3^5 - \left(6\log(x+1)\right)_2^5\right]$$

$$= \left[\frac{5}{2} + 4 - 6\log 2\right] = \frac{13}{2} - 6\ln 2 \text{ sq. units.}$$

**29. b.** $\max(|x|, |y|) \leq 1 \Rightarrow |x| \leq 1$, and $|y| \leq 1$

which represent square bounded by $x = \pm 1$ and $y = \pm 1$

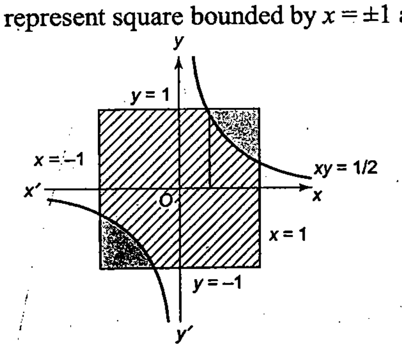



**Fig. 9.54**

Required area is lined area

Now, shaded area is

$$2\int_{1/2}^{1}\left(1 - \frac{1}{2x}\right)dx = 2\left(x - \frac{1}{2}\ln x\right)_{1/2}^{1}$$

$$= 2\left[(1-0) - \left(\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2}\right)\right]$$

$= 1 - \ln 2$ sq. units.

$\Rightarrow$ Horizontal lined area $= 4 - (1 - \ln 2) = 3 + \ln 2$ sq. units.

**30. c.** $(y - x)^2 = x^3$, where $x \geq 0 \Rightarrow y - x = \pm x^{3/2}$

$\Rightarrow y = x + x^{3/2}$ (1)

$y = x - x^{3/2}$ (2)

Function (1) is an increasing function.

Function (2) meets $x$-axis, when $x - x^{3/2} = 0$ or $x = 0, 1$.

Also, for $0 < x < 1, x - x^{3/2} > 0$ and for $x > 1, x - x^{3/2} < 0$.

When $x \to \infty, x - x^{3/2} \to -\infty$.

From these information, we can plot the graph as below:

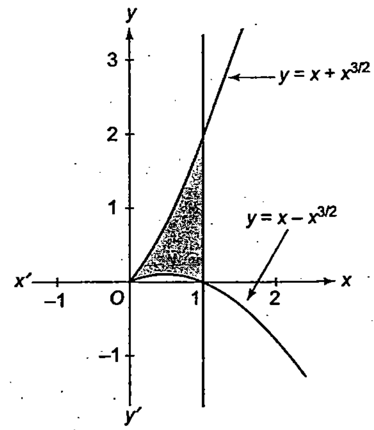



**Fig. 9.55**

Required area

$$= \int_0^1 \left[\left(x + x^{3/2}\right) - \left(x - x^{3/2}\right)\right]dx = 2\int_0^1 x^{3/2}dx$$

$$= 2\left[\frac{x^{5/2}}{5/2}\right]_0^1 = \frac{4}{5} \text{ sq. units.}$$

**31. b.** Given curves are $y = \log_e x$ and $y = (\log_e x)^2$

Solving $\log_e x = (\log_e x)^2 \Rightarrow \log_e x = 0, 1 \Rightarrow x = 1$ and $x = e$

Also, for $1 < x < e$, $0 < \log_e x < 1 \Rightarrow \log_e x > (\log_e x)^2$

For $x > e$, $\log_e x < (\log_e x)^2$

$y = (\log_e x)^2 > 0$ for all $x > 0$

and when $x \to 0$, $(\log_e x)^2 \to \infty$.

From these information, we can plot the graph of the functions.

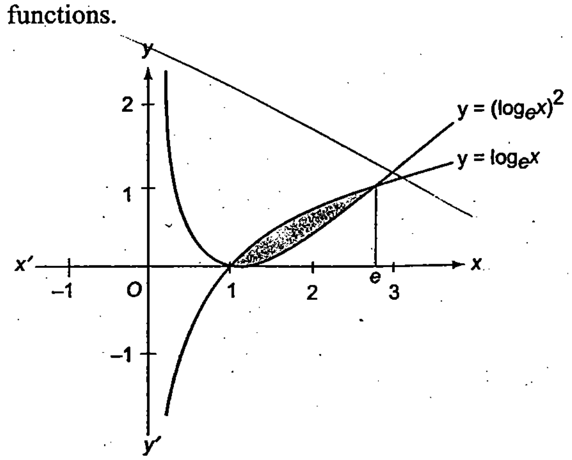



**Fig. 9.56**

Then the required area $= \int_1^e (\log x - (\log_e x)^2)dx$

$$= \int_1^e \log x dx - \int_1^e (\log_e x)^2 dx$$

$$= \left[x\log_e x - x\right]_1^e - \left[x(\log_e x)^2\right]_1^e + \int_1^e \frac{2\log_e x}{x} x dx$$

$$= 1 - e + 2\left[x\log_e x - x\right]_1^e = 3 - e \text{ sq. units.}$$

**32. a.** The points in the required region satisfy

$4 \le x^2 + y^2 \le 2(|x| + |y|)$ (1)

Since the curve (1) is symmetrical about both the axes, the required area is 4 times the area of the region in the first quadrant. Therefore, it is sufficient to sketch the region and to find the area in the first quadrant.

In the first quadrant, the curve (1) consist of two curves

$x^2 + y^2 \ge 4$, and ($C_1$)

$x^2 + y^2 - 2x - 2y \ge 0$ ($C_2$)

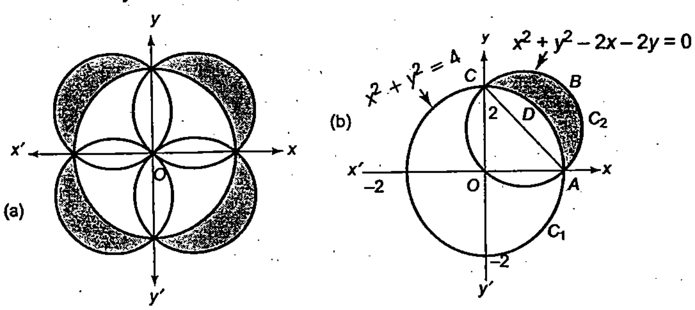



**Fig. 9.57**

$\therefore$ Required area = 4 area $ABCDA$

= 4(area of semi-circle $ABCA$) – (area of sector $ADCA$)

= 4(area of semi-circle $ABCA$) – (area of sector $OADCO$ – area of triangle $OAC$)

$= 4\{\pi - (\pi - 2\} = 8$ sq. units.

**33. c.** The required area will be equal to the area enclosed by $y = f(x)$, $y$-axis between the abscissa at $y = -2$ and $y = 6$

$$\text{Hence, } A = \int_0^1 \left(6 - f(x)\right) dx + \int_{-1}^0 \left(f(x) - (-2)\right) dx$$

$$= \int_0^1 \left(4 - x^3 - 3x\right) dx + \int_{-1}^0 \left(x^3 + 3x + 4\right) dx = \frac{5}{4} \text{ sq. units.}$$

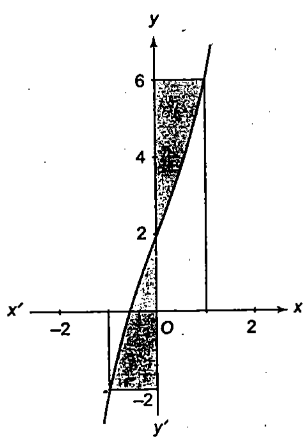



**Fig. 9.58**

**34. d.** Curve tracing : $y = x + \sin x$

$$\frac{dy}{dx} = 1 + \cos x \ge 0 \ \forall x$$

Also $\frac{d^2y}{dx^2} = -\sin x = 0$ when $x = n\pi$, $n \in Z$

Hence, $x = n\pi$ are points of inflection, where curve changes its concavity.

Also for $x \in (0, \pi)$, $\sin x > 0 \Rightarrow x + \sin x > x$.

And for $x \in (\pi, 2\pi)$, $\sin x < 0 \Rightarrow x + \sin x < x$.

From these information, we can plot the graph of $y = f(x)$ and its inverse.

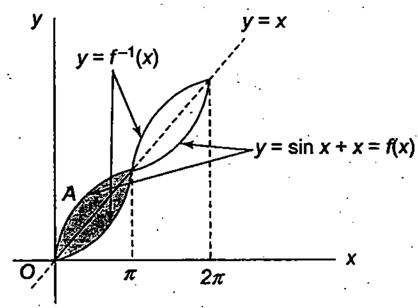



**Fig. 9.59**

Required area = $4A$, where

$$A = \int_0^\pi (x + \sin x) dx - \int_0^\pi x dx$$

$$= \int_0^\pi \sin x \, dx = 2 \text{ square units.}$$

**35. d.** Area $= \int_1^b f(x)\, dx = \sqrt{b^2 + 1} - \sqrt{2}$

$$= \sqrt{b^2 + 1} - \sqrt{1 + 1}$$

$$= \left|\sqrt{x^2 + 1}\right|_1^b$$

$$\therefore f(x) = \frac{d}{dx}\left(\sqrt{x^2 + 1}\right) = \frac{1}{2}\frac{2x}{\sqrt{x^2 + 1}} = \frac{x}{\sqrt{x^2 + 1}}.$$

**36. c.** $\int_{\pi/4}^{\beta} f(x) dx = \beta \sin\beta + \frac{\pi}{4}\cos\beta + \sqrt{2}\beta$

Differentiating both sides w.r.t. $\beta$, we get

$$\therefore f(\beta) = \beta\cos\beta + \sin\beta - \frac{\pi}{4}\sin\beta + \sqrt{2}$$

$$\Rightarrow f'(\beta) = -\beta\sin\beta + \cos\beta + \cos\beta - \frac{\pi}{4}\cos\beta$$

$$\Rightarrow f'\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$$

**37. d.** $y=\sin^{-1}|\sin x|=\begin{cases} x, & 0\le x<\dfrac{\pi}{2} \\ \pi-x, & \dfrac{\pi}{2}\le x<\pi \\ x-\pi, & \pi\le x<\dfrac{3\pi}{2} \\ 2\pi-x, & \dfrac{3\pi}{2}\le x<2\pi \end{cases}$

$$y=(\sin^{-1}|\sin x|)^2=\begin{cases} x^2, & 0\le x<\dfrac{\pi}{2} \\ (\pi-x)^2, & \dfrac{\pi}{2}\le x<\pi \\ (x-\pi)^2, & \pi\le x<\dfrac{3\pi}{2} \\ (2\pi-x)^2, & \dfrac{3\pi}{2}\le x<2\pi. \end{cases}$$

The required area $A$ is shown shaded in Fig.9.60.

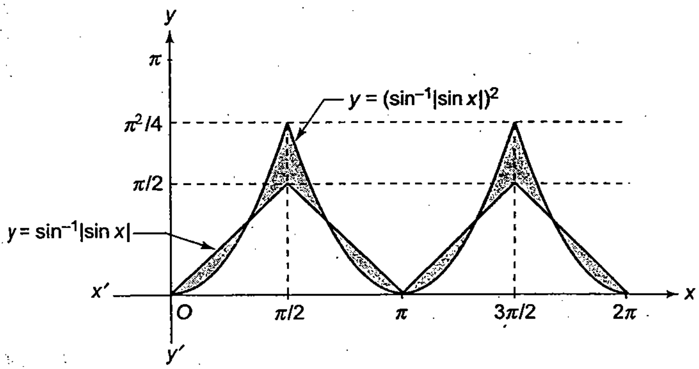



Fig. 9.60

$$\Rightarrow 4\int_0^1 (x-x^2)\,dx+4\int_1^{\pi/2}(x^2-x)\,dx$$

$$=\frac{4}{3}+\pi^2\left[\frac{\pi-3}{6}\right] \text{ sq. units.}$$

**38. c.** $y^2=4\left[\sqrt{y}\right]x$

For $y\in[1,4)$, $\left[\sqrt{y}\right]=1 \Rightarrow y^2=4x$.

Similarly, for $x\in[1,4)$, $\left\lfloor\sqrt{x}\right\rfloor=1$ and

$x^2=4\left\lfloor\sqrt{x}\right\rfloor y$ would transform into $x^2=4y$.

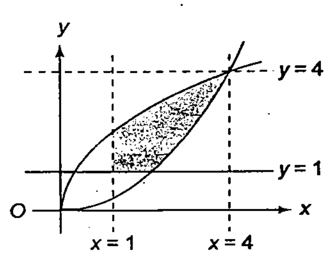



Fig. 9.61

The required area is being shaded.

$$A=\int_1^2(2\sqrt{x}-1)dx+\int_2^4\left(2\sqrt{x}-\frac{x^2}{4}\right)dx$$

$$=\left(\frac{4}{3}x^{3/2}-x\right)_1^2+\left(\frac{4}{3}x^{3/2}-\frac{x^3}{12}\right)_2^4=\frac{11}{3} \text{ sq. units.}$$

**39. b.** The required area $A=\displaystyle\int_0^{2a}\sqrt{\frac{x^3}{2a-x}}\,dx$

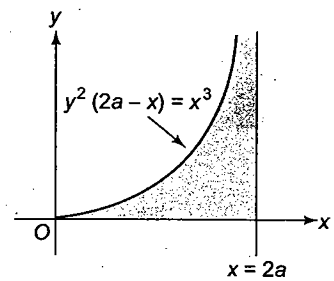



Fig. 9.62

Put $x=2a\sin^2\theta$

$\Rightarrow dx=2a\,2\sin\theta\cos\theta\,d\theta$

$$\Rightarrow A=8a^2\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta$$

$$=2a^2\int_0^{\pi}(1-2\cos 2\theta+\cos^2 2\theta)\,d\theta$$

$$=2a^2\int_0^{\pi}\left(1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta$$

$$=\frac{3\pi a^2}{2}$$

**40. a.**

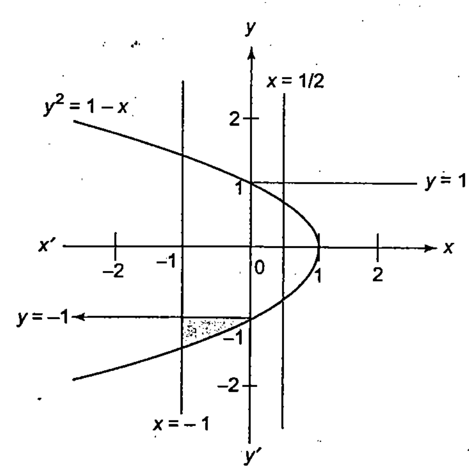



Fig. 9.63

From Fig. 9.63

$$A=\int_{-1}^{0}\left(-1-\left(-\sqrt{1-x}\right)\right)dx+\int_0^{1/2}(1-\sqrt{1-x})\,dx$$

$$=\left[-x-\frac{(1-x)^{3/2}}{3/2}\right]_{-1}^{0}+\left[x+\frac{(1-x)^{3/2}}{3/2}\right]_0^{1/2}$$

$$=\left[-\frac{2}{3}-\left(1-\frac{2\times 2^{3/2}}{3}\right)\right]+\left[\frac{1}{2}+\frac{2}{3\times 2^{3/2}}-\frac{2}{3}\right]$$

$$=\frac{2}{3\times 2^{3/2}}+\frac{2\times 2^{3/2}}{3}-\frac{4}{3}-\frac{1}{2}$$

$$=\frac{3}{\sqrt{2}}-\frac{4}{3}-\frac{1}{2}$$

$$=\frac{3}{\sqrt{2}}-\frac{11}{6} \text{ sq. units}$$

## Multiple Correct Answers Type

**1. b, c.**

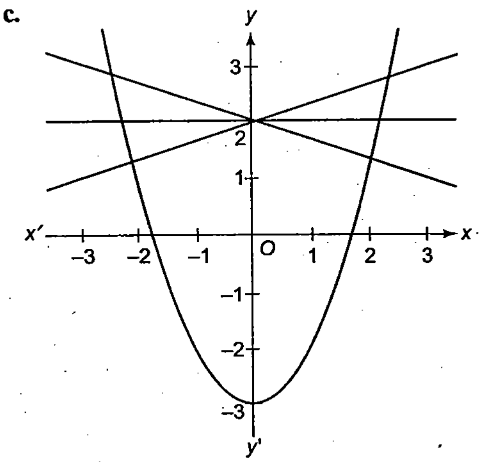



**Fig. 9.64**

Line $y = kx + 2$ passes through fixed point (0, 2) for different value of $k$.

Also, it is obvious that minimum $A(k)$ occurs when $k = 0$, as when line is rotated from this position about point (0, 2) the increased part of area is more than the decreased part of area.

$$\therefore \quad \text{Minimum area} = 2\int_0^{\sqrt{5}} (2-(x^2-3))dx$$

$$= 2\int_0^{\sqrt{5}} (5-x^2)dx$$

$$= 2\left[5x-\frac{x^3}{3}\right]_0^{\sqrt{5}}$$

$$= 2\left[5\sqrt{5}-\frac{5\sqrt{5}}{3}\right]$$

$$= \frac{20\sqrt{5}}{3} \text{ sq. units}$$

**2. a, c, d.**

$y^2 = 4x$ and $x^2 = 4y$ meet at $O(0, 0)$ and $A$ (4, 4).

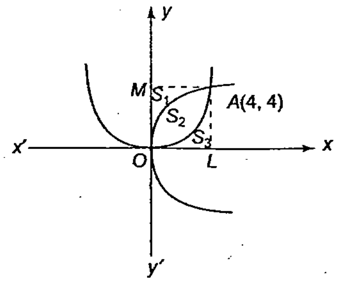



**Fig. 9.65**

$$\text{Now } S_3 = \int_0^4 \frac{x^2}{4}\,dx = \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4 = \frac{1}{12}\,[64-0] = \frac{16}{3}.$$

$$S_2 = \int_0^4 2\sqrt{x}\,dx - S_3 = 2\left[\frac{x^{3/2}}{3/2}\right]_0^4 - \frac{16}{3}$$

$$= \frac{4}{3}\,[8-0] - \frac{16}{3} = \frac{16}{3}.$$

And $\quad S_1 = 4\times 4-(S_2+S_3) = 16-\left(\frac{16}{3}+\frac{16}{3}\right) = \frac{16}{3}.$

Hence, $S_1 : S_2 : S_3 = 1:1:1$

**3. a, c, d.**

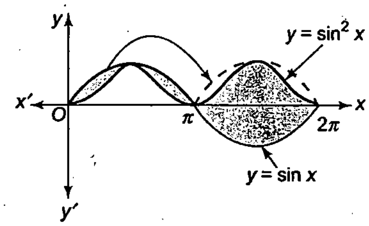



**Fig. 9.66**

We know that area bounded by $y = \sin x$ and $x$-axis for $x \in [0, \pi]$ is 2 sq. units.

Then area bounded by $y = \sin x$ and $y = \sin^2 x$ is 4 sq. units for $x \in [0, 2\pi]$.

Then for $x \in [0, 10\pi]$, the area bounded is 20 sq. units.

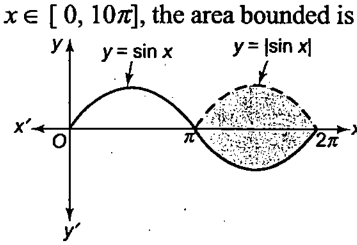



**Fig. 9.67**

The area bounded by $y = \sin x$ and $y = |\sin x|$ for $x \in [0, 2\pi]$ is 4 sq. units.

Then for $x \in [0, 20\pi]$, the area bounded is 40 sq. units.

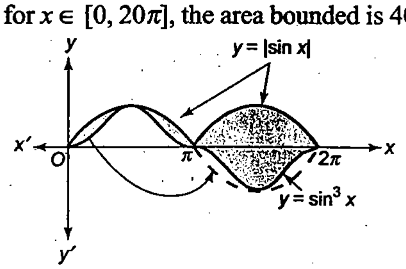



**Fig. 9.68**

The area bounded by $y = \sin x$ and $y = \sin^3 x$ for $x \in [0, 2\pi]$ is 4 sq. units.

Then for $x \in [0, 10\pi]$, the area bounded is 20 sq. units. Similarly, the area bounded by $y = \sin x$ and $y = \sin^4 x$ for $x \in [0, 10\pi]$ is 20 sq. units.

**4. c, d.**

Since the curve $y = ax^{1/2} + bx$ passes through the point (1, 2)

$$\therefore \quad 2 = a + b \qquad (1)$$

By observation the curve also passes through (0, 0). Therefore, the area enclosed by the curve, $x$-axis and $x = 4$ is given by

$$A = \int_0^4 (ax^{1/2} + bx)dx = 8 \Rightarrow \frac{2a}{3} \times 8 + \frac{b}{2} \times 16 = 8$$

$$\Rightarrow \quad \frac{2a}{3} + b = 1. \qquad (2)$$

Solving (1) and (2), we get $a = 3$, $b = -1$.

**5. a, c, d.**

Eliminating $t$, we have $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \Rightarrow y = \left(a^{2/3} - x^{2/3}\right)^{3/2}$.

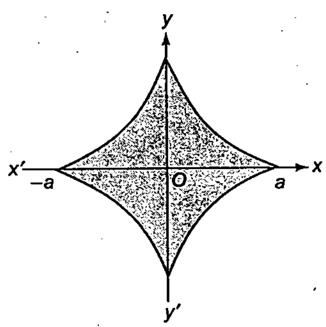



**Fig. 9.69**

From diagram,

$$\Rightarrow \quad A = 2\int_{-a}^{a} \left(a^{2/3} - x^{2/3}\right)^{3/2} dx = 4\int_0^a \left(a^{2/3} - x^{2/3}\right)^{3/2} dx$$

$$A = 4\int_0^a y\,dx$$

$$= 4a^2 \int_0^{\pi/2} 3\cos^3 t \sin^2 t \cos t \; dt.$$

**6. a, c.**

$$a_1 = 0, b_1 = 32, a_2 = a_1 + \frac{3}{2}b_1 = 48, b_2 = \frac{b_1}{2} = 16$$

$$a_3 = 48 + \frac{3}{2} \times 16 = 72, b_3 = \frac{16}{2} = 8$$

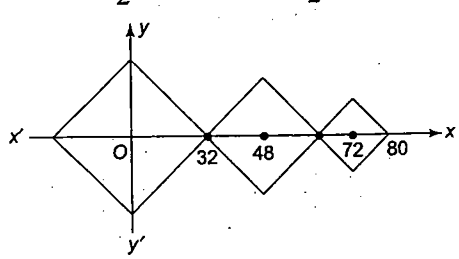



**Fig. 9.70**

So the three loops from $i = 1$ to $i = 3$ are alike.

Now area of $i$th loop (square) $= \frac{1}{2}$ (diagonal)$^2$

$$A_i = \frac{1}{2}(2b_i)^2 = 2(b_i)^2$$

So, $\frac{A_{i+1}}{A_i} = \frac{2(b_{i+1})^2}{2(b_i)^2} = \frac{1}{4}$.

So the areas form a G.P. series

So, the sum of the G.P. upto infinite terms

$$= A_1 \frac{1}{1-r} = 2(32)^2 \times \frac{1}{1-\frac{1}{4}}$$

$$= 2 \times (32)^2 \times \frac{4}{3}$$

$$= \frac{8}{3}(32)^2 \text{ square units.}$$

## Reasoning Type

**1. a.** Since $y = e^x$ and $y = \log_e x$ are inverse to each other.

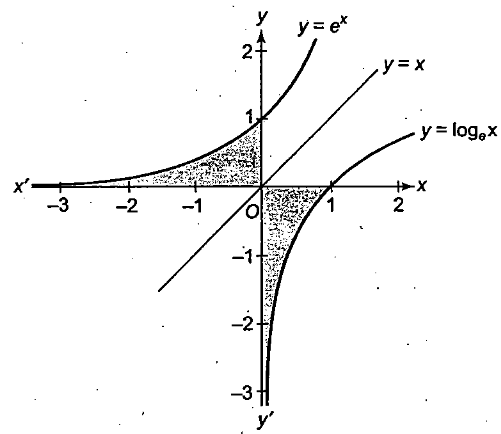



**Fig. 9.71**

**2. a.** Statement 2 is correct as $y = f(x)$ is odd and hence statement 1 is correct.

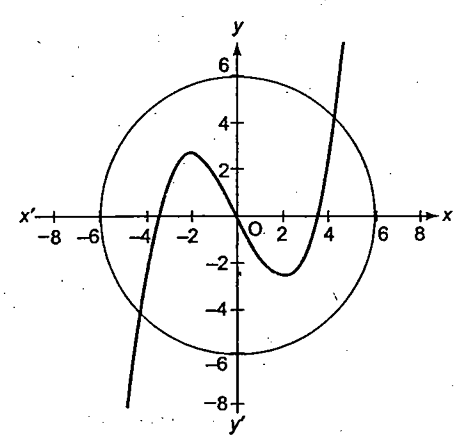



**Fig. 9.72**

**3. b.** Area $= \int_1^3 -\left(x^2 - 4x + 3\right) dx = -\left(\frac{x^3}{3} - \frac{4x^2}{2} + 3x\right)\Big|_1^3$

$= \frac{4}{3}$ sq. units.

∴ Statement 1 is true.

Obviously, statement 2 is true, but does not explain statement 1.

**4. a.** Given curves are $y^2 - 2y + 4x + 5 = 0$ and $x^2 + 2x - y + 2 = 0$ or $(y-1)^2 = -4(x+1)$ and $(x+1)^2 = y - 1$.

Shifting origin to (–1, 1), equation of given curves changes to $Y^2 = -4X$ and $X^2 = Y$.

Hence, statement 1 is true and statement 2 is correct explanation of statement 1.

**5. a.**

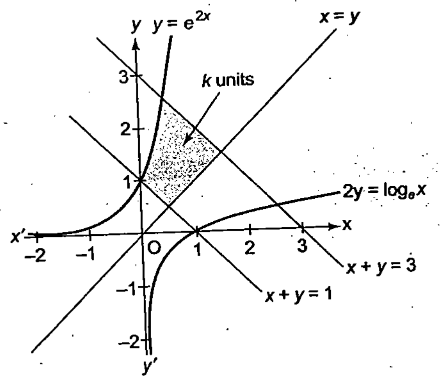



**Fig. 9.73**

$y = e^{2x}$ and $2y = \log_e x$ are inverse of each other

The shaded area is given as $k$ sq. units.

⇒ The required area is $2k$ sq. units.

**6. d.** $R_1$ : points $P(x, y)$ is nearer to (1, 0) than to $x = -1$

$\Rightarrow \sqrt{(x-1)^2 + y^2} < |x+1|$

$\Rightarrow y^2 < 4x$

⇒ Point $P$ lies inside parabola $y^2 = 4x$.

$R_2$ : Point $P(x, y)$ is nearer to (0, 0) than to (8, 0)

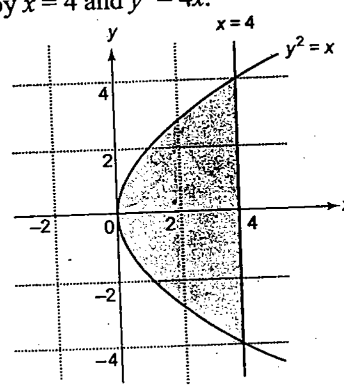
$\Rightarrow |x| < |x - 8|$

$\Rightarrow x^2 < x^2 - 16x + 64$

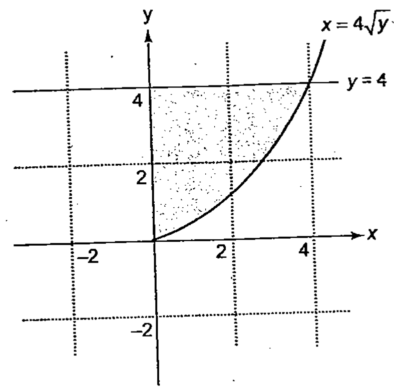
$\Rightarrow x < 4$

⇒ Point $P$ is towards left side of line $x = 4$.

The area of common region of $R_1$ and $R_2$ is the area bounded by $x = 4$ and $y^2 = 4x$.




(a)




(b)

**Fig. 9.74**

This area is twice the area bounded by $x = 4\sqrt{y}$ and $y = 4$.

Now, the area bounded by $x = 4\sqrt{y}$ and $y = 4$ is

$$A = \int_0^4 \left(4 - \frac{x^2}{4}\right) dx = \left[4x - \frac{x^3}{12}\right]_0^4 = \left[16 - \frac{64}{12}\right] = \frac{32}{3} \text{ sq. units}$$

. Hence, the area bounded by $R_1$ and $R_2$ is $\frac{64}{3}$ sq. units.

Thus, statement 1 is false but statement 2 is true.

**7. b.** $2 \geq \max.\{|x-y|, |x+y|\}$

$\Rightarrow |x - y| \leq 2$ and $|x + y| \leq 2$, which forms a square of diagonal length 4 units.

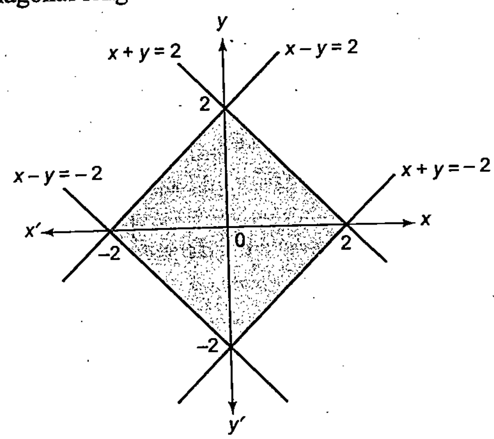



**Fig. 9.75**

⇒ The area of the region is $\frac{1}{2} \times 4 \times 4 = 8$ sq. units.

This is equal to the area of the square of side length $2\sqrt{2}$.

## Linked Comprehension Type

**For Problems 1–2**

**1. b, 2. c**

**Sol.**

Solving the two equations,

$$m^2x^2 = (e^{-kr})\, x$$

$$x_1 = 0, x_2 = \frac{e^{-kr}}{m^2}.$$

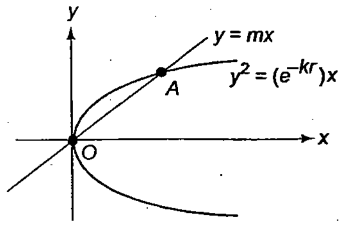



**Fig. 9.76**

So, $A_r = \int_0^{x_2} \left( e^{-\frac{kr}{2}} \sqrt{x} - mx \right) dx$

$= \frac{2}{3} e^{-kr/2} x_2^{3/2} - m \frac{x_2^2}{2}$

$= \frac{2}{3} e^{-kr/2} \frac{e^{-3kr/2}}{m^3} - \frac{m}{2} \frac{e^{-2kr}}{m^4} = \frac{e^{-2kr}}{6m^3}.$

Now, $\frac{A_{r+1}}{A_r} = \frac{e^{-2k(r+1)}}{e^{-2kr}} = e^{-2k} =$ constant.

So, the sequence $A_1, A_2, A_3, \ldots$ is in G.P.

Sum of $n$ terms $= \frac{e^{-2k}}{6m^3} \frac{e^{-2nk} - 1}{e^{-2k} - 1} = \frac{1}{6m^3} \frac{e^{-2nk} - 1}{1 - e^{2k}}$

Sum to infinite terms $= A_1 \frac{1}{1 - e^{-2k}}$

$= \frac{e^{-2k}}{6m^3} \times \frac{e^{2k}}{e^{2k} - 1} = \frac{1}{6m^3(e^{2k} - 1)}.$

**For Problems 3–5**

**3. d., 4. c., 5. a.**

**Sol.**

**3. d.** $f(x) = \frac{x^3}{3} - x^2 + a$

$f'(x) = x^2 - 2x = x(x-2) < 0$ (note that $f(x)$ is monotonic in $(0, 2)$)

Hence for the minimum and $f(x)$ must cross the $x$-axis at $\frac{0+2}{2} = 1$.

Hence, $f(1) = \frac{1}{3} - 1 + a = 0$

$\Rightarrow \quad a = \frac{2}{3}.$

**4. c.** $f(x) = x^3 + 3x^2 + x + a$

$f'(x) = 3x^2 + 6x + 1 = 0$

$\Rightarrow \quad x = -1 \pm \frac{\sqrt{6}}{3}.$

Hence, $f(x)$ cuts the $x$-axis at $\frac{1}{2}\left[\left(-1 + \frac{\sqrt{6}}{3}\right) + \left(-1 - \frac{\sqrt{6}}{3}\right)\right]$

$= -1.$

$f(-1) = -1 + 3 - 1 + a = 0$

$a = -1.$

**5. a.** $f(x) = \sin x + \cos x$

$\Rightarrow \frac{df(x)}{dx} = \cos x - \sin x$

If $\frac{df(x)}{dx} = 0$, then $\cos x = \sin x \Rightarrow x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$

(considering any two of consecutive points of extremum). For minimum area bounded by $y = f(x)$ and $y = a$, between $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$, graphs of $g(x)$ must cut $y = a$ at $c$

$= \frac{\frac{\pi}{4} + \frac{5\pi}{4}}{2} = \frac{3\pi}{4}.$

$a = f\left(\frac{3\pi}{4}\right) \Rightarrow a = \sin\left(\frac{3\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right) = 0.$

**For Problems 6–8**

**6. a, 7. c, 8. b**

**Sol.** Since $-1 \le \sin x \le 1$, the curve $y = e^{-x} \sin x$ is bounded by the curves $y = e^{-x}$ and $y = e^{-x}$.

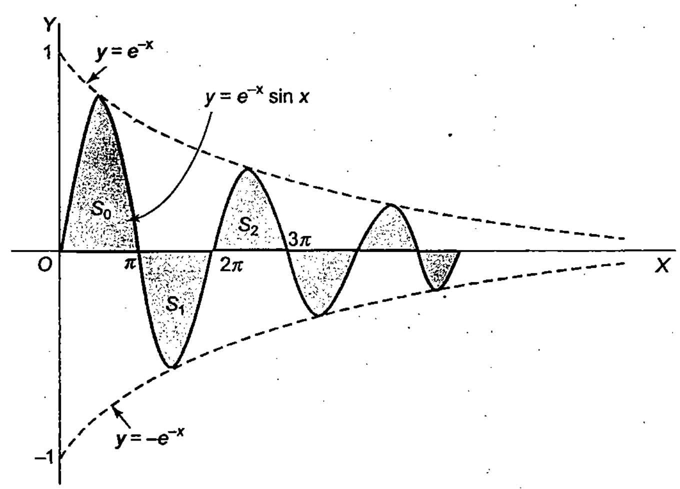



**Fig. 9.77**

Also, the curve $y = e^{-x} \sin x$ intersects the positive semi-axis $OX$ at the points where $\sin x = 0$, where $x_n = n\pi, n \in Z$. Also $|y_n| = |y$ coordinate in the half-wave $S_n|$

$= (-1)^n e^{-x} \sin x$, and

in $S_n$, $n\pi \le x \le (n+1)\pi$

$\therefore S_n = (-1)^n \int_{n\pi}^{(n+1)\pi} e^{-x} \sin x dx$

$= \frac{(-1)^{n+1}}{2} \left[ e^{-x}(-\sin x + \cos x) \right]_{n\pi}^{(n+1)\pi}$

$= \frac{(-1)^{n+1}}{2} \left[ e^{-(n+1)\pi}(-1)^{n+1} - e^{n\pi}(-1)^n \right]$

$$= \frac{e^{-n\pi}}{2}\left(1+e^{\pi}\right)$$

$$\Rightarrow \frac{S_{n+1}}{S_n} = e^{-\pi} \quad \text{and } S_0 = \frac{1}{2}\left(1+e^{\pi}\right).$$

$\therefore$ the sequence $S_0, S_1, S_2, \ldots$ forms an infinite G.P. with common ratio $e^{-\pi}$.

$$\therefore \sum_{n=0}^{\infty} S_n = \frac{\frac{1}{2}(1+e^{\pi})}{1-e^{-\pi}}$$

**For Problems 9–11**

**9. b, 10. a, 11. c**

**Sol.**

**9. b.** Given

$(x-y)f(x+y)-(x+y)f(x-y)=4xy(x^2-y^2)$

$=(x^2-y^2)\,[(x+y)^2-(x-y)^2]$

$=(x-y)\,(x+y)^3-(x+y)\,(x-y)^3$

$\Rightarrow f(x+y)=(x+y)^3 \Rightarrow f(x)=x^3, f(y)=y^3$

Now equations of given curves are

$y^2+x=0$ (1)

$x^2+y^2=12$ (2)

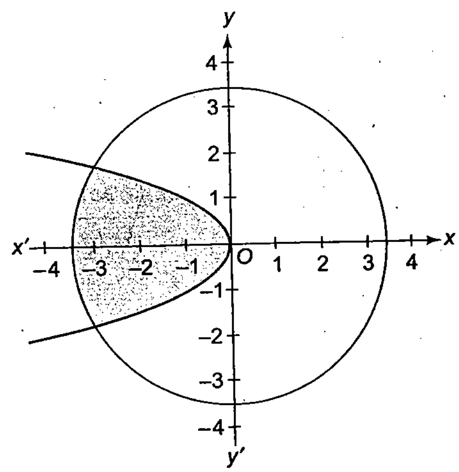



**Fig. 9.78**

Solving equations (1) and (2), we get $x=-3, y=\pm\sqrt{3}$

The area bounded by curves

$$A = 2\left[\left|\int_{-2\sqrt{3}}^{-3}\sqrt{12-x^2}\,dx\right| + \left|\int_{-3}^{0}\sqrt{-x}\,dx\right|\right]$$

$$I_1 = 2\int_{-2\sqrt{3}}^{-3}\sqrt{12-x^2}\,dx = 2\int_{-\pi/2}^{-\pi/3} 12\cos^2\theta\,d\theta$$

$$= 12\left[\int_{-\pi/2}^{-\pi/3}(1+\cos 2\theta)\,d\theta\right]$$

$$= 12\left[\theta + \frac{\sin\theta}{2}\right]_{-\pi/2}^{-\pi/3} = 12\left[-\frac{\pi}{3}-\frac{\sqrt{3}}{4}+\frac{\pi}{2}\right]$$

$$= 12\left[\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right] = 2\pi - 3\sqrt{3}.$$

$$I_2 = 2\int_{-3}^{0}\sqrt{-x}\,dx = \frac{2[(-x)^{3/2}]_{-3}^{0}}{-3/2} = -\frac{4}{3}[0-3^{3/2}]$$

$$= 4\sqrt{3}.$$

$A = 2\pi - 3\sqrt{3} + 4\sqrt{3} = 2\pi + \sqrt{3}$ sq. units.

**10. a.** The required area is = area of circle – area of square $= 12\pi - 24$ sq. units.

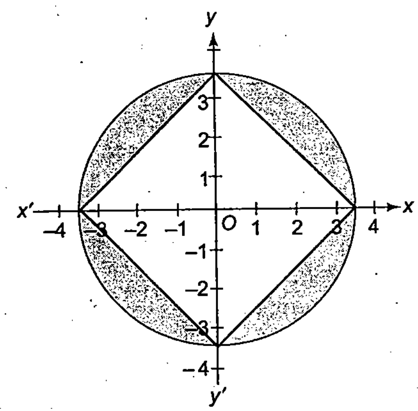



**Fig. 9.79**

**11. c.** The required area

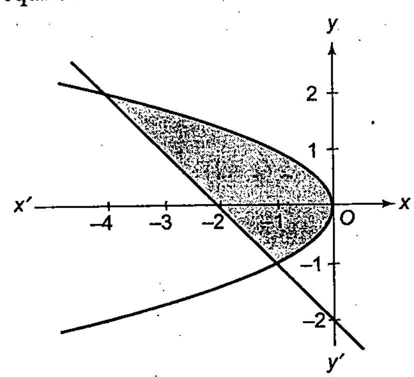



**Fig. 9.80**

$$= \int_{-1}^{2}(-y^2-(-y-2))\,dy$$

$$= \left[\frac{y^2}{2}+2y-\frac{y^3}{3}\right]_{-1}^{2}$$

$$= \left[\frac{4}{2}+4-\frac{8}{3}-\left(\frac{1}{2}-2+\frac{1}{3}\right)\right]$$

$= 9/2$ sq. units.

**For Problems 12–13**

**12. c., 13. b.**

**Sol.**

**12. c.**

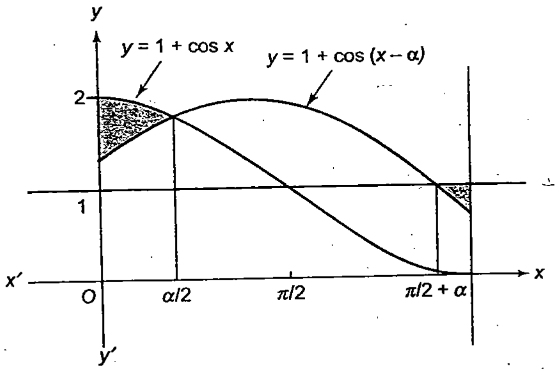



**Fig. 9.81**

$$1+\cos x = 1+\cos(x-\alpha)$$

$$x = \alpha - x \Rightarrow x = \frac{\alpha}{2}$$

$$\text{Now} \int_0^{\alpha/2}\left((1+\cos x)-(1+\cos(x-\alpha))\right)dx$$

$$= -\int_{\frac{\pi}{2}+\alpha}^{\pi}\left(1-(1+\cos(x-\alpha))\right)dx$$

$$\Rightarrow [\sin x - \sin(x-a)]_0^{\alpha/2} = [\sin(x-\alpha)]_{\pi}^{\frac{\pi}{2}+\alpha}$$

$$\Rightarrow \left[\sin\frac{\alpha}{2} - \sin\left(-\frac{\alpha}{2}\right)\right] - [0-\sin(-\alpha)]$$

$$= \sin\left(\frac{\pi}{2}\right) - \sin(\pi-\alpha)$$

$$\Rightarrow 2\sin\frac{\alpha}{2} - \sin\alpha = 1-\sin\alpha$$

Hence, $2\sin\frac{\alpha}{2} = 1 \Rightarrow \alpha = \frac{\pi}{3}$.

**13. b.** $$\int_0^{\pi/6}\left((1+\cos x)-\left(1+\cos\left(x-\frac{\pi}{3}\right)\right)\right)dx + \int_{\pi/6}^{\pi}\left(\left(1+\cos\left(x-\frac{\pi}{3}\right)\right)-(1+\cos x)\right)dx$$

$$= \left[\sin x - \sin\left(x-\frac{\pi}{3}\right)\right]_0^{\pi/6} + \left[\sin\left(x-\frac{\pi}{3}\right)-\sin x\right]_{\pi/6}^{\pi}$$

$$= \left[\left(\frac{1}{2}+\frac{1}{2}\right)-\frac{\sqrt{3}}{2}\right] + \left[\frac{\sqrt{3}}{2}-\left(-\frac{1}{2}-\frac{1}{2}\right)\right]$$

$= 2$ sq. units.

**For Problems 14–16**

**14. a., 15. d., 16. a.**

**Sol.**

**14. a.** For $-1 \le x < 0$

$$(y - e^{\sin^{-1}x})^2 = 2-x^2$$

$$y = e^{\sin^{-1}x} \pm \sqrt{2-x^2}$$

$$A = \int_{-1}^{0}\left(\left(e^{\sin^{-1}x}+\sqrt{2-x^2}\right)-\left(e^{\sin^{-1}x}-\sqrt{2-x^2}\right)\right)dx$$

$$= 2\int_{-1}^{0}\sqrt{2-x^2}\,dx$$

$$= 2\left(\frac{1}{2}x\sqrt{2-x^2}\Big|_{-1}^{0} + \frac{2}{2}\sin^{-1}\frac{x}{\sqrt{2}}\Big|_{-1}^{0}\right)$$

$$= \left[1+2\left(0-\left(-\frac{\pi}{4}\right)\right)\right]$$

$= \frac{\pi}{2}+1$ sq. units.

For $0 \le x < 1, y = \sin^{-1}x \pm \sqrt{1-x^2}$

$$A = 2\int_0^1 \sqrt{1-x^2}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{1-x^2}\Big|_0^1 + \frac{1}{2}\sin^{-1}\frac{x}{1}\Big|_0^1\right]$$

$= 0+\sin^{-1}(1) = \frac{\pi}{2}$ sq. units.

Total area $= \left(\frac{\pi}{2}+1\right)+\frac{\pi}{2} = \pi+1.$

**15. d.** Ratio $= \dfrac{\frac{\pi}{2}+1}{\frac{\pi}{2}} = \dfrac{\pi+2}{\pi}$.

**16. a.** $A = 2\int_0^{1/2}\sqrt{1-x^2}\,dx$

$$= 2\left[\frac{x}{2}\sqrt{1-x^2}\Big|_0^{\frac{1}{2}} + \frac{1}{2}\sin^{-1}x\Big|_0^{\frac{1}{2}}\right]$$

$= \frac{\sqrt{3}}{4}+\frac{\pi}{6}$ sq. unit.

**For Problems 17–19**

**17. b., 18. a., 19. c.**

**Sol.**

**17. b.** $S = \left|-\int_0^{2\pi} a(1-\cos t)a(1-\cos t)dt\right|$

$$= \left|-a^2\int_0^{2\pi}(1-2\cos t+\cos^2 t)dt\right|$$

$$= \left|-a^2\int_0^{2\pi}\left(1-2\cos t+\left(\frac{1+\cos 2t}{2}\right)\right)dt\right|$$

$$= \left|-\frac{a^2}{2}\int_0^{2\pi}(3-4\cos t+\cos 2t)dt\right|$$

$$= \left| -\frac{a^2}{2}[3t - 4\cos t + \cos 2t]_0^{2\pi} \right|$$

$$= |-3\pi a^2| = 3\pi a^2 \text{ sq. units.}$$

**18. a.** $\int_0^6 \left(\frac{3}{2}t^2 - \frac{1}{2}t^3 + \frac{1}{24}t^4\right)dt$

$$= \frac{3}{2}\frac{6^3}{3} - \frac{1}{2}\frac{6^4}{4} + \frac{1}{24}\frac{6^5}{5} = \frac{6^3}{2} - \frac{6^4}{8} + \frac{6^4}{20}$$

$$= 6^4\left(\frac{1}{12} - \frac{1}{8} + \frac{1}{20}\right) = \frac{54}{5}.$$

$$\therefore \frac{1}{2}\int_0^6 (xy' - yx')dx = \frac{1}{2} \times \frac{54}{5} = \frac{27}{5} \text{ sq. units.}$$

**19. c.** $\frac{dx}{dt} = 1 - 3t^2$ and $\frac{dy}{dt} = 1 - 4t^3$.

So, $x\frac{dy}{dt} - y\frac{dx}{dt}$

$$= (t - t^3)(1 - 4t^3) - (1 - t^4)(1 - 3t^2)$$

$$= t^6 - 3t^4 - t^3 + 3t^2 + t - 1$$

$$\therefore \text{ required area} = \frac{1}{2}\int_{-1}^{1}(t^6 - 3t^4 - t^3 + 3t^2 + t - 1)dt$$

$$= \frac{16}{35} \text{ sq. units (taking absolute value).}$$

## Matrix-Match Type

**1. a.→r; b.→p; c.→s; d.→q**

**Sol.**

**a.**

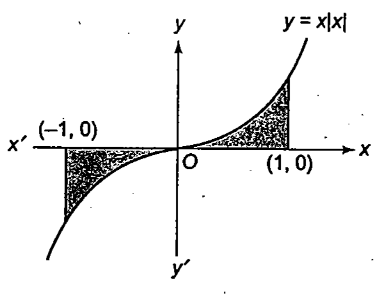



**Fig. 9.82**

$$\text{Required area} = 2\int_0^1 x|x|\,dx$$

$$= 2\left(\frac{x^3}{3}\right)_0^1 = \frac{2}{3}$$

**b.**

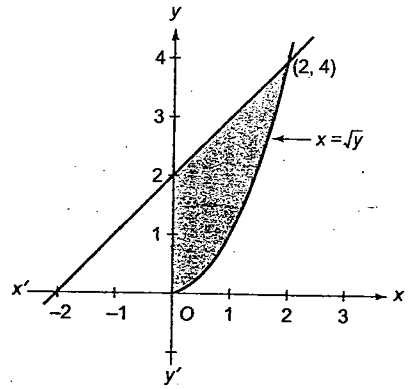



**Fig. 9.83**

$$= \int_0^2 [(x+2) - (x^2)]dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_0^2$$

$$= 2 + 4 - \frac{8}{3} = \frac{10}{3} \text{ sq. units.}$$

**c.** Reqd. area $= \int_0^1 \left(\sqrt{x} - x\right) dx = \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^1$

$$= \left(\frac{1}{3/2} - \frac{1}{2}\right) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6} \text{ sq. units.}$$

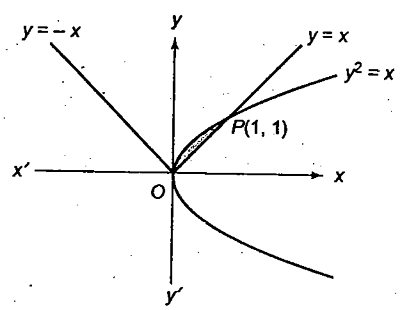



**Fig. 9.84**

**d.** $y = 4$ meets the parabola $y^2 = x$ at $A$ is $(16, 4)$

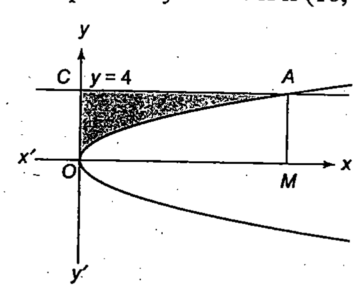



**Fig. 9.85**

Required area = Area of rectangle $OMAC$ – Area $OMA$

$$= 4 \times 16 - \int_0^{16} \sqrt{x}\,dx = 64 - \left|\frac{x^{3/2}}{3/2}\right|_0^{16}$$

$$= 64 - \frac{2}{3}(4)^3 = 64 - \frac{128}{3} = \frac{64}{3} \text{ sq. units.}$$

**2.** a. → q; b. → p; c. → s; d. → r

**Sol.**

a. Area $= 2\left(\frac{1}{2}1.1\right) = 1$ sq. units.

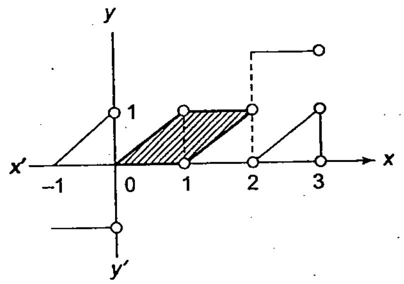



Fig. 9.86

b. $y^2 = x^3$ and $|y| = 2x$, both the curve are symmetric about $y$-axis

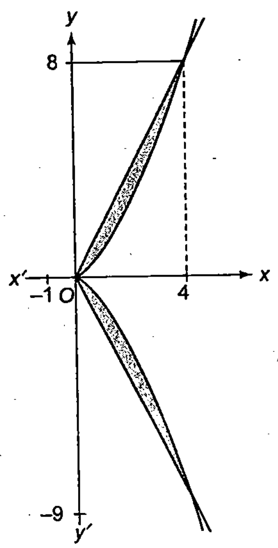



Fig. 9.87

$4x^2 = x^3 \Rightarrow x = 0, 4.$

The required area $= 2\int_0^4 \left(2x - x^{3/2}\right)dx = \frac{32}{5}$ sq. units.

c. $\sqrt{x} + \sqrt{|y|} = 1$

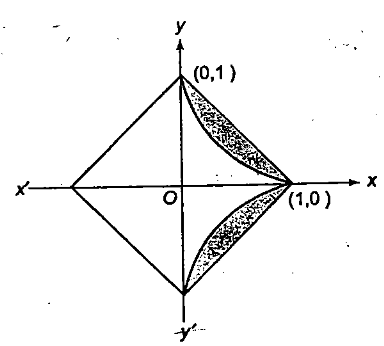



Fig. 9.88

The curve is symmetrical about $x$-axis

$\sqrt{|y|} = 1 - \sqrt{x}$ and $\sqrt{x} = 1 - \sqrt{|y|}$

$\Rightarrow$ for $x > 0, y > 0$ $\sqrt{y} = 1 - \sqrt{x}$

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$

$\frac{dy}{dx} < 0$, function is decreasing, the required area

$$= 2\int_0^1 \left((1-x) - (1 - 2\sqrt{x} + x)\right)dx$$

$$= 4\int_0^1 \left(\sqrt{x} - x\right)dx$$

$$= 4\left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^1$$

$$= 4\left[\frac{2}{3} - \frac{1}{2}\right]$$

$$= \frac{2}{3} \text{ sq. units.}$$

d. If $-8 < x < 8$, then $y = 2$.

If $x \in (-8\sqrt{2}, -8] \cup [8, 8\sqrt{2})$, then $y = 3$, and so on

Intersection of $y = x - 1$ and $y = 2$. We get $x = 3 \in (-8, 8)$.

Intersection of $y = x - 1$ and $y = 3$,

we get $x = 4 \notin (-8\sqrt{2}, -8] \cup [8, 8\sqrt{2})$.

Similarly, $y = x - 1$ will not intersect $y = \left[\frac{x^2}{64} + 2\right]$ at any other integral, except in the interval $x \in (-8, 8)$.

The required area (shaded region) $= 2 \times 3 - \frac{1}{2} \times 2 \times 2$

$= 4$ sq. units.

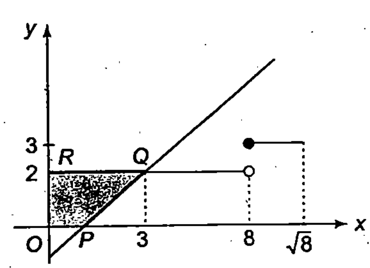



Fig. 9.89

**3. a. → q; b. → s; c. → p; d. → p**

**Sol.**

a. $[x]^2 = [y]^2$, where $1 \le x \le 4$

$\Rightarrow [x] = \pm [y]$

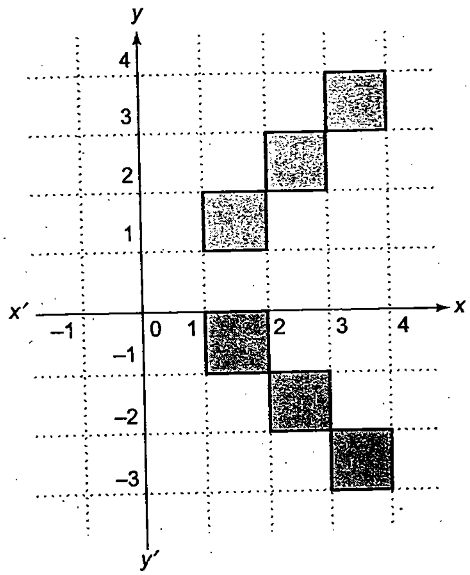



**Fig. 9.90**

b. $[|x|] + [|y|] = 2$

The graph is symmetrical about both $x$-axis and $y$-axis.

For $x, y > 0$; $[x] + [y] = 2$.

$\Rightarrow [x] = 0$ and $[y] = 2$, $[x] = 1$ and $[y] = 1$ or $[x] = 2$ and $[y] = 0$.

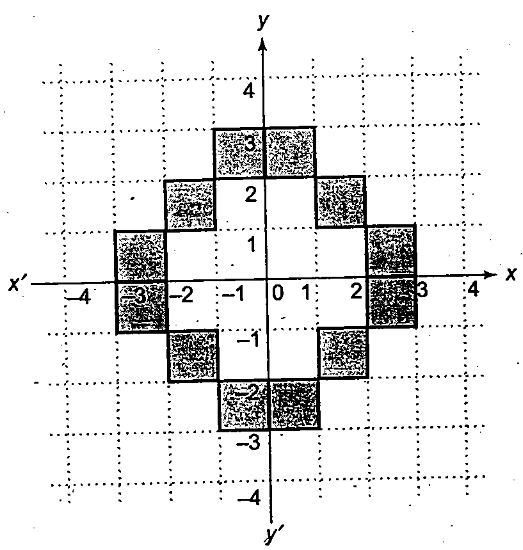



**Fig. 9.91**

c. $[|x|]\,[|y|] = 2$

The graph is symmetrical about both $x$-axis and $y$-axis.

For $x, y > 0$; $[x][y] = 2 \Rightarrow [x] = 1$ and $[y] = 2$ or $[x] = 2$ and $[y] = 1$.

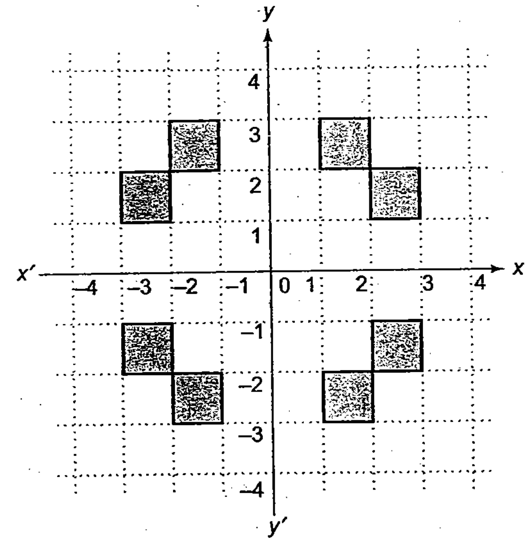



**Fig. 9.92**

d. $\dfrac{[|x|]}{[|y|]} = 2$, where $-5 \le x \le 5$.

The graph is symmetrical about both the axes.

For $x, y > 0$, $[x] = 2[y]$, $[y] \ne 0$.

$\Rightarrow [x] = 2$ and $[y] = 1$ or $[x] = 4$ and $[y] = 2$.

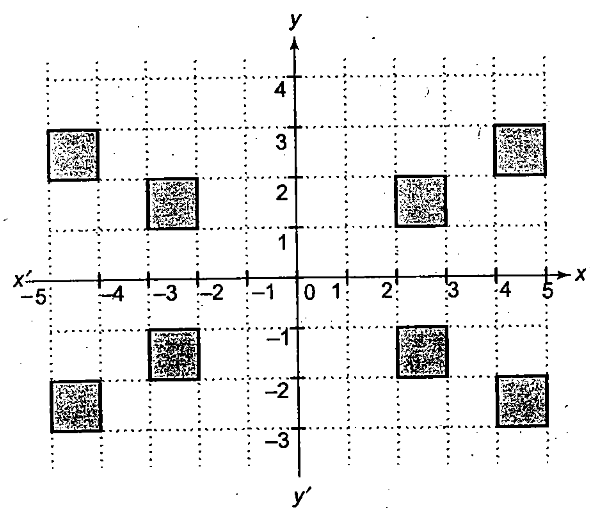



**Fig. 9.93**

## Integer Type

**1.(9)** Required area

$$A = \int_0^3 x\sqrt{9-x^2}\,dx\text{; Put } 9 - x^2 = t^2 \Rightarrow -2x\,dx = 2t\,dt$$

$$\therefore\ A = \int_0^3 t^2\,dt = 9$$

**2.(4)** We have $S = \int_0^{\pi} \sin x\,dx = 2$, so $T = \dfrac{2}{3}$, where $a > 0$.

Now $$T=\int_0^{\tan^{-1}a}\sin x\,dx+\int_{\tan^{-1}a}^{\pi/2}a\cos x\,dx=\frac{2}{3}$$

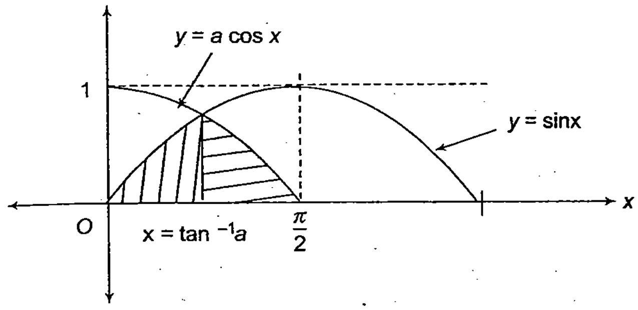



**Fig. 9.94**

i.e. $-\cos(\tan^{-1}a)+1+a\{[1-\sin(\tan^{-1}a)]\}=\frac{2}{3}$,

i.e. $-\frac{1}{\sqrt{1+a^2}}+1+a-\frac{a^2}{\sqrt{1+a^2}}=\frac{2}{3}$

$\Rightarrow\ (a+1)-\sqrt{a^2+1}=\frac{2}{3}\ \Rightarrow\ a+\frac{1}{3}=\sqrt{a^2+1}\ \Rightarrow\ a=\frac{4}{3}$

Hence $3a=4$

**3.(8)**

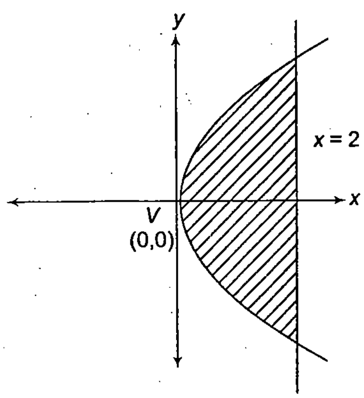



**Fig. 9.95**

Let $P(x, y)$ be any point on the curve $C$.

Now, $\frac{dy}{dx}=\frac{1}{y}$

$\Rightarrow ydy=dx\ \Rightarrow\ \frac{y^2}{2}=x+k$

Since the curve passes through $M(2,2)$, so $k=0$

$\Rightarrow y^2=2x$

Hence required area $=2\int_0^2\sqrt{2x}\,dx$

$=2\sqrt{2}\times\frac{2}{3}\left(x^{3/2}\right)_0^2$

$=\frac{4}{3}\sqrt{2}\times2\sqrt{2}$

$=\frac{16}{3}$ (square unit)

**4.(8)** $\int_0^3(-x^2+ax+12)dx=45$ gives $a=4$

hence $f(x)=12+4x-x^2=(2+x)(6-x)$

Hence $m=-2$ and $n=6$

$m+n+a=6-1+4=8$

**5.(9)** Graph of $f(x)$ is as

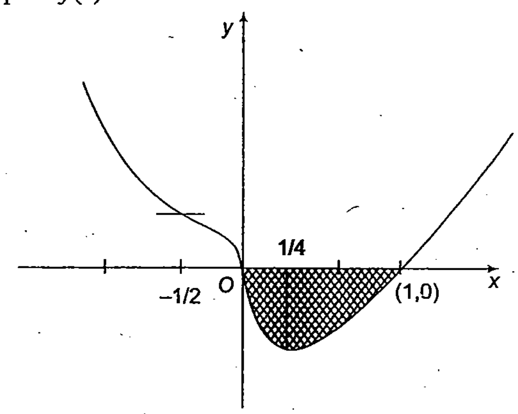



**Fig. 9.96**

$$A=\int_0^1(x^{4/3}-x^{1/3})dx=\left[\frac{3}{7}x^{3/7}-\frac{3}{4}x^{4/3}\right]_0^1$$

$$=\left|\frac{3}{7}-\frac{3}{4}\right|=3\left|\frac{4-7}{28}\right|=\frac{9}{28}$$

$\Rightarrow 28A=9$

**6.(2)** Let the point of the curve is $(x, x^2+1)$.

Now, the slope of tangent at this point is $2x$, which is equal to the slope of the line joining $(x, x^2+1)$ and $(0, 0)$.

Hence $2x=(x^2+1)/x\ \Rightarrow\ 2x^2=x^2+1$

$\Rightarrow x^2=1\ \Rightarrow\ x=\pm1$

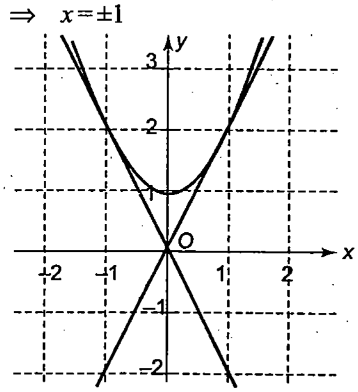



**Fig. 9.97**

Hence equation of tangent is $y=\pm2x$

Now area $2\int_0^1\left(x^2+1-2x\right)dx$

$=2\int_0^1(x-1)^2\,dx$

$=2\left[\frac{(x-1)^3}{3}\right]_0^1=2/3$

**7.(8)** Required area = area of one quadrant of the circle = $\pi/2$

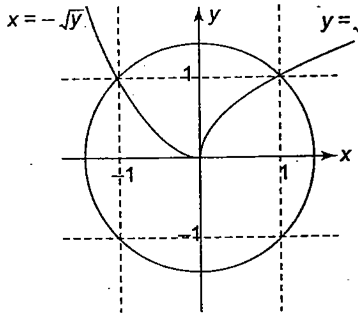



**Fig. 9.98**

**8.(1)** $f(a) = \int_a^{2a} \left(\frac{x}{6} + \frac{1}{x^2}\right) dx = \left(\frac{x^2}{12} - \frac{1}{x}\right)_a^{2a}$

$$= \left(\frac{4a^2}{12} - \frac{1}{2a} - \frac{a^2}{12} + \frac{1}{a}\right) = \frac{a^2}{4} + \frac{1}{2a}$$

Let $f'(a) = \frac{2a}{4} - \frac{1}{2a^2} = 0$

$\Rightarrow$ $a = 1$ which is point of minima.

**9.(3)** $[2x] = 0 \Rightarrow 2x \in [0, 1) \Rightarrow x \in [0, 1/2) \Rightarrow [y] = 5 \Rightarrow y \in [5, 6)$

Similarly we can consider $[2x] = 1, 2, 3, 4$ and $5$

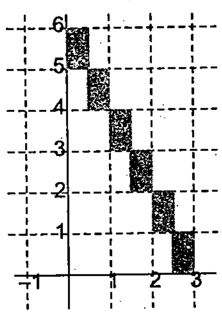



**Fig. 9.99**

From the graph, area is 3 sq. units

**10.(8)** Required area $= 2\int_0^2 \left(x(x-3)^2 - x\right) dx = 8$ sq. units

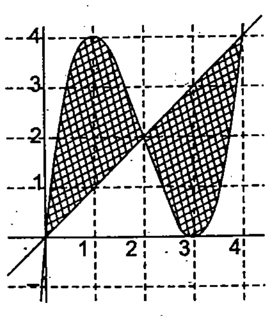



**Fig. 9.100**

**11.(6)** Draw the given region point of intersection of $y = x^2 + 1$

$y = x + 1$

$x + 1 = x^2 + 1$

$x = 0, 1$

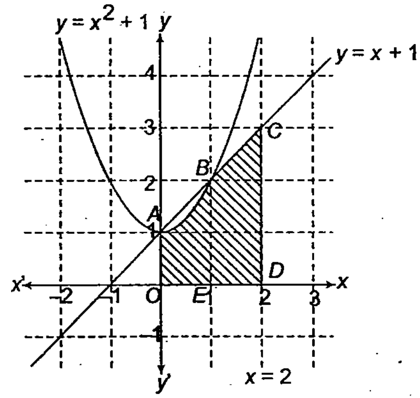



**Fig. 9.101**

Required area $OABCDE = \int_0^1 (x^2 + 1) dx + \int_1^2 (x + 1) dx$

$$= \left(\frac{x^3}{3} + x\right)_0^1 + \left(\frac{x^2}{2} + x\right)_1^2 = \frac{23}{6} \text{ sq. units}$$

**12.(6)**

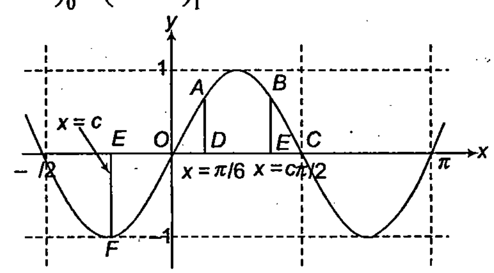



**Fig. 9.102**

Area $OABC = \int_0^{\pi/2} \sin 2x \, dx = 1$

Area $OAD = \int_0^{\pi/6} \sin 2x \, dx = \frac{1}{4}$

$\because$ $\sin 2x$ is symmetric about origin

so $c = -\frac{\pi}{6}$, because area $OAD$ = area $OEF$

$$\int_{\frac{\pi}{6}}^{c} \sin 2x \, dx = \frac{1}{2}$$

$\cos 2c = -\frac{1}{2}$ $\cos 2c = \frac{3}{2}$ (not posible)

$c = \frac{\pi}{3}$

so $c = -\frac{\pi}{6}, \frac{\pi}{3}$

3.(2) $y=\sqrt{1-x^2}$ (1)

$y=x^3-x$ (2)

$y=0$ in (2) $x=0, 1, -1$

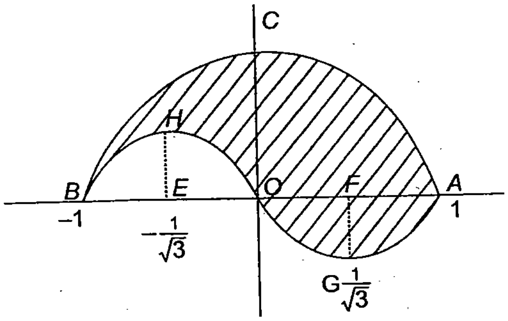



**Fig. 9.103**

Required area = area of region $BCAGOHB$

= Area of semi-circle $BCAOB$

$= \frac{\pi}{2}$

($\because$ area of $BHOEB$ = area of $OFAGO$)

**4.(1)** Given that $D_1 = D_2$

$$\int_1^c \left(\frac{1}{x}-\log x\right)dx=\int_c^a \left(\log x-\frac{1}{x}\right)dx$$

$$\left(\frac{-1}{x^2}-x(\log x-1)\right)_1^c=\left(x(\log x-1)+\frac{1}{x^2}\right)_c^a$$

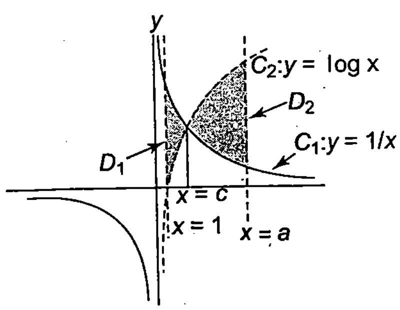



**Fig. 9.104**

$\therefore\ 0 = a(\log a-1)+\frac{1}{a^2}$

$\therefore\ a=1$

**15.(3)** $y=\frac{a^2-ax}{1+a^4}$ (1)

$y=\frac{x^2+2ax+3a^2}{1+a^4}$ (ii)

Point of intersection of (1) and (2)

$$\frac{a^2-ax}{1+a^4}=\frac{x^2+2ax+3a^2}{1+a^4}$$

$(x+a)(x+2a)=0$

$x=-a,-2a$

$$\text{Req. area} = \int_{-2a}^{-a}\left[\left(\frac{a^2-ax}{1+a^4}\right)-\left(\frac{x^2+2ax+3a^2}{1+a^4}\right)\right]$$

$\therefore\ f(a)=\frac{a^3}{6(1+a^4)}$

$f(a)$ is max is

then $f'(a)=0$

$3+3a^4-4a^4=0$

$a^4=3$

**16. (2)**

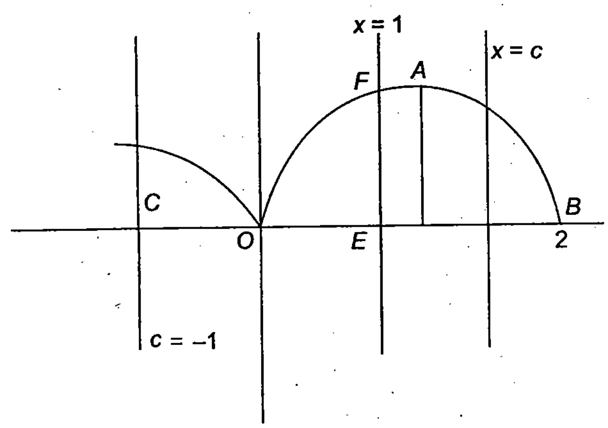



**Fig. 9.105**

given that $\int_1^c ydx=\frac{16}{3}$

$\Rightarrow \int_1^c (8x^2-x^5)dx=\frac{16}{3}$

$c=(8-\sqrt{17})^{1/3}$ $(c>0)$

area $OFE=\int_0^c (8x^2-x^5)dx=\frac{8}{3}$ $(c>0)$

so $c=-1$

Hence $c=-1$ and $(8-\sqrt{17})^{1/3}$

## Archives

### Subjective

**1.** Given curves $x^2 = 4y$ and $x = 4y - 2$ intersect, when

$x^2 = x + 2$

$\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow x = 2, -1$

$\Rightarrow y = 1, 1/4$

$\Rightarrow$ Points of intersection are $A(-1, 1/4), B(2, 1)$

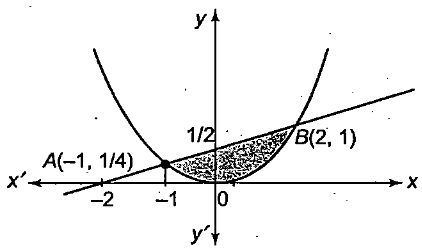



**Fig. 9.106**

Required area
= shaded region in the figure

$$= \int_{-1}^{2}\left(\frac{x+2}{4} - \frac{x^2}{4}\right)dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{1}{4}\left[\left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right]$$

$$= \frac{1}{4}\left[\frac{10}{3} - \left(\frac{-7}{6}\right)\right]$$

$$= \frac{1}{4}\left[\frac{27}{6}\right] = 9/8 \text{ sq. units.}$$

**2.**

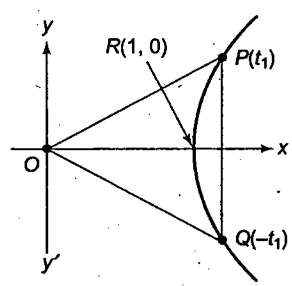



**Fig. 9.107**

$$x = \frac{e^t + e^{-t}}{2};\ y = \frac{e^t - e^{-t}}{2}$$

It is a point on hyperbola $x^2 - y^2 = 1$.

Then, the equation of line joining $t_1$ and $-t_1$, that is,

$$\left(\frac{e^{t_1} + e^{-t_1}}{2}, \frac{e^{t_1} - e^{-t_1}}{2}\right) \text{ and } \left(\frac{e^{-t_1} + e^{t_1}}{2}, \frac{e^{-t_1} - e^{t_1}}{2}\right) \text{ is}$$

$$x = \frac{e^{t_1} + e^{-t_1}}{2}$$

$$\therefore \text{area }(PQRP) = 2\int_1^{\frac{e^{t_1}+e^{-t_1}}{2}} y\,dx$$

$$= 2\int_1^{\frac{e^{t_1}+e^{-t_1}}{2}} \sqrt{x^2 - 1}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{x^2 - 1} - \frac{1}{2}\log|x + \sqrt{x^2 - 1}|\right]_1^{\frac{e^{t_1}+e^{-t_1}}{2}}$$

$$= \left(\frac{e^{t_1} + e^{-t_1}}{2}\right)\left(\frac{e^{t_1} - e^{-t_1}}{2}\right) - \log\left|\frac{e^{t_1} + e^{-t_1}}{2} + \frac{e^{t_1} - e^{-t_1}}{2}\right|$$

$$= \frac{e^{2t_1} - e^{-2t_1}}{4} - \log e^{t_1}$$

$$= \frac{e^{2t_1} - e^{-2t_1}}{4} - t_1 \qquad (1)$$

$$\text{Area of } \Delta OPQ = 2 \times \frac{1}{2}\left(\frac{e^{t_1} + e^{-t_1}}{2}\right)\left(\frac{e^{t_1} - e^{-t_1}}{2}\right)$$

$$= \frac{e^{2t_1} - e^{-2t_1}}{4} \qquad (2)$$

$\therefore$ required area = Ar $\Delta OPQ$ − Ar$(PQRP)$
= $t_1$ (using (1) and (2)).

**3.** Given $y = 1 + \frac{8}{x^2}$.

Here $y$ is always positive, hence curve is lying above the $x$-axis.

$$\Rightarrow \text{Req. area} = \int_2^4 y\,dx = \int_2^4\left(1 + \frac{8}{x^2}\right)dx$$

$$= \left[x - \frac{8}{x}\right]_2^4 = 4.$$

If $x = a$ bisects the area, then we have

$$\int_2^a\left(1 + \frac{8}{x^2}\right)dx = \left[x - \frac{8}{x}\right]_2^a = \left[a - \frac{8}{a} - 2 + 4\right] = \frac{4}{2}.$$

$\Rightarrow a - \frac{8}{a} = 0$

$\Rightarrow a^2 = 8$

$\Rightarrow a = \pm 2\sqrt{2}$

Since $a > 2$, $a = 2\sqrt{2}$.

**4.**

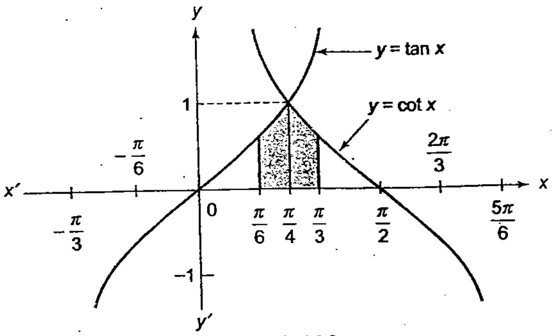



**Fig. 9.108**

The two curves are

$y = \tan x$, where $-\pi/3 \le x \le \pi/3$ (1)

$y = \cot x$, where $\pi/6 \le x \le 3\pi/2$ (2)

At the point of intersection of the two curves

$\tan x = \cot x$ or $\tan^2 x = 1$ or $\tan x = \pm 1, x = \pm\pi/4$

Thus, the curves intersect at $x = \pi/4$

The required area is the shaded area

$$A = \int_{\pi/6}^{\pi/4} \tan x\, dx + \int_{\pi/4}^{\pi/3} \cot x\, dx$$

$$= [\log \sec x]_{\pi/6}^{\pi/4} + [\log \sin x]_{\pi/4}^{\pi/3}$$

$$= \left(\log \sqrt{2} - \log \frac{2}{\sqrt{3}}\right) + \left(\log \frac{\sqrt{3}}{2} - \log \frac{1}{\sqrt{2}}\right)$$

$$= \log \sqrt{2} + \log \frac{\sqrt{3}}{2} + \log \frac{\sqrt{3}}{2} + \log \sqrt{2}$$

$$= 2\left(\log \sqrt{2}\, \frac{\sqrt{3}}{2}\right)$$

$$= 2 \log \sqrt{\frac{3}{2}} = \log 3/2 \text{ sq. units.}$$

**5.** The given curves are

$y = \sqrt{5 - x^2}$ (1)

$y = |x - 1|$ (2)

We can clearly see that (on squaring the both sides of (1), equation (2) represents a circle.

But as $y$ is + ve square root

∴ (1) represents the upper-half of the circle with centre (0, 0) and radius $\sqrt{5}$.

Equation (2) represents the curve

$$y = \begin{cases} -x + 1 & \text{if } x < 1 \\ x - 1 & \text{if } x \ge 1 \end{cases}$$

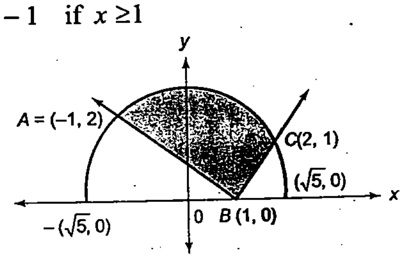



**Fig. 9.109**

Graph of these curves is as shown in the figure with point of intersection of

$y = \sqrt{5 - x^2}$ and $y = -x + 1$ as $A(-1, 2)$

and of $y = \sqrt{5 - x^2}$ and $y = x - 1$ as $C(2, 1)$

Thus the required area

= Shaded area

$$= \int_{-1}^{2} \sqrt{5 - x^2}\, dx - \int_{-1}^{2} |x - 1|\, dx$$

$$= \left[\frac{x}{2}\sqrt{5 - x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)\right]_{-1}^{2} - \int_{-1}^{1} -(x - 1)\, dx$$

$$- \int_{1}^{2} (x - 1)\, dx$$

$$= \left(\frac{2}{2}\sqrt{5 - 4} + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}\right) - \left(\frac{-1}{2}\sqrt{5 - 1}\right.$$

$$\left. + \frac{5}{2}\sin^{-1}\left(\frac{-1}{\sqrt{5}}\right)\right) - \left(\frac{-x^2}{2} + x\right)_{-1}^{1} - \left(\frac{x^2}{2} - x\right)_{1}^{2}$$

$$= 1 + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}} + 1 + \frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

$$- \left[\left(\frac{-1}{2} + 1\right) - \left(\frac{-1}{2} - 1\right)\right] - \left[(2 - 2) - \left(\frac{1}{2} - 1\right)\right]$$

$$= 2 + \frac{5}{2}\left[\sin^{-1}\frac{2}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{5}}\right] - 2 - \frac{1}{2}$$

$$= \frac{5}{2}\left[\sin^{-1}\frac{2}{\sqrt{5}} + \cos^{-1}\frac{2}{\sqrt{5}}\right] - \frac{1}{2} = \frac{5}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}$$

$$= \frac{5\pi - 2}{4} \text{ sq. units.}$$

**6.** The given curves are

$x^2 + y^2 = 4$ (circle) (1)

$x^2 = -\sqrt{2}y$ (parabola, concave downward) (2)

$x = y$ (straight line through origin) (3)

Solving equations (1) and (2), we get

$y^2 - \sqrt{2}y - 4 = 0$

$\Rightarrow y = \frac{4\sqrt{2}}{2}$ or $\frac{-2\sqrt{2}}{2}$

$\Rightarrow y = 2\sqrt{2}$ or $-\sqrt{2}$

$\Rightarrow x^2 = 2$ (rejecting $y = 2\sqrt{2}$ as $x^2$ is positive)

$\Rightarrow x = \pm\sqrt{2}$.

∴ Points of intersection of (1) and (2) are $B(\sqrt{2}, -\sqrt{2})$, $A(-\sqrt{2}, -\sqrt{2})$.

Solving (1) and (3), we get

$2x^2 = 4 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2} \Rightarrow y = \pm\sqrt{2}$.

∴ Points of intersection are $(-\sqrt{2}, -\sqrt{2})$, $(\sqrt{2}, \sqrt{2})$.

Thus, all the three curves pass through the same point $A(-\sqrt{2}, -\sqrt{2})$.

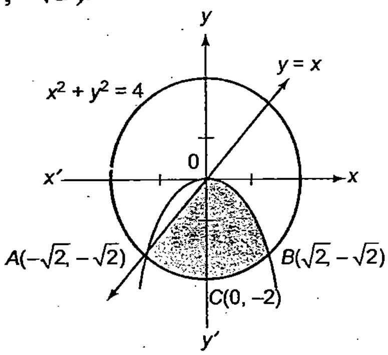



**Fig. 9.110**

Now the required area = the shaded area

$$= \int_{-\sqrt{2}}^{0}\left(x-\left(-\sqrt{4-x^2}\right)\right)dx + \int_{0}^{\sqrt{2}}\left(-\frac{x^2}{\sqrt{2}}-\left(-\sqrt{4-x^2}\right)\right)dx$$

$$= 2\int_0^{\sqrt{2}}\sqrt{4-x^2}\,dx + \int_{-\sqrt{2}}^{0} x\,dx - \int_0^{\sqrt{2}}\frac{x^2}{\sqrt{2}}dx$$

$$= 2\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{2}} + \left[\frac{x^2}{2}\right]_{-\sqrt{2}}^{0} - \left[\frac{x^3}{3\sqrt{2}}\right]_0^{\sqrt{2}}$$

$$= 2\left[\frac{\sqrt{2}}{2}\sqrt{4-2}+2\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right]+\left[\frac{-2}{2}\right]-\left[\frac{2\sqrt{2}}{3\sqrt{2}}\right]$$

$$= 2\left[1+2\frac{\pi}{4}\right]-1-\frac{2}{3} = \pi+\frac{1}{3} \text{ sq. units.}$$

**7.** Given curves are

$x^2+y^2=25$ (1)

$4y=|4-x^2|$ (2)

$x=0$ (3)

and above $x$-axis

Solving (1) and (2), we get

$4y+4+y^2=25$

$\Rightarrow (y+2)^2=5^2$

$\Rightarrow y=3,-7$

$y=-7$ is rejected, $y=3$ gives the points above $x$-axis.

When $y=3, x=\pm 4$.

Hence, the points of intersection are $P(4, 3)$ and $Q(-4, 3)$.

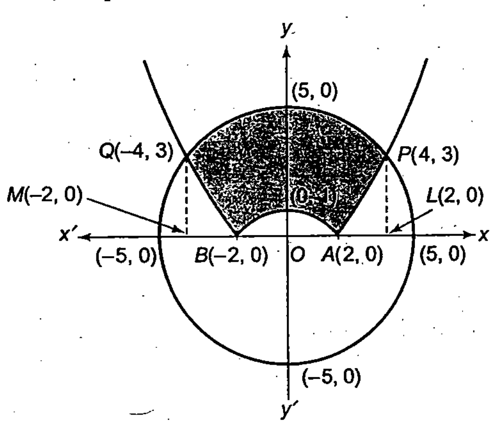



**Fig. 9.111**

The required area is

$$= 2\left[\int_0^4\sqrt{25-x^2}\,dx - \frac{1}{4}\int_0^2\left(4-x^2\right)dx - \frac{1}{4}\int_2^4\left(x^2-4\right)dx\right]$$

$$= 2\left[\left(\frac{x}{2}\sqrt{25-x^2}+\frac{25}{2}\sin^{-1}\frac{x}{5}\right)_0^4 - \frac{1}{4}\left(4x-\frac{x^3}{3}\right)_0^2 - \frac{1}{4}\left(\frac{x^3}{3}-4x\right)_2^4\right]$$

$$= 2\left[6+\frac{25}{2}\sin^{-1}\frac{4}{5}\right] - \frac{1}{4}\left[8-\frac{8}{3}\right] - \frac{1}{4}\left[\left(\frac{64}{3}-16\right)-\left(\frac{8}{3}-8\right)\right]$$

$$= 4 + 25\sin^{-1}\frac{4}{5} \text{ sq. units.}$$

**8.** The given curve is $y = \tan x$ (1)

When $x = \pi/4, y = 1$

i.e., co-ordinates of $P$ are $(\pi/4, 1)$

$\therefore$ equation of tangent at $P$ is $y-1 = \left(\sec^2\frac{\pi}{4}\right)(x-\pi/4)$

or $y = 2x+1-\pi/2$ (2)

The graphs of (1) and (2) are as shown in Fig. 9.112.

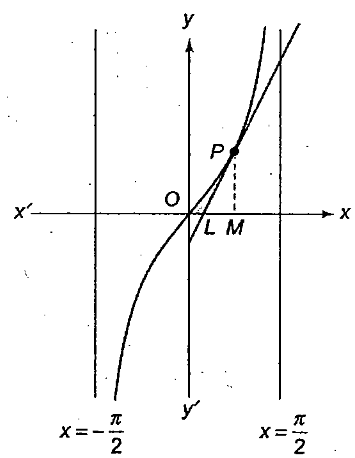



**Fig. 9.112**

Tangent (2) meets $x$-axis at $L\left(\frac{\pi-2}{4}, 0\right)$

Now the required area = the shaded area

= Area $OPMO$ − Ar($\Delta PLM$)

$$= \int_0^{\pi/4}\tan x\,dx - \frac{1}{2}(OM-OL)PM$$

$$= [\log\sec x]_0^{\pi/4} - \frac{1}{2}\left\{\frac{\pi}{4}-\frac{\pi-2}{4}\right\}1$$

$$= \frac{1}{2}\left[\log 2-\frac{1}{2}\right] \text{sq. units.}$$

**9.** The given curves are

$y = ex\log_e x$ (1)

$$y=\frac{\log x}{ex} \qquad (2)$$

The two curves intersect where $ex\log x=\dfrac{\log x}{ex}$

$$\Rightarrow\left(ex-\frac{1}{ex}\right)\log x=0$$

$$\Rightarrow x=\frac{1}{e}\text{ or } x=1$$

At $x=1/e, y=-1$ (from (1))

At $x=1, y=0$ (from (1))

So points of intersection are $\left(\frac{1}{e},-1\right)$ and $(1, 0)$.

**Curve Tracing**

| Curve 1 | Curve 2 |
|---|---|
| For $0<x<1, y<0$ | For $0<x<1, y<0$ |
| For $x>1, y>0$ | For $x>1, y>0$ |
| When $x\to 0, y\to -\infty$ | When $x\to 0, y\to -\infty$ |
| When $x\to\infty, y\to\infty$ | When $x\to\infty, y\to 0$ |
| $\frac{dy}{dx}=e(\log x+1)$ | $\frac{dy}{dx}=\frac{(1-\log x)}{ex^2}$ |
| $x=\frac{1}{e}$ is a point of min. | $x=e$ is a point of max. |

From the above information, the rough sketch of two curves is as shown in Fig. 9.113 and shaded area is the required area.

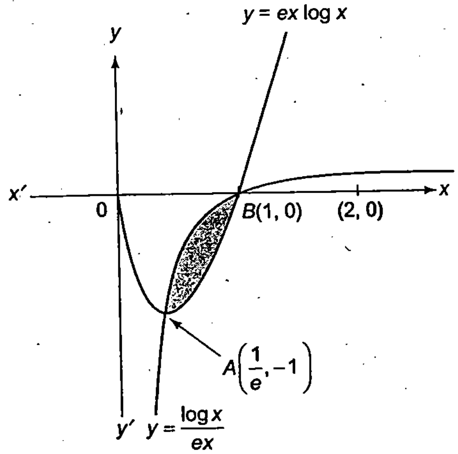



**Fig. 9.113**

∴ the required area = the shaded area

$$=\left|\int_{1/e}^{1}\left[ex\log x-\frac{\log x}{ex}\right]dx\right|$$

$$=\left|e\left[\frac{x^2}{2}\log x-\frac{x^2}{4}\right]_{1/e}^{1}-\frac{1}{e}\left[\frac{(\log x)^2}{2}\right]_{1/e}^{1}\right|$$

$$=\left|e\left[\left(-\frac{1}{4}\right)-\left(-\frac{1}{2e^2}-\frac{1}{4e^2}\right)\right]-\frac{1}{e}\left[0-\frac{1}{2}\right]\right|$$

$$=\left|e\left[-\frac{1}{4}+\frac{3}{4e^2}\right]+\frac{1}{2e}\right|$$

$$=\left|\frac{5-e^2}{4e}\right|$$

$$=\frac{e^2-5}{4e}\text{ sq. units.}$$

**10.** The given curves are

$$x=\frac{1}{2} \qquad (1)$$

$$x=2 \qquad (2)$$

$$y=\log_e x \qquad (3)$$

$$y=2^x \qquad (4)$$

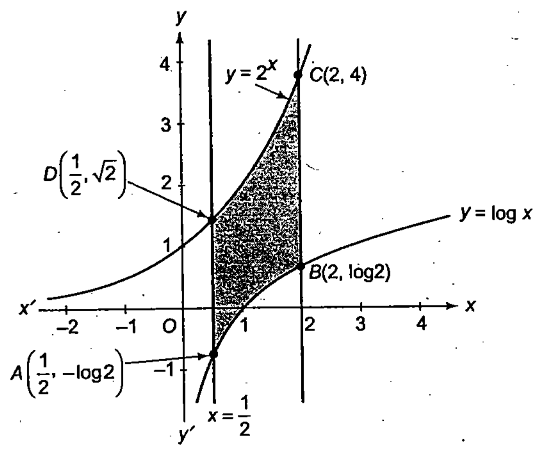



**Fig. 9.114**

Required area = $ABCDA$

$$=\int_{1/2}^{2}\left(2^x-\log x\right)dx$$

$$=\left[\frac{2^x}{\log 2}-(x\log x-x)\right]_{1/2}^{2}$$

$$=\left(\frac{4}{\log 2}-2\log 2+2\right)-\left(\frac{\sqrt{2}}{\log 2}-\frac{1}{2}\log\frac{1}{2}+\frac{1}{2}\right)$$

$$=\left(\frac{4-\sqrt{2}}{\log 2}-\frac{5}{2}\log 2+\frac{3}{2}\right)\text{ sq. units.}$$

**11.** The given curves are $y=x^2$ (1)

and $y=\dfrac{2}{1+x^2}$ (2)

Solving (1) and (2), we have

$$x^2=\frac{2}{1+x^2}$$

$$\Rightarrow x^4+x^2-2=0$$

$$\Rightarrow (x^2-1)(x^2+2)=0$$

$$\Rightarrow x=\pm 1$$

Also, $y = \dfrac{2}{1+x^2}$ is an even function.

Hence, its graph is symmetrical about $y$-axis.

At $x = 0$, $y = 2$, by increasing the values of $x$, $y$ decreases and when $x \to \infty, y \to 0$.

$\therefore$ $y = 0$ is an asymptote of the given curve.

Thus, the graph of the two curves is as follows

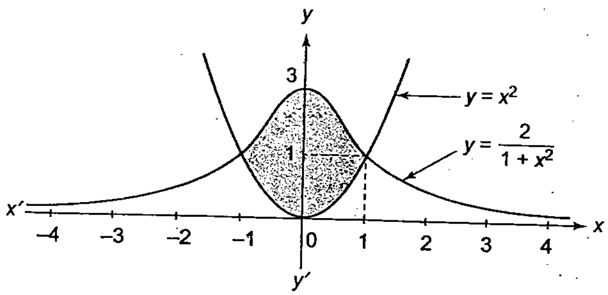



**Fig. 9.115**

$\therefore$ The required area

$$= 2\int_0^1 \left(\frac{2}{1+x^2} - x^2\right) dx$$

$$= \left(4\tan^{-1} x - \frac{2x^3}{3}\right)_0^1$$

$$= \pi - \frac{2}{3} \text{ sq. units.}$$

**12.** Both the given curves are parabola.

$y = 4x - x^2$ (1)

and $y = x^2 - x$ (2)

Solving (1) and (2), we get

$4x - x^2 = x^2 - x$

$\Rightarrow x = 0, x = \dfrac{5}{2}.$

$\Rightarrow$ Two curves intersect at $O(0, 0)$ and $A\left(\dfrac{5}{2}, \dfrac{15}{4}\right)$

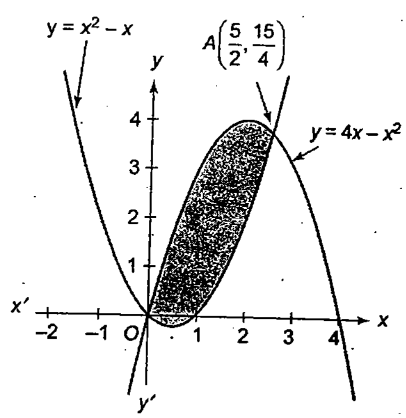



**Fig. 9.116**

Here the area below $x$-axis

$$A_1 = \int_0^1 \left(x - x^2\right) dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ sq. units}$$

Area above $x$-axis,

$$A_2 = \int_0^{5/2} (4x - x^2)\,dx - \int_1^{5/2} (x^2 - x)\,dx$$

$$= \left(2x^2 - \frac{x^3}{3}\right)_0^{5/2} - \left(\frac{x^3}{3} - \frac{x^2}{2}\right)_1^{5/2}$$

$$= \left(\frac{25}{2} - \frac{125}{24}\right) - \left[\left(\frac{125}{24} - \frac{25}{8}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right]$$

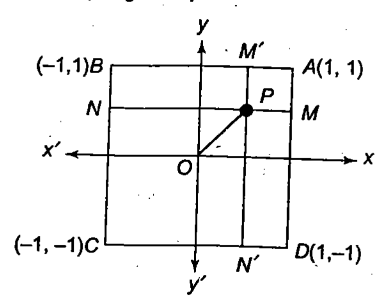
$$= \frac{25}{2} - \frac{125}{12} + \frac{25}{8} - \frac{1}{6}$$

$$= \frac{300 - 250 + 75 - 4}{24} = \frac{121}{24} \text{ sq. units.}$$

$\therefore$ ratio of area above $x$-axis to area below $x$-axis

$$A_2 : A_1 = \frac{121}{24} : \frac{1}{6} = \frac{121}{4} = 121 : 4.$$

**13.**




(a)

**Fig. 9.117(a)**

Let us consider any point $(x, y)$ inside the square such that its distance from origin $\le$ its distance from any of the edges, say $AD$.

$\therefore$ $OP < PM$

$$\Rightarrow \sqrt{(x^2 + y^2)} < 1 - x \Rightarrow y^2 \le -2\left(x - \frac{1}{2}\right) \quad (1)$$

Above represents all points within the parabola $P_1$.

If we consider the edge $BC$, then $OP < PN$ will imply

$$y^2 < 2\left(x + \frac{1}{2}\right) \quad (2)$$

Similarly, if we consider the edges $AB$ and $CD$, we will have

$$x^2 < -2\left(y - \frac{1}{2}\right) \quad (3)$$

$$x^2 < 2\left(y + \frac{1}{2}\right) \quad (4)$$

Hence, $S$ consists of the region bounded by four parabolas meeting the axes at $\left(\pm\dfrac{1}{2}, 0\right)$ and $\left(0, \pm\dfrac{1}{2}\right)$.

The point $L$ is intersection of $P_1$ and $P_3$ given by (1) and (3).

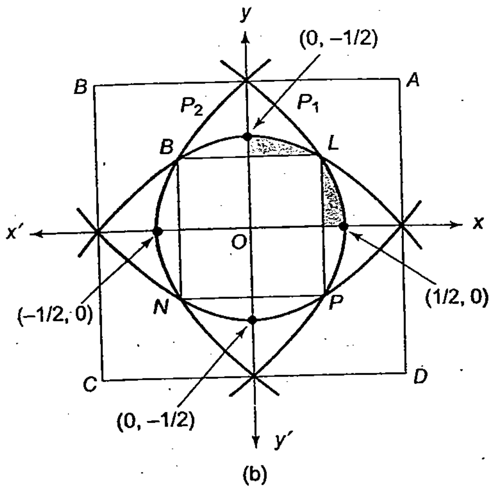



Fig. 9.117(b)

$y^2 - x^2 = -2(x-y) = 2(y-x)$

$\Rightarrow y - x = 0$

$\Rightarrow y = x$

$\Rightarrow x^2 + 2x - 1 = 0$

$\Rightarrow (x+1)^2 = 2$

$\Rightarrow x = \sqrt{2} - 1$ as $x$ is +ve

$\therefore L$ is $\left(\sqrt{2}-1, \sqrt{2}-1\right)$.

$\therefore$ The total area $= 4\Big[$square of side $\left(\sqrt{2}-1\right)$

$$+ 2\int_{\sqrt{2}-1}^{1/2} \sqrt{1-2x}\,dx\Big]$$

$$= 4\left\{\left(\sqrt{2}-1\right)^2 + 2\int_{\sqrt{2}-1}^{1/2} \sqrt{(1-2x)}\,dx\right\}$$

$$= 4\left[3 - 2\sqrt{2} - \frac{2}{3}\left\{(1-2x)^{3/2}\right\}_{\sqrt{2}-1}^{1/2}\right]$$

$$= 4\left[3 - 2\sqrt{2} - \frac{2}{3}\left\{0 - \left(1 - 2\sqrt{2} + 2\right)^{3/2}\right\}\right]$$

$$= 4\left[3 - 2\sqrt{2} + \frac{2}{3}\left(3 - 2\sqrt{2}\right)^{3/2}\right]$$

$$= 4\left(3 - 2\sqrt{2}\right)\left[1 + \frac{2}{3}\sqrt{\left(3 - 2\sqrt{2}\right)}\right]$$

$$= 4\left(3 - 2\sqrt{2}\right)\left[1 + \frac{2}{3}\left(\sqrt{2} - 1\right)\right]$$

$$= \frac{4}{3}\left(3 - 2\sqrt{2}\right)\left(1 + 2\sqrt{2}\right) = \frac{4}{3}\left[\left(4\sqrt{2} - 5\right)\right]$$

$$= \frac{16\sqrt{2} - 20}{3} \text{ sq. units.}$$

14. We have $A_n = \int_0^{\pi/4} (\tan x)^n\,dx$

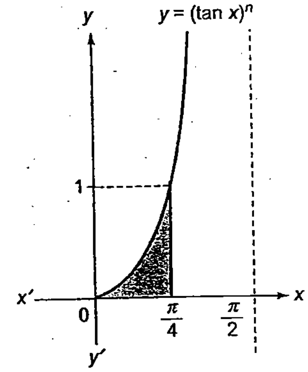



Fig. 9.118

Since $0 < \tan x < 1$, when $0 < x < \pi/4$,

we have $0 < (\tan x)^{n+1} < (\tan x)^n$ for each $n \in N$

$$\Rightarrow \int_0^{\pi/4} (\tan x)^{n+1}\,dx < \int_0^{\pi/4} (\tan x)^n\,dx$$

$\Rightarrow A_{n+1} < A_n$

Now, for $n > 2$,

$$A_n + A_{n+2} = \int_0^{\pi/4} [(\tan x)^n + (\tan x)^{n+2}]\,dx$$

$$= \int_0^{\pi/4} (\tan x)^n (1 + \tan^2 x)\,dx$$

$$= \int_0^{\pi/4} (\tan x)^n (\sec^2 x)\,dx$$

$$= \left[\frac{1}{(n+1)} (\tan x)^{n+1}\right]_0^{\pi/4}$$

$$\left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= \frac{1}{(n+1)}(1-0)$$

Since $A_{n+2} < A_{n+1} < A_n$, we get

$A_n + A_{n+2} < 2A_n$

$$\Rightarrow \frac{1}{n+1} < 2A_n \Rightarrow \frac{1}{2n+2} < A_n \qquad (1)$$

Also for $n > 2$, $A_n + A_n < A_n + A_{n-2} = \dfrac{1}{n-1}$

$$\Rightarrow 2A_n < \frac{1}{n-1}$$

$$\Rightarrow A_n < \frac{1}{2n-2} \qquad (2)$$

Combining (1) and (2), we get $\dfrac{1}{2n+2} < A_n < \dfrac{1}{2n-2}$.

15. The given curves are $y = x - bx^2$ (1)

and $y = x^2/b$ (2)

$$\Rightarrow \left(y - \frac{1}{4b}\right) = -b\left(x - \frac{1}{2b}\right)^2 \text{ and } x^2 = by$$

Here, clearly the first curve is a downward parabola which meets $x$-axis at (0, 0) and (1/$b$, 0), while the second is an upward parabola with vertex at (0, 0).
Solving (1) and (2), we get the intersection points of two curves at (0, 0) and $\left(\frac{b}{1+b^2}, \frac{b}{(1+b^2)^2}\right)$.

Hence, the graph of given curves is as below

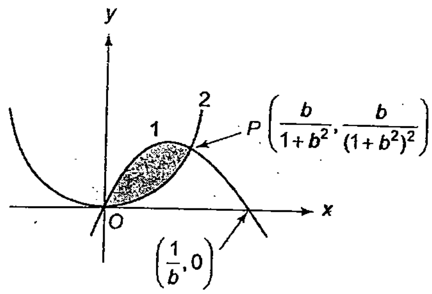



**Fig. 9.119**

Shaded portion represents the required area

$$A = \int_0^{\frac{b}{1+b^2}} \left(x - bx^2 - \frac{x^2}{b}\right)dx$$

$$= \left(\frac{x^2}{2} - \frac{bx^3}{3} - \frac{x^3}{3b}\right)_0^{\frac{b}{1+b^2}}$$

$$= \frac{b^2}{2(1+b^2)^2} - \frac{b^4}{3(1+b^2)^3} - \frac{b^2}{3(1+b^2)^3}$$

$$= \frac{b^4+b^2}{6(1+b^2)^3} = \frac{b^2}{6(1+b^2)^2}$$

$$= \frac{1}{6\left(\frac{1}{b}+b\right)^2} \text{ sq. units.}$$

Now, $\left(\frac{1}{b}+b\right) \geq 2$ or $\leq -2 \Rightarrow \left(\frac{1}{b}+b\right)^2 \geq 4$.

Hence, area is max. when $\left(\frac{1}{b}+b\right)^2_{\min} = 4$, for which $b = \pm 1$

but given that $b > 0$

$\therefore b = 1$.

**16.**

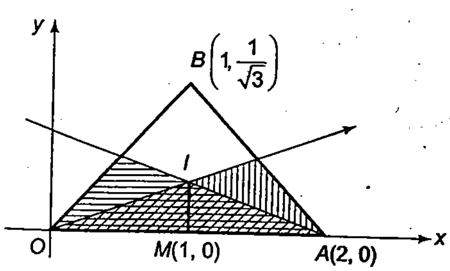



**Fig. 9.120**

$d(P, OA) \leq \min\,[d(P, OB), d(P, AB)]$
$\Rightarrow d(P, OA) \leq d(P, OB)$ and $d(P, OA) \leq d(P, AB)$

When $d(P, OA) = d(P, OB)$, $P$ is equidistant from $OA$ and $OB$, or $P$ lies on the angular bisector of lines $OA$ and $OB$.
Hence, when $d(P, OA) \leq d(P, OB)$, point $P$ is nearer to $OA$ than $OB$ or lies on or below the bisector of $OA$ and $OB$.
Similarly, when $d(P, OA) \leq d(P, AB)$, $P$ is nearer to $OA$ than $OB$, or lies on or below the bisector of $OA$ and $AB$.
$\therefore$ Req. area = Area or $\Delta\, OIA$.

Now, $\tan \angle BOA = \frac{1/\sqrt{3}}{1} = \frac{1}{\sqrt{3}}$

$\Rightarrow \angle BOA = 30° \Rightarrow \angle IOA = 15°$

$\Rightarrow IM = \tan 15° = 2 - \sqrt{3}$.

Hence, area of $\Delta\, OIA = \frac{1}{2} OA \times IM = \frac{1}{2} \times 2 \times (2 - \sqrt{3})$

$= 2 - \sqrt{3}$ sq. units

**17.** $f(x) = \text{Maximum}\,\{x^2, (1-x)^2, 2x(1-x)\}$

We draw the graphs of

$y = x^2$ (1)
$y = (1-x)^2$ (2)
$y = 2x(1-x)$ (3)

Solving (1) and (3), we get $x^2 = 2x(1-x)$
$\Rightarrow 3x^2 = 2x \Rightarrow x = 0$ or $x = 2/3$.
Solving (2) and (3) we get $(1-x)^2 = 2x(1-x)$
$\Rightarrow x = 1/3$ and $x = 1$,

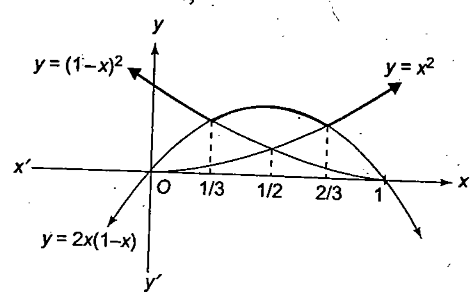



**Fig. 9.121**

From Fig. 9.121, it is clear that

$$f(x) = \begin{cases} (1-x)^2 & \text{for } 0 \leq x \leq 1/3 \\ 2x(1-x) & \text{for } 1/3 \leq x \leq 2/3 \\ x^2 & \text{for } 2/3 < x \leq 1 \end{cases}$$

The required area $A$ is given by

$$A = \int_0^1 f(x)\,dx$$

$$= \int_0^{1/3} (1-x)^2\,dx + \int_{1/3}^{2/3} 2x(1-x)\,dx + \int_{2/3}^{1} x^2 dx$$

$$= \left[-\frac{1}{3}(1-x)^3\right]_0^{1/3} + \left[\left(x^2 - \frac{2x^3}{3}\right)\right]_{1/3}^{2/3} + \left[\frac{x^3}{3}\right]_{2/3}^{1}$$

$$= -\frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{3} + \left(\frac{2}{3}\right)^2 - \frac{2}{3}\left(\frac{2}{3}\right)^3 - \left(\frac{1}{3}\right)^2 + \frac{2}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{3} - \frac{1}{3}\left(\frac{2}{3}\right)^3$$

$= \dfrac{17}{27}$ sq. units.

8. Let $P$ be on $C_1, y = x^2$ be $(t, t^2)$

$\therefore$ $y$ co-ordinate of $Q$ is also $t^2$

Now, $Q$ on $y = 2x, y = t^2$

$\therefore x = t^2/2$

$\therefore Q\left(\dfrac{t^2}{2}, t^2\right)$

For point $R$, $x = t$ and it is on $y = f(x)$

$\therefore R(t, f(t))$

Given that,

Area $OPQ$ = Area $OPR$

$\Rightarrow \int_0^{t^2}\left(\sqrt{y} - \dfrac{y}{2}\right)dy = \int_0^t \left(x^2 - f(x)\right)dx$

Diff. both sides w.r.t. $t$, we get

$\left(\sqrt{t^2} - \dfrac{t^2}{2}\right)(2t) = t^2 - f(t)$

$\Rightarrow f(t) = t^3 - t^2 \quad \Rightarrow \quad f(x) = x^3 - x^2$

9. $f(x) = \begin{cases} x^2 + ax + b; & x < -1 \\ 2x; & -1 \le x \le 1 \\ x^2 + ax + b; & x > 1 \end{cases}$

$\because f(x)$ is continuous at $x = -1$ and $x = 1$

$\therefore (-1)^2 + a(-1) + b = -2 \Rightarrow b - a = -3$ (1)

$2 = (1)^2 + a.1 + b \Rightarrow a + b = 1$ (2)

On solving, we get $a = 2, b = -1$

$\therefore f(x) = \begin{cases} x^2 + 2x - 1; & x < -1 \\ 2x; & -1 \le x \le 1 \\ x^2 + 2x - 1; & x > 1 \end{cases}$

Given curves are $y = f(x)$, $x = -2y^2$ and $8x + 1 = 0$

Solving $x = -2y^2$, $y = x^2 + 2x - 1$ (where $x < -1$), we get $x = -2$.

Also, $y = 2x$, $x = -2y^2$ meet at $(0, 0)$.

$y = 2x$ and $x = -1/8$ meet at $\left(-\dfrac{1}{8}, \dfrac{-1}{4}\right)$.

The required area is the shaded region in Fig. 9.122

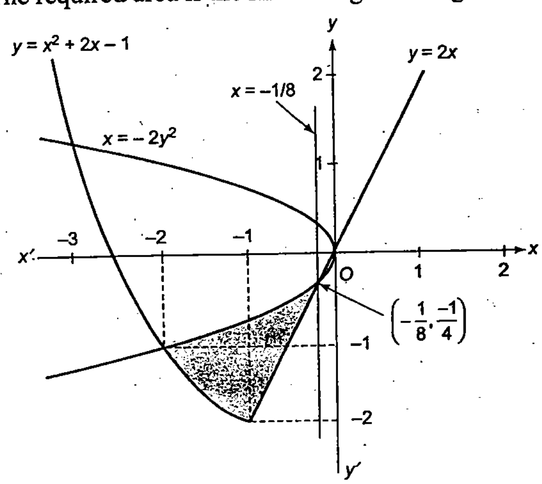



Fig. 9.122

$\therefore$ required area

$= \int_{-2}^{-1}\left[-\sqrt{\dfrac{-x}{2}} - \left(x^2 + 2x - 1\right)dx\right] + \int_{-1}^{-1/8}\left[-\sqrt{\dfrac{-x}{2}} - 2x\right]dx$

$= \left[\dfrac{1}{\sqrt{2}}\dfrac{2(-x)^{3/2}}{3} - \dfrac{x^3}{3} - x^2 + x\right]_{-2}^{-1} + \left[\dfrac{1}{\sqrt{2}}\dfrac{2(-x)^{3/2}}{3} - x^2\right]_{-1}^{-1/8}$

$= \left(\dfrac{\sqrt{2}}{3} + \dfrac{1}{3} - 1 - 1\right) - \left(\dfrac{4}{3} + \dfrac{8}{3} - 4 - 2\right) + \left(\dfrac{\sqrt{2}}{3} \times \dfrac{1}{16\sqrt{2}} - \dfrac{1}{64}\right) - \left(\dfrac{\sqrt{2}}{3} - 1\right)$

$= \left(\dfrac{\sqrt{2} - 5}{3}\right) - \left(\dfrac{4 + 8 - 18}{3}\right) + \left(\dfrac{4 - 3}{192}\right) - \left(\dfrac{\sqrt{2} - 3}{3}\right)$

$= \dfrac{257}{192}$ sq. units.

20. The given curves are

$y = x^2$ (1)

$y = |2 - x^2|$ (2)

The graph of these curves is as follows :

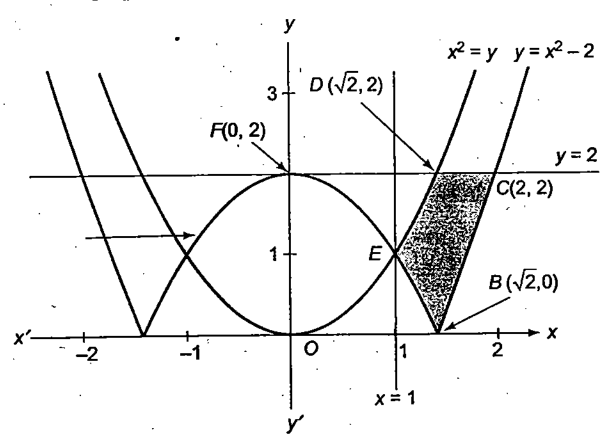



Fig. 9.123

$\therefore$ Required area = $BCDEB$

$= \int_1^{\sqrt{2}}\left[x^2 - \left(2 - x^2\right)\right]dx + \int_{\sqrt{2}}^2 [2 - (x^2 - 2)]\, dx$

$= \int_1^{\sqrt{2}}\left(2x^2 - 2\right)dx + \int_{\sqrt{2}}^2\left(4 - x^2\right)dx$

$= \left[\dfrac{2x^3}{3} - 2x\right]_1^{\sqrt{2}} + \left[4x - \dfrac{x^3}{3}\right]_{\sqrt{2}}^2$

$= \left(\dfrac{4\sqrt{2}}{3} - 2\sqrt{2} - \dfrac{2}{3} + 2\right) + \left(8 - \dfrac{8}{3} - 4\sqrt{2} + \dfrac{2\sqrt{2}}{3}\right)$

$= \left(\frac{20}{3} - 4\sqrt{2}\right)$ sq. units.

**21.** The given curves are,

$x^2 = y$ (1)

$x^2 = -y$ (2)

$y^2 = 4x - 3$ (3)

Clearly (1) and (2) meet at (0, 0).

Solving (1) and (3), we get $x^4 - 4x + 3 = 0$

$\Rightarrow (x-1)(x^3 + x^2 + x - 3) = 0$

$\Rightarrow (x-1)^2(x^2 + 2x + 3) = 0$

$\Rightarrow x = 1 \Rightarrow y = 1$

$\Rightarrow$ Point of intersection is (1, 1).

Similarly, point of intersection of (2) and (3) is (1, – 1).

The graph of three curves is as follow:

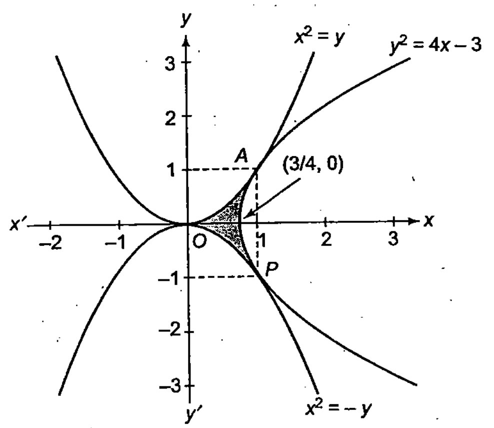



**Fig. 9.124**

We also observe that at $x = 1$ and $y = 1$, $\frac{dy}{dx}$ for (1) and (3) is same and hence the two curves touch each other at (1, 1).

Same is the case with (2) and (3) at (1, – 1).

Required area = shaded region in the figure

= 2 (Ar *OPA*)

$= 2\left[\int_0^1 x^2\, dx - \int_{3/4}^1 \sqrt{4x-3}\, dx\right]$

$= 2\left[\left(\frac{x^3}{3}\right)_0^1 - \left(\frac{2(4x-3)^{3/2}}{4 \times 3}\right)_{3/4}^1\right] = 2\left[\frac{1}{3} - \frac{1}{6}\right]$

$= \frac{1}{3}$ sq. units.

**22.** $f'(x) = g(x)$

$\int_0^3 g(x)dx = \int_0^3 f'(x)dx = [f(x)]_0^3 = [f(3) - f(0)] \in (-2, 2)$

$\int_{-3}^0 g(x)dx = \int_{-3}^0 f'(x)dx = [f(x)]_{-3}^0$

$= [f(0) - f(-3)] \in (-2, 2)$

$(f(0))^2 + (g(0))^2 = 9$

$\Rightarrow |g(0)| > 2\sqrt{2}$ $(\because |f(0)| < 1)$

**Case I**

$g(0) > 2\sqrt{2}$

Let $g''(x) \geq 0$ in (–3, 3)

One of the two situations is possible.

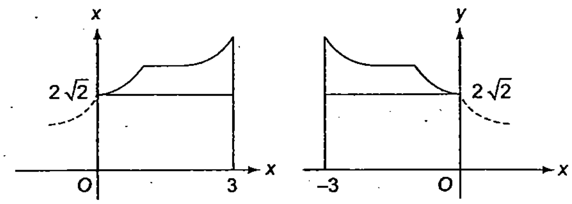



**Fig. 9.125**

$\int_0^3 g(x)dx > 6\sqrt{2} > 2$

So contradiction arises

So $g''(x)$ has to be negative somewhere in (0, 3) while $g(x) > 0$ in (0, 3)

$\int_{-3}^0 g(x)dx > 6\sqrt{2} > 2$

So contradiction arises

So $g''(x)$ has to be negative somewhere in (–3, 0) while $g(x) > 0$ in (–3, 0)

So at least somewhere $g''(x) < 0$, while $g(x) > 0$ in (–3, 3).

**Case II**

$g(0) < -2\sqrt{2}$

Let $g''(x) \leq 0$ in (–3, 3)

One of the two situations is possible.

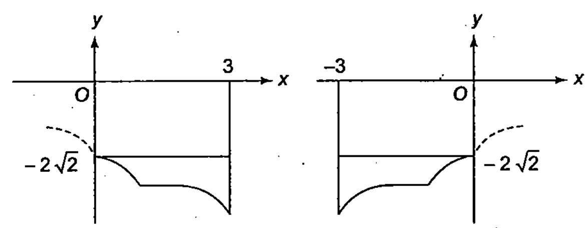



**Fig. 9.126**

$\int_0^3 g(x)dx < -6\sqrt{2} < -2$

So contradiction arises

So $g''(x)$ has to be positive somewhere in (0, 3) while $g(x) < 0$ in (0, 3)

$\int_{-3}^0 g(x)dx < -6\sqrt{2} < -2$

So contradiction arises

So $g''(x)$ has to be positive somewhere in (–3, 0) while $g(x) < 0$ in (–3, 0)

So at least somewhere $g''(x) > 0$ while $g(x) < 0$ in (–3, 3).

So at least at one point in (–3, 3).

**23.** $4a^2 f(-1) + 4af(1) + f(2) = 3a^2 + 3a$

$4b^2 f(-1) + 4bf(1) + f(2) = 3b^2 + 3b$

$4c^2 f(-1) + 4cf(1) + f(2) = 3c^2 + 3c$

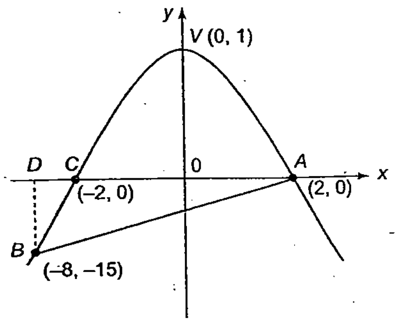



**Fig. 9.127**

Comparing coefficient of $a^2$, $a$ and constant term on both sides, we get

$$f(-1)=\frac{3}{4}=f(1) \text{ and } f(2)=0 \qquad (1)$$

Let $f(x)=Ax^2+Bx+C \qquad (2)$

From (1) and (2), $A=-\frac{1}{4}, B=0, C=1.$

$\therefore\ f(x)=-\frac{1}{4}x^2+1$

Let $B\left(t, 1-\frac{t^2}{4}\right)$ be any point on the parabola

$f(x)=y\ =-\frac{x^2}{4}+1$

As $AB$ chord subtends right angle at $V$

$\Rightarrow \left(-\frac{1}{2}\right)\times\left(\frac{\frac{t^2}{4}}{-t}\right)=-1 \quad \Rightarrow t=-8$

$\Rightarrow B=(-8,-15)$

$\Rightarrow$ Area ($BCVAB$)

$$=2\times\int_0^2\left(1-\frac{x^2}{4}\right)dx+\frac{1}{2}\times 10\times 15-\left|\int_{-8}^{-2}\left(1-\frac{x^2}{4}\right)dx\right|$$

$=\frac{125}{3}$ sq. units.

## Objective

*Multiple choice questions with one correct answer*

**1. c.** Given $\int_1^b f(x)\,dx=(b-1)\sin(3b+4)$

Differentiating both sides w.r.t. $b$, we get

$\Rightarrow f(b)=3(b-1)\cos(3b+4)+\sin(3b+4)$

$\Rightarrow f(x)=\sin(3x+4)+3(x-1)\cos(3x+4).$

**2. b.**

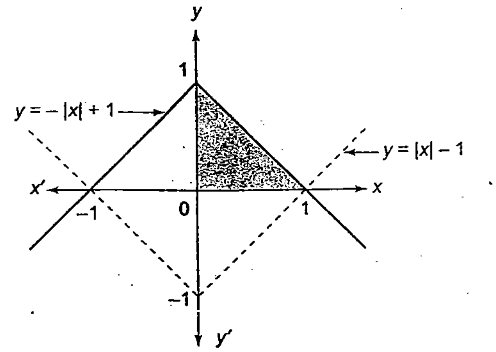



**Fig. 9.128**

Required area $=4\times$ (shaded area shown in Fig. 9.128)

$=4\times\frac{1}{2}$

$=2.$

**3. d.** To find the area between the curves $y=\sqrt{x}$ and $2y+3=x$ and $x$-axis in the 1st quadrant.

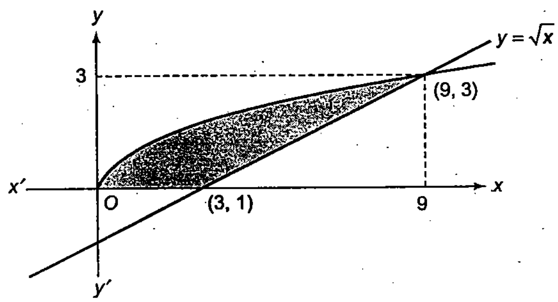



**Fig. 9.129**

Given curves intersect when $y^2=2y+3$

$\Rightarrow y^2-2y-3=0 \Rightarrow (y-3)(y+1)=0 \Rightarrow y=3,-1$

when $y=3, x=9$ (Ist quadrant)

$$\text{Required area}=\int_0^9 \sqrt{x}\,dx-\int_3^9\left(\frac{x-3}{2}\right)dx$$

$$=\left|\frac{x^{3/2}}{3/2}\right|_0^9-\frac{1}{2}\left|\frac{x^2}{2}-3x\right|_3^9$$

$$=\frac{2}{3}(27)-\frac{1}{2}\left[\left(\frac{81}{2}-27\right)-\left(\frac{9}{2}-9\right)\right]$$

$=9$ sq. units.

**4. d.** The given curves are $y=(x+1)^2$ and $y=(x-1)^2$ and $y=1/4$

The graph is as shown

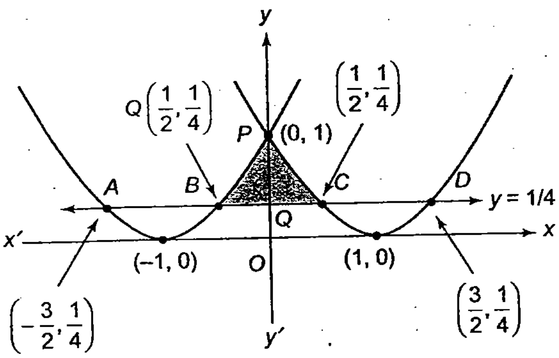



**Fig. 9.130**

The required area is the shaded portion,
given by Ar $(BPCQB) = 2\text{Ar}(PQCP)$ (by symmetry)

$$= 2\left[\int_0^{1/2}\left((x-1)^2 - \frac{1}{4}\right)dx\right] = 2\left[\left(\frac{(x-1)^3}{3} - \frac{x}{4}\right)_0^{1/2}\right]$$

$$= 2\left[\left(-\frac{1}{24} - \frac{1}{8}\right) - \left(-\frac{1}{3}\right)\right]$$

$$= \frac{1}{3} \text{ sq. units.}$$

**5. a** The area bounded by $y^2 = 4ax$ and $x^2 = 4by$ is $\frac{16ab}{3}$.

Then the area bounded by $y^2 = x/a$ and $x^2 = y/a$ is $\frac{1}{3a^2}$.

Given $\frac{1}{3a^2} = 1 \Rightarrow a = \pm\frac{1}{\sqrt{3}}$.

**6. b** $\because \int_0^b (1-x)^2\, dx - \int_b^1 (1-x)^2\, dx = \frac{1}{4}$

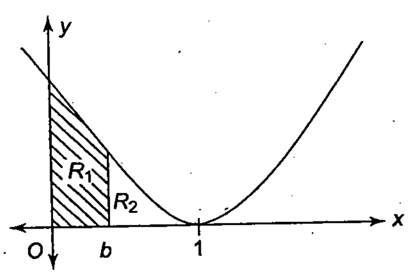



**Fig. 9.131**

$$\Rightarrow \left.\frac{(x-1)^3}{3}\right|_0^b - \left.\frac{(x-1)^3}{3}\right|_b^1 = \frac{1}{4}$$

$$\Rightarrow \frac{(b-1)^3}{3} + \frac{1}{3} - \left(0 - \frac{(b-1)^3}{3}\right) = \frac{1}{4}$$

$$\Rightarrow \frac{2(b-1)^3}{3} = -\frac{1}{12} \Rightarrow (b-1)^3 = -\frac{1}{8} \Rightarrow b = \frac{1}{2}$$

***Multiple choice questions with one or more than one correct answer***

**1. b,d.** The two curves meet at $mx = x - x^2$ or $x^2 = x(1-m)$

$\therefore x = 0, 1-m$

$$A = \int_0^{1-m}\left(x - x^2 - mx\right)dx$$

$$= \left[(1-m)\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{1-m} = \frac{9}{2} \text{ if } m < 1$$

$$\Rightarrow (1-m)^3\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{9}{2}$$

$$\Rightarrow (1-m)^3 = 27$$

$$\Rightarrow m = -2.$$

But if $m > 1$ and $1 - m$ is −ve, then

$$\left[(1-m)\frac{x^2}{2} - \frac{x^3}{3}\right]_{1-m}^0 = \frac{9}{2}.$$

$$\Rightarrow -(1-m)^3\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{9}{2}.$$

$$\Rightarrow -(1-m)^3 = -27.$$

$$\Rightarrow m = 4.$$

**2. b, c, d.**

Required Area $= \int_1^e \ln y\, dy$

$= (y\ln y - y)_1^e = (e - e) - \{-1\} = 1$

Also, $\int_1^e \ln y\, dy \int_1^e \ln(e+1-y)\, dy$

Further the required area $= e \times 1 - \int_0^1 e^x\, dx$

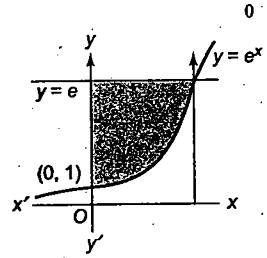



**Fig. 9.132**