CHAPTER XVI.

LOGARITHMS.

199. DEFINITION. The **logarithm** of any number to a given **base** is the index of the power to which the base must be raised in order to equal the given number. Thus if $a^x = N$, x is called the logarithm of N to the base a.

Examples. (1) Since $3^4 = 81$, the logarithm of 81 to base 3 is 4.

(2) Since $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$,......

the natural numbers 1, 2, 3,... are respectively the logarithms of 10, 100, 1000,...... to base 10.

200. The logarithm of N to base a is usually written $\log_a N$, so that the same meaning is expressed by the two equations

$$a^x = N; \quad x = \log_a N.$$

From these equations we deduce

$$N = a^{\log_a N},$$

an identity which is sometimes useful.

Example. Find the logarithm of $32\sqrt[5]{4}$ to base $2\sqrt{2}$.

Let x be the required logarithm; then,

by definition, $$(2\sqrt{2})^x = 32\sqrt[5]{4};$$

$$\therefore\ (2 \cdot 2^{\frac{1}{2}})^x = 2^5 \cdot 2^{\frac{2}{5}};$$

$$\therefore\ 2^{\frac{3}{2}x} = 2^{5+\frac{2}{5}};$$

hence, by equating the indices, $$\frac{3}{2}x = \frac{27}{5};$$

$$\therefore\ x = \frac{18}{5} = 3 \cdot 6.$$

201. When it is understood that a particular system of logarithms is in use, the suffix denoting the base is omitted. Thus in arithmetical calculations in which 10 is the base, we usually write log 2, log 3,...... instead of $\log_{10} 2$, $\log_{10} 3$,......

Any number might be taken as the base of logarithms, and corresponding to any such base a system of logarithms of all numbers could be found. But before discussing the logarithmic systems commonly used, we shall prove some general propositions which are true for all logarithms independently of any particular base.

202. *The logarithm of* 1 *is* 0.

For $a^0 = 1$ for all values of a; therefore $\log 1 = 0$, whatever the base may be.

203. *The logarithm of the base itself is* 1.

For $a^1 = a$; therefore $\log_a a = 1$.

204. *To find the logarithm of a product.*

Let MN be the product; let a be the base of the system, and suppose

$$x = \log_a M, \qquad y = \log_a N;$$

so that $$a^x = M, \qquad a^y = N.$$

Thus the product $$MN = a^x \times a^y$$
$$= a^{x+y};$$

whence, by definition, $$\log_a MN = x + y$$
$$= \log_a M + \log_a N.$$

Similarly, $\log_a MNP = \log_a M + \log_a N + \log_a P$;

and so on for any number of factors.

Example. $$\log 42 = \log (2 \times 3 \times 7)$$
$$= \log 2 + \log 3 + \log 7.$$

205. *To find the logarithm of a fraction.*

Let $\dfrac{M}{N}$ be the fraction, and suppose

$$x = \log_a M, \qquad y = \log_a N;$$

so that $$a^x = M, \qquad a^y = N.$$

Thus the fraction $\frac{M}{N} = \frac{a^x}{a^y}$

$= a^{x-y}$;

whence, by definition, $\log_a \frac{M}{N} = x - y$

$= \log_a M - \log_a N.$

Example. $\log(4\frac{2}{7}) = \log\frac{30}{7}$

$= \log 30 - \log 7$

$= \log(2 \times 3 \times 5) - \log 7$

$= \log 2 + \log 3 + \log 5 - \log 7.$

206. *To find the logarithm of a number raised to any power, integral or fractional.*

Let $\log_a(M^p)$ be required, and suppose

$x = \log_a M$, so that $a^x = M$;

then $M^p = (a^x)^p$

$= a^{px}$;

whence, by definition, $\log_a(M^p) = px$;

that is, $\log_a(M^p) = p \log_a M.$

Similarly, $\log_a(M^{\frac{1}{r}}) = \frac{1}{r}\log_a M.$

207. It follows from the results we have proved that

(1) the logarithm of a product is equal to the sum of the logarithms of its factors;

(2) the logarithm of a fraction is equal to the logarithm of the numerator diminished by the logarithm of the denominator;

(3) the logarithm of the p^{th} power of a number is p times the logarithm of the number;

(4) the logarithm of the r^{th} root of a number is equal to $\frac{1}{r}$th of the logarithm of the number.

Also we see that by the use of logarithms the operations of multiplication and division may be replaced by those of addition and subtraction; and the operations of involution and evolution by those of multiplication and division.

Example 1. Express the logarithm of $\frac{\sqrt{a^3}}{c^5b^2}$ in terms of $\log a$, $\log b$ and $\log c$.

$$\log\frac{\sqrt{a^3}}{c^5b^2}=\log\frac{a^{\frac{3}{2}}}{c^5b^2}$$

$$=\log a^{\frac{3}{2}}-\log(c^5b^2)$$

$$=\frac{3}{2}\log a-(\log c^5+\log b^2)$$

$$=\frac{3}{2}\log a-5\log c-2\log b.$$

Example 2. Find x from the equation $a^x.c^{-2x}=b^{3x+1}$.

Taking logarithms of both sides, we have

$$x\log a-2x\log c=(3x+1)\log b;$$

$$\therefore\ x(\log a-2\log c-3\log b)=\log b;$$

$$\therefore\ x=\frac{\log b}{\log a-2\log c-3\log b}.$$

EXAMPLES. XVI. a.

Find the logarithms of

1. 16 to base $\sqrt{2}$, and 1728 to base $2\sqrt{3}$.

2. 125 to base $5\sqrt{5}$, and $\cdot 25$ to base 4.

3. $\frac{1}{256}$ to base $2\sqrt{2}$, and $\cdot\dot{3}$ to base 9.

4. $\cdot 0625$ to base 2, and 1000 to base $\cdot 01$.

5. $\cdot 0001$ to base $\cdot 001$, and $\cdot\dot{1}$ to base $9\sqrt{3}$.

6. $\sqrt[4]{a^{\frac{8}{5}}}$, $\frac{1}{a^{-\frac{1}{2}}}$, $\sqrt[3]{a^{-\frac{15}{2}}}$ to base a.

7. Find the value of

$$\log_8 128,\quad \log_6\frac{1}{216},\quad \log_{27}\frac{1}{81},\quad \log_{343}49.$$

Express the following seven logarithms in terms of $\log a$, $\log b$, and $\log c$.

8. $\log(\sqrt{a^2b^3})^6$. 9. $\log(\sqrt[3]{a^2}\times\sqrt[2]{b^3})$. 10. $\log(\sqrt[9]{a^{-4}b^3})$.

11. $\log(\sqrt{a^{-2}b} \times \sqrt[3]{ab^{-3}})$.

12. $\log(\sqrt[3]{a^{-1}\sqrt{b^3}} \div \sqrt{b^3\sqrt{a}})$.

13. $\log \dfrac{\sqrt[3]{ab^{-1}c^{-2}}}{(a^{-1}b^{-2}c^{-4})^{\frac{1}{6}}}$.

14. $\log\left\{\left(\dfrac{bc^{-2}}{b^{-4}c^3}\right)^{-3} \div \left(\dfrac{b^{-1}c}{b^2c^{-3}}\right)^5\right\}$.

15. Shew that $\log \dfrac{\sqrt[4]{5} \,.\, \sqrt[10]{2}}{\sqrt[3]{18} \,.\, \sqrt{2}} = \dfrac{1}{4}\log 5 - \dfrac{2}{5}\log 2 - \dfrac{2}{3}\log 3$.

16. Simplify $\log \sqrt[4]{729\sqrt[3]{9^{-1} \,.\, 27^{-\frac{4}{3}}}}$.

17. Prove that $\log\dfrac{75}{16} - 2\log\dfrac{5}{9} + \log\dfrac{32}{243} = \log 2$.

Solve the following equations:

18. $a^x = cb^x$.

19. $a^{2x} \,.\, b^{3x} = c^5$.

20. $\dfrac{a^{x+1}}{b^{x-1}} = c^{2x}$.

21. $\left.\begin{array}{l} a^{2x} \,.\, b^{3y} = m^5 \\ a^{3x} \,.\, b^{2y} = m^{10} \end{array}\right\}$.

22. If $\log(x^2y^3) = a$, and $\log\dfrac{x}{y} = b$, find $\log x$ and $\log y$.

23. If $a^{3-x} \,.\, b^{5x} = a^{x+5} \,.\, b^{3x}$, shew that $x\log\left(\dfrac{b}{a}\right) = \log a$.

24. Solve the equation

$$(a^4 - 2a^2b^2 + b^4)^{x-1} = (a-b)^{2x}(a+b)^{-2}.$$

Common Logarithms.

208. Logarithms to the base 10 are called **Common Logarithms**; this system was first introduced, in 1615, by Briggs, a contemporary of Napier the inventor of logarithms.

From the equation $10^x = N$, it is evident that common logarithms will not in general be integral, and that they will not always be positive.

For instance $\quad 3154 > 10^3$ and $< 10^4$;

$$\therefore \log 3154 = 3 + \text{a fraction}.$$

Again, $\cdot 06 > 10^{-2}$ and $< 10^{-1}$;

$$\therefore \log \cdot 06 = -2 + \text{a fraction}.$$

209. Definition. The integral part of a logarithm is called the **characteristic**, and the decimal part is called the **mantissa**.

The characteristic of the logarithm of any number to the base 10 can be written down by inspection, as we shall now shew.

210. *To determine the characteristic of the logarithm of any number greater than unity.*

Since $10^1 = 10,$

$$10^2 = 100,$$

$$10^3 = 1000,$$

$$\cdots\cdots\cdots\cdots$$

it follows that a number with two digits in its integral part lies between 10^1 and 10^2; a number with three digits in its integral part lies between 10^2 and 10^3; and so on. Hence a number with n digits in its integral part lies between 10^{n-1} and 10^n.

Let N be a number whose integral part contains n digits; then

$$N = 10^{(n-1) + \text{a fraction}};$$

$$\therefore \log N = (n-1) + \text{a fraction}.$$

Hence the characteristic is $n-1$; that is, *the characteristic of the logarithm of a number greater than unity is less by one than the number of digits in its integral part, and is positive.*

211. *To determine the characteristic of the logarithm of a decimal fraction.*

Since $10^0 = 1,$

$$10^{-1} = \frac{1}{10} = \cdot 1,$$

$$10^{-2} = \frac{1}{100} = \cdot 01,$$

$$10^{-3} = \frac{1}{1000} = \cdot 001,$$

$$\cdots\cdots\cdots\cdots$$

it follows that a decimal with one cipher immediately after the decimal point, such as ·0324, being greater than ·01 and less than ·1, lies between 10^{-2} and 10^{-1}; a number with two ciphers after the decimal point lies between 10^{-3} and 10^{-2}; and so on. Hence a decimal fraction with n ciphers immediately after the decimal point lies between $10^{-(n+1)}$ and 10^{-n}.

Let D be a decimal beginning with n ciphers; then

$$D = 10^{-(n+1) + \text{a fraction}};$$

$$\therefore \log D = -(n+1) + \text{a fraction}.$$

Hence the characteristic is $-(n+1)$; that is, *the characteristic of the logarithm of a decimal fraction is greater by unity than the number of ciphers immediately after the decimal point, and is negative.*

212. The logarithms to base 10 of all integers from 1 to 200000 have been found and tabulated; in most Tables they are given to seven places of decimals. This is the system in practical use, and it has two great advantages:

(1) From the results already proved it is evident that the characteristics can be written down by inspection, so that only the mantissæ have to be registered in the Tables.

(2) The mantissæ are the same for the logarithms of all numbers which have the same significant digits; so that it is sufficient to tabulate the mantissæ of the logarithms of *integers.*

This proposition we proceed to prove.

213. Let N be any number, then since multiplying or dividing by a power of 10 merely alters the position of the decimal point without changing the sequence of figures, it follows that $N \times 10^p$, and $N \div 10^q$, where p and q are any integers, are numbers whose significant digits are the same as those of N.

Now $\log(N \times 10^p) = \log N + p \log 10$

$$= \log N + p \quad \text{..............................(1).}$$

Again, $\log(N \div 10^q) = \log N - q \log 10$

$$= \log N - q \quad \text{........................ (2).}$$

In (1) an integer is added to $\log N$, and in (2) an integer is subtracted from $\log N$; that is, the mantissa or decimal portion of the logarithm remains unaltered.

In this and the three preceding articles the mantissæ have been supposed positive. In order to secure the advantages of Briggs' system, we arrange our work so as *always to keep the mantissa positive*, so that when the mantissa of any logarithm has been taken from the Tables the characteristic is prefixed with its appropriate sign according to the rules already given.

214. In the case of a negative logarithm the minus sign is written *over the characteristic*, and not before it, to indicate that the characteristic alone is negative, and not the whole expression. Thus $\bar{4}\cdot30103$, the logarithm of $\cdot0002$, is equivalent to $-4+\cdot30103$, and must be distinguished from $-4\cdot30103$, an expression in which both the integer and the decimal are negative. In working with negative logarithms an arithmetical artifice will sometimes be necessary in order to make the mantissa positive. For instance, a result such as $-3\cdot69897$, in which the whole expression is negative, may be transformed by subtracting 1 from the characteristic and adding 1 to the mantissa. Thus

$$-3\cdot69897=-4+(1-\cdot69897)=\bar{4}\cdot30103.$$

Other cases will be noticed in the Examples.

Example 1. Required the logarithm of $\cdot0002432$.

In the Tables we find that 3859636 is the mantissa of log 2432 (the decimal point as well as the characteristic being omitted); and, by Art. 211, the characteristic of the logarithm of the given number is -4;

$$\therefore \log \cdot0002432=\bar{4}\cdot3859636.$$

Example 2. Find the value of $\sqrt[5]{\cdot00000165}$, given

$$\log 165=2\cdot2174839, \quad \log 697424=5\cdot8434968.$$

Let x denote the value required; then

$$\log x=\log (\cdot00000165)^{\frac{1}{5}}=\frac{1}{5}\log (\cdot00000165)$$

$$=\frac{1}{5}(\bar{6}\cdot2174839);$$

the *mantissa* of log $\cdot00000165$ being the same as that of log 165, and the *characteristic* being prefixed by the rule.

Now $$\frac{1}{5}(\bar{6}\cdot2174839)=\frac{1}{5}(\overline{10}+4\cdot2174839)$$

$$=\bar{2}\cdot8434968$$

and ·8434968 is the mantissa of log 697424; hence x is a number consisting of these same digits but with one cipher after the decimal point. [Art. 211.]

Thus $$x = \cdot 0697424.$$

215. The method of calculating logarithms will be explained in the next chapter, and it will there be seen that they are first found to another base, and then transformed into common logarithms to base 10.

It will therefore be necessary to investigate a method for transforming a system of logarithms having a given base to a new system with a different base.

216. Suppose that the logarithms of all numbers to base a are known and tabulated, it is required to find the logarithms to base b.

Let N be any number whose logarithm to base b is required.

Let $y = \log_b N$, so that $b^y = N$;

$$\therefore \quad \log_a (b^y) = \log_a N;$$

that is, $$y \log_a b = \log_a N;$$

$$\therefore \quad y = \frac{1}{\log_a b} \times \log_a N,$$

or $$\log_b N = \frac{1}{\log_a b} \times \log_a N \qquad \text{(1)}.$$

Now since N and b are given, $\log_a N$ and $\log_a b$ are known from the Tables, and thus $\log_b N$ may be found.

Hence it appears that to transform logarithms from base a to base b we have only to multiply them all by $\frac{1}{\log_a b}$; this is a constant quantity and is given by the Tables; it is known as the *modulus*.

217. In equation (1) of the preceding article put a for N; thus

$$\log_b a = \frac{1}{\log_a b} \times \log_a a = \frac{1}{\log_a b};$$

$$\therefore \quad \log_b a \times \log_a b = 1.$$

This result may also be proved directly as follows:

Let $\qquad x = \log_a b$, so that $a^x = b$;

then by taking logarithms to base b, we have

$$x \log_b a = \log_b b$$
$$= 1;$$
$$\therefore \quad \log_a b \times \log_b a = 1.$$

218. The following examples will illustrate the utility of logarithms in facilitating arithmetical calculation; but for information as to the use of Logarithmic Tables the reader is referred to works on Trigonometry.

Example 1. Given $\log 3 = \cdot 4771213$, find $\log \{(2\cdot 7)^3 \times (\cdot 81)^{\frac{4}{5}} \div (90)^{\frac{5}{4}}\}$.

The required value $= 3 \log \frac{27}{10} + \frac{4}{5} \log \frac{81}{100} - \frac{5}{4} \log 90$

$$= 3(\log 3^3 - 1) + \frac{4}{5}(\log 3^4 - 2) - \frac{5}{4}(\log 3^2 + 1)$$
$$= \left(9 + \frac{16}{5} - \frac{5}{2}\right) \log 3 - \left(3 + \frac{8}{5} + \frac{5}{4}\right)$$
$$= \frac{97}{10} \log 3 - 5\tfrac{17}{20}$$
$$= 4\cdot 6280766 - 5\cdot 85$$
$$= \bar{2}\cdot 7780766.$$

The student should notice that the logarithm of 5 and its powers can always be obtained from log 2; thus

$$\log 5 = \log \frac{10}{2} = \log 10 - \log 2 = 1 - \log 2.$$

Example 2. Find the number of digits in 875^{16}, given

$$\log 2 = \cdot 3010300, \quad \log 7 = \cdot 8450980.$$

$$\log(875^{16}) = 16 \log(7 \times 125)$$
$$= 16(\log 7 + 3 \log 5)$$
$$= 16(\log 7 + 3 - 3 \log 2)$$
$$= 16 \times 2\cdot 9420080$$
$$= 47\cdot 072128;$$

hence the number of digits is 48. [Art. 210.]

Example 3. Given log 2 and log 3, find to two places of decimals the value of x from the equation

$$6^{3-4x} \cdot 4^{x+5} = 8.$$

Taking logarithms of both sides, we have

$$(3-4x)\log 6 + (x+5)\log 4 = \log 8;$$

$$\therefore\ (3-4x)(\log 2 + \log 3) + (x+5)\, 2\log 2 = 3\log 2;$$

$$\therefore\ x(-4\log 2 - 4\log 3 + 2\log 2) = 3\log 2 - 3\log 2 - 3\log 3 - 10\log 2;$$

$$\therefore\ x = \frac{10\log 2 + 3\log 3}{2\log 2 + 4\log 3}$$

$$= \frac{4{\cdot}4416639}{2{\cdot}5105452}$$

$$= 1{\cdot}77\ldots$$

EXAMPLES. XVI. b.

1. Find, by inspection, the characteristics of the logarithms of 21735, 23·8, 350, ·035, ·2, ·87, ·875.

2. The mantissa of log 7623 is ·8821259; write down the logarithms of 7·623, 762·3, ·007623, 762300, ·000007623.

3. How many digits are there in the integral part of the numbers whose logarithms are respectively

4·30103, 1·4771213, 3·69897, ·56515?

4. Give the position of the first significant figure in the numbers whose logarithms are

$\bar{2}{\cdot}7781513$, ·6910815, $\bar{5}{\cdot}4871384$.

Given log 2 = ·3010300, log 3 = ·4771213, log 7 = ·8450980, find the value of

5. log 64. **6.** log 84. **7.** log ·128.

8. log ·0125. **9.** log 14·4. **10.** $\log 4\frac{2}{3}$.

11. $\log\sqrt[3]{12}$. **12.** $\log\sqrt{\frac{35}{27}}$. **13.** $\log\sqrt[4]{{\cdot}0105}$.

14. Find the seventh root of ·00324, having given that

log 44092388 = 7·6443636.

15. Given log 194·8445 = 2·2896883, find the eleventh root of $(39{\cdot}2)^2$.

16. Find the product of 37·203, 3·7203, ·0037203, 372030, having given that

$$\log 37{\cdot}203 = 1{\cdot}5705780, \text{ and } \log 1915631 = 6{\cdot}2823120.$$

17. Given log 2 and log 3, find $\log \sqrt[3]{\left(\frac{3^2 5^4}{\sqrt{2}}\right)}$.

18. Given log 2 and log 3, find $\log(\sqrt[3]{48} \times 108^{\frac{1}{4}} \div \sqrt[12]{6})$.

19. Calculate to six decimal places the value of

$$\sqrt[3]{\left(\frac{294 \times 125}{42 \times 32}\right)^2};$$

given log 2, log 3, log 7; also $\log 9076{\cdot}226 = 3{\cdot}9579053$.

20. Calculate to six places of decimals the value of

$$(330 \div 49)^4 \div \sqrt[3]{22 \times 70};$$

given log 2, log 3, log 7; also

$$\log 11 = 1{\cdot}0413927, \text{ and } \log 17814{\cdot}1516 = 4{\cdot}2507651.$$

21. Find the number of digits in $3^{12} \times 2^8$.

22. Shew that $\left(\frac{21}{20}\right)^{100}$ is greater than 100.

23. Determine how many ciphers there are between the decimal point and the first significant digit in $\left(\frac{1}{2}\right)^{1000}$.

Solve the following equations, having given log 2, log 3, and log 7.

24. $3^{x-2} = 5$.

25. $5^x = 10^3$.

26. $5^{5-3x} = 2^{x+2}$.

27. $21^x = 2^{2x+1} \cdot 5^x$.

28. $2^x \cdot 6^{x-2} = 5^{2x} \cdot 7^{1-x}$.

29. $\left.\begin{aligned} 2^{x+y} &= 6^y \\ 3^x &= 3 \cdot 2^{y+1} \end{aligned}\right\}$.

30. $\left.\begin{aligned} 3^{1-x-y} &= 4^{-y} \\ 2^{2x-1} &= 3^{3y-x} \end{aligned}\right\}$.

31. Given $\log_{10} 2 = {\cdot}30103$, find $\log_{25} 200$.

32. Given $\log_{10} 2 = {\cdot}30103$, $\log_{10} 7 = {\cdot}84509$, find $\log_7 \sqrt{2}$ and $\log_{\sqrt{2}} 7$.