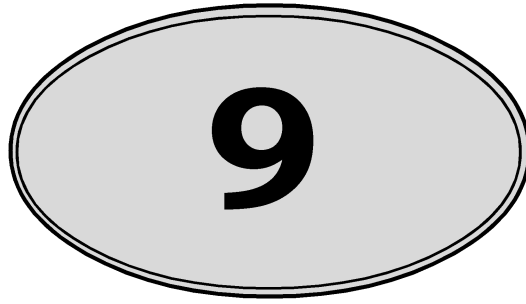




Government of Karnataka

MATHEMATICS



Ninth Standard

Part-I



राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्
NATIONAL COUNCIL OF EDUCATIONAL RESEARCH AND TRAINING

Karnataka Textbook Society (R.)

100 Feet Ring Road, Banashankari 3rd Stage, Bengaluru - 85

FOREWORD

The National Curriculum Framework (NCF), 2005, recommends that children's life at school must be linked to their life outside the school. This principle marks a departure from the legacy of bookish learning which continues to shape our system and causes a gap between the school, home and community. The syllabi and textbooks developed on the basis of NCF signify an attempt to implement this basic idea. They also attempt to discourage rote learning and the maintenance of sharp boundaries between different subject areas. We hope these measures will take us significantly further in the direction of a child-centred system of education outlined in the national Policy on Education (1986).

The success of this effort depends on the steps that school principals and teachers will take to encourage children to reflect on their own learning and to pursue imaginative activities and questions. We must recognize that, given space, time and freedom, children generate new knowledge by engaging with the information passed on to them by adults. Treating the prescribed textbook as the sole basis of examination is one of the key reasons why other resources and sites of learning are ignored. Inculcating creativity and initiative is possible if we perceive and treat children as participants in learning, not as receivers of a fixed body of knowledge.

This aims imply considerable change in school routines and mode of functioning. Flexibility in the daily time-table is as necessary as rigour in implementing the annual calendar so that the required number of teaching days are actually devoted to teaching. The methods used for teaching and evaluation will also determine how effective this textbook proves for making children's life at school a happy experience, rather than a source of stress or boredom. Syllabus designers have tried to address the problem of curricular burden by restructuring and reorienting knowledge at different stages with greater consideration for child psychology and the time available for teaching. The textbook attempts to enhance this endeavour by giving higher priority and space to opportunities for contemplation and wondering, discussion in small groups, and activities requiring hands-on experience.

The National Council of Educational Research and Training (NCERT) appreciates the hard work done by the textbook development committee responsible for this book. We wish to thank the Chairperson of the advisory group in science and mathematics, Professor J.V. Narlikar and the Chief Advisor for this book, Professor P. Sinclair of IGNOU, New Delhi for guiding the work of this committee. Several teachers contributed

to the development of this textbook; we are grateful to their principals for making this possible. We are indebted to the institutions and organizations which have generously permitted us to draw upon their resources, material and personnel. We are especially grateful to the members of the National Monitoring Committee, appointed by the Department of Secondary and Higher Education, Ministry of Human Resource Development under the Chairpersonship of Professor Mrinal Miri and Professor G.P. Deshpande, for their valuable time and contribution. As an organisation committed to systemic reform and continuous improvement in the quality of its products, NCERT welcomes comments and suggestions which will enable us to undertake further revision and refinement.

New Delhi
20 December 2005

Director
National Council of Educational
Research and Training

TEXTBOOK DEVELOPMENT COMMITTEE

CHAIRPERSON, ADVISORY GROUP IN SCIENCE AND MATHEMATICS

J.V. Narlikar, *Emeritus Professor, Chairman*, Advisory Committee, Inter University Centre for Astronomy & Astrophysics (IUCAA), Ganeshkhind, Pune University, Pune

CHIEF ADVISOR

P. Sinclair, Director, NCERT and *Professor of Mathematics*, IGNOU, New Delhi

CHIEF COORDINATOR

Hukum Singh, *Professor* (Retd.), DESM, NCERT

MEMBERS

A.K. Wazalwar, *Professor and Head*, DESM, NCERT

Anjali Lal, *PGT*, DAV Public School, Sector-14, Gurgaon

Anju Nirula, *PGT*, DAV Public School, Pushpanjali Enclave, Pitampura, Delhi

G.P. Dikshit, *Professor* (Retd.), Department of Mathematics & Astronomy, Lucknow University, Lucknow

K.A.S.S.V. Kameswara Rao, *Associate Professor*, Regional Institute of Education, Bhubaneswar

Mahendra R. Gajare, *TGT*, Atul Vidyalya, Atul, Dist. Valsad

Mahendra Shanker, *Lecturer* (S.G.) (Retd.), NCERT

Rama Balaji, *TGT*, K.V., MEG & Centre, ST. John's Road, Bangalore

Sanjay Mudgal, *Lecturer*, CIET, NCERT

Shashidhar Jagadeeshan, *Teacher and Member*, Governing Council, Centre for Learning, Bangalore

S. Venkataraman, *Lecturer*, School of Sciences, IGNOU, New Delhi

Uaday Singh, *Lecturer*, DESM, NCERT

Ved Dudeja, *Vice-Principal* (Retd.), Govt. Girls Sec. School, Sainik Vihar, Delhi

MEMBER-COORDINATOR

Ram Avtar, *Professor* (Retd.), DESM, NCERT (till December 2005)

R.P. Maurya, *Professor*, DESM, NCERT (Since January 2006)

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ಮುನ್ನುಡಿ

NCF-2005 ಆಧರಿಸಿ ರಚಿತವಾದ ರಾಷ್ಟ್ರೀಯ ಪಠ್ಯವಸ್ತುವಿನ ಆಧಾರದ ಮೇಲೆ ರಚಿಸಲಾದ ಎನ್.ಸಿ.ಇ.ಆರ್.ಟಿ 9ನೆಯ ತರಗತಿಯ ಆಂಗ್ಲ ಮಾಧ್ಯಮದ ಪಠ್ಯಪುಸ್ತಕವನ್ನು ಕರ್ನಾಟಕದಲ್ಲಿ ಯಥಾವತ್ತಾಗಿ 2017-18ನೇ ಸಾಲಿನಿಂದ ಜಾರಿಗೊಳಿಸಲಾಗಿದೆ. ಈ ಪಠ್ಯಪುಸ್ತಕವನ್ನು ಒಟ್ಟು 7 ಮಾಧ್ಯಮಗಳಲ್ಲಿ ಭಾಗ-1 ಮತ್ತು ಭಾಗ-2 ಎಂಬುದಾಗಿ ಎರಡು ಪುಸ್ತಕಗಳನ್ನು ಮುದ್ರಿಸಿ ಹೊರತರಲಾಗಿದೆ. NCF-2005ರ ಪಠ್ಯಕ್ರಮದ ಎಲ್ಲ ವೈಶಿಷ್ಟ್ಯಗಳನ್ನು ಈ ಪುಸ್ತಕ ಹೊಂದಿದೆ.

2005ರ ರಾಷ್ಟ್ರೀಯ ಪಠ್ಯಕ್ರಮವು ಈ ಕೆಳಗಿನ ವೈಶಿಷ್ಟ್ಯಗಳನ್ನು ಹೊಂದಿದೆ.

- ಕಲಿಕೆಯನ್ನು ಜೀವನದ ಅವಶ್ಯಕತೆಗಳೊಂದಿಗೆ ಜೋಡಿಸುವುದು.
- ಕಂಠಪಾಠ ವಿಧಾನದಿಂದ ಕಲಿಕೆಯನ್ನು ಮುಕ್ತಗೊಳಿಸುವುದು.
- ಪಠ್ಯಪುಸ್ತಕಗಳ ಹೊರತಾಗಿ ಪಠ್ಯಕ್ರಮವನ್ನು ಶ್ರೀಮಂತಗೊಳಿಸುವುದು.
- ಜ್ಞಾನದ ಅಭಿವೃದ್ಧಿಗೆ ಕಲಿಕಾ ಅನುಭವಗಳನ್ನು ಬಳಸುವುದು.
- ಭಾರತದ ಪ್ರಜಾಸತ್ತಾತ್ಮಕ ನೀತಿಯನ್ವಯ ಮಕ್ಕಳ ಅವಶ್ಯಕತೆಗಳಿಗೆ ತಕ್ಕಂತೆ ಸ್ಪಂದಿಸುವುದು.
- ಶಿಕ್ಷಣವನ್ನು ಇಂದಿನ ಹಾಗೂ ಭವಿಷ್ಯದ ಜೀವನಾವಶ್ಯಕತೆಗಳಿಗೆ ಹೊಂದುವಂತೆ ಮಾಡುವುದು.
- ವಿಷಯಗಳ ಮೇರೆಗಳನ್ನು ಮೀರಿ ಅವುಗಳಿಗೆ ಸಮಗ್ರ ದೃಷ್ಟಿಯ ಬೋಧನೆಯನ್ನು ಅಳವಡಿಸುವುದು.
- ಶಾಲೆಯ ಹೊರಗಿನ ಬದುಕಿಗೆ ಜ್ಞಾನ ಸಂಯೋಜನೆ.
- ಮಕ್ಕಳಿಂದಲೇ ಜ್ಞಾನವನ್ನು ಅಭಿವೃದ್ಧಿಪಡಿಸುವುದು.

9ನೇ ತರಗತಿಯ ಗಣಿತ ಪುಸ್ತಕದಲ್ಲಿ ಅಂತರ್ಗತ ವಿಧಾನ (Integrated Approach), ರಚನಾತ್ಮಕ ವಿಧಾನ (Constructive Approach) ಹಾಗೂ ಸುರುಳಿಯಾಕಾರದ ವಿಧಾನ (Spiral Approach) ಗಳನ್ನು ಅಳವಡಿಸಲಾಗಿದೆ.

ಪಠ್ಯಪುಸ್ತಕಗಳ ವಿಷಯ ಹಾಗೂ ಅಭ್ಯಾಸಗಳು ವಿದ್ಯಾರ್ಥಿಗಳನ್ನು ಯೋಚನೆ ಮಾಡುವಂತೆ ಮಾಡಿ, ಚಟುವಟಿಕೆಗಳ ಮೂಲಕ ಜ್ಞಾನ ಹಾಗೂ ಸಾಮರ್ಥ್ಯಗಳನ್ನು ಪಡೆಯುವಂತೆ ಮಾಡುವ ಪ್ರಯತ್ನ ಮಾಡಲಾಗಿದೆ. ಪಠ್ಯವಸ್ತುಗಳೊಂದಿಗೆ ಅತ್ಯಂತ ಅವಶ್ಯಕ ಜೀವನ ಮೌಲ್ಯಗಳನ್ನು ಅಂತರ್ಗತವಾಗಿ ಬಳಸಲಾಗಿದೆ. ಈ ನೂತನ ಪಠ್ಯಪುಸ್ತಕಗಳು ಪರೀಕ್ಷಾ ದೃಷ್ಟಿಯಿಂದ ರಚಿತವಾಗಿಲ್ಲ. ಬದಲಾಗಿ ಅವುಗಳು ವಿದ್ಯಾರ್ಥಿಗಳ ಸರ್ವಾಂಗೀಣ ವ್ಯಕ್ತಿತ್ವ ವಿಕಸನಕ್ಕೆ ಪೂರಕವಾಗಿವೆ. ತನ್ಮೂಲಕ ಅವರನ್ನು ಸ್ವತಂತ್ರ ಭಾರತದ ಸ್ವಸ್ಥಸಮಾಜದ ಉತ್ತಮ ಪ್ರಜೆಗಳನ್ನಾಗಿ ಮಾಡುವ ಪ್ರಯತ್ನ ನಡೆದಿದೆ.

ನಿತ್ಯ ಜೀವನದಲ್ಲಿ ಗಣಿತವು ಎಲ್ಲಾ ಹಂತಗಳಲ್ಲೂ ಯಶಸ್ಸಿಗೆ ಅತ್ಯವಶ್ಯಕವಾಗಿದೆ. ರಾಷ್ಟ್ರೀಯ ಪಠ್ಯಕ್ರಮ-2005ರಂತೆ ಗಣಿತವು ಕೆಲವು ಪರಿಕಲ್ಪನೆಗಳನ್ನು ಅರ್ಥಮಾಡಿಕೊಂಡು ಸಿದ್ಧಾಂತಗಳನ್ನು ನಿರೂಪಿಸಿ, ಪ್ರಮೇಯಗಳನ್ನು ಸಾಧಿಸಿ, ಲೆಕ್ಕಗಳನ್ನು ಮಾಡಿ ಪರೀಕ್ಷೆಯಲ್ಲಿ ಉತ್ತಮ ಅಂಕಗಳನ್ನು ಪಡೆಯುವುದರ ಜೊತೆಗೆ ಗಣಿತವನ್ನು ಜೀವನದ ಸಕಲ ಕ್ಷೇತ್ರಗಳಲ್ಲೂ ಬಳಸುವ ಸಾಮರ್ಥ್ಯವನ್ನು ಬೆಳೆಸಿಕೊಂಡು ಜೀವನದಲ್ಲಿ ಯಶಸ್ಸನ್ನು ಗಳಿಸುವಂತೆ ಮಾಡಬೇಕು. ಅದು ಸಹಕಾರಿ ಕಲಿಕೆಗೂ ಪೂರಕವಾಗಿರಬೇಕು.

ಪ್ರಾಚೀನ ಭಾರತದ ಶ್ರೇಷ್ಠ ಗಣಿತ ಶಾಸ್ತ್ರಜ್ಞರ ಕೊಡುಗೆಗಳನ್ನು ಸೂಕ್ತ ಸಂದರ್ಭಗಳಲ್ಲಿ ಕೊಡಲಾಗಿದೆ. 9ನೇ ತರಗತಿಯ ಪಠ್ಯಪುಸ್ತಕಗಳು ಶೈಕ್ಷಣಿಕವಾಗಿ ವೈಶಿಷ್ಟ್ಯಪೂರ್ಣವಾಗಿವೆ. ಇತರ ಪಠ್ಯಪುಸ್ತಕಗಳಂತೆಯೇ ಈ ಪಠ್ಯಪುಸ್ತಕಗಳು ವಿದ್ಯಾರ್ಥಿ/ವಿದ್ಯಾರ್ಥಿನಿಯರಿಗೆ ಸಾಮರ್ಥ್ಯ ಹಾಗೂ ಕೌಶಲ್ಯಗಳನ್ನು ಬೆಳೆಸಿಕೊಳ್ಳಲು ಸಹಾಯ ಮಾಡುತ್ತವೆ.

ರಾಷ್ಟ್ರೀಯ ಮಟ್ಟದ ಶೈಕ್ಷಣಿಕ ಸುಧಾರಣೆ ಹಾಗೂ ಅಭಿವೃದ್ಧಿಗಳನ್ನು ಗಮನದಲ್ಲಿರಿಸಿ ಕರ್ನಾಟಕ ರಾಜ್ಯದ ವಿದ್ಯಾರ್ಥಿಗಳೂ ಸಹ ರಾಷ್ಟ್ರೀಯ ಮಟ್ಟದಲ್ಲಿ ಶೈಕ್ಷಣಿಕವಾಗಿ ಹಾಗೂ ಸ್ಪರ್ಧಾತ್ಮಕ ಪರೀಕ್ಷೆಗಳಲ್ಲಿ ಯಶಸ್ವಿಯಾಗಿ ತಮ್ಮ ಸಾಧನೆಯನ್ನು ಮಾಡಲು ಈ ಪಠ್ಯಪುಸ್ತಕ ಸಹಕಾರಿಯಾಗಲಿ ಎಂಬುವುದೇ ಸಾರ್ವಜನಿಕ ಶಿಕ್ಷಣ ಇಲಾಖೆಯ ಪ್ರಮುಖ ಆಶಯವಾಗಿದೆ.

ಈ ಪಠ್ಯಪುಸ್ತಕವು ವಿದ್ಯಾರ್ಥಿ ಸ್ನೇಹಿ ಹಾಗೂ ಶಿಕ್ಷಕ ಸ್ನೇಹಿಯಾಗಿದೆ. ಗಣಿತ ಕಲಿಕೆ ಸಂತೋಷದಾಯಕ ಹಾಗೂ ಅರ್ಥಪೂರ್ಣವಾಗುವಂತೆ ಮಾಡಲು ಈ ಪಠ್ಯಪುಸ್ತಕವು ಸೂಕ್ತವಾದ ದಾರಿಯನ್ನು ರೂಪಿಸುತ್ತದೆಯೆಂದು ನಾವು ಭಾವಿಸುತ್ತೇವೆ.

9ನೇ ತರಗತಿಗೆ NCERT ಯ ಗಣಿತ ಪಠ್ಯಪುಸ್ತಕವನ್ನು ರಾಜ್ಯದಲ್ಲಿ ಪ್ರಥಮಭಾರಿಗೆ ಜಾರಿಗೆ ತರಲು ದಿಟ್ಟ ನಿರ್ದಾರ ತೆಗೆದುಕೊಂಡು ಸಹಕರಿಸಿದ ಸರ್ಕಾರ ಮತ್ತು ಇಲಾಖೆಯ ಉನ್ನತ ಅಧಿಕಾರಿಗಳಿಗೆ ಆಭಾರಿಯಾಗಿದ್ದೇನೆ. ಈ ಪಠ್ಯಪುಸ್ತಕವನ್ನು ನಮ್ಮ ರಾಜ್ಯದಲ್ಲಿ ಜಾರಿಗೆ ತರಲು ಅನುಮತಿ, ಸಹಕಾರ ಹಾಗೂ ಮಾರ್ಗದರ್ಶನವನ್ನು ನೀಡಿದ ಎನ್.ಸಿ.ಇ.ಆರ್.ಟಿ ಸಂಸ್ಥೆ ನವದೆಹಲಿ ಹಾಗೂ ಆ ಸಂಸ್ಥೆಯ ಅಧಿಕಾರಿಗಳಿಗೂ ಇಲಾಖೆ ತನ್ನ ಹೃತ್ಪೂರ್ವಕ ಕೃತಜ್ಞತೆಗಳನ್ನು ಅರ್ಪಿಸುತ್ತದೆ.

ನರಸಿಂಹಯ್ಯ

ವ್ಯವಸ್ಥಾಪಕ ನಿರ್ದೇಶಕರು

ಕರ್ನಾಟಕ ಪಠ್ಯಪುಸ್ತಕ ಸಂಘ (ರಿ)

ಬೆಂಗಳೂರು-85

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NUMBER SYSTEMS

1.1 Introduction

In your earlier classes, you have learnt about the number line and how to represent various types of numbers on it (see Fig. 1.1).

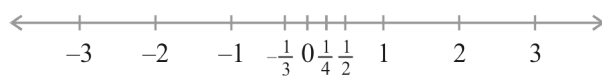


Fig. 1.1 : The number line

Just imagine you start from zero and go on walking along this number line in the positive direction. As far as your eyes can see, there are numbers, numbers and numbers!

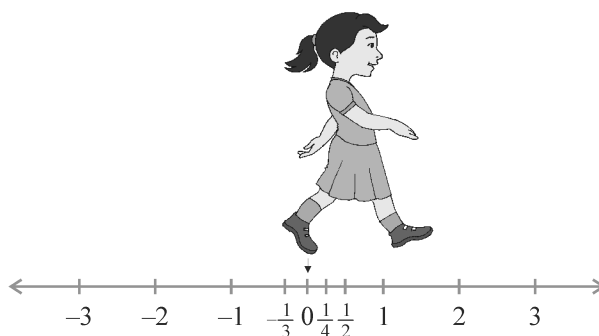


Fig. 1.2

Now suppose you start walking along the number line, and collecting some of the numbers. Get a bag ready to store them!

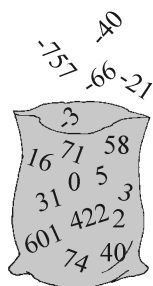
You might begin with picking up only natural numbers like 1, 2, 3, and so on. You know that this list goes on for ever. (Why is this true?) So, now your bag contains infinitely many natural numbers! Recall that we denote this collection by the symbol **N**.



Now turn and walk all the way back, pick up zero and put it into the bag. You now have the collection of *whole numbers* which is denoted by the symbol **W**.



Now, stretching in front of you are many, many negative integers. Put all the negative integers into your bag. What is your new collection? Recall that it is the collection of all *integers*, and it is denoted by the symbol **Z**.

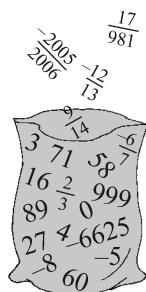


Why **Z** ?

Z comes from the German word "zahlen", which means "to count".



Are there some numbers still left on the line? Of course! There are numbers like $\frac{1}{2}$, $\frac{3}{4}$, or even $\frac{-2005}{2006}$. If you put all such numbers also into the bag, it will now be the



collection of *rational numbers*. The collection of rational numbers is denoted by **Q**. ‘Rational’ comes from the word ‘ratio’, and Q comes from the word ‘quotient’.

You may recall the definition of rational numbers:

A number ‘ r ’ is called a *rational number*, if it can be written in the form $\frac{p}{q}$,

where p and q are integers and $q \neq 0$. (Why do we insist that $q \neq 0$?)

Notice that all the numbers now in the bag can be written in the form $\frac{p}{q}$, where p

and q are integers and $q \neq 0$. For example, -25 can be written as $\frac{-25}{1}$; here $p = -25$ and $q = 1$. Therefore, the rational numbers also include the natural numbers, whole numbers and integers.

You also know that the rational numbers do not have a unique representation in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. For example, $\frac{1}{2} = \frac{2}{4} = \frac{10}{20} = \frac{25}{50} = \frac{47}{94}$, and so on. These are *equivalent rational numbers (or fractions)*. However,

when we say that $\frac{p}{q}$ is a rational number, or when we represent $\frac{p}{q}$ on the number line, we assume that $q \neq 0$ and that p and q have no common factors other than 1 (that is, p and q are *co-prime*). So, on the number line, among the infinitely many fractions equivalent to $\frac{1}{2}$, we will choose $\frac{1}{2}$ to represent all of them.

Now, let us solve some examples about the different types of numbers, which you have studied in earlier classes.

Example 1 : Are the following statements true or false? Give reasons for your answers.

- (i) Every whole number is a natural number.
- (ii) Every integer is a rational number.
- (iii) Every rational number is an integer.

Solution : (i) False, because zero is a whole number but not a natural number.

(ii) True, because every integer m can be expressed in the form $\frac{m}{1}$, and so it is a rational number.

(iii) False, because $\frac{3}{5}$ is not an integer.

Example 2 : Find five rational numbers between 1 and 2.

We can approach this problem in at least two ways.

Solution 1 : Recall that to find a rational number between r and s , you can add r and s and divide the sum by 2, that is $\frac{r+s}{2}$ lies between r and s . So, $\frac{3}{2}$ is a number between 1 and 2. You can proceed in this manner to find four more rational numbers between 1 and 2. These four numbers are $\frac{5}{4}$, $\frac{11}{8}$, $\frac{13}{8}$ and $\frac{7}{4}$.

Solution 2 : The other option is to find all the five rational numbers in one step. Since we want five numbers, we write 1 and 2 as rational numbers with denominator $5 + 1$, i.e., $1 = \frac{6}{6}$ and $2 = \frac{12}{6}$. Then you can check that $\frac{7}{6}$, $\frac{8}{6}$, $\frac{9}{6}$, $\frac{10}{6}$ and $\frac{11}{6}$ are all rational numbers between 1 and 2. So, the five numbers are $\frac{7}{6}$, $\frac{4}{3}$, $\frac{3}{2}$, $\frac{5}{3}$ and $\frac{11}{6}$.

Remark : Notice that in Example 2, you were asked to find five rational numbers between 1 and 2. But, you must have realised that in fact there are infinitely many rational numbers between 1 and 2. In general, **there are infinitely many rational numbers between any two given rational numbers.**

Let us take a look at the number line again. Have you picked up all the numbers? Not, yet. The fact is that there are infinitely many more numbers left on the number line! There are gaps in between the places of the numbers you picked up, and not just one or two but infinitely many. The amazing thing is that there are infinitely many numbers lying between any two of these gaps too!

So we are left with the following questions:

1. What are the numbers, that are left on the number line, called?
2. How do we recognise them? That is, how do we distinguish them from the rationals (rational numbers)?

These questions will be answered in the next section.



EXERCISE 1.1

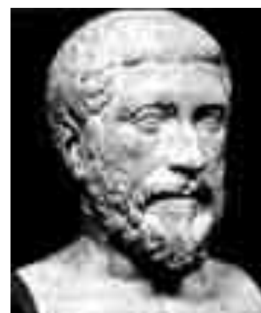
1. Is zero a rational number? Can you write it in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$?
2. Find six rational numbers between 3 and 4.
3. Find five rational numbers between $\frac{3}{5}$ and $\frac{4}{5}$.
4. State whether the following statements are true or false. Give reasons for your answers.
 - (i) Every natural number is a whole number.
 - (ii) Every integer is a whole number.
 - (iii) Every rational number is a whole number.

1.2 Irrational Numbers

We saw, in the previous section, that there may be numbers on the number line that are not rationals. In this section, we are going to investigate these numbers. So far, all

the numbers you have come across, are of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. So, you may ask: are there numbers which are not of this form? There are indeed such numbers.

The Pythagoreans in Greece, followers of the famous mathematician and philosopher Pythagoras, were the first to discover the numbers which were not rationals, around 400 BC. These numbers are called *irrational numbers* (*irrationals*), because they cannot be written in the form of a ratio of integers. There are many myths surrounding the discovery of irrational numbers by the Pythagorean, Hippacus of Croton. In all the myths, Hippacus has an unfortunate end, either for discovering that $\sqrt{2}$ is irrational or for disclosing the secret about $\sqrt{2}$ to people outside the secret Pythagorean sect!



Pythagoras
(569 BCE – 479 BCE)

Fig. 1.3

Let us formally define these numbers.

A number 's' is called *irrational*, if it cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

You already know that there are infinitely many rationals. It turns out that there are infinitely many irrational numbers too. Some examples are:

$$\sqrt{2}, \sqrt{3}, \sqrt{15}, \pi, 0.1011011101110...$$

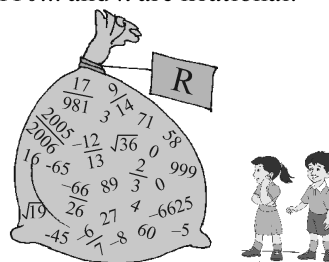
Remark : Recall that when we use the symbol $\sqrt{\quad}$, we assume that it is the positive square root of the number. So $\sqrt{4} = 2$, though both 2 and -2 are square roots of 4.

Some of the irrational numbers listed above are familiar to you. For example, you have already come across many of the square roots listed above and the number π .

The Pythagoreans proved that $\sqrt{2}$ is irrational. Later in approximately 425 BC, Theodorus of Cyrene showed that $\sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{10}, \sqrt{11}, \sqrt{12}, \sqrt{13}, \sqrt{14}, \sqrt{15}$ and $\sqrt{17}$ are also irrationals. Proofs of irrationality of $\sqrt{2}, \sqrt{3}, \sqrt{5}$, etc., shall be discussed in Class X. As to π , it was known to various cultures for thousands of years, it was proved to be irrational by Lambert and Legendre only in the late 1700s. In the next section, we will discuss why $0.1011011101110...$ and π are irrational.

Let us return to the questions raised at the end of the previous section. Remember the bag of rational numbers. If we now put all irrational numbers into the bag, will there be any number left on the number line? The answer is no! It turns out that the collection of all rational numbers and irrational numbers together make up what we call the collection of *real numbers*,

which is denoted by **R**. Therefore, a real number is either rational or irrational. So, we can say that **every real number is represented by a unique point on the number line. Also, every point on the number line represents a unique real number.** This is why we call the number line, the *real number line*.



R. Dedekind (1831-1916)
Fig. 1.4

In the 1870s two German mathematicians, Cantor and Dedekind, showed that : Corresponding to every real number, there is a point on the real number line, and corresponding to every point on the number line, there exists a unique real number.



G. Cantor (1845-1918)
Fig. 1.5

Let us see how we can locate some of the irrational numbers on the number line.

Example 3 : Locate $\sqrt{2}$ on the number line.

Solution : It is easy to see how the Greeks might have discovered $\sqrt{2}$. Consider a unit square OABC, with each side 1 unit in length (see Fig. 1.6). Then you can see by the Pythagoras theorem that $OB = \sqrt{1^2 + 1^2} = \sqrt{2}$. How do we represent $\sqrt{2}$ on the number line?

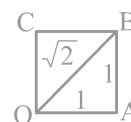


Fig. 1.6

This is easy. Transfer Fig. 1.6 onto the number line making sure that the vertex O coincides with zero (see Fig. 1.7).

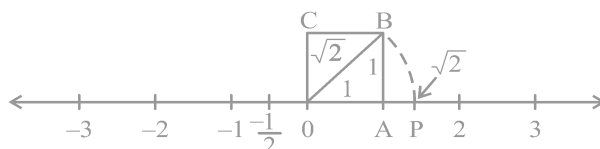


Fig. 1.7

We have just seen that $OB = \sqrt{2}$. Using a compass with centre O and radius OB, draw an arc intersecting the number line at the point P. Then P corresponds to $\sqrt{2}$ on the number line.

Example 4 : Locate $\sqrt{3}$ on the number line.

Solution : Let us return to Fig. 1.7.

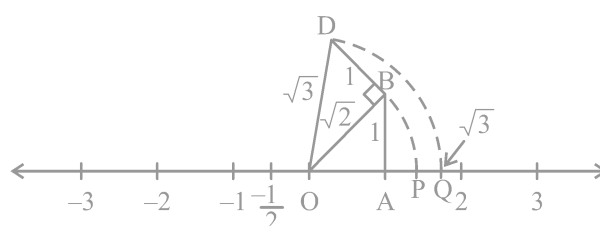


Fig. 1.8

Construct BD of unit length perpendicular to OB (as in Fig. 1.8). Then using the Pythagoras theorem, we see that $OD = \sqrt{(\sqrt{2})^2 + 1^2} = \sqrt{3}$. Using a compass, with centre O and radius OD, draw an arc which intersects the number line at the point Q. Then Q corresponds to $\sqrt{3}$.

In the same way, you can locate $\sqrt{n}$ for any positive integer n , after $\sqrt{n-1}$ has been located.

EXERCISE 1.2

- State whether the following statements are true or false. Justify your answers.
 - Every irrational number is a real number.
 - Every point on the number line is of the form $\sqrt{m}$, where m is a natural number.
 - Every real number is an irrational number.
- Are the square roots of all positive integers irrational? If not, give an example of the square root of a number that is a rational number.
- Show how $\sqrt{5}$ can be represented on the number line.
- Classroom activity (Constructing the ‘square root spiral’)** : Take a large sheet of paper and construct the ‘square root spiral’ in the following fashion. Start with a point O and draw a line segment OP_1 of unit length. Draw a line segment P_1P_2 perpendicular to OP_1 of unit length (see Fig. 1.9). Now draw a line segment P_2P_3 perpendicular to OP_2 . Then draw a line segment P_3P_4 perpendicular to OP_3 . Continuing in this manner, you can get the line segment $P_{n-1}P_n$ by drawing a line segment of unit length perpendicular to OP_{n-1} . In this manner, you will have created the points $P_2, P_3, \dots, P_n, \dots$, and joined them to create a beautiful spiral depicting $\sqrt{2}, \sqrt{3}, \sqrt{4}, \dots$

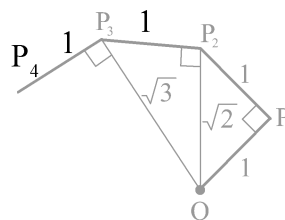


Fig. 1.9 : Constructing square root spiral

1.3 Real Numbers and their Decimal Expansions

In this section, we are going to study rational and irrational numbers from a different point of view. We will look at the decimal expansions of real numbers and see if we can use the expansions to distinguish between rationals and irrationals. We will also explain how to visualise the representation of real numbers on the number line using their decimal expansions. Since rationals are more familiar to us, let us start with

them. Let us take three examples : $\frac{10}{3}, \frac{7}{8}, \frac{1}{7}$.

Pay special attention to the remainders and see if you can find any pattern.

Solution :

$$\begin{array}{r} 3 \overline{) 3.333...} \\ \underline{10} \\ 9 \\ \underline{10} \\ 9 \\ \underline{10} \\ 9 \\ \underline{10} \\ 9 \\ \underline{10} \\ 9 \\ \underline{10} \\ 1 \end{array}$$

	0.875
8	7.0
	64
	60
	56
	40
	40
	0

7	0.142857...
	1.0
	7
	30
	28
	20
	14
	60
	56
	40
35	
50	
49	
1	

Remainders : 3, 2, 6, 4, 5, 1,
3, 2, 6, 4, 5, 1, ...
Divisor : 7

- (i) The remainders either become 0 after a certain stage, or start repeating themselves.
- (ii) The number of entries in the repeating string of remainders is less than the divisor
(in $\frac{1}{3}$ one number repeats itself and the divisor is 3, in $\frac{1}{7}$ there are six entries
326451 in the repeating string of remainders and 7 is the divisor).
- (iii) If the remainders repeat, then we get a repeating block of digits in the quotient
(for $\frac{1}{3}$, 3 repeats in the quotient and for $\frac{1}{7}$, we get the repeating block 142857 in
the quotient).

Although we have noticed this pattern using only the examples above, it is true for all rationals of the form $\frac{p}{q}$ ($q \neq 0$). On division of p by q , two main things happen – either the remainder becomes zero or never becomes zero and we get a repeating string of remainders. Let us look at each case separately.

Case (i) : The remainder becomes zero

In the example of $\frac{7}{8}$, we found that the remainder becomes zero after some steps and

the decimal expansion of $\frac{7}{8} = 0.875$. Other examples are $\frac{1}{2} = 0.5$, $\frac{639}{250} = 2.556$. In all these cases, the decimal expansion terminates or ends after a finite number of steps. We call the decimal expansion of such numbers *terminating*.

Case (ii) : The remainder never becomes zero

In the examples of $\frac{1}{3}$ and $\frac{1}{7}$, we notice that the remainders repeat after a certain stage forcing the decimal expansion to go on for ever. In other words, we have a repeating block of digits in the quotient. We say that this expansion is non-terminating recurring. For example, $\frac{1}{3} = 0.3333\dots$ and $\frac{1}{7} = 0.142857142857142857\dots$

The usual way of showing that 3 repeats in the quotient of $\frac{1}{3}$ is to write it as $0.\overline{3}$.

Similarly, since the block of digits 142857 repeats in the quotient of $\frac{1}{7}$, we write $\frac{1}{7}$ as $0.\overline{142857}$, where the bar above the digits indicates the block of digits that repeats. Also $3.57272\dots$ can be written as $3.5\overline{72}$. So, all these examples give us *non-terminating recurring (repeating)* decimal expansions.

Thus, we see that the decimal expansion of rational numbers have only two choices: either they are terminating or non-terminating recurring.

Now suppose, on the other hand, on your walk on the number line, you come across a number like 3.142678 whose decimal expansion is terminating or a number like 1.272727... that is, $1.\overline{27}$, whose decimal expansion is non-terminating recurring, can you conclude that it is a rational number? The answer is yes!

We will not prove it but illustrate this fact with a few examples. The terminating cases are easy.

Example 6 : Show that 3.142678 is a rational number. In other words, express 3.142678 in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Solution : We have $3.142678 = \frac{3142678}{1000000}$, and hence is a rational number.

Now, let us consider the case when the decimal expansion is non-terminating recurring.

Example 7 : Show that $0.3333... = 0.\overline{3}$ can be expressed in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Solution : Since we do not know what $0.\overline{3}$ is, let us call it ' x ' and so

$$x = 0.3333...$$

Now here is where the trick comes in. Look at

$$10x = 10 \times (0.3333...) = 3.333...$$

Now, $3.3333... = 3 + x$, since $x = 0.3333...$

Therefore, $10x = 3 + x$

Solving for x , we get

$$9x = 3, \text{ i.e., } x = \frac{1}{3}$$

Example 8 : Show that $1.272727... = 1.\overline{27}$ can be expressed in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Solution : Let $x = 1.272727...$ Since two digits are repeating, we multiply x by 100 to get

$$100x = 127.2727...$$

So, $100x = 126 + 1.272727... = 126 + x$

Therefore, $100x - x = 126$, i.e., $99x = 126$

i.e.,
$$x = \frac{126}{99} = \frac{14}{11}$$

You can check the reverse that $\frac{14}{11} = 1.\overline{27}$.

Example 9 : Show that $0.2353535\ldots = 0.2\overline{35}$ can be expressed in the form $\frac{p}{q}$,

where p and q are integers and $q \neq 0$.

Solution : Let $x = 0.2\overline{35}$. Over here, note that 2 does not repeat, but the block 35 repeats. Since two digits are repeating, we multiply x by 100 to get

$$100x = 23.53535\ldots$$

So,
$$100x = 23.3 + 0.23535\ldots = 23.3 + x$$

Therefore,
$$99x = 23.3$$

i.e.,
$$99x = \frac{233}{10}, \text{ which gives } x = \frac{233}{990}$$

You can also check the reverse that $\frac{233}{990} = 0.2\overline{35}$.

So, every number with a non-terminating recurring decimal expansion can be expressed

in the form $\frac{p}{q}$ ($q \neq 0$), where p and q are integers. Let us summarise our results in the

following form :

The decimal expansion of a rational number is either terminating or non-terminating recurring. Moreover, a number whose decimal expansion is terminating or non-terminating recurring is rational.

So, now we know what the decimal expansion of a rational number can be. What about the decimal expansion of irrational numbers? Because of the property above, we can conclude that their decimal expansions are *non-terminating non-recurring*.

So, the property for irrational numbers, similar to the property stated above for rational numbers, is

The decimal expansion of an irrational number is non-terminating non-recurring. Moreover, a number whose decimal expansion is non-terminating non-recurring is irrational.

Recall $s = 0.10110111011110\dots$ from the previous section. Notice that it is non-terminating and non-recurring. Therefore, from the property above, it is irrational. Moreover, notice that you can generate infinitely many irrationals similar to s .

What about the famous irrationals $\sqrt{2}$ and π ? Here are their decimal expansions up to a certain stage.

$$\sqrt{2} = 1.4142135623730950488016887242096\dots$$

$$\pi = 3.14159265358979323846264338327950\dots$$

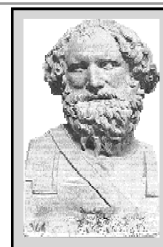
(Note that, we often take $\frac{22}{7}$ as an approximate value for π , but $\pi \neq \frac{22}{7}$.)

Over the years, mathematicians have developed various techniques to produce more and more digits in the decimal expansions of irrational numbers. For example, you might have learnt to find digits in the decimal expansion of $\sqrt{2}$ by the division method. Interestingly, in the Sulbasutras (rules of chord), a mathematical treatise of the Vedic period (800 BC - 500 BC), you find an approximation of $\sqrt{2}$ as follows:

$$\sqrt{2} = 1 + \frac{1}{3} + \left(\frac{1}{4} \times \frac{1}{3}\right) - \left(\frac{1}{34} \times \frac{1}{4} \times \frac{1}{3}\right) = 1.4142156$$

Notice that it is the same as the one given above for the first five decimal places. The history of the hunt for digits in the decimal expansion of π is very interesting.

The Greek genius Archimedes was the first to compute digits in the decimal expansion of π . He showed $3.140845 < \pi < 3.142857$. Aryabhatta (476 – 550 C.E.), the great Indian mathematician and astronomer, found the value of π correct to four decimal places (3.1416). Using high speed computers and advanced algorithms, π has been computed to over 1.24 trillion decimal places!



Archimedes (287 BCE – 212 BCE)

Fig. 1.10

Now, let us see how to obtain irrational numbers.

Example 10 : Find an irrational number between $\frac{1}{7}$ and $\frac{2}{7}$.

Solution : We saw that $\frac{1}{7} = 0.\overline{142857}$. So, you can easily calculate $\frac{2}{7} = 0.\overline{285714}$.

To find an irrational number between $\frac{1}{7}$ and $\frac{2}{7}$, we find a number which is

non-terminating non-recurring lying between them. Of course, you can find infinitely many such numbers.

An example of such a number is 0.150150015000150000...

EXERCISE 1.3

1. Write the following in decimal form and say what kind of decimal expansion each has :

(i) $\frac{36}{100}$

(ii) $\frac{1}{11}$

(iii) $4\frac{1}{8}$

(iv) $\frac{3}{13}$

(v) $\frac{2}{11}$

(vi) $\frac{329}{400}$

2. You know that $\frac{1}{7} = 0.\overline{142857}$. Can you predict what the decimal expansions of $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, $\frac{6}{7}$ are, without actually doing the long division? If so, how?

[Hint : Study the remainders while finding the value of $\frac{1}{7}$ carefully.]

3. Express the following in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.
- (i) $0.\overline{6}$ (ii) $0.4\overline{7}$ (iii) $0.0\overline{01}$
4. Express $0.99999 \dots$ in the form $\frac{p}{q}$. Are you surprised by your answer? With your teacher and classmates discuss why the answer makes sense.
5. What can the maximum number of digits be in the repeating block of digits in the decimal expansion of $\frac{1}{17}$? Perform the division to check your answer.
6. Look at several examples of rational numbers in the form $\frac{p}{q}$ ($q \neq 0$), where p and q are integers with no common factors other than 1 and having terminating decimal representations (expansions). Can you guess what property q must satisfy?
7. Write three numbers whose decimal expansions are non-terminating non-recurring.
8. Find three different irrational numbers between the rational numbers $\frac{5}{7}$ and $\frac{9}{11}$.
9. Classify the following numbers as rational or irrational :
- (i) $\sqrt{23}$ (ii) $\sqrt{225}$ (iii) 0.3796
- (iv) $7.478478\dots$ (v) $1.101001000100001\dots$

1.4 Representing Real Numbers on the Number Line

In the previous section, you have seen that any real number has a decimal expansion. This helps us to represent it on the number line. Let us see how.

Suppose we want to locate 2.665 on the number line. We know that this lies between 2 and 3.

So, let us look closely at the portion of the number line between 2 and 3. Suppose we divide this into 10 equal parts and mark each point of division as in Fig. 1.11 (i). Then the first mark to the right of 2 will represent 2.1, the second 2.2, and so on. You might be finding some difficulty in observing these points of division between 2 and 3 in Fig. 1.11 (i). To have a clear view of the same, you may take a magnifying glass and look at the portion between 2 and 3. It will look like what you see in Fig. 1.11 (ii). Now, 2.665 lies between 2.6 and 2.7. So, let us focus on the portion between 2.6 and 2.7 [See Fig. 1.12(i)]. We imagine to divide this again into ten equal parts. The first mark will represent 2.61, the next 2.62, and so on. To see this clearly, we magnify this as shown in Fig. 1.12 (ii).

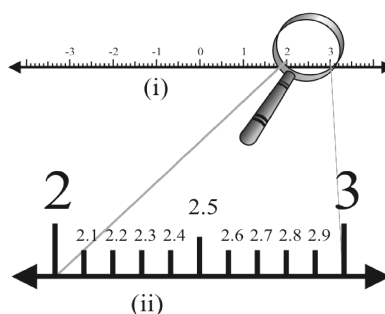


Fig. 1.11

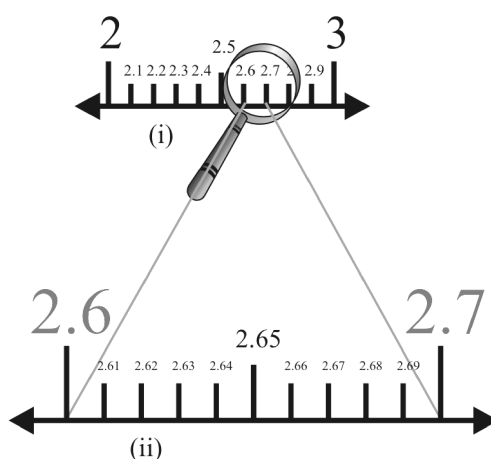


Fig. 1.12

Again, 2.665 lies between 2.66 and 2.67. So, let us focus on this portion of the number line [see Fig. 1.13(i)] and imagine to divide it again into ten equal parts. We magnify it to see it better, as in Fig. 1.13 (ii). The first mark represents 2.661, the next one represents 2.662, and so on. So, 2.665 is the 5th mark in these subdivisions.

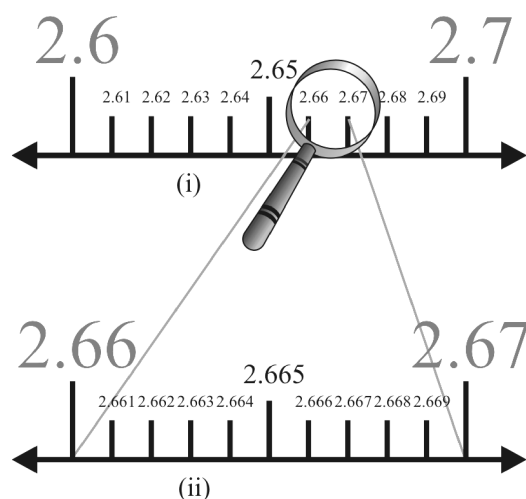


Fig. 1.13

We call this process of visualisation of representation of numbers on the number line, through a magnifying glass, as the **process of successive magnification**.

So, we have seen that it is possible by sufficient successive magnifications to visualise the position (or representation) of a real number with a terminating decimal expansion on the number line.

Let us now try and visualise the position (or representation) of a real number with a non-terminating recurring decimal expansion on the number line. We can look at appropriate intervals through a magnifying glass and by successive magnifications visualise the position of the number on the number line.

Example 11 : Visualize the representation of $5.\overline{37}$ on the number line upto 5 decimal places, that is, up to 5.37777 .

Solution : Once again we proceed by successive magnification, and successively decrease the lengths of the portions of the number line in which $5.\overline{37}$ is located. First, we see that $5.\overline{37}$ is located between 5 and 6. In the next step, we locate $5.\overline{37}$ between 5.3 and 5.4. To get a more accurate visualization of the representation, we divide this portion of the number line into 10 equal parts and use a magnifying glass to visualize that $5.\overline{37}$ lies between 5.37 and 5.38. To visualize $5.\overline{37}$ more accurately, we again divide the portion between 5.37 and 5.38 into ten equal parts and use a magnifying glass to visualize that $5.\overline{37}$ lies between 5.377 and 5.378. Now to visualize $5.\overline{37}$ still more accurately, we divide the portion between 5.377 and 5.378 into 10 equal parts, and

visualize the representation of $5.\overline{37}$ as in Fig. 1.14 (iv). Notice that $5.\overline{37}$ is located closer to 5.3778 than to 5.3777 [see Fig 1.14 (iv)].

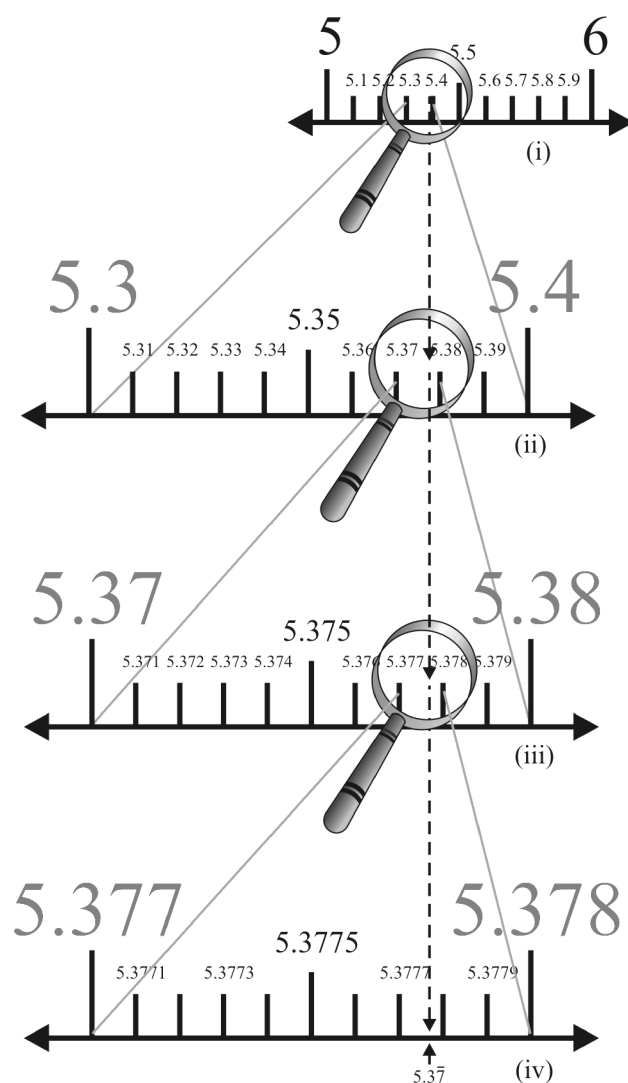


Fig. 1.14

Remark : We can proceed endlessly in this manner, successively viewing through a magnifying glass and simultaneously imagining the decrease in the length of the portion of the number line in which $5.\overline{37}$ is located. The size of the portion of the line we specify depends on the degree of accuracy we would like for the visualisation of the position of the number on the number line.

You might have realised by now that the same procedure can be used to visualise a real number with a non-terminating non-recurring decimal expansion on the number line.

In the light of the discussions above and visualisations, we can again say that *every real number is represented by a unique point on the number line. Further, every point on the number line represents one and only one real number.*

EXERCISE 1.4

1. Visualise 3.765 on the number line, using successive magnification.
2. Visualise $4.\overline{26}$ on the number line, up to 4 decimal places.

1.5 Operations on Real Numbers

You have learnt, in earlier classes, that rational numbers satisfy the commutative, associative and distributive laws for addition and multiplication. Moreover, if we add, subtract, multiply or divide (except by zero) two rational numbers, we still get a rational number (that is, rational numbers are ‘closed’ with respect to addition, subtraction, multiplication and division). It turns out that irrational numbers also satisfy the commutative, associative and distributive laws for addition and multiplication. However, the sum, difference, quotients and products of irrational numbers are not *always*

irrational. For example, $(\sqrt{6}) + (-\sqrt{6})$, $(\sqrt{2}) - (\sqrt{2})$, $(\sqrt{3}) \cdot (\sqrt{3})$ and $\frac{\sqrt{17}}{\sqrt{17}}$ are rationals.

Let us look at what happens when we add and multiply a rational number with an irrational number. For example, $\sqrt{3}$ is irrational. What about $2 + \sqrt{3}$ and $2\sqrt{3}$? Since $\sqrt{3}$ has a non-terminating non-recurring decimal expansion, the same is true for $2 + \sqrt{3}$ and $2\sqrt{3}$. Therefore, both $2 + \sqrt{3}$ and $2\sqrt{3}$ are also irrational numbers.

Example 12 : Check whether $7\sqrt{5}$, $\frac{7}{\sqrt{5}}$, $\sqrt{2} + 21$, $\pi - 2$ are irrational numbers or not.

Solution : $\sqrt{5} = 2.236...$, $\sqrt{2} = 1.4142...$, $\pi = 3.1415...$

Then $7\sqrt{5} = 15.652\dots$, $\frac{7}{\sqrt{5}} = \frac{7\sqrt{5}}{\sqrt{5}\sqrt{5}} = \frac{7\sqrt{5}}{5} = 3.1304\dots$

$$\sqrt{2} + 21 = 22.4142\dots, \pi - 2 = 1.1415\dots$$

All these are non-terminating non-recurring decimals. So, all these are irrational numbers. Now, let us see what generally happens if we add, subtract, multiply, divide, take square roots and even n th roots of these irrational numbers, where n is any natural number. Let us look at some examples.

Example 13 : Add $2\sqrt{2} + 5\sqrt{3}$ and $\sqrt{2} - 3\sqrt{3}$.

$$\begin{aligned} \text{Solution : } (2\sqrt{2} + 5\sqrt{3}) + (\sqrt{2} - 3\sqrt{3}) &= (2\sqrt{2} + \sqrt{2}) + (5\sqrt{3} - 3\sqrt{3}) \\ &= (2 + 1)\sqrt{2} + (5 - 3)\sqrt{3} = 3\sqrt{2} + 2\sqrt{3} \end{aligned}$$

Example 14 : Multiply $6\sqrt{5}$ by $2\sqrt{5}$.

$$\text{Solution : } 6\sqrt{5} \times 2\sqrt{5} = 6 \times 2 \times \sqrt{5} \times \sqrt{5} = 12 \times 5 = 60$$

Example 15 : Divide $8\sqrt{15}$ by $2\sqrt{3}$.

$$\text{Solution : } 8\sqrt{15} \div 2\sqrt{3} = \frac{8\sqrt{3} \times \sqrt{5}}{2\sqrt{3}} = 4\sqrt{5}$$

These examples may lead you to expect the following facts, **which are true**:

- (i) The sum or difference of a rational number and an irrational number is irrational.
- (ii) The product or quotient of a non-zero rational number with an irrational number is irrational.
- (iii) If we add, subtract, multiply or divide two irrationals, the result may be rational or irrational.

We now turn our attention to the operation of taking square roots of real numbers. Recall that, if a is a natural number, then $\sqrt{a} = b$ means $b^2 = a$ and $b > 0$. The same definition can be extended for positive real numbers.

Let $a > 0$ be a real number. Then $\sqrt{a} = b$ means $b^2 = a$ and $b > 0$.

In Section 1.2, we saw how to represent $\sqrt[n]{a}$ for any positive integer n on the number

line. We now show how to find $\sqrt{x}$ for any given positive real number x geometrically. For example, let us find it for $x = 3.5$, i.e., we find $\sqrt{3.5}$ geometrically.

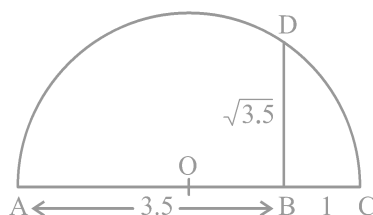


Fig. 1.15

Mark the distance 3.5 units from a fixed point A on a given line to obtain a point B such that $AB = 3.5$ units (see Fig. 1.15). From B, mark a distance of 1 unit and mark the new point as C. Find the mid-point of AC and mark that point as O. Draw a semicircle with centre O and radius OC. Draw a line perpendicular to AC passing through B and intersecting the semicircle at D. Then, $BD = \sqrt{3.5}$.

More generally, to find $\sqrt{x}$, for any positive real number x , we mark B so that $AB = x$ units, and, as in Fig. 1.16, mark C so that $BC = 1$ unit. Then, as we have done for the case $x = 3.5$, we find $BD = \sqrt{x}$ (see Fig. 1.16). We can prove this result using the Pythagoras Theorem.

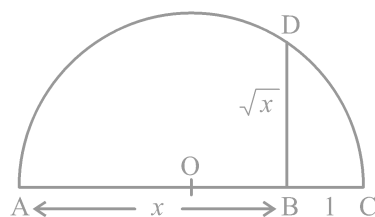


Fig. 1.16

Notice that, in Fig. 1.16, $\triangle OBD$ is a right-angled triangle. Also, the radius of the circle is $\frac{x+1}{2}$ units.

Therefore, $OC = OD = OA = \frac{x+1}{2}$ units.

Now, $OB = x - \left(\frac{x+1}{2}\right) = \frac{x-1}{2}$.

So, by the Pythagoras Theorem, we have

$$BD^2 = OD^2 - OB^2 = \left(\frac{x+1}{2}\right)^2 - \left(\frac{x-1}{2}\right)^2 = \frac{4x}{4} = x.$$

This shows that $BD = \sqrt{x}$.

This construction gives us a visual, and geometric way of showing that $\sqrt{x}$ exists for all real numbers $x > 0$. If you want to know the position of $\sqrt{x}$ on the number line, then let us treat the line BC as the number line, with B as zero, C as 1, and so on. Draw an arc with centre B and radius BD, which intersects the number line in E (see Fig. 1.17). Then, E represents $\sqrt{x}$.

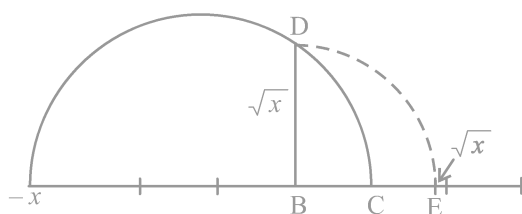


Fig. 1.17

We would like to now extend the idea of square roots to cube roots, fourth roots, and in general n th roots, where n is a positive integer. Recall your understanding of square roots and cube roots from earlier classes.

What is $\sqrt[3]{8}$? Well, we know it has to be some positive number whose cube is 8, and you must have guessed $\sqrt[3]{8} = 2$. Let us try $\sqrt[5]{243}$. Do you know some number b such that $b^5 = 243$? The answer is 3. Therefore, $\sqrt[5]{243} = 3$.

From these examples, can you define $\sqrt[n]{a}$ for a real number $a > 0$ and a positive integer n ?

Let $a > 0$ be a real number and n be a positive integer. Then $\sqrt[n]{a} = b$, if $b^n = a$ and $b > 0$. Note that the symbol ' $\sqrt{}$ ' used in $\sqrt{2}$, $\sqrt[3]{8}$, $\sqrt[n]{a}$, etc. is called the *radical sign*.

We now list some identities relating to square roots, which are useful in various ways. You are already familiar with some of these from your earlier classes. The remaining ones follow from the distributive law of multiplication over addition of real numbers, and from the identity $(x + y)(x - y) = x^2 - y^2$, for any real numbers x and y . Let a and b be positive real numbers. Then

$$(i) \quad \sqrt{ab} = \sqrt{a}\sqrt{b} \qquad (ii) \quad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

$$(iii) (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b \quad (iv) (a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$$

$$(v) (\sqrt{a} + \sqrt{b})(\sqrt{c} + \sqrt{d}) = \sqrt{ac} + \sqrt{ad} + \sqrt{bc} + \sqrt{bd}$$

$$(vi) (\sqrt{a} + \sqrt{b})^2 = a + 2\sqrt{ab} + b$$

Let us look at some particular cases of these identities.

Example 16 : Simplify the following expressions:

$$(i) (5 + \sqrt{7})(2 + \sqrt{5})$$

$$(ii) (5 + \sqrt{5})(5 - \sqrt{5})$$

$$(iii) (\sqrt{3} + \sqrt{7})^2$$

$$(iv) (\sqrt{11} - \sqrt{7})(\sqrt{11} + \sqrt{7})$$

Solution : (i) $(5 + \sqrt{7})(2 + \sqrt{5}) = 10 + 5\sqrt{5} + 2\sqrt{7} + \sqrt{35}$

$$(ii) (5 + \sqrt{5})(5 - \sqrt{5}) = 5^2 - (\sqrt{5})^2 = 25 - 5 = 20$$

$$(iii) (\sqrt{3} + \sqrt{7})^2 = (\sqrt{3})^2 + 2\sqrt{3}\sqrt{7} + (\sqrt{7})^2 = 3 + 2\sqrt{21} + 7 = 10 + 2\sqrt{21}$$

$$(iv) (\sqrt{11} - \sqrt{7})(\sqrt{11} + \sqrt{7}) = (\sqrt{11})^2 - (\sqrt{7})^2 = 11 - 7 = 4$$

Remark : Note that ‘simplify’ in the example above has been used to mean that the expression should be written as the sum of a rational and an irrational number.

We end this section by considering the following problem. Look at $\frac{1}{\sqrt{2}}$. Can you tell where it shows up on the number line? You know that it is irrational. May be it is easier to handle if the denominator is a rational number. Let us see, if we can ‘rationalise’ the denominator, that is, to make the denominator into a rational number. To do so, we need the identities involving square roots. Let us see how.

Example 17 : Rationalise the denominator of $\frac{1}{\sqrt{2}}$.

Solution : We want to write $\frac{1}{\sqrt{2}}$ as an equivalent expression in which the denominator

is a rational number. We know that $\sqrt{2} \cdot \sqrt{2}$ is rational. We also know that multiplying

$\frac{1}{\sqrt{2}}$ by $\frac{\sqrt{2}}{\sqrt{2}}$ will give us an equivalent expression, since $\frac{\sqrt{2}}{\sqrt{2}} = 1$. So, we put these two facts together to get

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

In this form, it is easy to locate $\frac{1}{\sqrt{2}}$ on the number line. It is half way between 0 and $\sqrt{2}$!

Example 18 : Rationalise the denominator of $\frac{1}{2 + \sqrt{3}}$.

Solution : We use the Identity (iv) given earlier. Multiply and divide $\frac{1}{2 + \sqrt{3}}$ by

$$2 - \sqrt{3} \text{ to get } \frac{1}{2 + \sqrt{3}} \times \frac{2 - \sqrt{3}}{2 - \sqrt{3}} = \frac{2 - \sqrt{3}}{4 - 3} = 2 - \sqrt{3}.$$

Example 19 : Rationalise the denominator of $\frac{5}{\sqrt{3} - \sqrt{5}}$.

Solution : Here we use the Identity (iii) given earlier.

$$\text{So, } \frac{5}{\sqrt{3} - \sqrt{5}} = \frac{5}{\sqrt{3} - \sqrt{5}} \times \frac{\sqrt{3} + \sqrt{5}}{\sqrt{3} + \sqrt{5}} = \frac{5(\sqrt{3} + \sqrt{5})}{3 - 5} = \left(\frac{-5}{2}\right)(\sqrt{3} + \sqrt{5})$$

Example 20 : Rationalise the denominator of $\frac{1}{7 + 3\sqrt{2}}$.

$$\text{Solution : } \frac{1}{7 + 3\sqrt{2}} = \frac{1}{7 + 3\sqrt{2}} \times \left(\frac{7 - 3\sqrt{2}}{7 - 3\sqrt{2}}\right) = \frac{7 - 3\sqrt{2}}{49 - 18} = \frac{7 - 3\sqrt{2}}{31}$$

So, when the denominator of an expression contains a term with a square root (or a number under a radical sign), the process of converting it to an equivalent expression whose denominator is a rational number is called *rationalising the denominator*.

EXERCISE 1.5

1. Classify the following numbers as rational or irrational:

(i) $2 - \sqrt{5}$ (ii) $(3 + \sqrt{23}) - \sqrt{23}$ (iii) $\frac{2\sqrt{7}}{7\sqrt{7}}$
 (iv) $\frac{1}{\sqrt{2}}$ (v) 2π

2. Simplify each of the following expressions:

(i) $(3 + \sqrt{3})(2 + \sqrt{2})$ (ii) $(3 + \sqrt{3})(3 - \sqrt{3})$
 (iii) $(\sqrt{5} + \sqrt{2})^2$ (iv) $(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2})$

3. Recall, π is defined as the ratio of the circumference (say c) of a circle to its diameter (say d). That is, $\pi = \frac{c}{d}$. This seems to contradict the fact that π is irrational. How will you resolve this contradiction?

4. Represent $\sqrt{9.3}$ on the number line.

5. Rationalise the denominators of the following:

(i) $\frac{1}{\sqrt{7}}$ (ii) $\frac{1}{\sqrt{7} - \sqrt{6}}$
 (iii) $\frac{1}{\sqrt{5} + \sqrt{2}}$ (iv) $\frac{1}{\sqrt{7} - 2}$

1.6 Laws of Exponents for Real Numbers

Do you remember how to simplify the following?

(i) $17^2 \cdot 17^5 =$ (ii) $(5^2)^7 =$
 (iii) $\frac{23^{10}}{23^7} =$ (iv) $7^3 \cdot 9^3 =$

Did you get these answers? They are as follows:

(i) $17^2 \cdot 17^5 = 17^7$ (ii) $(5^2)^7 = 5^{14}$
 (iii) $\frac{23^{10}}{23^7} = 23^3$ (iv) $7^3 \cdot 9^3 = 63^3$

To get these answers, you would have used the following laws of exponents, which you have learnt in your earlier classes. (Here a , n and m are natural numbers. Remember, a is called the base and m and n are the exponents.)

$$(i) \quad a^m \cdot a^n = a^{m+n}$$

$$(ii) \quad (a^m)^n = a^{mn}$$

$$(iii) \quad \frac{a^m}{a^n} = a^{m-n}, \quad m > n$$

$$(iv) \quad a^m b^m = (ab)^m$$

What is $(a)^0$? Yes, it is 1! So you have learnt that $(a)^0 = 1$. So, using (iii), we can get $\frac{1}{a^n} = a^{-n}$. We can now extend the laws to negative exponents too.

So, for example :

$$(i) \quad 17^2 \cdot 17^{-5} = 17^{-3} = \frac{1}{17^3}$$

$$(ii) \quad (5^2)^{-7} = 5^{-14}$$

$$(iii) \quad \frac{23^{-10}}{23^7} = 23^{-17}$$

$$(iv) \quad (7)^{-3} \cdot (9)^{-3} = (63)^{-3}$$

Suppose we want to do the following computations:

$$(i) \quad 2^{\frac{2}{3}} \cdot 2^{\frac{1}{3}}$$

$$(ii) \quad \left(3^{\frac{1}{5}}\right)^4$$

$$(iii) \quad \frac{7^{\frac{1}{5}}}{7^{\frac{1}{3}}}$$

$$(iv) \quad 13^{\frac{1}{5}} \cdot 17^{\frac{1}{5}}$$

How would we go about it? It turns out that we can extend the laws of exponents that we have studied earlier, even when the base is a positive real number and the exponents are rational numbers. (Later you will study that it can further to be extended when the exponents are real numbers.) But before we state these laws, and to even make sense of these laws, we need to first understand what, for example $4^{\frac{3}{2}}$ is. So, we have some work to do!

In Section 1.4, we defined $\sqrt[n]{a}$ for a real number $a > 0$ as follows:

Let $a > 0$ be a real number and n a positive integer. Then $\sqrt[n]{a} = b$, if $b^n = a$ and $b > 0$.

In the language of exponents, we define $\sqrt[n]{a} = a^{\frac{1}{n}}$. So, in particular, $\sqrt[3]{2} = 2^{\frac{1}{3}}$.

There are now two ways to look at $4^{\frac{3}{2}}$.

$$4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8$$

$$4^{\frac{3}{2}} = \left(4^3\right)^{\frac{1}{2}} = (64)^{\frac{1}{2}} = 8$$

Therefore, we have the following definition:

Let $a > 0$ be a real number. Let m and n be integers such that m and n have no common factors other than 1, and $n > 0$. Then,

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

We now have the following extended laws of exponents:

Let $a > 0$ be a real number and p and q be rational numbers. Then, we have

$$(i) \quad a^p \cdot a^q = a^{p+q}$$

$$(ii) \quad (a^p)^q = a^{pq}$$

$$(iii) \quad \frac{a^p}{a^q} = a^{p-q}$$

$$(iv) \quad a^p b^p = (ab)^p$$

You can now use these laws to answer the questions asked earlier.

Example 21 : Simplify (i) $2^{\frac{2}{3}} \cdot 2^{\frac{1}{3}}$

$$(ii) \quad \left(3^{\frac{1}{5}}\right)^4$$

$$(iii) \quad \frac{7^{\frac{1}{5}}}{7^{\frac{1}{3}}}$$

$$(iv) \quad 13^{\frac{1}{5}} \cdot 17^{\frac{1}{5}}$$

Solution :

$$(i) \quad 2^{\frac{2}{3}} \cdot 2^{\frac{1}{3}} = 2^{\left(\frac{2}{3} + \frac{1}{3}\right)} = 2^{\frac{3}{3}} = 2^1 = 2$$

$$(ii) \quad \left(3^{\frac{1}{5}}\right)^4 = 3^{\frac{4}{5}}$$

$$(iii) \quad \frac{7^{\frac{1}{5}}}{7^{\frac{1}{3}}} = 7^{\left(\frac{1}{5} - \frac{1}{3}\right)} = 7^{\frac{3-5}{15}} = 7^{\frac{-2}{15}}$$

$$(iv) \quad 13^{\frac{1}{5}} \cdot 17^{\frac{1}{5}} = (13 \times 17)^{\frac{1}{5}} = 221^{\frac{1}{5}}$$

EXERCISE 1.6

1. Find: (i) $64^{\frac{1}{2}}$

(ii) $32^{\frac{1}{5}}$

(iii) $125^{\frac{1}{3}}$

2. Find: (i) $9^{\frac{3}{2}}$

(ii) $32^{\frac{2}{5}}$

(iii) $16^{\frac{3}{4}}$

(iv) $125^{\frac{-1}{3}}$

3. Simplify: (i) $2^{\frac{2}{3}} \cdot 2^{\frac{1}{5}}$

(ii) $\left(\frac{1}{3^3}\right)^7$

(iii) $\frac{11^{\frac{1}{2}}}{11^{\frac{1}{4}}}$

(iv) $7^{\frac{1}{2}} \cdot 8^{\frac{1}{2}}$

1.7 Summary

In this chapter, you have studied the following points:

1. A number r is called a rational number, if it can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.
2. A number s is called a irrational number, if it cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.
3. The decimal expansion of a rational number is either terminating or non-terminating recurring. Moreover, a number whose decimal expansion is terminating or non-terminating recurring is rational.
4. The decimal expansion of an irrational number is non-terminating non-recurring. Moreover, a number whose decimal expansion is non-terminating non-recurring is irrational.
5. All the rational and irrational numbers make up the collection of real numbers.
6. There is a unique real number corresponding to every point on the number line. Also, corresponding to each real number, there is a unique point on the number line.
7. If r is rational and s is irrational, then $r + s$ and $r - s$ are irrational numbers, and rs and $\frac{r}{s}$ are irrational numbers, $r \neq 0$.
8. For positive real numbers a and b , the following identities hold:

(i) $\sqrt{ab} = \sqrt{a}\sqrt{b}$	(ii) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
(iii) $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$	(iv) $(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$
(v) $(\sqrt{a} + \sqrt{b})^2 = a + 2\sqrt{ab} + b$	
9. To rationalise the denominator of $\frac{1}{\sqrt{a} + b}$, we multiply this by $\frac{\sqrt{a} - b}{\sqrt{a} - b}$, where a and b are integers.
10. Let $a > 0$ be a real number and p and q be rational numbers. Then

(i) $a^p \cdot a^q = a^{p+q}$	(ii) $(a^p)^q = a^{pq}$
(iii) $\frac{a^p}{a^q} = a^{p-q}$	(iv) $a^p b^p = (ab)^p$

INTRODUCTION TO EUCLID'S GEOMETRY

2.1 Introduction

The word 'geometry' comes from the Greek words 'geo', meaning the 'earth', and 'metrein', meaning 'to measure'. Geometry appears to have originated from the need for measuring land. This branch of mathematics was studied in various forms in every ancient civilisation, be it in Egypt, Babylonia, China, India, Greece, the Incas, etc. The people of these civilisations faced several practical problems which required the development of geometry in various ways.

For example, whenever the river Nile overflowed, it wiped out the boundaries between the adjoining fields of different land owners. After such flooding, these boundaries had to be redrawn. For this purpose, the Egyptians developed a number of geometric techniques and rules for calculating simple areas and also for doing simple constructions. The knowledge of geometry was also used by them for computing volumes of granaries, and for constructing canals and pyramids. They also knew the correct formula to find the volume of a truncated pyramid (see Fig. 2.1). You know that a pyramid is a solid figure, the base of which is a triangle, or square, or some other polygon, and its side faces are triangles converging to a point at the top.

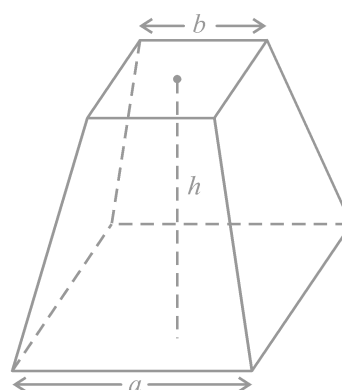


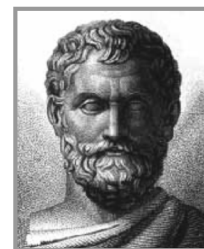
Fig. 2.1 : A Truncated Pyramid

In the Indian subcontinent, the excavations at Harappa and Mohenjo-Daro, etc. show that the Indus Valley Civilisation (about 3000 BCE) made extensive use of geometry. It was a highly organised society. The cities were highly developed and very well planned. For example, the roads were parallel to each other and there was an underground drainage system. The houses had many rooms of different types. This shows that the town dwellers were skilled in mensuration and practical arithmetic. The bricks used for constructions were kiln fired and the ratio length : breadth : thickness, of the bricks was found to be 4 : 2 : 1.

In ancient India, the *Sulbasutras* (800 BCE to 500 BCE) were the manuals of geometrical constructions. The geometry of the Vedic period originated with the construction of altars (or *vedis*) and fireplaces for performing Vedic rites. The location of the sacred fires had to be in accordance to the clearly laid down instructions about their shapes and areas, if they were to be effective instruments. Square and circular altars were used for household rituals, while altars whose shapes were combinations of rectangles, triangles and trapeziums were required for public worship. The *sriyantra* (given in the *Atharvaveda*) consists of nine interwoven isosceles triangles. These triangles are arranged in such a way that they produce 43 subsidiary triangles. Though accurate geometric methods were used for the constructions of altars, the principles behind them were not discussed.

These examples show that geometry was being developed and applied everywhere in the world. But this was happening in an unsystematic manner. What is interesting about these developments of geometry in the ancient world is that they were passed on from one generation to the next, either orally or through palm leaf messages, or by other ways. Also, we find that in some civilisations like Babylonia, geometry remained a very practical oriented discipline, as was the case in India and Rome. The geometry developed by Egyptians mainly consisted of the statements of results. There were no general rules of the procedure. In fact, Babylonians and Egyptians used geometry mostly for practical purposes and did very little to develop it as a systematic science. But in civilisations like Greece, the emphasis was on the *reasoning* behind why certain constructions work. The Greeks were interested in establishing the truth of the statements they discovered using deductive reasoning (see Appendix 1).

A Greek mathematician, Thales is credited with giving the first known proof. This proof was of the statement that a circle is bisected (i.e., cut into two equal parts) by its diameter. One of Thales' most famous pupils was Pythagoras (572 BCE), whom you have heard about. Pythagoras and his group discovered many geometric properties and developed the theory of geometry to a great extent. This process continued till 300 BCE. At that time Euclid, a teacher of mathematics at Alexandria in Egypt, collected all the known work and arranged it in his famous treatise,



Thales
(640 BCE – 546 BCE)

Fig. 2.2

called ‘Elements’. He divided the ‘Elements’ into thirteen chapters, each called a book. These books influenced the whole world’s understanding of geometry for generations to come.

In this chapter, we shall discuss Euclid’s approach to geometry and shall try to link it with the present day geometry.



Euclid (325 BCE – 265 BCE)

Fig. 2.3

2.2 Euclid’s Definitions, Axioms and Postulates

The Greek mathematicians of Euclid’s time thought of geometry as an abstract model of the world in which they lived. The notions of point, line, plane (or surface) and so on were derived from what was seen around them. From studies of the space and solids in the space around them, an abstract geometrical notion of a solid object was developed. A solid has shape, size, position, and can be moved from one place to another. Its boundaries are called **surfaces**. They separate one part of the space from another, and are said to have no thickness. The boundaries of the surfaces are **curves** or straight **lines**. These lines end in **points**.

Consider the three steps from solids to points (solids-surfaces-lines-points). In each step we lose one extension, also called a **dimension**. So, a solid has three dimensions, a surface has two, a line has one and a point has none. Euclid summarised these statements as definitions. He began his exposition by listing 23 definitions in Book 1 of the ‘Elements’. A few of them are given below :

1. A **point** is that which has no part.
2. A **line** is breadthless length.
3. The ends of a line are points.
4. A **straight line** is a line which lies evenly with the points on itself.
5. A **surface** is that which has length and breadth only.
6. The edges of a surface are lines.
7. A **plane surface** is a surface which lies evenly with the straight lines on itself.

If you carefully study these definitions, you find that some of the terms like part, breadth, length, evenly, etc. need to be further explained clearly. For example, consider his definition of a point. In this definition, ‘a part’ needs to be defined. Suppose if you define ‘a part’ to be that which occupies ‘area’, again ‘an area’ needs to be defined. So, to define one thing, you need to define many other things, and you may get a long chain of definitions without an end. For such reasons, mathematicians agree to leave

some geometric terms *undefined*. However, we do have a intuitive feeling for the geometric concept of a point than what the 'definition' above gives us. So, we represent a point as a dot, even though a dot has some dimension.

A similar problem arises in Definition 2 above, since it refers to breadth and length, neither of which has been defined. Because of this, a few terms are kept undefined while developing any course of study. So, in geometry, we *take a point, a line and a plane (in Euclid's words a plane surface) as undefined terms*. The only thing is that we can represent them intuitively, or explain them with the help of 'physical models'.

Starting with his definitions, Euclid assumed certain properties, which were not to be proved. These assumptions are actually 'obvious universal truths'. He divided them into two types: axioms and postulates. He used the term '**postulate**' for the assumptions that were specific to geometry. Common notions (often called **axioms**), on the other hand, were assumptions used throughout mathematics and not specifically linked to geometry. For details about axioms and postulates, refer to Appendix 1. Some of **Euclid's axioms**, not in his order, are given below :

- (1) Things which are equal to the same thing are equal to one another.
- (2) If equals are added to equals, the wholes are equal.
- (3) If equals are subtracted from equals, the remainders are equal.
- (4) Things which coincide with one another are equal to one another.
- (5) The whole is greater than the part.
- (6) Things which are double of the same things are equal to one another.
- (7) Things which are halves of the same things are equal to one another.

These 'common notions' refer to magnitudes of some kind. The first common notion could be applied to plane figures. For example, if an area of a triangle equals the area of a rectangle and the area of the rectangle equals that of a square, then the area of the triangle also equals the area of the square.

Magnitudes of the same kind can be compared and added, but magnitudes of different kinds cannot be compared. For example, a line cannot be added to a rectangle, nor can an angle be compared to a pentagon.

The 4th axiom given above seems to say that if two things are identical (that is, they are the same), then they are equal. In other words, everything equals itself. It is the justification of the principle of superposition. Axiom (5) gives us the definition of 'greater than'. For example, if a quantity B is a part of another quantity A, then A can be written as the sum of B and some third quantity C. Symbolically, $A > B$ means that there is some C such that $A = B + C$.

Now let us discuss **Euclid's five postulates**. They are :

Postulate 1 : *A straight line may be drawn from any one point to any other point.*

Note that this postulate tells us that at least one straight line passes through two distinct points, but it does not say that there cannot be more than one such line. However, in his work, Euclid has frequently assumed, without mentioning, that there is a *unique* line joining two distinct points. We state this result in the form of an axiom as follows:

Axiom 2.1 : *Given two distinct points, there is a unique line that passes through them.*

How many lines passing through P also pass through Q (see Fig. 2.4)? Only one, that is, the line PQ. How many lines passing through Q also pass through P? Only one, that is, the line PQ. Thus, the statement above is self-evident, and so is taken as an axiom.

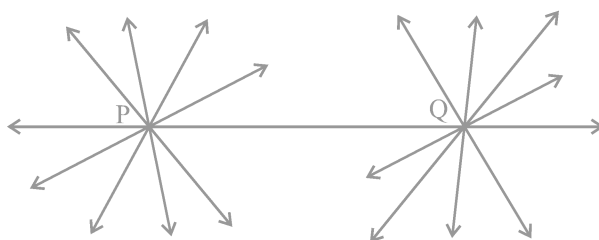


Fig. 2.4

Postulate 2 : *A terminated line can be produced indefinitely.*

Note that what we call a line segment now-a-days is what Euclid called a terminated line. So, according to the present day terms, the second postulate says that a line segment can be extended on either side to form a line (see Fig. 2.5).



Fig. 2.5

Postulate 3 : *A circle can be drawn with any centre and any radius.*

Postulate 4 : *All right angles are equal to one another.*

Postulate 5 : *If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the sum of angles is less than two right angles.*

For example, the line PQ in Fig. 2.6 falls on lines AB and CD such that the sum of the interior angles 1 and 2 is less than 180° on the left side of PQ. Therefore, the lines AB and CD will eventually intersect on the left side of PQ.

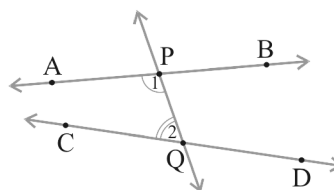


Fig. 2.6

A brief look at the five postulates brings to your notice that Postulate 5 is far more complex than any other postulate. On the other hand, Postulates 1 through 4 are so simple and obvious that these are taken as ‘self-evident truths’. However, it is not possible to prove them. So, these statements are accepted without any proof (see Appendix 1). Because of its complexity, the fifth postulate will be given more attention in the next section.

Now-a-days, ‘postulates’ and ‘axioms’ are terms that are used interchangeably and in the same sense. ‘Postulate’ is actually a verb. When we say “let us postulate”, we mean, “let us make some statement based on the observed phenomenon in the Universe”. Its truth/validity is checked afterwards. If it is true, then it is accepted as a ‘Postulate’.

A system of axioms is called **consistent** (see Appendix 1), if it is impossible to deduce from these axioms a statement that contradicts any axiom or previously proved statement. So, when any system of axioms is given, it needs to be ensured that the system is consistent.

After Euclid stated his postulates and axioms, he used them to prove other results. Then using these results, he proved some more results by applying deductive reasoning. The statements that were proved are called **propositions or theorems**. Euclid deduced 465 propositions in a logical chain using his axioms, postulates, definitions and theorems proved earlier in the chain. In the next few chapters on geometry, you will be using these axioms to prove some theorems.

Now, let us see in the following examples how Euclid used his axioms and postulates for proving some of the results:

Example 1 : If A, B and C are three points on a line, and B lies between A and C (see Fig. 2.7), then prove that $AB + BC = AC$.



Fig. 2.7

Solution : In the figure given above, AC coincides with AB + BC.

Also, Euclid's Axiom (4) says that things which coincide with one another are equal to one another. So, it can be deduced that

$$AB + BC = AC$$

Note that in this solution, it has been assumed that there is a unique line passing through two points.

Example 2 : Prove that an equilateral triangle can be constructed on any given line segment.

Solution : In the statement above, a line segment of any length is given, say AB [see Fig. 5.8(i)].

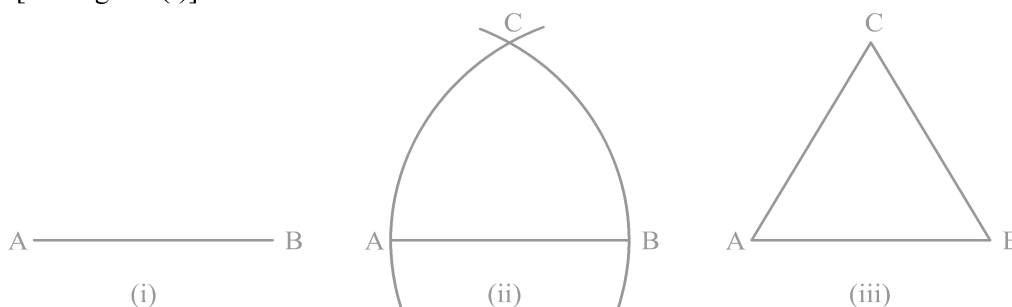


Fig. 2.8

Here, you need to do some construction. Using Euclid's Postulate 3, you can draw a circle with point A as the centre and AB as the radius [see Fig. 2.8(ii)]. Similarly, draw another circle with point B as the centre and BA as the radius. The two circles meet at a point, say C. Now, draw the line segments AC and BC to form $\triangle ABC$ [see Fig. 2.8 (iii)].

So, you have to prove that this triangle is equilateral, i.e., $AB = AC = BC$.

Now, $AB = AC$, since they are the radii of the same circle (1)

Similarly, $AB = BC$ (Radii of the same circle) (2)

From these two facts, and Euclid's axiom that things which are equal to the same thing are equal to one another, you can conclude that $AB = BC = AC$.

So, $\triangle ABC$ is an equilateral triangle.

Note that here Euclid has assumed, without mentioning anywhere, that the two circles drawn with centres A and B will meet each other at a point.

Now we prove a theorem, which is frequently used in different results: