Mathematics & Statistics

Academic Year: 2016-2017 Marks: 70
Date: July 2017

Question 1: [12]

Question 1: Select and write the correct answer from the given alternatives in each of the following sub-questions: [6]

Question 1.1.1: [3]

The inverse of the matrix $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ is ....

(A) $\frac{1}{5}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

(B) $\frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

(C) $\frac{1}{5}\begin{bmatrix} -3 & 1 \\ -2 & 1 \end{bmatrix}$

(D) $\frac{1}{5}\begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$

Solution:

(B) $\frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\therefore A = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

Question 1.1.2: [3]

If $\bar{a} = 3\hat{i} - \hat{j} + 4\hat{k}, \bar{b} = 2\hat{i} + 3\hat{j} - \hat{k}, \bar{c} = -5\hat{i} + 2\hat{j} + 3\hat{k}$ then $\bar{a}.\left(\bar{b} \times \bar{c}\right) =$

(A) 100

(B) 101

(C) 110

(D) 109

Solution:

(C) 110

$$\bar{a}.\left(\bar{b}\times\bar{c}\right)=\begin{vmatrix}3 & -1 & 4\\ 2 & 3 & -1\\ -5 & 2 & 3\end{vmatrix}$$

$$=3(9+2)+1(6-5)+4(4+15)$$

$$=33+1+76$$

$$=110$$

Question 1.1.3: If a line makes angles 90°, 135°, 45° with the X, Y, and Z axes respectively, then its direction cosines are _______. **[3]**

(A) $0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$
(B) $0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$
(C) $1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$
(D) $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

Solution:

(D) $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

Let α, β, γ be the angles made by the line with positive directions of X, Y, Z axes respectively.
α = 90°, β = 135°, γ = 45°
l = cos 90°, m = cos 135°, n = cos 45°
Now, m = cos 135° = cos(180° – 45°)

$$=-\cos 45^{\circ}=-\frac{1}{\sqrt{2}}$$

$$l=0, m=-\frac{1}{\sqrt{2}}, n=\frac{1}{\sqrt{2}}$$

Direction cosines of the line are $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

Question 1.2: Attempt any THREE of the following: **[6]**

Question 1.2.1: **[2]**

$r=\left(\hat{i}-2\hat{j}+3\hat{k}\right)+\lambda\left(2\hat{i}+\hat{j}+2\hat{k}\right)$ is parallel to the plane $r.\left(3\hat{i}-2\hat{j}+p\hat{k}\right)=10$, find the value of p.

Solution:

$\bar{b}=2\hat{i}+\hat{j}+2\hat{k}$

$\widehat{n}=3\hat{i}-2\hat{j}+p\hat{k}$

Since, line is parallel to the plane

$\bar{b}.\widehat{n}=0$

$\left(2\hat{i}+\hat{j}+2\hat{k}\right).\left(3\hat{i}-2\hat{j}+p\hat{k}\right)=0$

$(6-2+2p)=0$

$p=-2$

Question 1.2.2: If a line makes angles α, β, γ with co-ordinate axes, prove that cos 2α + cos2β + cos2γ+ 1 = 0. [2]

Solution 1: Consider cos 2α + cos2β + cos2γ+ 1

$=\left(2\cos^2\alpha-1\right)+\left(2\cos^2\beta-1\right)+\left(2\cos^2\gamma-1\right)$

$=2\left(\cos^2\alpha+\cos^2\beta+\cos^2\gamma\right)-3$

$=2(1)-3 \quad [\because \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1]$

$=-1$

$\therefore \cos 2\alpha+\cos 2\beta+\cos 2\gamma=-1$

$\therefore \cos 2\alpha+\cos 2\beta+\cos 2\gamma+1=0$

Solution 2: L.H.S: cos 2α + cos2β + cos2γ + 1

$=2\cos^2\alpha-1+2\cos^2\beta-1+2\cos^2\gamma-1+1$

$=2\left(\cos^2\alpha+\cos^2\beta+\cos^2\gamma\right)-2$

= 2 x 1 - 2

= 2- 2

= 0

= R.H.S

Question 1.2.3: Write the negations of the following statements: [2]

a. $\forall n \in N, n+7>6$
b. The kitchen is neat and tidy.

Solution:

(a). $\exists n \in N$ such that $n+7 \leq 6$

(b) The kitchen is not neat or it is not tidy.

Question 1.2.4: Find the angle between the lines whose direction ratios are 4, −3, 5 and 3, 4, 5. **[2]**

Solution: Let θ be the acute angle between the lines whose direction ratios are 4, −3, 5 and 3, 4, 5.
Then,

$$\cos\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}.\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

$$\cos\theta = \left|\frac{4(3)+(-3)(4)+5(5)}{\sqrt{4^2+(3)^2+5^2}.\sqrt{3^2+4^2+5^2}}\right|$$

$$= \left|\frac{12-12+25}{\sqrt{16+9+25}.\sqrt{9+16+25}}\right|$$

$$= \left|\frac{25}{50}\right| = \frac{1}{2}$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

The angle between the lines is $\frac{\pi}{3}$

Question 1.2.5: If $\bar{a}, \bar{b}, \bar{c}$ are position vectors of the points A, B, C respectively such that $3\bar{a}+5\bar{b}-8\bar{c}=0$, find the ratio in which A divides BC. **[2]**

Solution:

Given : $3\bar{a}+5\bar{b}-8\bar{c}=0$

$3\bar{a}=8\bar{c}-5\bar{b}$

$\bar{a}=\dfrac{8\bar{c}-5\bar{b}}{3}$

$\bar{a}=\dfrac{8\bar{c}-5\bar{b}}{8-5}$ $\qquad [\because 3=8-5]$

$A(\bar{a})$ divides BC externally in the ratio 8 : 5.

Question 2: [14]

Question 2.1 | Attempt any TWO of the following: [6]

Question 2.1.1: [3]

If $\tan^{-1}(2x)+\tan^{-1}(3x)=\dfrac{\pi}{4}$, then find the value of 'x'.

Solution:

$\tan^{-1}(2x)+\tan^{-1}(3x)=\dfrac{\pi}{4}$

$\tan^{-1}\left(\dfrac{2x+3x}{1-(2x)(3x)}\right)=\dfrac{\pi}{4}$

$\therefore \dfrac{5x}{1-6x^2}=\tan\left(\dfrac{\pi}{4}\right)$

$\dfrac{5x}{1-6x^2}=1$

$5x=1-6x^2$

$6x^2+5x-1=0$

$6x^2+6x-x-1=0$

$6x(x+1)-1(x+1)=0$

$(x+1)(6x-1)=0$

$x=-1 \text{ or } x=\dfrac{1}{6}$

But x = -1 does not satisfy $\tan^{-1}(2x)+\tan^{-1}(3x)=\dfrac{\pi}{4}$

$x=\dfrac{1}{6}$

Question 2.1.2: Write the converse, inverse and contrapositive of the following statement. "If it rains then the match will be cancelled." **[3]**

Solution: Let

p : It rains,

q : the match will be cancelled.

The symbolic form of the given statement is p → q.
Converse: q → p
i.e., If the match is cancelled then it rains.
Inverse: ~p → ~q
i.e., If it does not rain then the match will not be cancelled.
Contrapositive: ~q → ~p
i.e. If the match is not cancelled then it does not rain.

Question 2.1.3: Find p and q, if the equation $px^2 - 8xy + 3y^2 + 14x + 2y + q = 0$ represents a pair of perpendicular lines. **[3]**

Solution:

Given equation is $px^2 - 8xy + 3y^2 + 14x + 2y + q = 0$ Comparing with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we get

$a = p, h = -4, b = 3, g = 7, f = 1, c = q$

The given equation represents a pair of lines perpendicular to each other

$a + b = 0$

$p + 3 = 0$

$p = -3$

Also, the given equation represents a pair of lines

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\begin{vmatrix} -3 & -4 & 7 \\ -4 & 3 & 1 \\ 7 & 1 & q \end{vmatrix} = 0$$

$-3(3q - 1) + 4(-4q - 7) + 7(-4 - 21) = 0$

$-9q + 3 - 16q - 28 - 175 = 0$

$-9q + 3 - 16q - 28 - 175 = 0$

$-25q - 200 = 0$

$-25q = 200$

$q = -8$

$p = -3 \text{ and } q = -8$

Question 2.2 | Attempt any TWO of the following: [8]

Question 2.2.1: Find the equation of the plane passing through the intersection of the planes 3x + 2y – z + 1 = 0 and x + y + z – 2 = 0 and the point (2, 2, 1). [4]

Solution: The equation of plane passing through the intersection of the planes 3x + 2y – z + 1 = 0 and x + y + z – 2 = 0 is

$3x + 2y - z + 1) + \lambda(x + y + z - 2) = 0$ ....(1)

It passes through the point (2, 2, 1)

$(6 + 4 - 1 + 1) + \lambda(2 + 2 + 1 - 2) = 0$

$10 + 3\lambda = 0$

$\lambda = -\frac{10}{3}$

Now,

$(3x + 2y - z + 1) + \left(-\frac{10}{3}\right)(x + y + z - 2) = 0$ ....[from(1)]

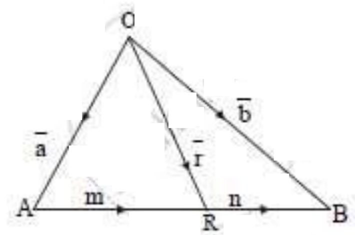
$9x + 6y - 3z + 3 - 10x - 10y - 10z + 20 = 0$

$-x - 4y - 13z + 23 = 0$

The equation of plane is $x + 4y + 13z = 23$

Question 2.2.2: Let $A(\bar{a})$ and $B(\bar{b})$ be any two points in the space and $R(\bar{r})$ be a point on the line segment AB dividing it internally in the ratio m : n, then prove that $\bar{r} = \frac{m\bar{b} + n\bar{a}}{m + n}$. Hence find the position vector of R which divides the line segment joining the points A(1, –2, 1) and B(1, 4, –2) internally in the ratio 2 : 1. [4]

Solution:



Consider a line segment AB.

Let R be any point on it such that point R divides AB internally in the ratio m : n.
bar(OA) = bara, bar(OR) = bar and bar(OB) = barb are the position vectors of points A, R, B respectively. Since point R divides AB internally in the ratio m : n,

$$\frac{AR}{RB}=\frac{m}{n}$$

$$n(AR)=m(RB)$$

$\overline{AR}$ and $\overline{RB}$ are in the same direction.

$$\therefore n\left(\overline{AR}\right)=m\left(\overline{RB}\right)$$

$$n(\bar{r}-\bar{a})=m\left(\bar{b}-\bar{r}\right)$$

$$n\bar{r}-n\bar{a}=m\bar{b}-m\bar{r}$$

$$n\bar{r}+m\bar{r}=m\bar{b}+m\bar{a}$$

$$\bar{r}(m+n)=m\bar{b}+n\bar{a}$$

$$\bar{r}=\frac{m\bar{b}+n\bar{a}}{m+n}\ \text{.........(1)}$$

This is the section formula for internal division

Let P. V. of point $A\bar{a}=\hat{i}-2\hat{j}+\hat{k}$

P. V. of point $B\bar{b}=\hat{i}+4\hat{j}-2\hat{k}$

Given, $\dfrac{m}{n}=\dfrac{2}{1}$

Now, $\bar{r}=\dfrac{2\left(\hat{i}+4\hat{j}-2\hat{k}\right)+1\left(\hat{i}-2\hat{j}+\hat{k}\right)}{2+1}$ ....from (1)

$$\bar{r}=\frac{3\hat{i}+6\hat{j}-3\hat{k}}{3}$$

P. V. of R is $\bar{r}=\hat{i}+2\hat{j}-\hat{k}$

Question 2.2.3: The angles of the ΔABC are in A.P. and b:c= $\sqrt{3}:\sqrt{2}$ then find $\angle A, \angle B, \angle C$
[4]

Solution:

$\angle A, \angle B, \angle C$ are in A.P and $b : c = \sqrt{3} : \sqrt{2}$

$\therefore 2B = A + C$

$2B = 180^\circ - B$ ....$[\because A + B + C = 180^\circ]$

$3B = 180^\circ$

$\angle B = 60^\circ$

In ΔABC by sine rule, we have

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin B}{\sin C} = \frac{b}{c}$$

$$\frac{\sin 60^\circ}{\sin c} = \frac{\sqrt{3}}{\sqrt{2}}$$

$$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{\sqrt{2}} = \sin C$$

$$\sin C = \frac{1}{\sqrt{2}}$$

$\angle C = 45^\circ$

$\angle A = 180^\circ - 60^\circ - 45^\circ = 75^\circ$

Thus, the angles of ΔABC are $\angle A = 75^\circ, \angle B = 60^\circ, \angle C = 45^\circ$

Question 3: [14]

Question 3.1: Attempt any TWO of the following: [6]

Question 3.1.1: Find the cartesian equation of the line passing through the points A(3, 4, -7) and B(6,-1, 1). [3]

Solution:

Equation of line passing through the point $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Equation of line passing through the point A(3, 4,-7) and B(6,-1,1) is

$$\frac{x-3}{6-3}=\frac{y-4}{-1-4}=\frac{z-(-7)}{1-(-7)}$$

$$\frac{x-3}{3}=\frac{y-4}{-5}=\frac{z+7}{8}$$

Question 3.1.1: Find the vector equation of a line passing through the points A(3, 4, –7) and B(6, –1, 1).

Solution: The vector equation of a line passing through the points having position vectors $\bar{a}$ and $\bar{b}$ is given

by $\bar{r}=\bar{a}+\lambda(\bar{b}-\bar{a})$

$Here, \bar{a}=3\hat{i}+4\hat{j}-7\hat{k}$ and $\bar{b}=6\hat{i}-\hat{j}+\hat{k}$

the vector equation of the line passing through A (3, 4, -7) and B (6,-1, 1) is

$$\bar{r}=\left(3\hat{i}+4\hat{j}-7\hat{k}\right)+\lambda\left[\left(6\hat{i}-\hat{j}+\hat{k}\right)-\left(3\hat{i}+4\hat{j}-7\hat{k}\right)\right]$$

$$\bar{r}=\left(3\hat{i}+4\hat{j}-7\hat{k}\right)+\lambda\left(3\hat{i}-5\hat{j}+8\hat{k}\right)$$

Question 3.1.2: Find the general solution of the equation sin 2x + sin 4x + sin 6x = 0 **[3]**

Solution:

$$(\sin 2x + \sin 6x) + \sin 4x = 0$$

$$2\sin 4x.\cos 2x + \sin 4x = 0$$

$$\sin 4x(2\cos 2x + 1) = 0$$

$$\sin 4x = 0 \text{ or } 2\cos 2x + 1 = 0$$

$$\sin 4x = 0 \text{ or } \cos 2x = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right)$$

Using $\sin x = 0 \Rightarrow x = n\pi$

$$\sin 4x = 0$$

$$4x = n\pi$$

The genral solution is x

$$x = \frac{n\pi}{4}$$

using $\cos x = \cos\alpha \Rightarrow x = 2mx \pm \alpha$

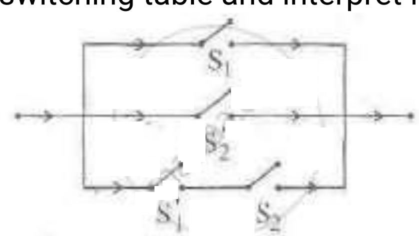
$$\cos 2x = \cos\left(\frac{2\pi}{3}\right)$$

$$2x = 2m\pi \pm \frac{2\pi}{3}$$

The genral solution is x

$x = m\pi \pm \frac{\pi}{3}$ where $m, n \in z$

Question 3.1.3: find the symbolic form of the following switching circuit, construct its switching table and interpret it. **[3]**



Solution: Let

p: The switch S_1 is closed,
q: The switch S_2 is closed.

Switching circuit is (p∨~q)∨(~p∧q)

The switching table

p	q	~p	~q	pv~q	~p∧ q	(pv~q)v(~p∧q)
1	1	0	0	1	0	1
1	0	0	1	1	0	1
0	1	1	0	0	1	1
0	0	1	1	1	0	1

From the last column of switching table we conclude that the current will always flow through the circuit.

Question 3.2: Attempt any TWO of the following: [8]

Question 3.2.1: [4]

If $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$ verify that A (adj A) = |A| I.

Solution:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{vmatrix}$$

$$= 1(0) + 1(9+2) + 2(0)$$

$$= 0 + 11 + 0$$

$$\therefore |A| = 11$$

$$A_{11} = (-1)^{1+1} M_{11} = 1\begin{vmatrix} 0 & -2 \\ 0 & 3 \end{vmatrix} = 1(0-0) = 1 \times 0 = 0$$

$$A_{12} = (-1)^{1+2} M_{12} = -1\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = -1(9+2) = -11$$

$$A_{13} = (-1)^{1+3} M_{13} = 1\begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} = 1(0-0) = 1 \times 0 = 0$$

$$A_{21} = (-1)^{2+1} M_{21} = -1\begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} = -1(-3-0) = 1 \times 0 = 3$$

$$A_{22} = (-1)^{2+2}M_{22} = 1\begin{vmatrix}1 & 2\\1 & 3\end{vmatrix} = 1(3-2) = 1$$

$$A_{23} = (-1)^{2+3}M_{23} = -1\begin{vmatrix}1 & -1\\1 & 0\end{vmatrix} = -1(0+1) = -1$$

$$A_{31} = (-1)^{3+1}M_{31} = 1\begin{vmatrix}-1 & 2\\0 & -2\end{vmatrix} = 1(2-0) = 1 \times 2 = 2$$

$$A_{32} = (-1)^{3+2}M_{32} = -1\begin{vmatrix}1 & 2\\3 & -2\end{vmatrix} = -1(-2-6) = 8$$

$$A_{33} = (-1)^{3+3}M_{33} = 1\begin{vmatrix}1 & -1\\3 & 0\end{vmatrix} = 1(0+3) = 1 \times 3 = 3$$

Hence, matrix of the co-factors is

$$\begin{bmatrix}A_{11} & A_{12} & A_{13}\\A_{21} & A_{22} & A_{23}\\A_{31} & A_{32} & A_{33}\end{bmatrix} = \begin{bmatrix}0 & -11 & 0\\3 & 1 & -1\\2 & 8 & 3\end{bmatrix} = [A_{ij}]_{3\times3}$$

$$Now, adjA = [A_{ij}]^T_{3\times3} = \begin{bmatrix}0 & 3 & 2\\-11 & 1 & 8\\0 & -1 & 3\end{bmatrix}$$

$$A(adjA) = \begin{bmatrix}1 & -1 & 2\\3 & 0 & -2\\1 & 0 & 3\end{bmatrix}\begin{bmatrix}0 & 3 & 2\\-11 & 1 & 8\\0 & -1 & 3\end{bmatrix}$$

Now, $\bar{r} = \dfrac{2\left(\hat{i} + 4\hat{j} - 2\hat{k}\right) + 1\left(\hat{i} - 2\hat{j} + \hat{k}\right)}{2+1}$ ....from (1)

$$\bar{r} = \frac{3\hat{i} + 6\hat{j} - 3\hat{k}}{3}$$

P. V. of R is $\bar{r} = \hat{i} + 2\hat{j} - \hat{k}$

Question 3.2.2: A company manufactures bicycles and tricycles each of which must be processed through machines A and B. Machine A has maximum of 120 hours available and machine B has maximum of 180 hours available. Manufacturing a bicycle requires 6 hours on machine A and 3 hours on machine B. Manufacturing a tricycle requires 4 hours on machine A and 10 hours on machine B.
If profits are Rs. 180 for a bicycle and Rs. 220 for a tricycle, formulate and solve the L.P.P. to determine the number of bicycles and tricycles that should be manufactured in order to maximize the profit. **[4]**

Solution: Let x number of bicycles and y number of tricycles be manufactured by the company.
Total profit Z = 180x + 220y
This is the objective function to be maximized.
The given information can be tabulated as shown below:

	Bicycles (x)	Tricycles (y)	Maximum availability of time (hrs)
Machine A	6	4	120
Machine B	3	10	180

The constraints are $6x + 4y \le 120$, $3x + 10y \le 180$, $x \ge 0$, $y \ge 0$

Given problem can be formulated as

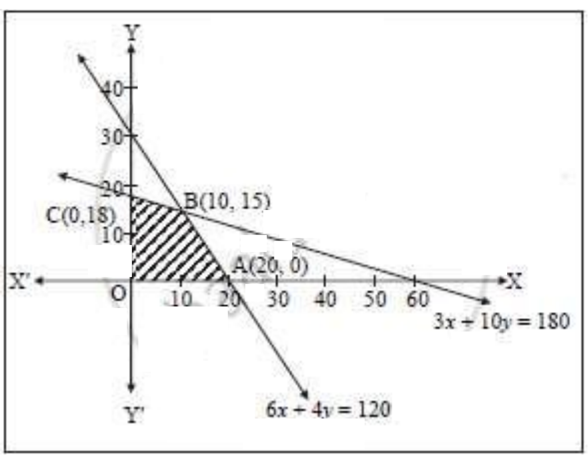
Maximize Z = 180x + 220y

Subject to, $6x + 4y \le 120$, $3x + 10y \le 180$, $x \ge 0$, $y \ge 0$.

To draw the feasible region, construct the table as follows:

Inequality	$6x + 4y \le 120$	$3x + 10y \le 180$
Corresponding equation (of line)	6x + 4y = 120	3x + 10y = 180
Intersection of line with X-axis	(20, 0)	(60, 0)
Intersection of line with Y-axis	(0, 30)	(0, 18)
Region	Origin side	Origin side

Shaded portion OABC is the feasible region, whose vertices are O=(0, 0), A =(20, 0), B and C = (0, 18)



B is the point of intersection of the lines 3x + 10y = 180 and 6x + 4y = 120.

Solving the above equations, we get
B = (10, 15) Here the objective function is,

Z = 180x + 220y

Z at O(0, 0) = 180(0) + 220(0) = 0
Z at A(20, 0) = 180(20) + 220(0) = 3600
Z at B(10, 15) = 180(10) + 220(15) = 5100
Z at C(0, 18) = 180(0) + 220(18) = 3960

Z has maximum value 5100 at B(10, 15)
Z is maximum when x = 10, y = 15

Thus, the company should manufacture 10 bicycles and 15 tricycles to gain maximum profit of Rs.5100.

Question 3.2.3: [4]

If θ is the measure of acute angle between the pair of lines given by $ax^2 + 2hxy + by^2 = 0$, then prove that $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|, a + b \neq 0$

Solution:

Let m_1 and m_2 be the slopes of the lines represented by the equation

$ax^2 + 2hxy + by^2 = 0$ ....(1)

Then their separate equation are

$y = m_1x$ and $y = m_2x$

therefore their combined equation is

$(y - m_1x)(y - m_2x) = 0$

i.e $m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0$ ....(2)

Since (1) and (2) represent the same two lines, comparing the coefficients, we get

$$\frac{m_1m_2}{a} = \frac{1}{b} = \frac{m_1 + m_2}{2h}$$

$$\therefore m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$

Thus $(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2$

$$(m_1 - m_2)^2 = \left(-\frac{2h}{b}\right)^2 - 4\left(\frac{a}{b}\right)$$

$$= (m_1 - m_2)^2 = \frac{4(h^2 - ab)}{b^2}$$

Let the angle between y = m_1x and y = m_2x be θ .

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|,$$

$$= \left|\frac{\sqrt{m_1 - m_2}^2}{1 + m_1m_2}\right|$$

$$= \left| \frac{\frac{4\sqrt{h^2-ab}}{b^2}}{1+\frac{a}{b}} \right|,$$

$$\therefore \tan\theta = \left| \frac{2\sqrt{h^2-ab}}{a+b} \right|, \quad \text{if} \quad a+b \neq 0$$

Hence to proved.

Question 3.2.3: find the acute angle between the lines [4]
$x^2 - 4xy + y^2 = 0$.

Solution:

The given pair of lines are $x^2 - 4xy + y^2 = 0$

Comparing with $ax^2 + 2hxy + by^2 = 0$, we get

$a = 1, h = -2, b = 1$

$$\tan\theta = \left| \frac{2\sqrt{h^2-ab}}{a+b} \right| = \left| \frac{2\sqrt{4-1}}{1+1} \right|$$

$$= \left| \frac{2\sqrt{3}}{2} \right|$$

$$\tan\theta = \sqrt{3}$$

$$\theta = 60^\circ \text{ or } \frac{\pi}{3}$$

Question 4: [12]

Question 4.1 | Select and write the correct answer from the given alternatives in each of the following sub-questions: [6]

Question 4.1.1: [3]

Given f (x) = 2x, x < 0
= 0, x ≥ 0
then f (x) is ______ .
discontinuous and not differentiable at x = 0
continuous and differentiable at x = 0
discontinuous and differentiable at x = 0
continuous and not differentiable at x = 0

Solution: continuous and not differentiable at x = 0

solution:

f (x) = 2x, x < 0

= 0, x ≥ 0

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 2x = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 0 = 0$$

and f(0) = 0

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

Hence, f(x) is continuous at x = 0.

Now we find left hand derivative and right hand derivative of f(0) at x = 0

Right hand derivative at x = 0

i.e $f\prime(0^+) = \lim_{h \to 0^+} \dfrac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \dfrac{0-0}{h} = 0$

Left hand derivative at x = 0

i.e $f\prime(0^-) = \lim_{h \to 0^-} \dfrac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \dfrac{h-0}{h} = 1$

$f\prime(0^+) \neq f\prime(0^-)$ Hence, f(x) is not differentiable at x = 0.

Question 4.1.2: **[2]**

If $\int_0^{\alpha} (3x^2 + 2x + 1) dx = 14$ then $\alpha =$

(A) 1

(B) 2

(C) –1

(D) –2

Solution:

(B) 2

$$\int_0^{\alpha} (3x^2 + 2x + 1) dx = 14$$

$$[x^3 + x^2 + x]_0^{\alpha} = 14$$

$$\alpha^3 + \alpha^2 + \alpha - 14 = 0$$

$$(\alpha - 2)(\alpha^2 + 3\alpha + 7) = 0$$

But $\alpha^2 + 3\alpha + 7 = 0$ does not have real roots

$$\alpha = 2$$

Question 4.1.3: [2]

The function f (x) = $x^3 - 3x^2 + 3x - 100$, x∈ R is _______.

(A) increasing
(B) decreasing
(C) increasing and decreasing
(D) neither increasing nor decreasing

Solution:

(A) increasing

$$f(x) = x^3 - 3x^2 + 3x - 100, x \in R$$

$$f\prime(x) = 3x^2 - 6x + 3$$

$$= 3\left(x^2 - 2x + 1\right)$$

$$= 3(x-1)^2$$

Since, $(x - 1)^2$ is always positive x ≠ 1

f'(x) > 0 for all x ∈ R, x ≠ 1

Hence, f (x) is an increasing function, for all x ∈ R, x ≠ 1

Question 4.2: Attempt any THREE of the following: [6]

Question 4.2.1: Differentiate 3^x w.r.t. $\log_3 x$ [2]

Solution:

Let $u = 3^x$

$$\frac{\mathrm{du}}{dx} = 3^x \log 3$$

Let $v = \log_3 x = \frac{\log x}{\log 3}$

$$\frac{\mathrm{dv}}{dx} = \frac{1}{\log 3} \frac{d}{dx}(\log x)$$

$$= \frac{1}{\log 3} \frac{1}{x} = \frac{1}{x \log 3}$$

$$\frac{\mathrm{du}}{\mathrm{dv}} = \frac{\frac{\mathrm{du}}{dx}}{\frac{\mathrm{dv}}{dx}} = \frac{3^x \log 3}{\frac{1}{x \log 3}} = 3^x . x(\log 3)\text{^}2$$

Question 4.2.2: Check whether the conditions of Rolle's theorem are satisfied by the function f (x) = (x - 1) (x - 2) (x - 3), x ∈ [1, 3] [2]

Solution:

$$f(x) = (x-1)(x-2)(x-3), \qquad x \in [1,3]$$
$$= x^3 - 6x^2 + 11x - 6$$

As f(x) is a polynomial function, it is continuous and differentiable everywhere on its domain. Thus,
a. f(x) is continuous on [1, 3]
b. f(x) is differentiable on (1, 3)

Further, f(1) = 0 and f(3) = 0
∴ f(1) = f(3)

Thus, all the conditions of Rolle's theorem are satisfied.

Question 4.2.3: [2]

Evaluate: $\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$

Solution:

$$I = \int \frac{\sqrt{\tan x}}{\sin x . \cos x} \text{ dx}$$

Dividing numerator and denominator by cosx.

$$= \int \frac{\frac{\sqrt{\tan x}}{\cos x}}{\sin x \cos x . \cos x} \text{ dx}$$

$$= \int \frac{\left[\sqrt{\tan x}\left(\frac{1}{\cos x}\right)\right]}{\frac{\sin x}{\cos x} . \cos x} \text{ dx}$$

$$= \int \frac{\sqrt{\tan x}}{\frac{\sin x}{\cos x} .}\left(\frac{1}{\cos^2 x}\right) \text{ dx}$$

$$= \int \frac{\sqrt{\tan x}}{\tan x}\left(\frac{1}{\cos^2 x}\right) \text{ dx}$$

$$= \int \frac{\sqrt{\tan x}}{\tan x}\left(\sec^2 x\right) \text{ dx}$$

Put, tan x = t

$\text{Sec}^2 x$ dx = dt

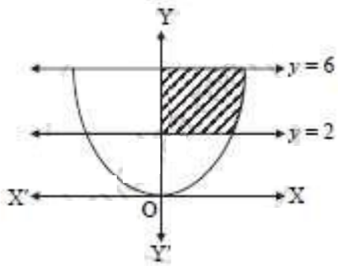
$$= \int \frac{1}{\sqrt{t}} dt$$

$= 2\tan^{\frac{1}{2}} + c$

$= 2\sqrt{\tan x} + c$

Question 4.2.4: Find the area of the region bounded by the curve $x^2 = 16y$, lines y = 2, y = 6 and Y-axis lying in the first quadrant. **[2]**

Solution:



y varies from y = 2 to y = 6.
Equation of parabola $x^2 = 16y$

$$x = 4\sqrt{y}$$

$$\text{Required area} = \int_a^b x\,dy$$

$$= \int_2^{64} \sqrt{y}\,dy$$

$$= 4\left[\frac{y^{\frac{1}{2}}}{\frac{3}{2}}\right]_2^6$$

$$= 4 \times \frac{2}{3}\left[(6)^{\frac{3}{2}} - 2^{\frac{3}{2}}\right]$$

$$= \frac{8}{3}\left[6^{\frac{3}{2}} - 2^{\frac{3}{2}}\right] \text{sq. units}$$

Question 4.2.5: Given X ~ B (n, p) **[2]**
If n = 10 and p = 0.4, find E(X) and var (X).

Solution: Given, n = 10, p = 0.4
q = 1 − p = 1 − 0.4 = 0.6
Now, E(X) = np = 10 x 0.4 = 4
Var(X) = npq = 10 x 0.4 x 0.6 = 2.4

Question 5: **[15]**

Question 5.1 | Attempt any TWO of the following: **[6]**

Question 5.1.1: **[3]**

If the function $f(x) = \dfrac{\left(5^{\sin x} - 1\right)^2}{x\log(1 + 2x)}$ for x ≠ 0 is continuous at x = 0, find f (0).

Solution:

f is continuous at x = 0.

$$f(0)=\lim_{x\to 0} f(x)$$

$$f(0)=\lim_{x\to 0}\frac{\left(5^{\sin x}-1\right)^2}{x\log(1+2x)}=\lim_{x\to 0}\frac{\frac{\left(5^{\sin x}-1\right)^2}{x^2}}{\frac{x\log(1+2x)}{x^2}}$$

$$=\lim_{x\to 0}\frac{\left(\frac{5^{\sin x}-1}{\sin x}\right)^2\cdot\frac{\sin^2 x}{x^2}}{\frac{2\log(1+2x)}{2x}}$$

$$=\frac{\left(\lim_{x\to 0}\frac{5^{\sin x}-1}{\sin x}\times.\lim_{x\to 0}\frac{\sin x}{x}\right)^2}{2\left(\frac{\lim_{x\to 0}\log(1+2x)}{2x}\right)}$$

$$f(0)=\frac{(\log 5)^2}{2}$$

Question 5.1.2: The probability mass function for X = number of major defects in a randomly selected appliance of a certain type is **[3]**

X = x	0	1	2	3	4
P(X = x)	0.08	0.15	0.45	0.27	0.05

Find the expected value and variance of X.

Solution:

$$E(X)=\sum x_i.\,P(x_i)$$

$$=0(0.08)+1(0.15)+2(0.45)+3(0.27)+4(0.05)$$

$$=0+0.15+0.9+0.81+0.2=2.06$$

$$E(X^2)=\sum x_i^2.\,P(x_i)$$

$$=0(0.08)+1^2(0.15)+2^2(0.45)+3^2(0.27)+4^2(0.05)$$

$$=0+0.15+1.8+2.43+0.8=5.18$$

$$Var(X)=E(X^2)-[E(X)]^2$$

$$=5.18-(2.06)^2$$

$$=5.18-4.2436$$

$$=0.9364$$

Question 5.1.3: Suppose that 80% of all families own a television set. If 5 families are interviewed at random, find the probability that **[3]**
a. three families own a television set.
b. at least two families own a television set.

Solution: X = Number of families who own a television set.
P = Probability of families who own a television set.

$$P = 80\% = \frac{80}{100} = \frac{4}{5}$$

$$q = 1 - p = 1 - \frac{4}{5} = \frac{1}{5}$$

Given $n = 5, X \sim B\left(5, \frac{4}{5}\right)$

The p.m.f. or X is given as

$$P(X = x) = {}^nC_x p^x q^{n-x}$$

$$= {}^nC_x p^x q^{5-x}$$

a. P(families own television set)

$$= P(X = 3)$$

$$= {}^5C_3\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)^{5-3}$$

$$= \frac{128}{625}$$

$$= 0.2048$$

b. P(At least two families own television set)

$$P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - [P(X = 0) + P(X = 1)]$$

$$= 1 - \left[{}^5C_0\left(\frac{4}{5}\right)^0\left(\frac{1}{5}\right)^5 + {}^5C_1\left(\frac{4}{5}\right)^1 \times \left(\frac{1}{5}\right)^4\right]$$

$$= 1 - \left(\frac{1}{55} + \frac{20}{55}\right)$$

$$= 1 - \left(\frac{21}{55}\right) = \frac{34}{55}.$$

Question 5.2 | Attempt any TWO of the following: **[8]**

Question 5.2.1: Find the approximate value of cos (60° 30'). **[4]**
(Given: 1° = 0.0175c, sin 60° = 0.8660)

Solution:

Let $f(X) = \cos x$

$f\prime(x) = -\sin x$

$x = 60^{\circ}30\prime = 60^{\circ} + \left(\frac{1}{2}\right)^{\circ} = a + h$

Here, $a = 60^{\circ} = \frac{\pi}{3}$

and $h = \left(\frac{1}{2}\right)^{\circ} = \frac{0.0175}{2} = 0.00875$

$f(a) = f\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} = 0.5$

$f\prime(a) = f\prime\left(\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -0.8660$

$f(a+h) \approx f(a) + hf\prime(a)$

$\cos(60^{\circ}30\prime) \approx 0.5 + (0.00875)(-0.8660)$

$\approx 0.5 - 0.0075775$

≈ 0.4924

Question 5.2.2: The rate of growth of bacteria is proportional to the number present. If, initially, there were 1000 bacteria and the number doubles in one hour, find the number of bacteria after 2½ hours. **[4]**

[Take $\sqrt{2}$ = 1.414]

Solution: Let 'N' be the number of bacteria at time 't '

$$\therefore \frac{\mathrm{dN}}{dt} \infty N$$

$$\frac{\mathrm{dN}}{dt} = kN$$

$$\frac{\mathrm{dN}}{N} = kdt$$

Integrating on both sides, we get

$$\int \frac{\mathrm{dN}}{N} = k \int dt$$

$$\log N = kt + c$$

$$when t = 0, N = 1000$$

$$c = \log 1000$$

$$\log N = kt + \log 1000$$

$$\log\left(\frac{N}{1000}\right) = kt$$

$N = 1000e^{kt}$ ....(1)

when t = 1 hour, N = 2000

$$e^{k} = 2$$

$N = 1000 \times (2)^{t}$ ....from (1)

when $t = 2\frac{1}{2}$ hours, we get

$$N = 1000 \times (2)^{\frac{5}{2}}$$

$$= 1000 \times 4 \times \sqrt{2} = 4000 \times 1.414$$

$$N = 5656$$

Number of bacteria present after $2\frac{1}{2}$ hours is 5656

Question 5.2.3: **[4]**

Prove that : $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$, if f (x) is an even function.
= 0, if f (x) is an odd function.

Solution 1:

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

$$\int_{a}^{a} f(x)dx = I + \int_{0}^{a} f(x)dx$$

$$Now I = \int_{-a}^{0} f(x)dx$$

Put x=-t

dx = - dt

When x = -a, t = a and when x = 0, t = 0

$$I = \int_{a}^{0} f(-t)(-dt)$$

$$= -\int_{a}^{0} f(-t)dt$$

$$= \int_{0}^{a} f(-t)dt.......... \left[\because \int_{a}^{b} f(x)dx = -\int_{b}^{a} f(x)dx\right]$$

$$= \int_{0}^{a} f(-x)dt.......... \left[\because \int_{a}^{b} f(x)dx = -\int_{b}^{a} f(t)dx\right]$$

Equation (i) becomes

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(-x)dx + \int_{0}^{a} f(x)dx$$

$$= \int_{0}^{a} [f(-x) + f(x)]dx.......(ii)$$

case 1: If f(x) is an even function, then f(-x) = f(x).

Thus, equation (ii) becomes

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} [f(x) + f(x)]dx = 2\int_{0}^{a} f(x)dx$$

Case 2: If f(x) is an odd function, then f(-x) = -f(x).

Thus, equation (ii) becomes

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} [-f(x) + f(x)]dx = 0$$

Solution 2: We shall use the following results :

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \qquad ....(1)$$

$$\int_a^b f(x)dx = \int_a^b f(t)\,dt \qquad ....(2)$$

If c is between a and b, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \qquad ....(3)$$

Since 0 lies between -a and a, by (3), we have,

$$\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x) = I_1 + I_2 \qquad ....(\text{Say})$$

In I_1, put $x = -t$. Then $dx = -dt$

When $x = -a, -t = -a$ $\quad \therefore t = a$

When $x = 0, -t = 0$ $\quad \therefore t = 0$

$$\therefore \int_{-a}^0 f(x)\,dx = \int_a^0 f(-t)(-dt) = \int_a^0 f(-t)\,dt$$

$$= \int_0^a f(-t)\,dt \qquad ....[By\ (1)]$$

$$= \int_0^a f(-x)\,dx \qquad ....[By(2)]$$

$$\therefore \int_{-a}^a f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx$$

(i) If f is an even function, then $f(-x) = f(x)$ $\therefore$ in this case,

$$\int_{-a}^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)dx = 2\int_0^a f(x)\,dx$$

(ii) If f is an odd function, then $f(-x) = -f(x)$ $\therefore$ in this case,

$$\int_{-a}^a f(x)\,dx = \int_0^a -f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 0.$$

Question 6: [12]

Question 6.1 | Attempt any TWO of the following [6]

Question 6.1.1: [3]

If f (x) is continuous on [–4, 2] defined as

f (x) = 6b – 3ax, for -4 ≤ x < –2

= 4x + 1, for –2 ≤ x ≤ 2

Show that a + b = $-\frac{7}{6}$

Solution: Since f is continuous on [-4, 2],

f is continuous on x = -2

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x)$$

$$\lim_{x \to -2^-} 6b - 3ax = \lim_{x \to -2^+} 4x + 1$$

$$6b - 3a(-2) = 4(-2) + 1$$

$$6b + 6a = -7$$

$$(a + b) = -\frac{7}{6}$$

Question 6.1.2: [3]

If u and v are two functions of x then prove that

$$\int uv dx = u \int v dx - \int \left[d\frac{u}{dx} \int v dx \right] dx$$

Hence evaluate, $\int xe^x dx$

Solution:

Let $\int v dx = w \ldots..(1)$

$then \quad \frac{dw}{dx} = v \ldots..(2)$

$$Now \frac{d}{dx}(u, w) = u.\frac{d}{dx}(w) + w\frac{d}{dx}(u)$$

$$= u.v + w\frac{du}{dx} \ldots\ldots. \text{from}(2)$$

By definition of integration.

$$u.w = \int \left[u.v + w\frac{du}{dx}\right] dx$$

$$= \int u.vdx + \int w.\frac{du}{dx} dx$$

$$\int u.vdx = u.w - \int w\frac{du}{dx} dx$$

$$= u\int vdx - \int \left[\frac{du}{dx}\int v.dx\right] dx$$

[next section only required for question 2]

Hence, $\int xe^x dx = x.\int e^x dx - \int \left[\frac{d}{dx}x.\int e^x dx\right] dx$

$$= xe^x - \int 1 \times e^x dx$$

$$= xe^x - e^x + c$$

Question 6.1.3: Probability distribution of X is given by **[3]**

X = x	1	2	3	4
P(X = x)	0.1	0.3	0.4	0.2

Find P(X ≥ 2) and obtain cumulative distribution function of X

Solution: By definition cumulative distribution function at x is

$P(x \geq 2) = 0.3 + 0.4 + 0.2 = 0.9$

$f(x_i) = P_1 + P_2 + P_3 + \ldots\ldots. + P_i$ where, i = 1, ..., x

Thus $f(x_1) = P_1 = 0.1$

$f(x_2) = P_1 = 0.1$

$f(x_2) = P_1 + P_2 = 0.1 + 0.3 = 0.4$

$f(x_3) = P_1 + P_2 + P_3 = 0.1 + 0.3 + 0.4 = 0.8$

$f(x_4) = P_1 + P_2 + P_3 + P_4 = 0.1 + 0.3 + 0.4 + 0.2 = 1$

$\therefore f(x_4) = \sum_{i=1}^{4} P_i = 1$

X = x	1	2	3	4
P(X = x)	0.1	0.4	0.8	1

Question 6.2 | Attempt any TWO of the following **[8]**

Question 6.2.1: If y = f (x) is a differentiable function of x such that inverse function x = f $^{-1}$(y) exists, then prove that x is a differentiable function of y and **[4]**

$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$, Where $\frac{dy}{dx} \neq 0$

Hence if $y = \sin^{-1} x, -1 \leq x \leq 1, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

then show that $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$, where $|x| < 1$

Solution: 'y' is a differentiable function of 'x'.
Let there be a small change δx in the value of 'x'.
Correspondingly, there should be a small change δy in the value of 'y'.

As $\delta x \to 0, \delta y \to 0$

Consider $\frac{\delta x}{\delta y} \times \frac{\delta y}{\delta x} = 1$

$$\frac{\delta x}{\delta y} = \frac{1}{\frac{\delta y}{\delta x}}, \frac{\delta y}{\delta x} \neq 0$$

Taking $\lim_{\delta x \to 0}$ on both sides, we get

$$\lim_{\delta x \to 0}\left(\frac{\delta x}{\delta y}\right) = \frac{1}{\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)}$$

Since 'y' is a differentiable function of 'x',

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)$$

As $\delta x \to 0, \delta y \to 0$

$$\lim_{\delta y \to 0}\left(\frac{\delta x}{\delta y}\right) = \frac{1}{\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)} \quad \text{..................(i)}$$

limits on R.H.S. of (i) exist and are finite.

Hence, limits on L.H.S. of (i) also should exist and be finite.

$\lim_{\delta y \to 0}\left(\frac{\delta x}{\delta y}\right) = \frac{dx}{dy}$ exists and its finite

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}, \frac{dy}{dx} \neq 0$$

$y=\sin^{-1}x, -1\le x\le 1, -\frac{\pi}{2}\le y\le \frac{\pi}{2}$

$x=\sin y$

Differentiating w.r.t. y, we get

$\frac{dx}{dy}=\cos y$

$\frac{dy}{dx}=\frac{1}{\cos y}$

$\frac{dy}{dx}=\frac{1}{\pm\sqrt{1-\sin^2 y}}$

$\frac{dy}{dx}=\frac{1}{\pm\sqrt{1-x^2}}$

since $-\frac{\pi}{2}\le y\le \frac{\pi}{2}$, y lies in I or IV quadrant.

cosy is positive

$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}, |x|<1$

Question 6.2.2: **[4]**

Solve the differential equation $\frac{dy}{dx}-y=e^x$

Solution:

$\frac{dy}{dx}-y=e^x$

The given equation is of the form $\frac{dy}{dx}+Py=Q$

Where, $P=-1$ and $Q=e^x$

$I.F=e^{\int \mathrm{p}dx}=e^{\int -1dx}=e^{-x}$

Solution of the given equation is

$y(I.F)=\int Q(I.F)dx+c$

$$y.e^{-x} = \int e^{x}.e^{-x}dx + c$$

$$ye^{-x} = x + c$$

we get c = 1

$$y.e^{-x} = x + 1$$

$y = (x+1)e^{x}$ is a particular solution of D.E.

Question 6.2.3: [3]

Evaluate: $\int \frac{8}{(x+2)(x^2+4)}dx$

Solution:

Let $I = \int \frac{8}{(x+2)(x^2+4)}dx$

Let $\frac{8}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$

$$8 = A(x^2+4) + (Bx+C)(x+2)$$

$$8 = A(x^2+4) + Bx^2 + 2Bx + Cx + 2C$$

$$8 = (A+B)x^2 + (2B+c)x + (4A+2C)$$

Comparing the coefficients of x^2 , x and the constant term, we get

A + B = 0, 2B + C = 0 and 4A + 2C = 8

On solving these equations, we get

A = 1, B = −1, C = 2

$$\frac{8}{(x+2)(x^2+4)} = \frac{1}{x+2} + \frac{-x+2}{x^2+4}$$

$$I = \int\left[\frac{1}{x+2} + \frac{-x+2}{x^2+4}\right]dx$$

$$= \int \frac{1}{x+2}dx - \frac{1}{2}\int \frac{2x}{x^2+4}dx + 2\int \frac{1}{x^2+2^2}dx$$

$$= \log|x+2| - \frac{1}{2}\log|x^2+4| + \tan^{-1}\left(\frac{x}{2}\right) + c$$

$$= \log\left|\frac{x+2}{\sqrt{x^2+4}}\right| + \tan^{-1}\left(\frac{x}{2}\right) + c$$