Equation Of The Gas State Processes (Part - 1)

Q. 1. A vessel of volume V = 30 l contains ideal gas at the temperature 0°C. After a portion of the gas has been let out, the pressure in the vessel decreased by Δp = 0.78 atm (the temperature remaining constant). Find the mass of the released gas. The gas density under the normal conditions p = 1.3 g/I.

Solution. 1. Let m_1 and m_2 be the masses of the gas in the vessel before and after the gas is released. Hence mass of the gas released,

$$\Delta m = m_1 - m_2$$

Now from ideal gas equation

$$p_1 V = m_1 \frac{R}{M} T_0 \text{ and } p_2 V = m_2 \frac{R}{M} T_0$$

As V and T are same before and after the release of the gas.

So, $(p_1 - p_2) V = (m_1 - m_2) \frac{R}{M} T_0 = \Delta m \frac{R}{M} T_0$

Or, $\Delta m = \frac{(p_1 - p_2) V M}{R T_0} = \frac{\Delta p V M}{R T_0}$ (1)

We also know $p = \rho \frac{R}{M} T$ so, $\frac{M}{R T_0} = \frac{\rho}{p_0}$ (2)

(Where p_0 = standard atmospheric pressure and $T_0 = 273$ K)

From Eqs. (1) And (2) we get

$$\Delta m = \rho V \frac{\Delta p}{p_0} = 1{\cdot}3 \times 30 \times \frac{0{\cdot}78}{1} = 30 \text{ g}$$

Q. 2. Two identical vessels are connected by a tube with a valve letting the gas pass from one vessel into the other if the pressure difference $\Delta p \geqslant 1.10$ atm. Initially there was a vacuum in one vessel while the other contained ideal gas at a temperature t_1 = 27°C and pressure p_1 = 1.00 atm. Then both vessels were heated to a temperature t_2 =107°C. Up to what value will the pressure in the first vessel (which had vacuum initially) increase?

Solution. 2. Let m_1 be the mass of the gas enclosed.

Then, $p_1 V = \nu_1 R T_1$

When heated, some gas, passes into the evacuated vessel till pressure difference

becomes Δp. Let p'_1 and p'_2 be the pressure on the two sides of the valve.

Then $p'_1 V = \nu'_1 R T_2$ and $p'_2 V = \nu'_2 R T_2 = (\nu_1 - \nu'_1) R T_2$

$$p'_2 V = \left(\frac{p_1 V}{R T_1} - \frac{p'_1 V}{R T_2}\right) \text{ or } p'_2 = \left(\frac{p_1}{T_1} - \frac{p'_1}{T_2}\right) T_2$$

But, $p'_1 - p'_2 = \Delta p$

So, $$p'_2 = \left(\frac{p_1}{T_1} - \frac{p'_2 + \Delta p}{T_2}\right) T_2$$

$$= \frac{p_1 T_2}{T_1} - p'_2 - \Delta p$$

Or, $$p_2' = \frac{1}{2}\left(\frac{p_1 T_2}{T_1} - \Delta p\right) = 0{\cdot}08 \text{ atm}$$

Q. 3. A vessel of volume V = 20 l contains a mixture of hydrogen and helium at a temperature t = 20°C and pressure p = 2.0 atm. The mass of the mixture is equal to m = 5.0 g. Find the ratio of the mass of hydrogen to that of helium in the given mixture.

Solution. 3. Let the mixture contain v_1 and v_2 moles of H_2 and H_e respectively. If molecular weights of H_2 and H_e are M_1 and M_2, then respective masses in the mixture are equal to

$m_1 = v_1 M_1$ and $m_2 = v_2 M_2$

Therefore, for the total mass of the mixture we get,

$m = m_1 + m_2$ or $m = v_1 M_1 + v_2 M_2$ (1)

Also, if v is the total number of moles of the mixture in the vessels, then we know,

$V = V_1 + V_2$ (2)

Solving (1) and (2) for v_1 and v_2, we get,

$$v_1 = \frac{(v M_2 - m)}{M_2 - M_1}, \quad v_2 = \frac{m - v M_1}{M_2 - M_1}$$

Therefore, we get $m_1 = M_1 \cdot \frac{(v M_2 - m)}{M_2 - M_1}$ and $m_2 = M_2 \frac{(m - v M_1)}{M_2 - M_1}$

Or, $\frac{m_1}{m_2} = \frac{M_1}{M_2} \frac{(v M_2 - m)}{(m - v M_1)}$

One can also express the above result in terms of the effective molecular weight M of the mixture, defined as,

$$M = \frac{m}{v} = m \frac{RT}{pV}$$

Thus, $\frac{m_1}{m_2} = \frac{M_1}{M_2} \cdot \frac{M_2 - M}{M - M_1} = \frac{1 - M/M_2}{M/M_1 - 1}$

Using the data and table, we get:

$$M = 3{\cdot}0 \text{ g and, } \frac{m_1}{m_2} = 0{\cdot}50$$

Q. 4. A vessel contains a mixture of nitrogen (m_1 = 7.0 g) and carbon dioxide (m_2 = 11 g) at a temperature T = 290 K and pressure p_0, = 1.0 atm. Find the density of this mixture, assuming the gases to be ideal.

Solution. 4. We know, for the mixture, N_2 and CO_2 (being regarded as ideal gases, their mixture too behaves like an ideal gas)

$$pV = vRT, \quad \text{so} \quad p_0 V = vRT$$

Where, v is the total number of moles of the gases (mixture) present and V is the volume of the vessel. If v_1 and v_2 are number of moles of N_2 and CO_2 respectively present in the mixture, then

$v = v_1 + v_2$

Now number of moles of N_2 and CO_2 is, by definition, given by

$$v_1 = \frac{m_1}{M_1} \text{ and, } v_2 = \frac{m_2}{M_2}$$

Where, m_1 is the mass of N_2 (Molecular weight = M_1) in the mixture and m_2 is the mass of CO_2 (Molecular weight = M_2) in the mixture.

Therefore density of the mixture is given by

$$\rho = \frac{m_1 + m_2}{V} = \frac{m_1 + m_2}{(vRT/P_0)}$$

$$= \frac{p_0}{RT} \cdot \frac{m_1 + m_2}{v_1 + v_2} = \frac{p_0 (m_1 + m_2) M_1 M_2}{RT (m_1 M_2 + m_2 M_1)}$$

$$= 1{\cdot}5 \text{ kg/m}^3 \text{ on substitution}$$

Q. 5. A vessel of volume V = 7.5 1 contains a mixture of ideal gases at a temperature T = 300 K: v_1 = 0.10 mole of oxygen, v_2 = 0.20 mole of nitrogen, and v_3 = 0.30 mole of carbon dioxide. Assuming the gases to be ideal, find:
(a) the pressure of the mixture;
(b) the mean molar mass M of the given mixture which enters its equation of state pV = (m/M) RT, where m is the mass of the mixture.

Solution. 5. (a) The mixture contains v_1, v_2 and v_3 moles of O_2 > N_2 and CO_2 respectively. Then the total number of moles of the mixture

$v = v_1 + v_2 + v_3$

We know, ideal gas equation for the mixture

$$pV = vRT \quad \text{or} \quad p = \frac{vRT}{V}$$

Or,

$$p = \frac{(v_1 + v_2 + v_3) RT}{V} = 1{\cdot}968 \text{ atm on substitution}$$

(b) M ass of oxygen (O_2) present in the mixture: $m_1 = v_1 M_1$

Mass of nitrogen (N_2) present in the mixture: $m_2 = v_2M_2$

Mass of carbon dioxide (CO_2) present in the mixture: $m_3 = v_3M_3$

So, mass of the mixture

$$m = m_1 + m_2 + m_3 = \nu_1 M_1 + \nu_2 M_2 + \nu_3 M_3$$

Molecular mass of the mixture $: M = \frac{\text{mass of the mixture}}{\text{total number of moles}}$

$$= \frac{\nu_1 M_1 + \nu_2 M_2 + \nu_3 M_3}{\nu_1 + \nu_2 + \nu_3} = 36{\cdot}7 \text{ g/mol. on substitution}$$

Q. 6. A vertical cylinder closed from both ends is equipped with an easily moving piston dividing the volume into two parts, each containing one mole of air. In equilibrium at T_0 = 300 K the volume of the upper part is η = 4.0 times greater than that of the lower part. At what temperature will the ratio of these volumes be equal η' = 3.0?

Solution. 6. Let p_1 and p_2 be the pressure in the upper and lower part of the cylinder respectively at temperature T_0. At the equilibrium position for the piston:

$$p_1 S + mg = p_2 S \quad \text{or,} \quad p_1 + \frac{mg}{S} = p_2$$ (m is the mass of the piston.)

But $p_1 = \frac{R T_0}{\eta V_0}$ (where V_0 is the initial volume of the lower part)

So, $$\frac{RT_0}{\eta V_0} + \frac{mg}{S} = \frac{RT_0}{V_0} \quad \text{or,} \quad \frac{mg}{S} = \frac{RT_0}{V_0}\left(1 - \frac{1}{\eta}\right) \quad (1)$$

Let T be the sought temperature and at this temperature the volume of the lower part becomes V, then according to the problem the volume of the upper part becomes η'V'

Hence, $$\frac{mg}{S} = \frac{RT}{V'}\left(1 - \frac{1}{\eta'}\right) \quad (2)$$

From (1) and (2).

$$\frac{RT_0}{V_0}\left(1-\frac{1}{\eta}\right)=\frac{RT'}{V'}\left(1-\frac{1}{\eta'}\right) \quad \text{or,} \quad T'=\frac{T_0\left(1-\frac{1}{\eta}\right)V'}{V_0\left(1-\frac{1}{\eta'}\right)}$$

As, the total volume must be constant,

$$V_0(1+\eta)=V'(1+\eta') \quad \text{or,} \quad V'=\frac{V_0(1+\eta)}{(1+\eta')}$$

Putting the value of V in Eq. (3), we get

$$T'=\frac{T_0\left(1-\frac{1}{\eta}\right)V_0\frac{(1+\eta)}{(1+\eta')}}{V_0\left(1-\frac{1}{\eta'}\right)}$$

$$=\frac{T_0(\eta^2-1)\eta'}{(\eta'^2-1)\eta}=0{\cdot}42\text{ k K}$$

Q. 7. A vessel of volume V is evacuated by means of a piston air pump. One piston stroke captures the volume ΔV. How many strokes are needed to reduce the pressure in the vessel η times? The process is assumed to be isothermal, and the gas ideal.

Solution. 7. Let P_1 be the density after the first stroke. The mass remains constant

$$V\rho=(V+\Delta V)\rho_1, \quad \text{or,} \quad \rho_1=\frac{V\rho}{(V+\Delta V)}$$

Similarly, if p_2 is the density after second stroke

$$V\rho_1=(V+\Delta V)\rho_2 \quad \text{or,} \quad \rho_2=\left(\frac{V}{V+\Delta V}\right)\rho_1=\left(\frac{V}{V+\Delta V}\right)^2\rho_0$$

In this way after nth stroke.

$$\rho_n=\left(\frac{V}{V+\Delta V}\right)^n\rho_0$$

Since pressure α density,

$$p_n=\left(\frac{V}{V+\Delta V}\right)^n p_0$$ (Because temperature is constant)

It is required by $\frac{p_n}{p_0}$ to be $\frac{1}{\eta}$

So, $\frac{1}{\eta} = \left(\frac{V}{V+\Delta V}\right)^n$ or, $\eta = \left(\frac{V+\Delta V}{V}\right)^n$

Hence $n = \frac{\ln \eta}{\ln\left(1+\frac{\Delta V}{V}\right)}$

Q. 8. Find the pressure of air in a vessel being evacuated as a function of evacuation time t. The vessel volume is V, the initial pressure is P_0. The process is assumed to be isothermal, and the evacuation rate equal to C and independent of pressure.
Note. The evacuation rate is the gas volume being evacuated per unit time, with that volume being measured under the gas pressure attained by that moment.

Solution. 8. From the ideal gas equation $p = \frac{m}{M}\frac{RT}{V}$

$$\frac{dp}{dt} = \frac{RT}{MV}\frac{dm}{dt} \qquad (1)$$

In each stroke, volume v of the gas is ejected, where v is given by

$$v = \frac{V}{m_N}\left[m_{N-1} - m_N\right]$$

In case of continuous ejection, if (m_{N-1}) corresponds to mass of gas in the vessel at time t, then m_N is the mass at time t + Δt, where Δt, is the time in which volume v of the gas has come out. The rate of evacuation is therefore $\frac{v}{\Delta t}$ i.e.

$$C = \frac{v}{\Delta t} = -\frac{V}{m(t+\Delta t)}\cdot\frac{m(t+\Delta t)-m(t)}{\Delta t}$$

In the limit Δt - 0, we get

$$C = \frac{V}{m}\frac{dm}{dt} \qquad (2)$$

From (1) and (2)

$$\frac{dp}{dt} = -\frac{C}{V}\frac{mRT}{MV} = -\frac{C}{V}p \quad \text{or} \quad \frac{dp}{p} = -\frac{C}{V}dt$$

Integrating $\int_{p}^{p_0} \frac{dp}{p} = -\frac{C}{V}\int_{t}^{0} dt \quad \text{or} \quad \ln\frac{p}{p_0} = -\frac{C}{V}t$

Thus $p = p_0 e^{-Ct/V}$

Q. 9. A chamber of volume V = 87 l is evacuated by a pump whose evacuation rate (see Note to the foregoing problem) equals C = 10 1/s. How soon will the pressure in the chamber decrease by η = 1000 times?

Solution. 9. Let p be the instantaneous density, then instantaneous mass = V_p. In a short interval dt the volume is increased by Cdt.

So, $V\rho = (V + Cdt)(\rho + d\rho)$

(Because mass remains constant in a short interval dt)

So, $\frac{d\rho}{\rho} = -\frac{C}{V}dt$

Since pressure α density $\frac{dp}{p} = -\frac{C}{V}dt$

Or $\int_{p_1}^{p_2} -\frac{dp}{p} = \frac{C}{V}t,$

Or $t = \frac{V}{C}\ln\frac{p_1}{p_2} = \frac{V}{C}\ln\frac{1}{\eta}\ 1{\cdot}0 \text{ min}$

Q. 10. A smooth vertical tube having two different sections is open from both ends and equipped with two pistons of different areas (Fig. 2.1). Each piston slides within a respective tube section. One mole of ideal gas is enclosed between the pistons tied with a non-stretchable thread. The cross-sectional area of the upper piston is ΔS = 10 cm^2 greater than that of the lower one. The combined mass of the

two pistons is equal to m = 5.0 kg. The outside air pressure is P_0 = 1.0 atm. By how many kelvins must the gas between the pistons be heated to shift the pistons through l = 5.0 cm?

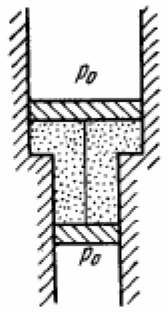


Fig. 2.1.

Solution. 10. The physical system consists of one mole of gas confined in the smooth vertical tube. Let m_1 and m_2 be the masses of upper and lower pistons and S_1 and S_2 are their respective areas.

For the lower piston

$$p S_2 + m_2 g = p_0 S_2 + T,$$

Or, $T = (p - p_0) S_2 + m_2 g$ (1)

Similarly for the upper piston

$$p_0 S_1 + T + m_1 g = p S_1 ,$$

Or, $T = (p - p_0) S_1 - m_1 g$ (2)

From (1) and (2)

$$(p - p_0)(S_1 - S_2) = (m_1 + m_2) g$$

Or, $(p - p_0) \Delta S = mg$

So, $p = \frac{mg}{\Delta S} + p_0 = \text{constant}$

From the gas law, p V = v R T

$p \Delta V = \nu R \Delta T$ (Because p is constant)

So, $\left(p_0 + \frac{mg}{\Delta S}\right)\Delta S\, l = R\,\Delta T,$

Hence, $\Delta T = \frac{1}{R}(p_0\,\Delta S + mg)\, l = 0{\cdot}9\ \text{K}$

Q. 11. Find the maximum attainable temperature of ideal gas in each of the following processes:

(a) $p = p_0 - \alpha V^2$; (b) $p = p_0 e^{-\beta V}$,

Where p_0,,α and β are positive constants, and V is the volume of one mole of gas.

Solution. 11. (a) $p = p_0 - \alpha V^2 = p_0 - \alpha\left(\frac{RT}{p}\right)^2$

(as, $V = RT/p$ for one mole of gas)

Thus, $T = \frac{1}{R\sqrt{\alpha}}\, p\sqrt{p_0 - p} = \frac{1}{R\sqrt{\alpha}}\sqrt{p_0 p^2 - p^3}$ (1)

For T_{max}, $\frac{d}{dp}\left(p_0 p^2 - p^3\right)$ must be zero

Which yields, $p = \frac{2}{3}p_0$ (2)

Hence, $T_{max} = \frac{1}{R\sqrt{\alpha}}\cdot\frac{2}{3}p_0\sqrt{p_0 - \frac{2}{3}p_0} = \frac{2}{3}\left(\frac{p_0}{R}\right)\sqrt{\frac{p_0}{3\alpha}}$

(b) $p = p_0 e^{-\beta V} = p_0 e^{-\beta RT/p}$

So $\frac{\beta RT}{p} = \ln\frac{p_o}{p}$, and $T = \frac{p}{\beta R}\ln\frac{p_0}{p}$ (1)

For T_{max} the condition is $\frac{dT}{dp} = 0$, which yields

$p = \frac{p_0}{e}$

Hence using this value of p in Eq. (1), we get

$$T_{max} = \frac{p_o}{e\beta R}$$

Q. 12. Find the minimum attainable pressure of ideal gas in the process T = T_0 + α V^2, where T_0 and α are positive constants, and V is the volume of one mole of gas. Draw the approximate p vs V plot of this process.

Solution. 12. $T = T_0 + \alpha V^2 = T_0 + \alpha \frac{R^2 T^2}{p^2}$

(as, V = RT/p for one mole of gas)

So, $p = \sqrt{\alpha}\, RT\, (T - T_0)^{1/2}$ (1)

For p_{min}, $\frac{dp}{dT} = 0$, which gives

$T = 2\,T_0$ (2)

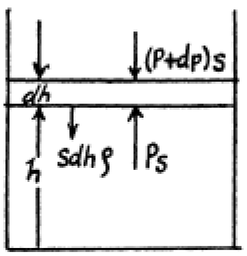
From (1) and (2), we get,

$$p_{min} = \sqrt{\alpha}\, R\, 2T_0\, (2T_0 - T_0)^{-1/2} = 2R\sqrt{\alpha T_0}$$

Equation Of The Gas State Processes (Part - 2)

Q. 13. A tall cylindrical vessel with gaseous nitrogen is located in a uniform gravitational field in which the free-fall acceleration is equal to g. The temperature of the nitrogen varies along the height h so that its density is the same throughout the volume. Find the temperature gradient dT/dh.

Solution. 13. Consider a thin layer at a height h and thickness dh. Let p and dp+p be the pressure on the two sides of the layer. The mass of the layer is Sdhp. Equating vertical downward force to the upward force acting on the layer.



So, $\frac{dp}{dh} = -\rho g$ (1)

But, $p = \frac{\rho}{M} RT$, we have $dp = \frac{\rho R}{M} dT$,

or, $-\frac{\rho R}{M} dT = \rho g \, dh$

So, $\frac{dT}{dh} = -\frac{gM}{R} = -34 \text{ K/km}$

That means, temperature of air drops by 34°C at a height of 1 km above bottom.

Q. 14. Suppose the pressure p and the density p of air are related as p/p^n = const regardless of height (n is a constant here). Find the corresponding temperature gradient.

Solution. 14. We have, $\frac{dp}{dh} = -\rho g$ (See 2.13) (1)

But, from $p = C\rho^n$ (where C is, a const) $\frac{dp}{d\rho} = Cn\,\rho^{n-1}$ (2)

We have from gas low $p = \rho \frac{R}{M} T$, so using (2)

$$C\rho^n = \rho \frac{R}{M} \cdot T, \quad \text{or} \quad T = \frac{M}{R} C \rho^{n-1}$$

Thus, $\frac{dT}{d\rho} = \frac{M}{R} \cdot C(n-1)\rho^{n-2}$

But, $\frac{dT}{dh} = \frac{dT}{d\rho} \cdot \frac{d\rho}{dp} \cdot \frac{dp}{dh}$

So, $\frac{dT}{dh} = \frac{M}{R} C(n-1)\rho^{n-2} \frac{1}{Cn\rho^{n-1}}(-\rho g) = \frac{-Mg(n-1)}{nR}$

Q. 15. Let us assume that air is under standard conditions close to the Earth's surface. Presuming that the temperature and the molar mass of air are independent of height, find the air pressure at the height 5.0 km over the surface and in a mine at the depth 5.0 km below the surface.

Solution. 15. We have, dp = - p g dh and from gas law $\rho = \frac{M}{RT} p$

$$\frac{dp}{p} = -\frac{Mg}{RT} dh$$

Integrating, we get

Or, $\int_{p_0}^{p} \frac{dp}{p} = -\frac{Mg}{RT} \int_0^h dh \quad \text{or,} \quad \ln \frac{p}{p_0} = -\frac{Mg}{RT} h,$

(Where p_0 is the pressure at the surface of the Earth.)

$$p = p_0 e^{-Mgh/RT},$$

[Under standard condition, $p_0 = 1 \text{ atm}, \; T = 273 \text{ K}$

Pressure at a height of 5 atm $= 1 \times e^{-28 \times 9 \cdot 81 \times 5000/8314 \times 273} = 0 \cdot 5 \text{ atm.}$

Pressure in a mine at a depth of 5 km $= 1 \times e^{-28 \times 9 \cdot 81 \times (-5000)/8314 \times 273} = 2 \text{ atm.}]$

Q. 16. Assuming the temperature and the molar mass of air, as well as the free-fall

acceleration, to be independent of the height, find the difference in heights at which the air densities at the temperature 0°C differ
(a) e times; (b) by η = 1.0%.

Solution. 16. We have dp = - ρg dh but from gas law $p = \frac{\rho}{M} RT,$

Thus $dp = \frac{d\rho}{M} RT$ at const, temperature

So, $\frac{d\rho}{\rho} = \frac{gM}{RT} dh$

Integrating within limits $\int_{\rho_0}^{\rho} \frac{d\rho}{\rho} = \int_0^h \frac{gM}{RT} dh$

Or, $\ln \frac{\rho}{\rho_0} = -\frac{gM}{RT} h$

So, $\rho = \rho_0 e^{-Mgh/RT}$ and $h = -\frac{RT}{Mg} \ln \frac{\rho}{\rho_0}$

(a) Given $T = 273°K$, $\frac{\rho_0}{\rho} = e$

Thus $h = -\frac{RT}{Mg} \ln e^{-1} = 8$ km.

(b) $T = 273°$ K and

$\frac{\rho_0 - \rho}{\rho_0} = 0.01$ or $\frac{\rho}{\rho_0} = 0.99$

Thus $h = -\frac{RT}{Mg} \ln \frac{\rho}{\rho_0} = 0.09$ km on substitution

Q. 17. An ideal gas of molar mass M is contained in a tall vertical cylindrical vessel whose base area is S and height h. The temperature of the gas is T, its pressure on the bottom base is P_0. Assuming the temperature and the free-fall acceleration g to be independent of the height, find the mass of gas in the vessel.

Solution. 17. From the Barometric formula, we have

$p = p_0 e^{-Mgh/RT}$

And from gas law

$$\rho = \frac{pM}{RT}$$

So, at constant temperature from these two Eqs.

$$\rho = \frac{Mp_0}{RT} e^{-Mgh/RT} = \rho_0 e^{-Mgh/RT} \qquad (1)$$

Eq. (1) shows that density varies with height in the same manner as pressure. Let us consider the mass element of the gas contained in the coltimn.

$$dm = \rho\,(Sdh) = \frac{Mp_0}{RT} e^{-Mgh/RT} Sdh$$

Hence the sought mass,

$$m = \frac{Mp_0 S}{RT} \int_0^h e^{-Mgh/RT} dh = \frac{p_0 S}{g}\left(1 - e^{-Mgh/RT}\right)$$

Q. 18. An ideal gas of molar mass M is contained in a very tall vertical cylindrical vessel in the uniform gravitational field in which the free-fall acceleration equals g. assuming the gas temperature to be the same and equal to T, find the height at which the centre of gravity of the gas is located.

Solution. 18. As the gravitational field is constant the centre of gravity and the centre of mass are same. The location of C.M.

$$h = \frac{\int_0^\infty h\,dm}{\int_0^\infty dm} = \frac{\int_0^\infty h\,\rho\,dh}{\int_0^\infty \rho\,dh}$$

But from Barometric formula and gas law $\rho = \rho_0 e^{-Mgh/RT}$

So, $$h = \frac{\int_0^{\alpha} h \left(e^{-Mg\,h/RT}\right) dh}{\int_0^{\alpha} \left(e^{-Mg\,h/RT}\right) dh} = \frac{RT}{Mg}$$

Q. 19. An ideal gas of molar mass /If is located in the uniform gravitational field in which the free-fall acceleration is equal to g. Find the gas pressure as a function of height h, if p = p_0 at h = 0, and the temperature varies with height as
(a) T = T_0 (1 — ah); (b) T = T_0 (1 + ah),
where a is a positive constant.

Solution. 19. (a) We know that the variation of pressure with height of a fluid is given

by :

dp = - p g dh

But from gas law $p = \frac{\rho}{M} RT$ or, $\rho = \frac{pM}{RT}$

From these two Eqs.

$$dp = -\frac{p\,Mg}{RT}\,dh \qquad (1)$$

Or, $\frac{dp}{p} = \frac{-M\,g\,dh}{R\,T_0\,(1 - ah)}$

Integrating, $\int_{p_0}^{p} \frac{dp}{p} = \frac{-Mg}{R\,T_0} \int_0^{h} \frac{dh}{(1 - ah)}$, we get

$$\ln \frac{p}{p_0} = \ln (1 - ah)^{Mg/aRT_0}$$

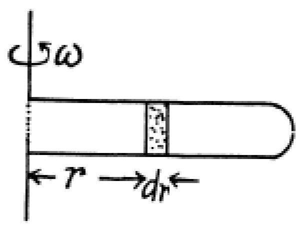
Hence, $p = p_0\,(1 - ah)^{Mg/aRT_0}$. Obvionsly $h < \frac{1}{a}$

(b) Proceed up to Eq. (1) of part (a), and then put $T = T_0\,(1 + a\,h)$ and proceed further in the same fashion to get

$$p = \frac{p_0}{(1 + ah)^{Mg/aRT_0}}$$

Q. 20. A horizontal cylinder closed from one end is rotated with a constant angular velocity ω about a vertical axis passing through the open end of the cylinder. The outside air pressure is equal to P_0, the temperature to T, and the molar mass of air to M. Find the air pressure as a function of the distance r from the rotation axis. The molar mass is assumed to be independent of r.

Solution. 20. Let us consider the mass element of the gas (thin layer) in the cylinder at a distance r from its open end as shown in the figure.



Using Newton's second law for the element

$$F_n = mw_n:$$

$$(p + dp)\,S - pS = (\rho\, S\, dr)\,\omega^2 r$$

$$\text{or,} \quad dp = \rho\,\omega^2 r\,dr = \frac{pM}{RT}\omega^2 r\,dr$$

So,

$$\frac{dp}{p} = \frac{M\omega^2}{RT} r\,dr \quad \text{or,} \quad \int_{p_0}^{p} \frac{dp}{p} = \frac{M\omega^2}{RT}\int_0^r r\,dr,$$

Thus,

$$\ln\frac{p}{p_0} = \frac{M\omega^2}{2RT} r^2 \quad \text{or,} \quad p = p_0\, e^{M\omega^2 r^2/2RT}$$

Q. 21. Under what pressure will carbon dioxide have the density p = 500 g/1 at the temperature T = 300 K? Carry out the calculations both for an ideal and for a Van der Waals gas.

Solution. 21. For an ideal gas law

$$p = \frac{\rho}{M} R\,T$$

So, $$p = 0{\cdot}082 \times 300 \times \frac{500}{44} \text{ atms} = 279{\cdot}5 \text{ atmosphere}$$

For Vander Waal gas Eq.

$$\left(p + \frac{\nu^2 a}{V^2}\right)(V - \nu b) = \nu R T, \quad \text{where } V = \nu V_M$$

Or, $$p = \frac{\nu R T}{V - \nu b} - \frac{a \nu^2}{V^2} = \frac{m R T / M}{V - \frac{mb}{M}} - \frac{a m^2}{V^2 M^2}$$

$$= \frac{\rho R T}{M - \rho b} - \frac{a \rho^2}{M^2} = 79{\cdot}2 \text{ atm}$$

Q. 22. One mole of nitrogen is contained in a vessel of volume V = 1.00 1. Find:
(a) the temperature of the nitrogen at which the pressure can be calculated from an ideal gas law with an error η = 10% (as compared with the pressure calculated from the Van der Waals equation of state);
(b) the gas pressure at this temperature.

Solution. 22. $$\text{(a)} \quad p = \left[\frac{R T}{V_M - b} - \frac{a}{V_M^2}\right](1 + \eta) = \frac{R T}{V_M}$$

(The pressure is less for a Vander Waal gas than for an ideal gas)

Or, $$\frac{a(1 + \eta)}{V_M^2} = R T\left[\frac{-1}{V_M} + \frac{1 + \eta}{V_M - b}\right] = R T \frac{\eta V_M + b}{V_M (V_M - b)}$$

Or, $$T = \frac{a(1 + \eta)(V_M - b)}{R V_M (\eta V_M + b)}, \quad \text{(here } V_M \text{ is the molar volume.)}$$

$$= \frac{1{\cdot}35 \times 1{\cdot}1 \times (1 - 0{\cdot}039)}{0{\cdot}082 \times (0{\cdot}139)} \approx 125 \text{ K}$$

(b) The corresponding pressure is

$$p = \frac{RT}{V_M - b} - \frac{a}{V_M^2} = \frac{a(1+\eta)}{V_M(\eta V_M + b)} - \frac{a}{V_M^2}$$

$$= \frac{a}{V_M^2}\frac{(V_M + \eta V_M - \eta V_M - b)}{(\eta V_M + b)} = \frac{a}{V_M^2}\frac{(V_M - b)}{(V_M + b)}$$

$$= \frac{1{\cdot}35}{1} \times \frac{0{\cdot}961}{0{\cdot}139} = 9{\cdot}3 \text{ atm}$$

Q. 23. One mole of a certain gas is contained in a vessel of volume V = 0.250 1. At a temperature T_1 = 300 K the gas pressure is p_1 = 90 atm, and at a temperature T_2 = 350 K the pressure is p_2 = 110 atm. Find the Van der Waals parameters for this gas.

Solution. 23. $p_1 = RT_1\frac{1}{V-b} - \frac{a}{V^2}, \; p_2 = RT_2\frac{1}{V-b} - \frac{a}{V^2}$

So, $p_2 - p_1 = \frac{R(T_2 - T_1)}{V - b}$

Or. $V - b = \frac{R(T_2 - T_1)}{p_2 - p_1}$ or, $b = V - \frac{R(T_2 - T_1)}{p_2 - p_1}$

Also, $p_1 = T_1\frac{p_2 - p_1}{T_2 - T_1} - \frac{a}{V^2},$

$$\frac{a}{V^2} = \frac{T_1(p_2 - p_1)}{T_2 - T_1} - p_1 = \frac{T_1 p_2 - p_1 T_2}{T_2 - T_1}$$

Or, $a = V^2\frac{T_1 p_2 - p_1 T_2}{T_2 - T_1}$

Using T_1 = 300 K, p_1 = 90 atoms, T_2 = 350 K, p_2 = 110 atm, V = 0.250 litre

a = 1.87 atm. litre^2/mole^2, b = 0.045 litre/mole

Q. 24. Find the isothermal compressibility x of a Van der Waals gas as a function of volume V at temperature T.

Note. By definition $\varkappa = -\frac{1}{V}\frac{\partial V}{\partial p}$.

Solution. 24. $p = \frac{RT}{V-b} - \frac{a}{V^2} - V\left(\frac{\partial p}{\partial V}\right)_T = \frac{RTV}{(V-b)^2} - \frac{2a}{V^2}$

Or, $\kappa = \frac{-1}{V}\left(\frac{\partial V}{\partial p}\right)_T$

$$= \left[\frac{R T V^3 - 2a(V-b)^2}{V^2 (V-b)^2}\right]^{-1} = \frac{V^2 (V-b)}{\left[RTV^3 - 2a(V-b)^2\right]}$$

Q. 25. Making use of the result obtained in the foregoing problem, find at what temperature the isothermal compressibility x of a Van der Waals gas is greater than that of an ideal gas. Examine the case when the molar volume is much greater than the parameter b.

Solution. 25. For an ideal gas $\kappa_0 = \frac{V}{RT}$

$$\text{Now } \kappa = \frac{(V-b)^2}{RTV}\left\{1 - \frac{2a(V-b)^2}{RTV^3}\right\}^{-1} = \kappa_0\left(1 - \frac{b}{V}\right)^2\left\{1 - \frac{2a}{RTV}\left(1 - \frac{b}{V}\right)^2\right\}^{-1}$$

$$= \kappa_0\left\{1 - \frac{2b}{V} + \frac{2a}{RTV}\right\}, \text{ to leading order in } a, b$$

Now $\kappa > \kappa_0$ if $\frac{2a}{RTV} > \frac{2b}{V}$ or $T < \frac{a}{bR}$

If a, b do not vary much with temperature, then the effect at high temperature is clearly determined by b and its effect is repulsive so compressibility is less.