CHAPTER 4
MOTION IN A PLANE

In the preceding two lessons you have studied the concepts related to motion in a straight line. Can you describe the motion of objects moving in a plane, i.e, in two dimensions, using the concepts discussed so far. To do so, we have to introduce certain new concepts. An interesting example of motion in two dimensions is the motion of a ball thrown at an angle to the horizontal. This motions is called a *projectile motion.*

In this lesson you will learn to answer questions like : What should be the position and speed of an aircraft so that food or medicine packets dropped from it reach the people affected by floods or an earthquake? How should an athlete throw a discuss or a javelin so that it covers the maximum horizontal distance? How should roads be designed so that cars taking a turn around a curve do not go off the road? What should be the speed of a satellite so that it moves in a circular orbit around the earth? And so on.

Such situations arise in projectile motion and circular motion. Generally, circular motion refers to motion in a horizontal circle. However, besides moving in a horizontal circle, the body may also move in a vertical circle.We will introduce the concepts of angular speed, centripetal acceleration, and centripetal force to explain this kind of motion.

OBJECTIVES

After studying this lesson, you should be able to :

- *explain projectile motion and circular motion and give their examples;*
- *explain the motion of a body in a vertical circle;*
- *derive expressions for the time of flight, range and maximum height of a projectile;*
- *derive the equation of the trajectory of a projectile;*
- *derive expressions for velocity and acceleration of a particle in circular motion; and*
- *define radial and tangential acceleration.*

4.1 PROJECTILE MOTION

The first breakthrough in the description of projectile motion was made by Galileo. He showed that the horizontal and vertical motions of a slow moving projectile are mutually independent. This can be understood by doing the following activity.

Take two cricket balls. Project one of them horizontally from the top of building. At the same time drop the other ball downward from the same height. What will you notice?

You will find that both the balls hit the ground at the same time. This shows that the downward acceleration of a projectile is the same as that of a freely falling body. Moreover, this takes place independent of its horizontal motion. Further, measurement of time and distance will show that the horizontal velocity continues unchanged and takes place independent of the vertical motion.

In other words, the two important properties of a projectile motion are :

(i) a constant horizontal velocity component

(ii) a constant vertically downward acceleration component.

The combination of these two motions results in the curved path of the projectile.

Refer to Fig. 4.1. Suppose a boy at A throws a ball with an initial horizontal speed. According to Newton's second law there will be no acceleration in the horizontal direction unless a horizontally directed force acts on the ball. Ignoring friction of air, the only force acting on the ball once it is free from the hand of the boy is the force of gravity.

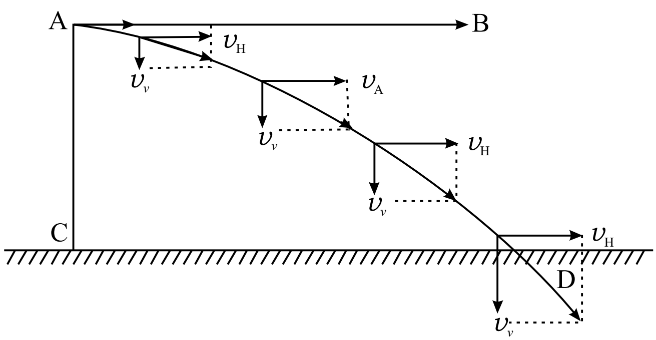

Fig. 4.1: Curved path of a projectile

Hence the horizontal speed υ_H of the ball does not change. But as the ball moves with this speed to the right, it also falls under the action of gravity as shown by the vector's υ_v representing the vertical component of the velocity. Note that $\upsilon = \sqrt{\upsilon_H^2 + \upsilon_v^2}$ and is tangential to the trajectory.

Having defined projectile motion, we would like to determine how high and how far does a projectile go and for how long does it remain in air. These factors are important if we want to launch a projectile to land at a certain target - for instance, a football in the goal, a cricket ball beyond the boundary and relief packets in the reach of people marooned by floods or other natural disasters.

4.1.1 Maximum Height, Time of Flight and Range of a Projectile

Let us analyse projectile motion to determine its maximum height, time of flight and range. In doing so, we will ignore effects such as wind or air resistance. We can characterise the initial velocity of an object in projectile motion by its vertical and horizontal components. Let us take the positive *x*-axis in the horizontal direction and the positive *y*-axis in the vertical direction (Fig. 4.2).

Let us assume that the initial position of the projectile is at the origin O at $t = 0$. As you know, the coordinates of the origin are $x = 0$, $y = 0$. Now suppose the projectile is launched with an initial velocity υ_0 at an angle θ_0, known as the ***angle of elevation,*** to the x-axis. Its components in the x and y directions are,

$$\upsilon_{ox} = \upsilon_o \cos \theta_0 \quad (4.1\ a)$$

and

$$\upsilon_{oy} = \upsilon_o \sin \theta_0 \quad (4.1\ b)$$

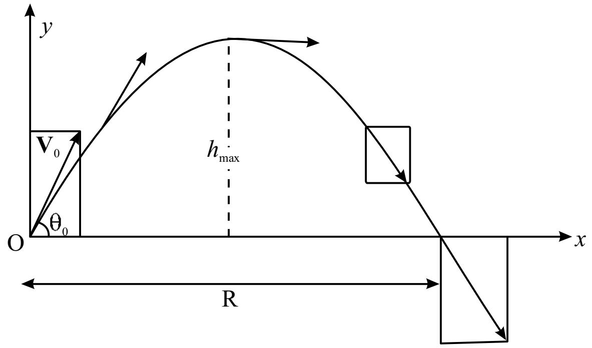

Fig 4.2 : Maximum height, time of flight and range of a projectile

Let a_x and a_y be the horizontal and vertical components, respectively, of the projectile's acceleration. Then

$$a_x = 0; a_y = -g = -9.8 \text{ m s}^{-2} \quad (4.2)$$

The negative sign for a_y appears as the acceleration due to gravity is always in the negative y direction in the chosen coordinate system.

Notice that a_y is constant. Therefore, we can use Eqns. (2.6) and (2.9) to write expressions for the horizontal and vertical components of the projectile's velocity and position at time t. These are given by

Horizontal motion	$v_x = v_{ox}$,	since $a_x = 0$	(4.3a)
	$x = v_{ox}t = v_0 \cos\theta_0 t$		(4.3b)
Vertical motion	$v_y = v_{oy} - g\,t = v_0 \sin\theta_0 - gt$		(4.3c)
	$y = v_{oy}t - ½g\,t^2 = v_0 \sin\theta_0 t - ½g\,t^2$		(4.3d)

The vertical position and velocity components are also related through Eqn. (2.10) as

$$-g\,y = ½\,(v_y^2 - v_{oy}^2) \qquad (4.3e)$$

You will note that the horizontal motion, given by Eqns. (4.3a and b), is motion with constant velocity. And the vertical motion, given by Eqns. (4.3c and d), is motion with constant (downward) acceleration. The vector sum of the two respective components would give us the velocity and position of the projectile at any instant of time.

Now, let us make use of these equations to know the maximum height, time of flight and range of a projectile.

(a) Maximum height : As the projectile travels through air, it climbs upto some maximum height (h) and then begins to come down. ***At the instant when the projectile is at the maximum height, the vertical component of its velocity is zero.*** This is the instant when the projectile stops to move upward and does not yet begin to move downward. Thus, putting $v_y = 0$ in Eqns. (4.3c and e), we get

$$0 = v_{oy} - g\,t,$$

Thus the time taken to rise taken to the maximum height is given by

$$t = \frac{v_{oy}}{g} = \frac{v_0 \sin\theta_0}{g} \qquad (4.4)$$

At the maximum height h attained by the projectile, the vertical velocity is zero. Therefore, applying $v^2 - u^2 = 2$ a s $= 2\,g\,h$, we get the expression for maximum height:

$$h = \frac{v_0^2 \sin^2\theta_0}{2g} \qquad \text{(as } v = 0 \text{ and } u = v_0 \sin\theta) \qquad (4.5)$$

Note that in our calculation we have ignored the effects of air resistance. This is a good approximation for a projectile with a fairly low velocity.

Using Eqn.(4.4) we can also determine the total time for which the projectile is in the air. This is termed as the ***time of flight.***

(b) Time of flight : ***The time of flight of a projectile is the time interval between the instant of its launch and the instant when it hits the ground.*** The time t given by Eq.(4.4) is the time for half the flight of the ball. Therefore, the total time of flight is given by

$$T = 2t = \frac{2\,\upsilon_0 \sin \theta_0}{g} \tag{4.6}$$

Finally we calculate the distance travelled horizontally by the projectile. This is also called its ***range.***

(c) Range : The range R of a projectile is calculated simply by multiplying its time of flight and horizontal velocity. Thus using Eqns. (4.3b) and (4.4), we get

$$R = (\upsilon_{ox})\ (2\,t)$$

$$= (\upsilon_0 \cos \theta_0)\frac{(2\,\upsilon_0 \sin \theta_0)}{g}$$

$$= \upsilon_0^2\,\frac{(2 \sin \theta_0 \cos \theta_0)}{g}$$

Since $2 \sin \theta \cos \theta = \sin 2\theta$, the range R is given by

$$R = \frac{\upsilon_0^2 \sin 2\theta_0}{g} \tag{4.7}$$

From Eqn. (4.7) you can see that the range of a projectile depends on

- its initial speed υ_0, and
- its direction given by θ_0.

Now can you determine the angle at which a disc, a hammer or a javelin should be thrown so that it covers maximum distance horizontally? In other words, let us find out the angle for which the range would be maximum?

Clearly, R will be maximum for any given speed when $\sin 2\theta_0 = 1$ or $2\theta_0 = 90^0$.

Thus, for *R to be maximum at a given speed* υ_0, θ_0 should be equal to 45^0.

Let us determine these quantities for a particular case.

Example 4.1 : In the centennial (on the occasion of its centenary) Olympics held at Atlanta in 1996, the gold medallist hammer thrower threw the hammer to a distance of 19.6m. Assuming this to be the maximum range, calculate the initial speed with which the hammer was thrown. What was the maximum height of the hammer? How long did it remain in the air? Ignore the height of the thrower's hand above the ground.

Solution : Since we can ignore the height of the thrower's hand above the ground, the launch point and the point of impact can be taken to be at the same height. We take the origin of the coordinate axes at the launch point. Since the distance covered by the hammer is the range, it is equal to the hammer's range for $\theta_0 = 45^0$. Thus we have from Eqn.(4.7):

$$R = \frac{v_0^2}{g}$$

or $$v_0 = \sqrt{Rg}$$

It is given that $R = 19.6$ m. Putting $g = 9.8 \text{ ms}^{-2}$ we get

$$v_0 = \sqrt{(19.6\text{m}) \times (9.8 \text{ ms}^{-2})} = 9.8\sqrt{2} \text{ ms}^{-1} = 14.01\text{ms}^{-1}$$

The maximum height and time of flight are given by Eqns. (4.5) and (4.6), respectively. Putting the value of v_0 and $\sin\theta_0$ in Eqns. (4.5) and (4.6), we get

$$\text{Maximum height, } h = \frac{\left(9.8\sqrt{2}\right)^2 \text{m}^2\text{s}^{-2} \times \left(\frac{1}{2}\right)^2}{2 \times 9.8\text{ms}^{-2}} = 4.9 \text{ m}$$

$$\text{Time of flight, } T = \frac{2 \times \left(9.8\sqrt{2}\right) \text{m s}^{-1}}{9.8 \text{ m s}^{-2}} \times \sqrt{\frac{1}{2}} = 2 \text{ s}$$

Now that you have studied some concepts related to projectile motion and their applications, you may like to check your understanding. Solve the following problems.

INTEXT QUESTIONS 4.1

1. Identify examples of projectile motion from among the following situations:
   (a) An archer shoots an arrow at a target
   (b) Rocks are ejected from an exploding volcano
   (c) A truck moves on a mountainous road
   (d) A bomb is released from a bomber plane.
   [**Hint** : Remember that at the time of release the bomb shares the horizontal motion of the plane.]
   (e) A boat sails in a river.
2. Three balls thrown at different angles reach the same maximum height (Fig. 4.3):
   (a) Are the vertical components of the initial velocity the same for all the balls? If not, which one has the least vertical component?

(b) Will they all have the same time of flight?

(c) Which one has the greatest horizontal velocity component?

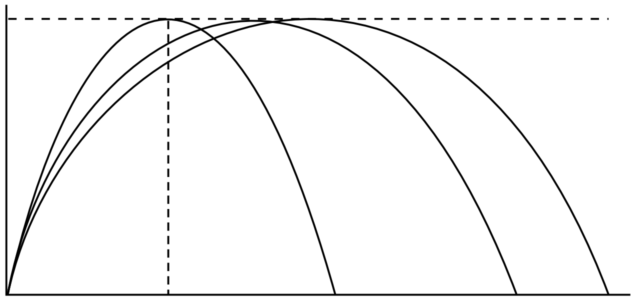

Fig. 4.3 : Trajectories of a projectile

3. An athelete set the record for the long jump with a jump of 8.90 m. Assume his initial speed on take off to be 9.5 ms^{-1}. How close did he come to the maximum possible range in the absence of air resistance?

   Take g = 9.78 ms^{-2}.

4.2 THE TRAJECTORY OF A PROJECTILE

The path followed by a projectile is called its trajectory. Can you recognise the shapes of the trajectories of projectiles shown in Fig. 4.1, 4.2 and 4.3.

Although we have discussed quite a few things about projectile motion, we have still not answered the question: What is the path or trajectory of a projectile? So let us determine the equation for the trajectory of a projectile.

It is easy to determine the equation for the path or trajectory of a projectile. You just have to eliminate *t* from Eqns. (4.3b) and (4.3d) for *x* and *y*. Substituting the value of *t* from Eqn. (4.3b) in Eqn.(4.3d) we get

$$y = v_{oy} \frac{x}{v_{ox}} - \frac{1}{2} \frac{g x^2}{v_{ox}^2} \quad \left(\text{as } t = \frac{x}{v_{ox}} \right) \tag{4.8 a}$$

Using Eqns. (4.1 a and b), Eqn (4.8a) becomes

$$y = (\tan \theta_0)\, x - \frac{g}{2(v_0 \cos \theta_0)^2} x^2 \tag{4.8 b}$$

as $v_{oy} = v_0 \sin \theta$ and $v_{ox} = v_0 \cos \theta$.

Eqn. (4.8) is of the form $y = a x + b x^2$, which is the equation of a **parabola.** Thus, if air resistance is negligible, ***the path of any projectile launched at an angle to the horizontal is a parabola or a portion of a parabola.*** In Fig 4.3 you can see some trajectories of a projectile at different angles of elevation.

Eqns. (4.5) to (4.7) are often handy for solving problems of projectile motion. For example, these equations are used to calculate the launch speed and the angle of elevation required to hit a target at a known range. However, these equations do not give us complete description of projectile motion, if distance covered are very large. To get a complete description, we must include the rotation of the earth also. This is beyond the scope of this course.

Now, let us summarise the important equations describing projectile motion launched from a point (x_0, y_0) with a velocity υ_0 at an angle of elevation, θ_0.

Equations of Projectile Motion:

$$a_x = 0 \qquad a_y = -g \tag{4.9 a}$$

$$\upsilon_x = \upsilon_0 \cos \theta_0 \qquad \upsilon_y = \upsilon_0 \sin \theta - g\,t \tag{4.9 b}$$

$$x = x_0 + (\upsilon_0 \cos \theta_0)t \qquad y = y_0 + (\upsilon_0 \sin \theta)\, t - (½)\, g\, t^2 \tag{4.9 c}$$

Equation of trajectory:

$$y = y_0 + (\tan \theta)\,(x - x_0) - \frac{g}{2(\upsilon_0 \cos \theta_0)^2}\,(x - x_0)^2 \tag{4.9 d}$$

Note that these equations are more general than the ones discussed earlier. The initial coordinates are left unspecified as (x_0, y_0) rather than being placed at (0,0). Can you derive this general equation of the projectile trajectory? Do it before proceeding further?

Thus far you have studied motion of objects in a plane, which can be placed in the category of projectile motion. In projectile motion, the acceleration is constant both in magnitude and direction. There is another kind of two-dimensional motion in which acceleration is constant in magnitude but not in direction. This is uniform circular motion. Generally, circular motion refers to motion in a horizontal circle. However, motion in a vertical circle is also possible. You willl earn about them in the following section

Evangelista Torricelli
(1608 – 1647)

Italian mathematician and a student of Galelio Galili, he invented mercury barometer, investigated theory of projectiles, improved telescope and invented a primitive microscope. Disproved that nature abhors vacuum, presented torricellis theorem.

4.3 CIRCULAR MOTION

Look at Fig. 4.4a. It shows the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ of a particle in uniform circular motion at two different instants of time t_1 and t_2, respectively. The word ‘uniform’ refers to constant speed. We have said that the speed of the particle is constant. What about its velocity? To find out velocity, recall the definition of average velocity and apply it to points P_1 and P_2:

$$\boldsymbol{v}_{av} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1} = \frac{\Delta \mathbf{r}}{\Delta t} \qquad (4.10\ a)$$

The motion of a gramophone record, a grinding wheel at constant speed, the moving hands of an ordinary clock, a vehicle turning around a corner are examples of circular motion. The movement of gears, pulleys and wheels also involve circular motion. The simplest kind of circular motion is uniform circular motion. The most familiar example of uniform circular motion are a point on a rotating fan blade or a grinding wheel moving at constant speed.

One of the example of uniform circular motion is an artificial satellite in circular orbit around the earth. We have been benefitted immensely by the INSAT series of satellites and other artificial satellites. So let us now learn about uniform circular motion.

4.3.1 Uniform Circular Motion

By definition, ***uniform circular motion is motion with constant speed in a circle.***

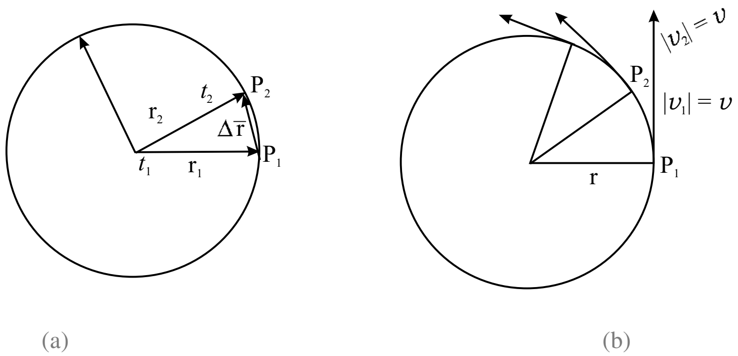

Fig. 4.4: (a) Positions of a particle in uniform circular motion; (b) Uniform circular motion

The vector $\Delta\mathbf{r}$ is shown in Fig. 4.4a. Now suppose you make the time interval Δt smaller and smaller so that it approaches zero. What happens to $\Delta\mathbf{r}$? In particular, what is the direction of $\Delta\mathbf{r}$? It approaches the tangent to the circle at point P_1 as Δt tends to zero. Mathematically, we define the instantaneous velocity at point P_1 as

$$\boldsymbol{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

Thus, in uniform circular motion, the velocity vector changes continuously. Can you say why? This is because the direction of velocity is not constant. It goes on changing continuously as the particle travels around the circle (Fig. 4.4b). Because of ***this change in velocity, uniform circular motion is accelerated motion.*** The acceleration of a particle in uniform circular motion is termed as centripetal acceleration. Let us learn about it in some detail.

Centripetal acceleration : Consider a particle of mass m moving with a ***uniform speed*** υ in a circle. Suppose at any instant its position is at A and its motion is directed along AX. After a small time Δt, the particle reaches B and its velocity is represented by the tangent at B directed along BY.

Let $\mathbf{r}$ and $\mathbf{r'}$ be the position vectors and $\boldsymbol{\upsilon}$ and $\boldsymbol{\upsilon'}$; the velocities of the particle at A and B respectively as shown in Fig. 4.5 (a). The change in velocity $\Delta\boldsymbol{\upsilon}$ is obtained using the triangle law of vectors. As the path of the particle is circular and velocity is along its tangent, $\boldsymbol{\upsilon}$ is perpendicular to $\mathbf{r}$ and $\boldsymbol{\upsilon'}$ is perpendicular to $\Delta\mathbf{r}$. As the average acceleration $\left(\mathbf{a} = \frac{\Delta\boldsymbol{\upsilon}}{\Delta t}\right)$ is along $\Delta\boldsymbol{\upsilon}$, it (i.e., the average acceleration) is perpendicular to $\Delta\mathbf{r}$.

Let the angle between the position vectors $\mathbf{r}$ and $\mathbf{r'}$ be $\Delta\theta$. Then the angle between velocity vectors $\boldsymbol{\upsilon}$ and $\boldsymbol{\upsilon'}$ will also be $\Delta\theta$ as the velocity vectors are always perpendicular to the position vectors.

To determine the change in velocity $\Delta\boldsymbol{\upsilon}$ due to the change in direction, consider a point O outside the circle. Draw a line OP parallel to and equal to AX (or $\boldsymbol{\upsilon}$) and a line OQ parallel to and equal to BY (or $\boldsymbol{\upsilon'}$). As $|\boldsymbol{\upsilon}| = |\boldsymbol{\upsilon'}|$, OP = OQ. Join PQ. You get a triangle OPQ (Fig. 4.5b)

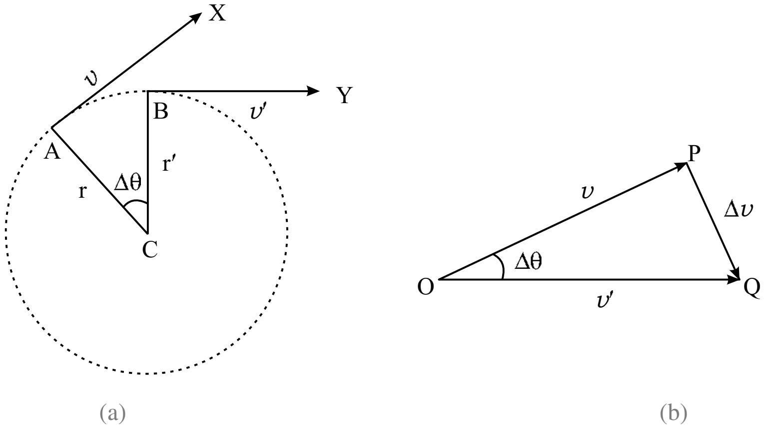

Fig. 4.5

Now in triangle OPQ, sides OP and OQ represent velocity vectors $\boldsymbol{\upsilon}$ and $\boldsymbol{\upsilon'}$ at A and B respectively. Hence, their difference is represented by the side PQ in magnitude and direction. In other words the change in the velocity equal to PQ in magnitude and direction takes place as the particle moves from A to B in time Δt.

$\therefore$ Acceleration = Rate of change of velocity

$$\mathbf{a} = \frac{\mathbf{PQ}}{\Delta t} = \frac{\Delta \boldsymbol{v}}{\Delta t}$$

As Δt is very small AB is also very small and is nearly a straight line. Then Δ ACB and ΔPOQ are isosceles triangles having their included angles equal. The triangles are, therefore, similar and hence,

$$\frac{PQ}{AB} = \frac{OP}{CA}$$

or

$$\frac{\Delta v}{v.\Delta t} = \frac{v}{r}$$

[as magnitudes of velocity vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2 = v$ (say)]

or

$$\frac{\Delta v}{\Delta t} = \frac{v^2}{r}$$

But $\frac{\Delta v}{\Delta t}$ is the acceleration of the particle. Hence

Centripetal acceleration, $a = \frac{v^2}{r}$

Since $v = r\omega$, the magnitude of centripetal force in given by

$$F = m\,a = \frac{m\,v^2}{r} = m\,r\omega^2.$$

As Δt is very small, $\Delta\theta$ is also very small and $\angle$OPQ = $\angle$OQP = 1 right angle.

Thus PQ is perpendicular to OP, which is parallel to the tangent AX at A. Now AC is also perpendicular to AX. Therefore AC is parallel to PQ. It shows that the contripetal force at any point acts towards the centre along the radius.

It shows that some minimum centripetal force has to be applied on a body to make it move in a circular path. In the absence of such a force, the body will move in a straight line path.

To experience this, you can perform a simple activity.

ACTIVITY 4.1

Take a small piece of stone and tie it to one end of a string. Hold the other end with your fingers and then try to whirl the stone in a horizontal or vertical circle. Start with a small speed of rotation and increase it gradually. What happens when the speed of rotation is low? Do you feel any pull on your fingers when the stone is whirling. What happens to the stone when you leave the end of the string you were holding? How do you explain this?

ACTIVITY 4.2

Take an aluminium channel of length one metre and bend it in the form shown in the diagram with a circular loop in the middle. Take help of some technical preson if required.

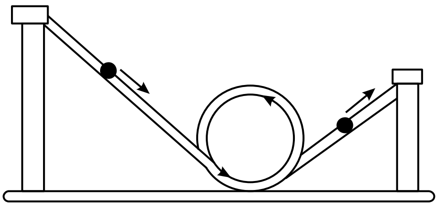

Fig. 4.6: The ball will loop if it starts rolling from a point high enough on the incline

Roll down a glass marble from different heights of the channel on the right hand side, and see whether the marble is able to loop the loop in each case or does it need some minimum height (hence velocity) below which the marble will not be able to complete the loop and fall down. How do you explain it?

Some Applications of Centripetal Force

(i) **Centrifuges :** These are spinning devices used for separating materials having different densities. When a mixture of two materials of different densities placed in a vessel is rotated at high speed, the centripetal force on the heavier material will be more. Therefore, it will move to outermost position in the vessel and hence can be separated. These devises are being used for uranium enrichment. In a chemistry laboratory these are used for chemical analysis.

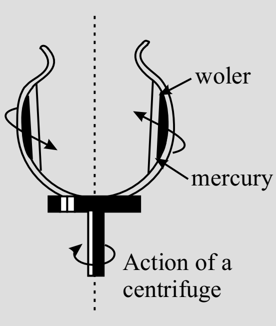

Fig. 4.7: When mercury and water are rotated in a dish, the water stays inside. Centripetal force, like gravitational force, is greater for the more dense substance.

(ii) Mud clings to an automobile tyre until the speed becomes too high and then it flies off tangentially (Fig. 4.8).

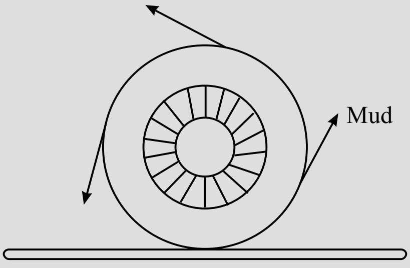

Fig. 4.8: Mud or water on a fast-turning wheel flies off tangentially

(iii) **Planetary motion :** The Earth and the other planets revolving round the sun get necessary centripetal force from the gravitational force between them and the sun.

Example 4.2 : Astronauts experience high acceleration in their flights in space. In the training centres for such situations, they are placed in a closed capsule, which is fixed at the end of a revolving arm of radius 15 m. The capsule is whirled around in a circular path, just like the way we whirl a stone tied to a string in a horizontal circle. If the arm revolves at a rate of 24 revolutions per minute, calculate the centripetal acceleration of the capsule.

Solution : The circumference of the circular path is $2\pi \times$ (radius) = $2\pi \times 15$ m. Since the capsule makes 24 revolutions per minute or 60 s, the time it takes to go once around this circumference is $\frac{60}{24}$ s. Therefore,

speed of the capsule, $v = \frac{2\pi r}{T} = \frac{2\pi \times 15 \text{ m}}{(60/24) \text{ s}} = 38 \text{ ms}^{-1}$

The magnitude of the centripetal acceleration

$$a = \frac{v^2}{r} = \frac{(38 \text{ ms}^{-1})^2}{15 \text{ m}} = 96 \text{ ms}^{-2}$$

Note that centripetal acceleration is about 10 times the acceleration due to gravity.

4.3.2 Motion In a Vertical Circle

When a body moves in a horizontal circle, the direction of its linear velocity goes on changing but the angular velocity remains constant. But, when a body moves in a vertical circle, the angular velocity, too, cannot remain constant on account of the acceleration due to gravity.

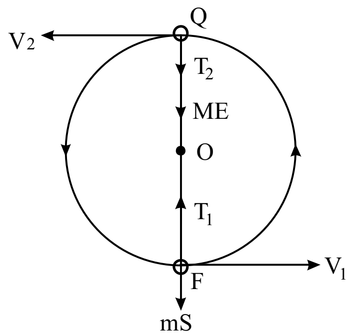

Fig. 4.9

Let a body of mass m tied to a string be rotated anticlockwise in a vertical circle of radius r about a point *O*. As the body rotates in the vertical circle, its speed is maximum at the lowest point *P*. It goes on decreasing as the body moves up to *Q*, and is minimum at the highest point *Q*. The speed goes on increasing as the body falls from *Q* to *P* along the circular path.

The forces acting on the body at *P* are weight of the body 'mg' and the tension T_1 of the string in the direction as shown in Fig. 4.9. Similarly, the forces acting on the body at *Q* are mg and the tension T_2 in the direction shown in Fig. 4.9. If v_1 and v_2 be the velocities of the body at *P* and *Q*, respectively, we have at *P*:

$$T_1 - mg = \frac{mv_1^2}{r}$$

or $$T_1 = \frac{mv_1^2}{r} + mg$$

Note that at *P*, the force $(T_1 - mg)$ acts along *PO* and provides the centripetal force.

Similarly at *Q*,

$$T_2 + mg = \frac{mv_2^2}{r}$$

or $$T_2 = \frac{mv_2^2}{r} - mg$$

For the body to move along the circle without any slaking of the string,

$$T_2 \geq 0$$

i.e. the minimum value of the tension should be zero at *Q*.

When, $T_2 = 0$, $$mg = \frac{mv_2^2}{r}$$

i.e. the minimum velocity at the highest point of the circle is, $\sqrt{gr}$

$$\because \qquad \omega_2 = \frac{v}{r} = \sqrt{g/r}$$

The minimum velocity (v_1) at the lowest point (P) of the circle should be such that the velocity (v_2) at the highest point (Q) becomes $\sqrt{gr}$

Using the relation, $v^2 - u^2 = 2as$, we have

$$v_2^2 - v_1^2 = -2g(2r) \qquad (s = 2r \text{ and g is negative})$$

or
$$v_1^2 = v_2^2 + 4gr$$

$$v_1^2 = gr + 4gr = 5gr$$

or
$$v_1 = \sqrt{5gr}$$

Hence, for a body to go around a vertical circle completely minimum velocity at the lowest point should be $\sqrt{5gr}$.

or
$$w_1 = \sqrt{5g/r}$$

which shows that the angular velocity is also changing as the body moves in a vertical circle.

INTEXT QUESTIONS 4.2

1. In uniform circular motion, (a) Is the speed constant? (b) Is the velocity constant? (c) Is the magnitude of the acceleration constant? (d) Is acceleration constant? Explain.
2. In a vertical motion does the angular velocity of the body change? Explain.
3. An athlete runs around a circular track with a speed of 9.0 ms^{-1} and a centripetal acceleration of 3 ms^{-2}. What is the radius of the track?
4. The Fermi lab accelerator is one of the largest particle accelerators. In this accelerator, protons are forced to travel in an evacuated tube in a circular orbit of diameter 2.0 km at a speed which is nearly equal to 99.99995% of the speed of light. What is the centripetal acceleration of these protons?

   Take $c = 3 \times 10^8$ ms^{-1}.

4.4 APPLICATIONS OF UNIFORM CIRCULAR MOTION

So far you have studied that an object moving in a circle is accelerating. You have also studied Newton's laws in the previous lesson. From Newton's second law we can say that as the object in circular motion is accelerating, a net force must be acting on it.

What is the direction and magnitude of this force? This is what you will learn in this section. Then we will apply Newton's laws of motion to uniform circular motion. This helps us to explain why roads are banked, or why pilots feel pressed to their seats when they fly aircrafts in vertical loops.

Let us first determine the force acting on a particle that keeps it in uniform circular motion. Consider a particle moving with constant speed v in a circle of radius r. From Newton's second law, the net external force acting on a particle is related to its acceleration by

$$\mathbf{F} = -\frac{mv^2}{r}\hat{r}, \ |\mathbf{F}| = \frac{mv^2}{r} \quad (4.19)$$

This net external force directed towards the centre of the circle with magnitude given by Eqn. (4.19) is called ***centripetal force***. ***An important thing to understand and remember is that the term 'centripetal force' does not refer to a type of force of interaction like the force of gravitation or electrical force.*** This term only tells us that the net force of a certain magnitude acting on a particle in uniform circular motion is directed towards the centre. It does not tell us how this force is provided.

Thus, the force may be provided by the gravitational attraction between two bodies. For example, in the motion of a planet around the sun, the centripetal force is provided by the gravitational force between the two. Similarly, the centripetal force for a car travelling around a bend is provided by the force of friction between the road and the car's tyres and/or by the horizontal component of normal reaction of banked road. You will understand these ideas better when we apply them in certain concrete situations.

4.4.1 Banking of Roads

While riding a bicycle and taking a sharp turn, you may have felt that something is trying to throw you away from your path. Have you ever thought as to why does it happen?

You tend to be thrown out because enough centripetal force has not been provided to keep you in the circular path. Some force is provided by the friction between the tyres and the road, but that may not be sufficient. When you slow down, the needed centripetal force decreases and you manage to complete this turn.

Consider now a car of mass m, travelling with speed v on a curved section of a highway (Fig. 4.10). To keep the car moving uniformly on the circular path, a force must act on it directed towards the centre of the circle and its magnitude must be equal to mv^2/r. Here r is the radius of curvature of the curved section.

Now if the road is levelled, the force of friction between the road and the tyres provides the necessary centripetal force to keep the car in circular path. This causes a lot of wear and tear in the tyre and may not be enough to give it a safe turn. The roads at curves are, therefore, banked, where banking means the raising of the outer edge of the road above the level of the inner edge (Fig. 4.10). As a matter of fact, roads are designed to minimise reliance on friction. For example, when car tyres are smooth or there is water or snow on roads, the coefficient of friction becomes negligible. Roads are banked at curves so that cars can keep on track even when friction is negligible.

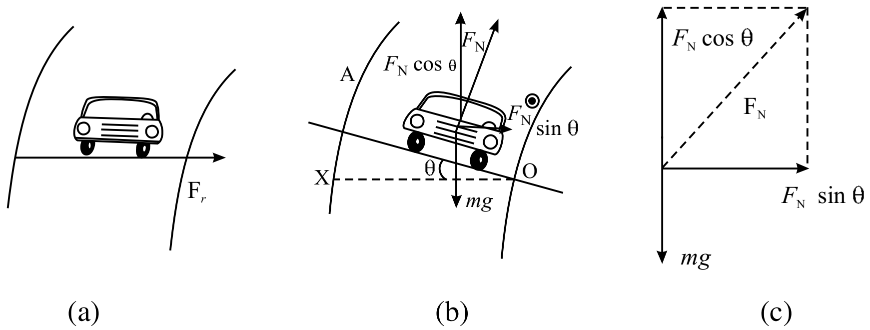

Fig. 4.10 : A car taking a turn (a) on a level road; (b) on a banked road; and (c) Forces on the car with F_N resolved into its rectangular components. Generally θ is not as large as shown here in the diagram.

Let us now analyse the free body diagram for the car to obtain an expression for the angle of banking, θ, which is adjusted for the sharpness of the curve and the maximum allowed speed.

Consider the case when there is no frictional force acting between the car tyres and the road. The forces acting on the car are the car's weight *mg* and $\mathbf{F}_N$, the force of normal reaction. The centripetal force is provided by the horizontal component of $\mathbf{F}_N$. Thus, resolving the force $\mathbf{F}_N$ into its horizontal and vertical components, we can write

$$F_N \sin\theta = \frac{m v^2}{r} \qquad (4.20a)$$

Since there is no vertical acceleration, the vertical component of $\mathbf{F}_N$ is equal to the car's weight:

$$F_N \cos\theta = m g \qquad (4.20b)$$

We have two equations with two unknowns, i.e., F_N and θ. To determine θ, we eliminate F_N. Dividing Eqn. (4.20 a) by Eqn. (4.20 b), we get

$$\tan\theta = \frac{m v^2 / r}{m g} = \frac{v^2}{r g}$$

or

$$\theta = \tan^{-1} \frac{v^2}{rg} \qquad (4.21)$$

How do we interpret Eqn. (4.21) for limits on υ and choice of θ? Firstly, Eqn.(4.21) tells us that the angle of banking is independent of the mass of the vehicle. So even large trucks and other heavy vehicles can ply on banked roads.

Secondly, θ should be greater for high speeds and for sharp curves (i.e., for lower values of r). For a given θ, if the speed is more than υ, it will tend to move towards the outer edge of the curved road. So a vehicle driver must drive within prescribed speed limits on curves. Otherwise, the will be pushed off the road. Hence, there may be accidents.

Usually, due to frictional forces, there is a range of speeds on either side of υ. Vehicles can maintain a stable circular path around curves, if their speed remains within this range. To get a feel of actual numbers, consider a curved path of radius 300 m. Let the typical speed of a vehicle be 50 ms^{-1}. What should the angle of banking be? You may like to quickly use Eqn.(4.21) and calculate θ.

$$\theta = \tan^{-1} \frac{(50 \text{ m s}^{-1})^2}{(300 \text{ m})(9.8 \text{ m s}^{-2})} = \tan^{-1}(0.017) = 1^0$$

You may like to consider another application.

4.4.2 Aircrafts in vertical loops

On Republic Day and other shows by the Indian Air Force, you might have seen pilots flying aircrafts in loops (Fig. 4.11a). In such situations, at the bottom of the loop, the pilots feel as if they are being pressed to their seats by a force several times the force of gravity. Let us understand as to why this happens. Fig. 4.11b shows the 'free body' diagram for the pilot of mass m at the bottom of the loop.

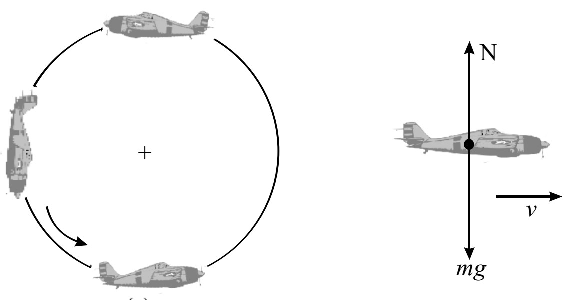

Fig. 4.11: (a) Aircrafts in vertical loops, (b) Free-body diagram for the pilot at the lowest point.

The forces acting on him are mg and the normal force N exerted by the seat. The net vertically upward force is $N - mg$ and this provides the centripetal acceleration:

$$N - mg = m a$$

or $$N - mg = m v^2/r$$

or $$N = m (g + v^2/r)$$

In actual situations, if $v = 200 \text{ ms}^{-1}$ and $r = 1500$ m, we get

$$N = m g \left[1 + \frac{(200 \text{ m s}^{-1})^2}{(9.8 \text{ m s}^{-2} \times 1500 \text{ m})}\right] = m g \times 3.7$$

So the pilots feel as though force of gravity has been magnified by a factor of 3.7. If this force exceeds set limits, pilots may even black out for a while and it could be dangerous for them and for the aircraft.

INTEXT QUESTIONS 4.3

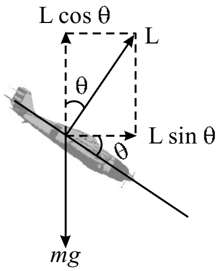

Fig. 4.12

1. Aircrafts usually bank while taking a turn when flying at a constant speed (Fig. 4.12). Explain why aircrafts do bank? Draw a free body diagram for this aircraft. ($\mathbf{F}_a$ is the force exerted by the air on the aircraft). Suppose an aircraft travelling at a speed $v = 100 \text{ ms}^{-1}$ makes a turn at a banking angle of 30^0. What is the radius of curvature of the turn? Take $g = 10 \text{ ms}^{-2}$.

2. Calculate the maximum speed of a car which makes a turn of radius 100 m on a horizontal road. The coefficient of friction between the tyres and the road is 0.90. Take $g = 10 \text{ ms}^{-2}$.

3. An interesting act performed at variety shows is to swing a bucket of water in a vertical circle such that water does not spill out while the bucket is inverted at the top of the circle. For this trick to be performed sucessfully, the speed of the bucket must be larger than a certain minimum value. Derive an expression for the minimum speed of the bucket at the top of the circle in terms of its radius R. Calculate the speed for R = 1.0 m.

WHAT YOU HAVE LEARNT

- **Projectile motion** is defined as the motion which has constant velocity in a certain direction and constant acceleration in a direction perpendicular to that of velocity:

$a_x = 0$ $\qquad$ $a_y = - g$

$v_x = v_0 \cos\theta$ $\qquad$ $v_y = v_0 \sin\theta - g\,t$

$x = x_0 + (v_0 \cos\theta)\,t$ $\qquad$ $y = y_0 + (v_0 \sin\theta) - ½ - g\,t^2$

- Height $h = \dfrac{v_0^2 \sin 2\theta}{g}$
- Time of flight $T = \dfrac{2v_0 \sin\theta}{g}$
- Range of the projectile $R = \dfrac{v_0^2 \sin 2\theta}{g}$
- Equation of the Trajectory of a projectile $y = (\tan\theta_0)\,x - \dfrac{g}{2(v_0 \cos\theta_0)^2}\,x^2$
- **Circular motion** is uniform when the speed of the particle is constant. A particle undergoing ***uniform circular motion*** in a circle of radius r at constant speed v has a ***centripetal acceleration*** given by

$$\mathbf{a}_r = -\frac{v^2}{r}\,\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is the unit vector directed from the centre of the circle to the particle. The speed v of the particle is related to its angular speed ω by $v = r\,\omega$.

- The ***centripetal force*** acting on the particle is given by

$$\mathbf{F} = m\,\mathbf{a}_r = \frac{m\,v^2}{r}\,\hat{\mathbf{r}} = m\,r\,\omega^2$$

- When a body moves in a vertical circle, its angular velocity cannot remain constant.
- The minimum velocities at the highest and lowest points of a vetical circle are $\sqrt{gr}$ and $\sqrt{5gr}$ respectively

ANSWERS TO INTEXT QUESTIONS

4.1

(1) (a), (b), (d)

(2) (a) Yes (b) Yes (c) The ball with the maximum range.

(3) Maximum Range

$$\frac{v_0^2}{g} = \frac{(9.5\ \text{ms}^{-1})^2}{9.78\ \text{ms}^{-2}} = 9.23\ \text{m}$$

Thus, the difference is 9.23 m – 8.90 m = 0.33 m.

4.2

(1) (a) Yes (b) No (c) Yes (d) No

The velocity and acceleration are not constant because their directions are changing continuously.

(2) Yes. The angular velocity changes because of acceleration due to gravity

(3) Since

$$a = \frac{v^2}{r},\ r = \frac{v^2}{a} = \frac{(9.0\ \text{ms}^{-1})^2}{3\ \text{ms}^{-2}} = 27\ \text{m}$$

(4) $$a = \frac{c^2}{r} = \frac{(3\times10^8\ \text{ms}^{-1})^2}{10\times10^3\ \text{m}}$$

$$= 9 \times 10^{13}\ \text{ms}^{-2}$$

4.3

(1) This is similar to the case of banking of roads. If the aircraft banks, there is a component of the force L exerted by the air along the radius of the circle to provide the centripetal acceleration. Fig. 4.15 shows the free body diagram. The radius of curvature is

$$\text{R} = \frac{v^2}{g\tan\theta_0} = \left(\frac{100\ \text{ms}^{-1}}{10\ \text{ms}^{-2}\times\tan 30^\circ}\right)^2 = 10\sqrt{3}\ \text{m} = 17.3\ \text{m}$$

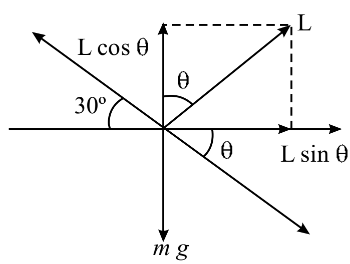

Fig. 4.15

(2) The force of friction provides the necessary centripetal acceleration :

$$F_s = \mu_s N = \frac{mv^2}{r}$$

Since the road is horizontal N – mg

Thus $$\mu_s\, mg = \frac{mv^2}{r}$$

or $$v^2 = \mu_s g\, r$$

or $$v = (0.9 \times 10 \text{ m s}^{-2} \times 100 \text{ m})^{1/2}$$

$$v = 30 \text{ ms}^{-1}.$$

(3) Refer to Fig. 4.16 showing the free body diagram for the bucket at the top of the circle. In order that water in the bucket does not spill but keeps moving in the circle, the force *mg* should provide the centripetal acceleration. At the top of the circle.

$$mg = \frac{mv^2}{r}$$

or $$v^2 = Rg$$

∴ $$v = \sqrt{Rg}$$

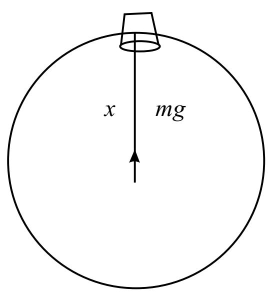

Fig. 4.16

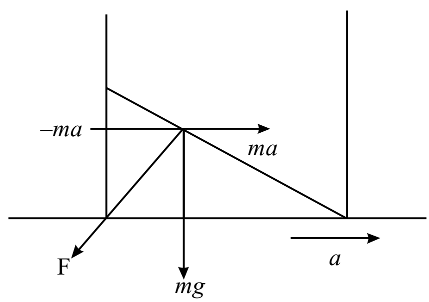

Fig. 4.17

This is the minimum value of the bucket's speed at the top of the vertical circle. For $R = 1.0$ m and taking $g = 10$ ms^{-2} we get

$v = 10$ m s^{-1} = 3.2 ms^{-1}