Number Systems

Real Numbers

Introduction

We are familiar with the number systems such as:

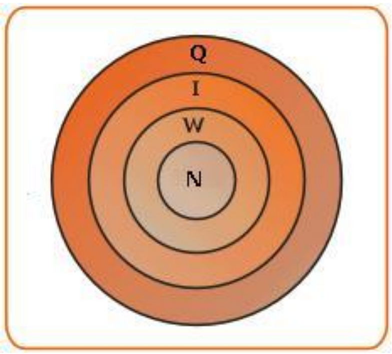
N = set of natural numbers = {1,2,3,4,5,6,7,8,9,.....}

W = set of whole numbers = {0,1,2,3……,9,....}

I or Z = set of Integers = {….-3,-2,-1,0,1,2,3,….}

Q = set of rational numbers $= \{p/q \mid p, q \in I \text{ and } q \neq 0\}$

The relationship of these sets of numbers can be pictorially represented as shown below:



Rational Numbers

A number that is in the form $\frac{p}{q}$ or can be put in the form $\frac{p}{q}$ where p and q are integers, $q \neq 0$ is called a rational number.

Rational numbers can be expressed in the decimal form as terminating decimals or non-terminating recurring decimals.

Some Properties of Rational Numbers

a) Every rational number is either a terminating decimal or a repeating decimal.

Example:

$\frac{3}{5} = 0.6$ (Terminating)

$\frac{6}{7} = 0.857142\overline{857142}$ (Repeating)

$\frac{9}{11} = 0.\overline{81}$ (Repeating)

$\frac{5}{8} = 0.625$ (Terminating)

b) Two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are equal if and only if $a \times d = b \times c$.

Example:

$\frac{3}{5}$ and $\frac{9}{15}$ are equal

implies that $3 \times 15 = 5 \times 9$

c) The above property also means that $\frac{a}{b} = \frac{c}{d}$ if and only if

$a \times d - b \times c = 0$.

Therefore, $\frac{a}{b} > \frac{c}{d}$ if and only if $a \times d - b \times c > 0$.

And $\frac{a}{b} < \frac{c}{d}$ if and only if $a \times d - b \times c < 0$.

d) For any three rational numbers p, q and r, the following order properties are true:

- Either $p > q$ or $q > p$ or $p = q$
- If $p > q$ and $q > r$, then $p > r$
- If $p > q$, then $p + r > q + r$
- If $p > q$ and $r > 0$, then $pr > qr$

Irrational Numbers

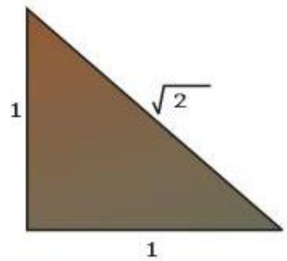
The square root of a non-perfect square number is called an irrational number. An irrational number is a non-terminating and non-recurring decimal,

that is, it cannot be written in the form $\frac{p}{q}$, where p and q are both

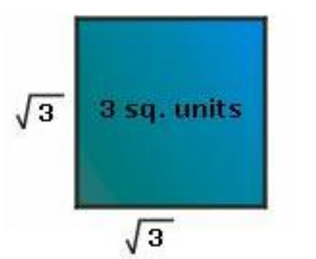
integers and $p \neq q$.

For example $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6}$ are irrational.

Find the hypotenuse of a right-angled triangle of sides 1 unit each.



- The sides of a square of area 3 sq. units.



B_1, B_2, B_3, B_4 etc., on the number line correspond to irrational numbers. Hence, every point on the number line does not represent a rational number, though; every rational number corresponds to a unique point on the number line.

Irrational number as a non-terminating non-repeating decimal.

Consider the following square roots of non-perfect square numbers (Irrational numbers).

$\sqrt{2} = 1.4142135 \ldots\ldots$

$\sqrt{3} = 1.730508 \ldots\ldots$

$\sqrt{5} = 2.2360680 \ldots\ldots$

All these are non-repeating, non-terminating decimals.

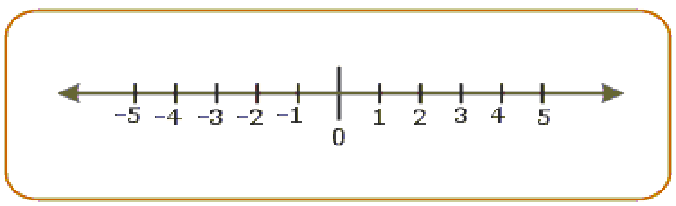
Hence, an alternate definition for an irrational number is:

Any number that cannot be expressed as a decimal with a finite number of digits is called an irrational number.

$\sqrt[3]{4}$, $\sqrt[3]{2}$, $\sqrt{6}$ etc., are some more examples of irrational numbers.

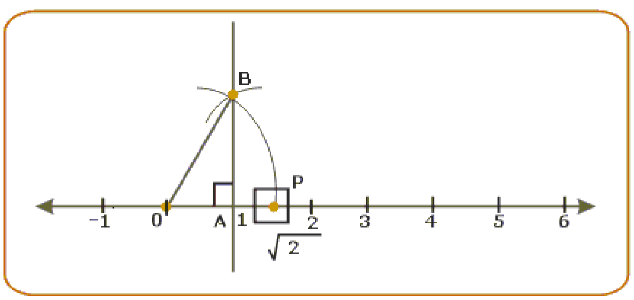
Geometrical Representation of Irrational Numbers

Every rational number has a unique position on the number line.



But does every point on the number line represent a rational number?

To verify let us draw a number line.



Consider a ∠ OAB such that

OA = 1 unit

AB = 1 unit

$\angle OAB = 90^0$

According to Pythagoras theorem,

$OB^2 = OA^2 + AB^2$

$= 1^2 + 1^2$

$= 1 + 1$

$= 2$

$\therefore\ OB = \sqrt{2}$

With O as center, OB as radius, draw an arc cutting the number line at P.

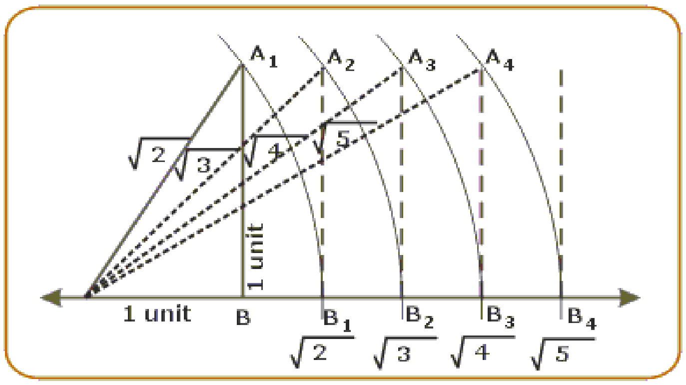
$OB = OP = \sqrt{2}$

Hence point P represents $\sqrt{2}$ on the number ($\sqrt{2}$ is irrational).

⇒There are many points on the number line which are irrational.

Thus, we have learnt that every irrational number can be represented on the number line.

This method is recalled here through the diagram.



Accuracy of this representation depends on the accuracy of construction.

Real Numbers

The set of rational numbers along with the set of irrational numbers, form the real number system. It is denoted by 'R'.

Properties of Real Numbers

Real numbers follow certain basic properties:

For all x, y, z and r belonging to R, (i.e., for any real number x, y, z and r),

- $x + y = y + x$ (Commutative property of addition)
- $(x + y) + z = x + (y + z)$ (Associative property of addition)
- $x \times y = y \times x$ (Commutative property of multiplication)
- $(x \times y) \times z = x \times (y \times z)$ (Associative property of multiplication)
- $x \times (y + z) = (x \times y) + (x \times z)$ (Distributive property of addition over multiplication)
- If x = z and y = r, then

a) $x + y = z + r$ (Addition of equals)

b) $x \times y = z \times r$ (Multiplication of equals)

c) x – y = z − r (Subtraction of equals)

d) $\frac{x}{y} = \frac{z}{r}$ (Division of equals)

All these properties are made use of in solving many arithmetical and algebraic problems in Ratio and Proportions, Linear equations etc.

Surds

If 'a' is a positive rational number which cannot be expressed as the n^{th} power of some other rational number then the irrational number $\sqrt[n]{a}$ or $a^{\frac{1}{n}}$, the positive n^{th} root of a is called a surd or a radical.

The symbol $\sqrt[n]{\ }$ is called the radical sign, n is called the order of the surd and a is called the radicand.

Thus $\sqrt[n]{a}$ is a surd if

- a is a rational number
- $\sqrt[n]{a}$ is an irrational number

Example:

$\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[4]{10}$, $\sqrt{7}$, $\sqrt[3]{9}$ are surds because 2,5,10,7 and 9 are rational numbers.

$\sqrt{\sqrt{2}}$, $\sqrt{2+\sqrt{3}}$ are not surds as $\sqrt{2}$, $2+\sqrt{3}$ are not rational numbers.

Also $\sqrt[3]{27}$ and $\sqrt[4]{625}$ are not surds as $\sqrt[3]{27} = \sqrt[3]{3^3} = 3$ and $\sqrt[4]{625} = \sqrt[4]{5^4} = 5$ are not irrational.

A surd of order 2 is called a quadratic surd. $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ are quadratic surds.

A surd of order 3 is called a cubic surd. $\sqrt[3]{2}$, $\sqrt[3]{3}$, $\sqrt[3]{4}$, $\sqrt[3]{9}$ are cubic surds.

Law of Surds

Since surds can be expressed as fractional exponents of rational numbers, laws of indices are applicable to surds also. ($\sqrt[n]{a} = a^{\frac{1}{n}}$)

- $\sqrt[n]{a^n} = a$ as $[a^n]^{\frac{1}{n}} = a^{n \times \frac{1}{n}} = a$
- $\sqrt[n]{a} . \sqrt[n]{b} = \sqrt[n]{ab}$ as $a^{\frac{1}{n}} . b^{\frac{1}{n}} = (a.b)^{\frac{1}{n}}$

- $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$ as $a^{\frac{1}{n}} \div b^{\frac{1}{n}} = \left(\frac{a}{b}\right)^{\frac{1}{n}}$

Types of Surds

Monomial Surd

A surd consisting of only one term is called monomial surd.

$\sqrt{2}, \sqrt[3]{4}, 3\sqrt[5]{7}$ are monomial surds.

Binomial Surd

An expression consisting of the sum or difference of two monomial surds or the sum of a monomial surd and a rational number is called a binomial surd.

Example:

$2 + \sqrt{3},\ 5 - \sqrt{3},\ \sqrt{2} + \sqrt{3},\ \sqrt{2} - \sqrt{3}, 2\sqrt{2} + \sqrt{5},\ 3\sqrt{5} - 4$

Conjugate Surds

Two binomial surds which differ only in sign (+ or -) between the terms connecting them are called conjugate surds.

$a + \sqrt{b},\ a - \sqrt{b}$ are conjugate surds.

$\sqrt{a} + \sqrt{b},\ \sqrt{a} - \sqrt{b}$ are conjugate surds.

The simplest rationalizing factor of a binomial surd is its conjugate as

$(a + \sqrt{b})(a - \sqrt{b}) = a^2 - (\sqrt{b})^2 = a^2 - b^2$ which is rational.

Example 1:

The rationalising factor of $2 + \sqrt{3}$ is $2 - \sqrt{3}$ because

$(2 + \sqrt{3})\ (2 - \sqrt{3}) = 2^2 - (\sqrt{3})^2$

$= 4 - 3 = 1$

Example 2:

The rationalising factor of $\sqrt{3} - \sqrt{5}$ is $\sqrt{3} + \sqrt{5}$ because

$(\sqrt{3} - \sqrt{5})\ (\sqrt{3} + \sqrt{5}) = (\sqrt{3})^2 - (\sqrt{5})^2$

$= 3 – 5 = - 2$

Rationalization of Surds

If the denominator of an expression is a surd, it can be reduced to an expression with rational denominator.

This process is known as rationalizing of the denominator.

We rationalize the denominator for simplification of surds.

Rationalizing Factor

If the product of two surds is a rational number, then each one of them is a rationalizing factor (R.F) of the other.

Example 1

$\sqrt{5}$ is the rationalising factor of $2\sqrt{5}$ because $2\sqrt{5} \times \sqrt{5} = 2 \times 5 = 10$ which is rational.

$2\sqrt{5}$ is also a rationalising factor of $2\sqrt{5}$ as $2\sqrt{5} \times 2\sqrt{5} = 4 \times 5 = 20$ which is rational.

In fact, $p\sqrt{5}$ is a rationalising factor of $\sqrt{5}$ where p is a non-zero rational number.

Example 2

$\sqrt[3]{3^2}$ is the rationalising factor of $\sqrt[3]{3}$ as $\sqrt[3]{3} \times \sqrt[3]{3^2} = \sqrt[3]{3^3} = 3$ $\left[\sqrt[n]{a^n} = a\right]$

Note:

In monomial surds, the R.F of the surd $\sqrt[n]{a}$ $[a^{\frac{1}{n}}]$ is $a^{1-\frac{1}{n}}$.