CHAPTER 09

Trigonometric Equation and Identities

We studied trigonometric ratios $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$ and $\text{cosec}\theta$ in class 9^{th}. These ratios can be found for any angle, but here we will discuss trigonometric ratios of acute angles only.

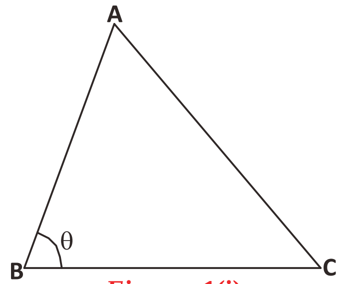


Figure - 1(i)

Look at figure-1(i). Here, in ΔABC consider angle B. Is it possible to identify all the trigonometric ratios at $\angle B = \theta$?

To find all the trigonometric ratios for angle θ, we have to make a right angle triangle including the angle θ.

How can we make right angle triangle ΔABC which includes angle θ?

In ΔABC we will draw the perpendicular AD on side BC from the vertex A. In the right angle triangles ADB and ADC so obtained (*figure*-1(ii)), complete the following table for acute angles θ and θ_1.

$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\text{cosec}\theta$
$\frac{AD}{AB}$					
$\sin\theta_1$	$\cos\theta_1$	$\tan\theta_1$	$\cot\theta_1$	$\sec\theta_1$	$\text{cosec}\theta_1$
$\frac{CD}{AC}$					

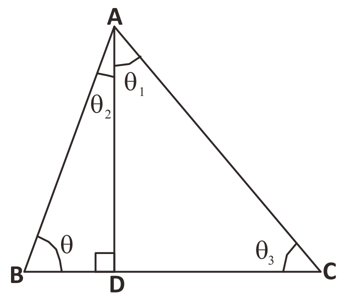


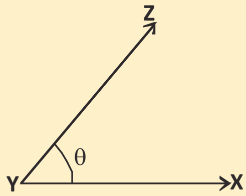
Figure - 1(ii)

Try These

Find all the trigonometric ratios at angle θ_2 and θ_3 in ΔABC.

Think and Discuss

In the adjacent figure, how will you find the trigonometric ratios for angle $\angle XYZ = \theta$?



Relations Between Trigonometric Ratios

In previous classes we learnt about some relations between the trigonometric ratios. Let us find some more relations between these trigonometric ratios.

We have right angle ΔABC in which $\angle C$ is the right angle. By Pythagoras theorem (figure-2),

$$AC^2 + BC^2 = AB^2 \qquad .....(1)$$

Dividing the above equation by AB^2

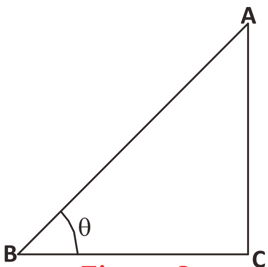


Figure - 2

$$\frac{AC^2}{AB^2} + \frac{BC^2}{AB^2} = \frac{AB^2}{AB^2}$$

$$\left(\frac{AC}{AB}\right)^2 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AB}{AB}\right)^2$$

$$(\sin\theta)^2 + (\cos\theta)^2 = 1$$

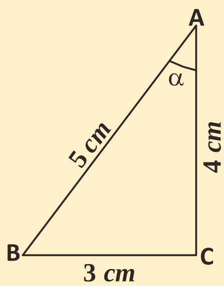
$$\sin^2\theta + \cos^2\theta = 1 \qquad .....(2)$$

Is the obtained relation between $\sin\theta$ and $\cos\theta$ true for all the values of θ between $0°$ to $90°$? Give reasons for your answer.

Try These

(i) Verify $\sin^2\theta + \cos^2\theta = 1$ for $\theta = 30°, 45°, 60°$.

(ii) For the given figure, check whether $\sin^2\alpha + \cos^2\alpha = 1$ is true or not.



You will find that $\sin^2\theta + \cos^2\theta = 1$ is true for all the value of θ which lie between $0°$ to $90°$.

Are, some other relations possible between the trigonometric ratios? Let us see.

Dividing equation (1) by BC^2

$$\frac{AC^2}{BC^2}+\frac{BC^2}{BC^2}=\frac{AB^2}{BC^2}$$

$$\left(\frac{AC}{BC}\right)^2+\left(\frac{BC}{BC}\right)^2=\left(\frac{AB}{BC}\right)^2$$

$$(\tan\theta)^2 + 1 = (\sec\theta)^2$$

$$\tan^2\theta + 1 = \sec^2\theta \qquad .....(2)$$

Is the above relation true for all the angles which lie between 0° and 90°? Let us check for some angles starting with when $\theta = 0°$

L.H.S.	=	$1 + \tan^2\theta$
	=	$1 + \tan^2 0°$
	=	$1 + 0$
	=	1
R.H.S.	=	$\sec^2\theta$
	=	$\sec^2 0°$
	=	1

Hence, it is true at $\theta = 0°$.

Is it also true at $\theta = 90°$? But at $\theta = 90°$ the value of $\tan\theta$ or $\sec\theta$ is not defined therefore we can say that $1 + \tan^2\theta = \sec^2\theta$ is true for all the values of θ except $\theta = 90°$ where θ lies between 0° and 90° i.e. $0 \leq \theta < 90°$.

Let us look at another relation between trigonometric ratios. Dividing equation (1) by AC^2, we get the following relation:

$$\frac{AC^2}{AC^2}+\frac{BC^2}{AC^2}=\frac{AB^2}{AC^2}$$

$$\left(\frac{AC}{AC}\right)^2+\left(\frac{BC}{AC}\right)^2=\left(\frac{AB}{AC}\right)^2$$

$$1 + \cot^2\theta = \mathrm{cosec}^2\theta \qquad .....(3)$$

We know that at $\theta = 0°$, $\cot\theta$ and $\mathrm{cosec}\theta$ are not defined so $1 + \cot^2\theta = \mathrm{cosec}^2\theta$ where $0 < \theta \leq 90°$.

Expressing all trigonometric ratios as any one trigonometric ratio

We have seen the relationship between different trigonometric ratios. Can we convert any trigonometric ratio into another trigonometric ratio? Suppose, we have to express cosA and tanA in terms of sinA, then

$$\because \sin^2 A + \cos^2 A = 1$$

$$\therefore \quad \cos^2 A = 1 - \sin^2 A$$

$$\cos A = \sqrt{1-\sin^2 A}$$

If one trigonometric ratio is known then we can determine the other trigonometric ratios.

And $\tan A = \dfrac{\sin A}{\cos A}$

$$= \frac{\sin A}{\sqrt{1-\sin^2 A}}$$

Try These

1. Express secA in terms of sinA.
2. Express all trigonometric ratios in terms of cosA.

We have studied about different trigonometric identities. Let us think about the relation given below:

$\cot\theta + \tan\theta = \operatorname{cosec}\theta . \sec\theta$

Is this correct? How can we verify? Let's try-

$\cot\theta + \tan\theta = \operatorname{cosec}\theta . \sec\theta$

L.H.S. $= \cot\theta + \tan\theta$

$$= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta . \cos\theta}$$

$$= \frac{1}{\sin\theta.\cos\theta} \qquad [\because \cos^2\theta + \sin^2\theta = 1]$$

$$= \frac{1}{\sin\theta} \cdot \frac{1}{\operatorname{cosec}\theta}$$

$= \operatorname{cosec}\theta \cdot \sec\theta$

= R.H.S.

Let us look at some more examples.

Example-1. Prove that

$\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$

Solution : L.H.S. $= \sin^4\theta - \cos^4\theta$

$= (\sin^2\theta)^2 - (\cos^2\theta)^2$ $\quad [\because a^2 - b^2 = (a-b)(a+b)]$

$= (\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)$ $\quad [\because \cos^2\theta + \sin^2\theta = 1]$

$= (\sin^2\theta - \cos^2\theta) \cdot 1$

$= \sin^2\theta - \cos^2\theta$

= R.H.S.

Example-2. Prove that-

$$\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \frac{1+\sin\theta}{\cos\theta}$$

Solution : L.H.S. $= \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$

$$= \sqrt{\frac{1+\sin\theta}{1-\sin\theta} \times \frac{1+\sin\theta}{1+\sin\theta}}$$

$$= \sqrt{\frac{(1+\sin\theta)^2}{1-\sin^2\theta}}$$

$$= \sqrt{\left(\frac{1+\sin\theta}{\cos\theta}\right)^2}$$

$$= \frac{1+\sin\theta}{\cos\theta}$$

= R.H.S.

Example-3. Prove that-

$$\frac{\cos A}{1-\tan A}+\frac{\sin A}{1-\cot A}=\sin A+\cos A$$

Solution : L.H.S.

$$=\frac{\cos A}{1-\frac{\sin A}{\cos A}}+\frac{\sin A}{1-\frac{\cos A}{\sin A}}$$

$$=\frac{\cos A\,.\,\cos A}{\cos A-\sin A}+\frac{\sin A\,.\,\sin A}{\sin A-\cos A}$$

$$=\frac{\cos^2 A}{\cos A-\sin A}-\frac{\sin^2 A}{\cos A-\sin A}$$

$$=\frac{\cos^2 A-\sin^2 A}{\cos A-\sin A}$$

$$=\frac{(\cos A-\sin A)(\cos A+\sin A)}{\cos A-\sin A}$$

$$=\sin A+\cos A \qquad = \text{R.H.S.}$$

Example-4. Prove that-

$$\frac{1+\cos\theta-\sin^2\theta}{\sin\theta+\sin\theta\,.\,\cos\theta}=\cot\theta$$

Solution : L.H.S. $=\dfrac{1+\cos\theta-\sin^2\theta}{\sin\theta+\sin\theta\,.\,\cos\theta}$

$$=\frac{\cos\theta+1-\sin^2\theta}{\sin\theta(1+\cos\theta)}$$

$$=\frac{\cos\theta+\cos^2\theta}{\sin\theta(1+\cos\theta)}$$

$$=\frac{\cos\theta(1+\cos\theta)}{\sin\theta(1+\cos\theta)}$$

$$=\frac{\cos\theta}{\sin\theta}$$

$$=\cot\theta \qquad = \text{R.H.S.}$$

Sometimes, we have to prove some relationships with the help of given identities. Let us understand through some examples.

Example-5. If $\sin\theta + \cos\theta = 1$ then prove that $\sin\theta - \cos\theta = \pm 1$

Solution : Given that $\sin\theta + \cos\theta = 1$

$(\sin\theta + \cos\theta)^2 = 1$

$\sin^2\theta + \cos^2\theta + 2\sin\theta \, . \cos\theta = 1$

$1 + 2\sin\theta \, . \cos\theta = 1$ $\quad [\because \sin^2\theta + \cos^2\theta = 1]$

$2\sin\theta \, . \cos\theta = 1 - 1$

$\sin\theta \, . \cos\theta = 0$ .....(1)

Again $(\sin\theta - \cos\theta)^2 = 1 - 2\sin\theta \, . \cos\theta$

$(\sin\theta - \cos\theta)^2 = 1 - 2 \times 0$ From equation (1)

$(\sin\theta - \cos\theta)^2 = 1$

$\therefore \sin\theta - \cos\theta = \pm 1$

Hence proved.

Example-6. If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$

then prove that $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$

Solution : Given that $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$

$\sin\theta = \sqrt{2}\cos\theta - \cos\theta$

$\sin\theta = \cos\theta(\sqrt{2} - 1)$

$$\frac{\sin\theta}{\sqrt{2}-1} = \cos\theta$$

$$\cos\theta = \frac{\sin\theta}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$$

$$\cos\theta = \frac{\sqrt{2}\sin\theta + \sin\theta}{2-1}$$

$\cos\theta = \sqrt{2}\sin\theta + \sin\theta$

$\cos\theta - \sin\theta = \sqrt{2}\sin\theta$

Hence proved.

Forming new Identities

If $x = \sin\theta$

$y = \cos\theta$

Then, how can we find the relation between x and y?

We can determine the relation between x and y by eliminating θ, using the trigonometric identities. For example:-

$$x^2 + y^2 = \sin^2\theta + \cos^2\theta$$

$$x^2 + y^2 = 1$$

Let us understand through some more examples.

Example-7. If $x = a\cos\theta - b\sin\theta$ and $y = a\sin\theta + b\cos\theta$ then prove that $x^2 + y^2 = a^2 + b^2$

Solution : Given $x = a\cos\theta - b\sin\theta$ .....(1)

$y = a\sin\theta + b\cos\theta$ .....(2)

By squaring equations (1) and (2)

$$x^2 = (a\cos\theta - b\sin\theta)^2$$

$$y^2 = (a\sin\theta + b\cos\theta)^2$$

$$x^2 = a^2\cos^2\theta + b^2\sin^2\theta - 2ab\cos\theta\,.\,\sin\theta \quad .....(3)$$

$$y^2 = a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\,.\,\cos\theta \quad .....(4)$$

By adding equations (3) and (4)

$$x^2 + y^2 = a^2\cos^2\theta + b^2\sin^2\theta - 2ab\cos\theta\,.\,\sin\theta$$
$$+ a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\,.\,\cos\theta$$
$$= a^2(\sin^2\theta + \cos^2\theta) + b^2(\sin^2\theta + \cos^2\theta)$$
$$= a^2 + b^2 \quad [\because \sin^2\theta + \cos^2\theta = 1]$$

Example-8. If $\tan\theta + \sin\theta = m$ and $\tan\theta - \sin\theta = n$ then prove that-

$$m^2 - n^2 = 4\sqrt{mn}$$

Solution : Given $m = \tan\theta + \sin\theta$

$n = \tan\theta - \sin\theta$

$m + n = 2\tan\theta$

$m - n = 2\sin\theta$

Now, $(m-n)(m+n) = 4\sin\theta \cdot \tan\theta$

$$m^2 - n^2 = 4\sin\theta \cdot \tan\theta \qquad .....(1)$$

$$m \cdot n = (\tan\theta + \sin\theta)(\tan\theta - \sin\theta)$$

$$= \tan^2\theta - \sin^2\theta$$

$$= \frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta$$

$$= \frac{\sin^2\theta - \sin^2\theta \cdot \cos^2\theta}{\cos^2\theta}$$

$$= \frac{\sin^2\theta[1-\cos^2\theta]}{\cos^2\theta}$$

$$= \sin^2\theta \cdot \frac{\sin^2\theta}{\cos^2\theta}$$

$$= \sin^2\theta \cdot \tan^2\theta$$

$$4\sqrt{mn} = 4\sqrt{\sin^2\theta \cdot \tan^2\theta}$$

$$= 4\sin\theta \cdot \tan\theta$$

$$4\sqrt{mn} = m^2 - n^2 \qquad \text{From equation (1)}$$

Or $m^2 - n^2 = 4\sqrt{mn}$ Hence proved.

Exercise - 1

Prove the following identities:-

1. $\dfrac{1}{\sec\theta - 1} - \dfrac{1}{\sec\theta + 1} = 2\cot^2\theta$
2. $\sec^2\theta + \mathrm{cosec}^2\theta = \sec^2\theta \cdot \mathrm{cosec}^2\theta$
3. $\sin^4 A + \cos^4 A = 1 - 2\sin^2 A \cdot \cos^2 A$
4. $\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \mathrm{cosec}\theta - \cot\theta$
5. $(1 + \cot\theta - \mathrm{cosec}\theta)(1 + \tan\theta + \sec\theta) = 2$

6. $\frac{1+\cos\theta}{1-\cos\theta}-\frac{1-\cos\theta}{1+\cos\theta} = 4\cot\theta\ \text{cosec}\theta$

7. $\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta} = 2\ \text{cosec}\theta$

8. If $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$ then prove that $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$

9. If $\tan\theta = n\tan\phi$ and $\sin\theta = m\sin\phi$ then prove that $\cos^2\theta = \frac{m^2-1}{n^2-1}$

10. If $x = a\ \text{cosec}\theta$ तथा $y = b\cot\theta$ then prove that $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$

11. If $x = r\sin A\cos C$, $y = r\sin A\sin C$ and $z = r\cos A$ then prove that

$r^2 = x^2 + y^2 + z^2$

Trigonometric equations and identities

We learned the relations among various trigonometric ratios like $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\text{cosec}\theta$, $\cot\theta$. Among these we saw one relation $\sin^2\theta + \cos^2\theta = 1$ which is true for all values of θ. Such relations between different trigonometric ratios which are true for all values taken by the angle are known as trigonometric identities.

Is the relation $\sin\theta + \cos\theta = 1$ a trigonometric identity?

Let us see.

At $\theta = 0°$

$= \sin 0° + \cos 0°$

$= 0 + 1$

$= 1$

At $\theta = 30°$

$= \sin 30° + \cos 30°$

$= \frac{1}{2}+\frac{\sqrt{3}}{2}$

$= \frac{\sqrt{3}+1}{2}$

$\neq 1$

We find that the relation is true for $\theta = 0°$ but not for $\theta = 30°$. Therefore, $\sin\theta + \cos\theta = 1$ cannot be called an identity.

Some trigonometric relations are true for certain values of angles given in the form of variables. These are known as trigonometric equations. Can we say that $\sin\theta + \cos\theta = 1$ is a trigonometric equation? We saw that the relation $\sin\theta + \cos\theta = 1$ is true for $\theta = 0°$ but not for $\theta = 30°$. Therefore, $\sin\theta + \cos\theta = 1$ is a trigonometric equation.

Try These

Put $\theta = 0°, 30°, 45°, 60°, 90°$ in the given relations and check which value of θ satisfies these relations.

1. $\cos\theta + \sin\theta = \sqrt{2}$
2. $\tan^2\theta + \cot^2\theta = 2$
3. $2\cos^2\theta = 3\sin\theta$
4. $\tan\theta . \sec\theta = 2\sqrt{3}$

The values of θ for which the equations are true or valid are called solutions of the trigonometric equation.

Let us solve the following trigonometric equations:-

Example-9. Solve $\sqrt{3}\tan\theta - 2\sin\theta = 0$

Solution : $\sqrt{3}\dfrac{\sin\theta}{\cos\theta} - 2\sin\theta = 0$ $\quad\left[\because \tan\theta = \dfrac{\sin\theta}{\cos\theta}\right]$

$\sqrt{3}\sin\theta - 2\sin\theta . \cos\theta = 0$

$\sin\theta(\sqrt{3} - 2\cos\theta) = 0$

$\sin\theta = 0$

$\theta = 0°$

Now, $\sqrt{3} - 2\cos\theta = 0$

$\Rightarrow \quad -2\cos\theta = -\sqrt{3}$

$\Rightarrow \quad \cos\theta = \dfrac{\sqrt{3}}{2}$

$\theta = 30°$

$\therefore\ \theta = 0°, 30°$

Example-10. Solve $\cos^2 x + \cos x = \sin^2 x$, where $0° \leq x \leq 90°$

Solution : $\cos^2 x + \cos x = \sin^2 x$

$\Rightarrow \quad \cos^2 x + \cos x = 1 - \cos^2 x$

$\Rightarrow \quad \cos^2 x + \cos^2 x + \cos x - 1 = 0$

$\Rightarrow \quad 2\cos^2 x + \cos x - 1 = 0$

$\Rightarrow \quad 2\cos^2 x + 2\cos x - \cos x - 1 = 0$

$\Rightarrow \quad 2\cos x(\cos x + 1) - 1(\cos x + 1) = 0$

$\Rightarrow \quad (2\cos x - 1)(\cos x + 1) = 0$

$2\cos x - 1 = 0$

$$\cos x = \frac{1}{2}$$

$x = 60°$

Also, $(\cos x + 1) = 0$

$\Rightarrow \quad \cos x + 1 = 0$

$\cos x = -1$

Since $\cos x$ cannot be negative where $0 \leq x \leq 90°$, therefore we will ignore $\cos x = -1$. Hence, the solution for given equation is $x = 60°$.

Example-11. Solve the given trigonometric equation where $0° \leq \theta \leq 90°$

$$\frac{\cos\theta}{\operatorname{cosec}\theta + 1} + \frac{\cos\theta}{\operatorname{cosec}\theta - 1} = 2$$

Solution :
$$\frac{\cos\theta}{\operatorname{cosec}\theta + 1} + \frac{\cos\theta}{\operatorname{cosec}\theta - 1} = 2$$

$$\Rightarrow \quad \frac{\cos\theta(\operatorname{cosec}\theta - 1) + \cos\theta(\operatorname{cosec}\theta + 1)}{\operatorname{cosec}^2\theta - 1} = 2$$

$$\Rightarrow \quad \frac{\cos\theta[\operatorname{cosec}\theta - 1 + \operatorname{cosec}\theta + 1]}{\cot^2\theta} = 2 \qquad [\because \operatorname{cosec}^2\theta - 1 = \cot^2\theta]$$

$$\Rightarrow \quad \frac{\cos\theta \cdot 2\operatorname{cosec}\theta}{\cot^2\theta} = 2$$

$$\Rightarrow \quad \frac{\cos\theta \cdot 2 \cdot \frac{1}{\sin\theta}}{\cot^2\theta} = 2$$

$$\Rightarrow \quad \frac{2\dfrac{\cos\theta}{\sin\theta}}{\cot^2\theta} = 2$$

$$\Rightarrow \quad \frac{2\cot\theta}{\cot^2\theta} = 2$$

$$\Rightarrow \quad \frac{2}{\cot\theta} = 2$$

$$\Rightarrow \quad 2\tan\theta = 2$$

$$\Rightarrow \quad \tan\theta = 1$$

$$\Rightarrow \quad \tan\theta = \tan 45^\circ$$

$$\therefore \quad \theta = 45^\circ$$

Exercise - 2

1. Solve the follow trigonometric equation where $0^\circ \le \theta \le 90^\circ$

(i) $2\cos^2\theta - \sqrt{3}\cos\theta = 0$ (ii) $2\sin^2\theta - \cos\theta = 1$

(iii) $3\tan^2\theta = 2\sec^2\theta + 1$ (iv) $\cos^2\theta - 3\cos\theta + 2 = \sin^2\theta$

(v) $\dfrac{\cos\theta}{1-\sin\theta} + \dfrac{\cos\theta}{1+\sin\theta} = 4$

Trigonometric Ratios of Complementary Angles

In right angle triangle ΔABC if $\angle A = 30^\circ$ then what will be the value of $\angle C$ (figure-3)? And is it possible to find the value of $\angle A$ if we know that $\angle C = 60^\circ$ (figure-4)?

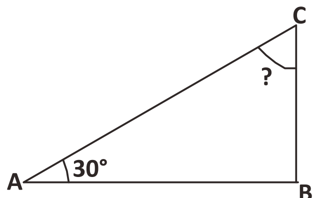


Figure - 3

Is there a relation between $\angle A$ and $\angle C$ so that if we know the value of one angle we can find the value of the other angle?

We know that in ΔABC,

$$\angle A + \angle B + \angle C = 180^\circ$$

$$\because \angle B = 90^\circ$$

$$\therefore \quad \angle A + \angle C = 90^\circ$$

This means that $\angle A$ and $\angle C$ are complementary angles.

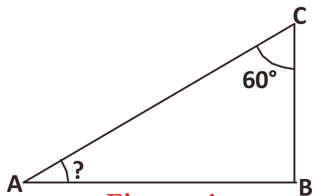


Figure - 4

Now, in triangle ABC (figure-5)

If $\angle A = \theta$ then $\angle C = 90^\circ - \theta$

Then, is there any relation between the trigonometric ratios of $\angle A$ and $\angle C$?

Is it possible in the given triangle to convert the trigonometric ratios of angle $(90°-\theta)$ into trigonometric ratios of angle θ? How?

$$\sin\theta = \frac{BC}{AC} \qquad \cos\theta = \frac{AB}{AC} \qquad \tan\theta = \frac{BC}{AB}$$

$$\operatorname{cosec}\theta = \frac{AC}{BC} \qquad \sec\theta = \frac{AC}{AB} \qquad \cot\theta = \frac{AB}{AC}$$

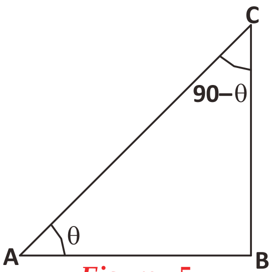


Figure - 5

If angle $\angle C = 90°-\theta$ in ΔABC, then

Trigonometric ratios are:-

$$\sin(90°-\theta) = \frac{AB}{AC}, \quad \cos(90°-\theta) = \frac{BC}{AC}, \quad \tan(90°-\theta) = \frac{AB}{BC},$$

$$\operatorname{cosec}(90°-\theta) = \frac{AC}{AB}, \quad \sec(90°-\theta) = \frac{AC}{BC}, \quad \cot(90°-\theta) = \frac{BC}{AB}$$

When we compare the trigonometric ratios for angles θ and $(90°-\theta)$ we get the following relations:-

$$\sin(90°-\theta) = \frac{AB}{AC} = \cos\theta, \quad \cos(90°-\theta) = \frac{BC}{AC} = \sin\theta$$

$\tan(90°-\theta) = \cot\theta$ and $\cot(90°-\theta) = \tan\theta$

$\sec(90°-\theta) = \operatorname{cosec}\theta$ और $\operatorname{cosec}(90°-\theta) = \sec\theta$

Think and Discuss

Is the above relation true for all values θ where $0° \le \theta \le 90°$.

Try These

Complete the following table by using the relations between trigonometric ratios of complementary angles

θ	0°	30°	45°	60°	90°
sinθ	0	....	$\frac{1}{\sqrt{2}}$	....	1
cosθ	....	$\frac{\sqrt{3}}{2}$	....	$\frac{1}{2}$	....

Applications of Trigonometric Ratios of Complementary Angles

Let us see how to find values using trigonometric ratios of complementary angles without using the trigonometric table. Can we use them to find trigonometric ratios of non-simple angles, for example, θ=31° or ϕ = 20° or 43°?

Now we will try to find the value of $\frac{2\sin 30^\circ}{\cos 60^\circ}$ without using the trigonometric table.

$$\frac{2\sin 30^\circ}{\cos 60^\circ}$$

$$= \frac{2\sin 30^\circ}{\cos(90^\circ - 30^\circ)} \qquad [\because \cos(90^\circ - \theta) = \sin\theta]$$

$$= 2\frac{\sin 30^\circ}{\sin 30^\circ}$$

$$= 2$$

Similarly, if we have to find out the value of $\frac{3\tan 15^\circ}{\cot 75^\circ}$

$$\frac{3\tan 15^\circ}{\cot 75^\circ}$$

$$= \frac{3\tan 15^\circ}{\cot(90^\circ - 15^\circ)}$$

$$= \frac{3\tan 15^\circ}{\tan 15^\circ}$$

$$= 3$$

Let's try to understand through examples.

Example-12. Find the value of the following:-

(a) $\frac{\sin 31^\circ}{2\cos 59^\circ}$ (b) $\frac{\sec 70^\circ}{\operatorname{cosec} 20^\circ} + \frac{\sin 59^\circ}{\cos 31^\circ}$

Soluiton : (a) $\frac{\sin 31^\circ}{2\cos 59^\circ}$

$$= \frac{\sin(90^0 - 59^0)}{2\cos 59^0}$$

$$= \frac{\cos 59°}{2\cos 59°} \qquad [\because \quad \sin(90°–\theta) = \cos\theta]$$

$$= \frac{1}{2}$$

(b) $\frac{\sec 70°}{\operatorname{cosec} 20°} + \frac{\sin 59°}{\cos 31°}$

$$= \frac{\sec(90° - 20°)}{\operatorname{cosec} 20°} + \frac{\sin(90° - 31°)}{\cos 31°} \qquad \left[\begin{array}{l}\therefore \sec(90° - \theta) = \operatorname{cosec}\theta \\ \sin(90° - \theta) = \cos\theta\end{array}\right]$$

$$= \frac{\operatorname{cosec} 70°}{\operatorname{cosec} 70°} + \frac{\cos 31°}{\cos 31°}$$

$$= 1 + 1$$

$$= 2$$

Example-13. Find the value of $\left(\frac{\sin 47°}{\cos 43°}\right)^2 + \left(\frac{\cos 43°}{\sin 47°}\right)^2 - 4\cos^2 45°$.

Solution : $\left(\frac{\sin 47°}{\cos 43°}\right)^2 + \left(\frac{\cos 43°}{\sin 47°}\right)^2 - 4\cos^2 45°$

$$= \left(\frac{\sin(90° - 43°)}{\cos 43°}\right)^2 + \left(\frac{\cos(90° - 47°)}{\sin 47°}\right)^2 - 4\left(\frac{1}{\sqrt{2}}\right)^2$$

$$= \left(\frac{\cos 43°}{\cos 43°}\right)^2 + \left(\frac{\sin 47°}{\sin 47°}\right)^2 - 4 \times \frac{1}{2}$$

$$= 1 + 1 - 2$$

$$= 0$$

Example-14. Prove that-

$$\tan 7° \tan 23° \tan 60° \tan 67° \tan 83° = \sqrt{3}$$

Solution : L.H.S. $= \tan 7° \tan 23° \tan 60° \tan 67° \tan 83°$

$= \tan (90° - 83°) \tan (90° - 67°) \tan 60° \tan 67° \tan 83°$

$= \cot 83° \cot 67° \tan 60° \tan 67° \tan 83°$

$= \cot 83° \tan 83° \cot 67° \tan 67° \tan 60°$

$$= \cot 83° \times \frac{1}{\cot 83°} \times \cot 67° \times \frac{1}{\cot 67°} \times \sqrt{3}$$

$= \sqrt{3}$

Solving trigonometric equations

Let us consider the given equation:

$$\cos(90° - \theta) = \frac{1}{2}$$

If we have to identify the value of the unknown θ, what should we do?

$$\cos (90° - \theta) = \frac{1}{2}$$

$\Rightarrow \quad \sin\theta = \sin 30°$

$\theta = 30°$

Let us understand through some more examples.

Example-15. If $\sin 55° \operatorname{cosec} (90° - \theta) = 1$, then find the value of θ where $0° \le \theta \le 90°$

Solution : $\sin 55° \operatorname{cosec} (90° - \theta) = 1$

$\Rightarrow \quad \sin (90° - 35°) \sec\theta = 1$

$\Rightarrow \quad \cos 35° \, . \, \sec\theta = 1$

$$\Rightarrow \quad \sec\theta = \frac{1}{\cos 35°}$$

$\Rightarrow \quad \sec\theta = \sec 35°$

$\therefore \quad \theta = 35°$

Example-16. If $\sin 34° = p$ then find the value of $\cot 56°$.

Solution : $\sin 34° = p$

$\sin(90° - 56°) = p$

$\cos 56° = p$ .....(1)

We know that $\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \sin^2\theta = 1 - \cos^2\theta$

$\Rightarrow \sin^2 56° = 1 - \cos^2 56°$

$\Rightarrow \sin^2 56° = 1 - p^2$ .....(2)

$\Rightarrow \sin 56° = \sqrt{1-p^2}$

With the help of equations (1) & (2)

$$\cot 56° = \frac{\cos 56°}{\sin 56°}$$

$$= \frac{p}{\sqrt{1-p^2}}$$

Example-17. If $\cot 3A = \tan(A - 22°)$ where $3A$ is an acute angle, then find the value of A.

Solution : Given that- $\cot 3A = \tan(A - 22°)$

$\Rightarrow$ $\tan(90° - 3A) = \tan(A - 22°)$

$\Rightarrow$ $90° - 3A = A - 22°$

$\Rightarrow$ $90° + 22° = A + 3A$

$\Rightarrow$ $112° = 4A$

$\Rightarrow$ $A = \frac{112°}{4}$

$\therefore$ $A = 28°$

With the help of the trigonometric ratios of complementary angles, it is possible to prove relations between trigonometric ratios. Let us see:-

Example-18. Prove that- $\frac{\sin(90° - \theta)\ \cos(90° - \theta)}{\tan\theta} = \cos^2\theta$

Solution : L.H.S. $= \frac{\sin(90° - \theta)\ \cos(90° - \theta)}{\tan\theta}$

$$= \frac{\cos\theta \sin\theta}{\tan\theta}$$

$$= \frac{\cos\theta \sin\theta}{\frac{\sin\theta}{\cos\theta}}$$

$$= \frac{\cos^2\theta \sin\theta}{\sin\theta}$$

$= \cos^2\theta$

= R.H.S.

Example-19. Prove that- $\sin(90° - \theta)\sec\theta + \cos(90° - \theta)\operatorname{cosec}\theta = 2$

Solution : L.H.S. $= \sin(90° - \theta)\sec\theta + \cos(90° - \theta)\operatorname{cosec}\theta$

$= \cos\theta\sec\theta + \sin\theta\operatorname{cosec}\theta$

$$= \cos\theta \times \frac{1}{\cos\theta} + \sin\theta \times \frac{1}{\sin\theta}$$

$= 1 + 1$

$= 2$

= R.H.S.

Example-20. If $\angle A, \angle B$ and $\angle C$ are the interior angle of ΔABC, then prove that-

$$\sin\left(\frac{A+B}{2}\right) = \cos\frac{C}{2}$$

Solution : Given that- A, B and C are the interior angle of ΔABC

Then, $A + B + C = 180°$

$A + B = 180° - C$ .....(1)

Again, L.H.S. $= \sin\left(\frac{A+B}{2}\right)$

$$= \sin\left(\frac{180° - C}{2}\right)$$

$$= \sin\left(\frac{180°}{2} - \frac{C}{2}\right)$$

$$= \sin\left(90° - \frac{C}{2}\right)$$

$$= \cos\frac{C}{2}$$

= R.H.S.

Now, we will learn how to convert trigonometric ratio of a given angle to a trigonometric ratio of an angle which lies between 0° and 45°.

Example-21. Express tan 59° + cot 75° as trigonometric ratios of angles between 0° and 45°.

Solution : tan 59° + cot 75° = tan (90° – 31°) + cot (90° – 15°)

= cot 31° + tan 15°

[∵ tan (90° – θ) = cotθ

cot (90° – θ) = tanθ]

Exercise-3

1. Express the following as trigonometric ratios of angles which lie between 0° and 45°.

   (i) sin 56° (ii) tan 81° (iii) sec 73°

2. Find the value of the following:-

   (i) $\dfrac{\cos 80°}{\sin 10°}$ (ii) $\dfrac{\sin 37°}{2\cos 53°}$ (iii) 3 sin 17° sec 73°

3. Find the value of the following:-

   (i) sin 64° – cos 26°

   (ii) 3 cos 80° cosec 10° + 2 cos 59° cosec 31°

   (iii) $2\dfrac{\cos 67°}{\sin 23°} - \dfrac{\tan 40°}{\cot 50°} + \cos 0°$

   (iv) $\sin^2 35° + \sin^2 55°$

   (v) $\left(\dfrac{5\sin 35°}{\cos 55°}\right) + \left(\dfrac{\cos 55°}{2\sin 35°}\right) - 2\cos 60°$

4. Prove that-

(i) $\sin 63° \cos 27° + \cos 63° \sin 27° = 1$

(ii) $\tan 15° \tan 36° \tan 45° \tan 54° \tan 75° = 1$

(iii) $\sin^2 85° + \sin^2 80° + \sin^2 10° + \sin^2 5° = 2$

5. Prove that-

$$\sin(90° - \theta)\cos(90° - \theta) = \frac{\tan\theta}{1 + \cot^2(90° - \theta)}$$

6. Prove that-

$$\frac{\cos\theta}{\sec\ (90° - \theta) + 1} + \frac{\sin(90° - \theta)}{\operatorname{cosec}\theta - 1} = 2\cot(90° - \theta)$$

7. Prove that-

$$\frac{\tan(90° - \theta)}{\operatorname{cosec}^2\theta . \tan\theta} = \cos^2\theta$$

8. If $\sin A = \cos B$ then prove that $A + B = 90°$

9. If cosec 2A = sec (A – 36°), where 2A is an acute angle, then fine the value of A.

10. If A + B = 90°, secA = a, cotB = b then prove that $a^2 - b^2 = 1$

11. If A, B and C are interior angle of a triangle ABC, then prove that-

$$\tan\left(\frac{B + C}{2}\right) = \cot\left(\frac{A}{2}\right)$$

12. If sec 34° = x then find the value of $\cot^2 56° + \operatorname{cosec} 56°$.

What We Have Learnt

1. The following relations are observed between trigonometric ratios:

$\sin^2\theta + \cos^2\theta = 1$ where $0° \le \theta \le 90°$

$1 + \tan^2\theta = \sec^2\theta$ where $0° \le \theta < 90°$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$ where $0° < \theta \le 90°$

2. Any trigonometric ratio can be written in terms of any other trigonometric ratio.

3. Identities are those trigonometric equations that are true for all values of the angles.

4. For a value of angle θ, if one trigonometric ratio is known then the other trigonometric ratios can be found.

5. The following relations are seen between trigonometric ratios of complementary angles:

$\sin(90°-\theta) = \cos\theta$, $\cos(90°-\theta) = \sin\theta$

$\tan(90°-\theta) = \cot\theta$, $\cot(90°-\theta) = \tan\theta$

$\sec(90°-\theta) = \text{coesc}\theta$, $\text{cosec}(90°-\theta) = \sec\theta$

6. Identities can't be proved or verified using only a few values of angle, they have to be true for all values.

ANSWER KEY

Exercise - 2

1(i).	$\theta = 30°, 90°$	1(ii).	$\theta = 60°$	1(iii).	$\theta = 60°$
1(iv).	$\theta = 0, 60°$	1(v).	$\theta = 60°$		

Exercise - 3

1.	(i)	cos 34°	(ii)	cot 9°	(iii)	cosec 17°
2.	(i)	1	(ii)	$\frac{1}{2}$	(iii)	3
3.	(i)	0	(ii)	5	(iii)	2
	(iv)	1	(v)	$\frac{9}{2}$		

9. 42°

12. $x^2 + x - 1$