MODEL PAPER - I

MATHEMATICS

Time allowed : 3 hours ***Maximum marks : 100***

General Instructions

1. All questions are compulsory.
2. The question paper consists of 29 questions divided into three sections A, B and C. Section A comprises of 10 questions of one mark each, Section B comprises of 12 questions of four marks each and Section C comprises of 7 questions of six marks each.
3. All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
4. There is no overall choice. However, internal choice has been provided in 4 questions of four marks each and 2 questions of six marks each. You have to attempt only one of the alternatives in all such questions.
5. Use of calculators is not permitted.

SECTION A

Question number 1 to 10 carry one mark each.

1. Find the value of x, if

$$\begin{pmatrix} 5x + y & -y \\ 2y - x & 3 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ -3 & 3 \end{pmatrix}$$

2. Let * be a binary operation on N given by a * b = HCF (a, b), a, b $\in$ N. Write the value of 6 * 4.

3. Evaluate : $\displaystyle\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1 - x^2}} dx$

4. Evaluate : $\int \frac{\sec^2(\log x)}{x} dx$

5. Write the principal value of $\cos^{-1}\left(\cos \frac{7\pi}{6}\right)$.

6. Write the value of the determinant :

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

7. Find the value of x from the following :

$$\begin{vmatrix} x & 4 \\ 2 & 2x \end{vmatrix} = 0$$

8. Find the value of $(\hat{i} \times \hat{j}) \cdot \hat{k} + (\hat{j} \times \hat{k}) \cdot \hat{i} + (\hat{k} \times \hat{i}) \cdot \hat{j}$

9. Write the direction cosines of the line equally inclined to the three coordinate axes.

10. If $\overrightarrow{p}$ is a unit vector and $(\overrightarrow{x} - \overrightarrow{p}).(\overrightarrow{x} + \overrightarrow{p}) = 80$, then find $|\overrightarrow{x}|$.

SECTION B

Question numbers 11 to 22 carry 4 marks each.

11. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When x = 8 cm and y = 6 cm, find the rate of change of (a) the perimeter, (b) the area of the rectangle.

OR

Find the intervals in which the function f given by f(x) = sin x + cos x, $0 \le x \le 2\pi$ is strictly increasing or strictly decreasing.

OR

12. If $(\cos x)^y = (\sin y)^x$, find $\frac{dy}{dx}$.

13. Consider $f : R - \left\{\frac{-4}{3}\right\} \to R - \left\{\frac{4}{3}\right\}$ defined as $f(x) = \frac{4x}{3x + 4}$

Show that f is invertible. Hence find f^{-1}.

14. Evaluate : $\int \frac{dx}{\sqrt{5 - 4x - 2x^2}}$.

OR

Evaluate : $\int x \sin^{-1} x \, dx$.

15. If $y = \frac{\sin^{-1} x}{\sqrt{1 - x^2}}$, show that $(1 - x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$.

16. The probability of student A passing an examination is $\frac{3}{5}$ and of student B passing is $\frac{4}{5}$. Find the probability of passing the examination by

   (i) both the students A and B

   (ii) atleast one of the students A and B.

   What ideal conditions a student should keep in mind while appearing in an examination?

17. Using properties of determinants, prove the following :

$$\begin{vmatrix} a + b + c & -c & -b \\ -c & a + b + c & -a \\ -b & -a & a + b + c \end{vmatrix} = 2(a + b)(b + c)(c + a)$$

18. Solve the following differential equation :

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right).$$

19. Solve the following differential equation :

$$\cos^2 x \cdot \frac{dy}{dx} + y = \tan x.$$

20. Find the shortest distance between the lines

$$\vec{r} = (1+\lambda)\hat{i} + (2-\lambda)\hat{j} + (\lambda+1)\hat{k}$$

$$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu\left(2\hat{i} + \hat{j} + 2\hat{k}\right).$$

21. Prove the following :

$$\cot^{-1}\left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right] = \frac{x}{2},\ x \in \left(0, \frac{\pi}{4}\right).$$

OR

Solve for x :

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$$

22. The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ.

OR

$\vec{a}$, $\vec{b}$ and $\vec{c}$ are three coplanar vectors. Show that $\left(\vec{a} + \vec{b}\right)$, $\left(\vec{b} + \vec{c}\right)$ and $\left(\vec{c} + \vec{a}\right)$ are also coplanar.

SECTION C

Question number 23 to 29 carry 6 marks each.

23. Find the equation of the plane determined by the points A (3, –1, 2), B (5, 2, 4) and C (–1, –1, 6). Also find the distance of the point P(6, 5, 9) from the plane.

24. Find the area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$.

25. Evaluate : $\displaystyle\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$.

26. Two schools A and B decided to award prizes to their students for three values – honesty, regularity and discipline. School A decided to award Rs. 11000 for three values to 5, 4 and 3 students respectively while school B decided to award Rs. 10700 for three values to 4, 3 and 5 students respectively. If all three prizes together amount to Rs. 2700, then

(i) Represent the above situation in matrix form and solve it by matrix method.

(ii) Which value you prefer to be awarded most and why?

OR

Obtain the inverse of the following matrix using elementary operations:

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}.$$

27. Coloured balls are distributed in three bags as shown in the following table :

Bag	*Colour of the Ball*		
	Red	***White***	***Black***
I	1	2	3
II	2	4	1
III	4	5	3

A bag is selected at random and then two balls are randomly drawn from the selected bag. They happen to be black and red. What is the probability that they came from bag I?

28. A dealer wishes to purchase a number of fans and sewing machines. He has only Rs. 5760 to invest and has space for at most 20 items. A fan costs him Rs. 360 and a sewing machine Rs. 240. His expectation is that he can sell a fan at a profit of Rs. 22 and a sewing machine at a profit of Rs. 18. Assuming that be can sell all the items that he can buy, how should he invest his money in order to maximise the profit? Formulate this as a linear programming problem and solve it graphically. What values are being promoted?

29. If the sum of the lengths of the hypotenuse and a side of a right-angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$.

OR

A tank with rectangular base and rectangular sides open at the top is to be constructed so that it's depts is 2m and volume is 8m^3. If building of tank cost Rs. 70 per sq metre for the base and Rs. 45 per sq metre for the sides. What is the cost of least expensive tank.

MODEL PAPER - I

SOLUTIONS AND MARKING SCHEME

SECTION A

Note : For 1 mark questions in Section A, full marks are given if answer is correct (i.e. the last step of the solution). Here, solution is given for your help.

Marks

1. We are given

$$\begin{bmatrix} 5x + y & -y \\ 2y - x & 3 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -5 & 3 \end{bmatrix}$$

$\therefore$ $5x + y = 4$ and $-y = 1$

$\therefore$ $y = -1$ and $5x - 1 = 4$

or $5x = 5$

$\therefore$ $x = 1$ ...(1)

2. 6 * 4 = HCF of 6 and 4 = 2. ...(1)

3. $$\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1-x^2}}\,dx = \left|\sin^{-1}x\right|_0^{1/\sqrt{2}}$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}0$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$ ...(1)

4. Let $$I = \int \frac{\sec^2(\log x)}{x}\,dx$$

Let $\log x = t$

then $\frac{1}{x}dx = dt$

Marks

or $\quad dx = x\,dt$

$\therefore \quad I = \int \sec^2 t\ dt$

$= \tan t + c$

$= \tan(\log x) + c$ ...(1)

5. $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \dfrac{5\pi}{6}\right)\right]$

$= \cos^{-1}\left[\cos\left(\dfrac{5\pi}{6}\right)\right]$

$= \dfrac{5\pi}{6}$ ...(1)

6. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix}$

$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix}$

$= 0$ ...(1)

7. Here $\begin{vmatrix} x & 4 \\ 2 & 2x \end{vmatrix} = 0$

or $\quad 2x^2 - 8 = 0$

or $\quad x^2 - 4 = 0$

$x = \pm 2$ ...(1)

8. $(\hat{i} \times \hat{j}) \cdot \hat{k} + (\hat{j} \times \hat{k}) \cdot \hat{i} + (\hat{k} \times \hat{i}) \cdot \hat{j}$

$= \hat{k} \cdot \hat{k} + \hat{i} \cdot \hat{i} + \hat{j} \cdot \hat{j}$

$= 1 + 1 + 1$

$= 3$ ...(1)

Marks

9. The d.c. of a line equally inclined to the coordinate axes are $\left(\frac{\pm 1}{\sqrt{3}}, \frac{\pm 1}{\sqrt{3}}, \frac{\pm 1}{\sqrt{3}}\right)$. ...(1)

10. $$(\overrightarrow{x} - \overrightarrow{p}).(\overrightarrow{x} + \overrightarrow{p}) = 80$$

$\therefore$ $$|\overrightarrow{x}|^2 - |\overrightarrow{p}|^2 = 80$$

As $\overrightarrow{p}$ is a unit vector,

$$|\overrightarrow{p}| = 1$$

$\therefore$ $$|\overrightarrow{x}|^2 - 1 = 80$$

or $$|\overrightarrow{x}|^2 = 81$$

$\therefore$ $$|x| = 9$$ ...(1)

SECTION B

11. Let P be the perimeter and A be the area of the rectangle at any time t, then

$$P = 2(x + y) \text{ and } A = xy$$

It is given that $\frac{dx}{dt} = -5$ cm/minute

and $\frac{dy}{dt} = 4$ cm/minute ...(1)

(i) We have $P = 2(x + y)$

$\therefore$ $$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

Marks

$$= 2\ (-5 + 4)$$

$$= -\ 2 \text{ cm/minute} \qquad \ldots(1\tfrac{1}{2})$$

(ii) We have $A = xy$

$$\therefore \quad \frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$= [8 \times 4 + 6\ (-5)] \qquad \ldots\ldots(\because\ x = 8 \text{ and } y = 6)$$

$$= (32 - 30)$$

$$= 2 \text{ cm}^2\text{/minute} \qquad \ldots(1\tfrac{1}{2})$$

OR

The given function is

$$f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$$

$$\therefore \quad f'(x) = \cos x - \sin x$$

$$= -\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x\right)$$

$$= -\sqrt{2}\left(\sin x \cos\frac{\pi}{4} - \cos x \sin\frac{\pi}{4}\right)$$

$$= -\sqrt{2}\sin\left(x - \frac{\pi}{4}\right) \qquad \ldots(1)$$

For strictly decreasing function,

$$f'(x) < 0$$

$$\therefore \quad -\sqrt{2}\sin\left(x - \frac{\pi}{4}\right) < 0$$

$$\text{or} \quad \sin\left(x - \frac{\pi}{4}\right) > 0$$

$$\text{or} \quad 0 < x - \frac{\pi}{4} < \pi$$

Marks

or $\frac{\pi}{4} < x < \pi + \frac{\pi}{4}$

or $\frac{\pi}{4} < x < \frac{5\pi}{4}$

Thus f(x) is a strictly decreasing function on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ ...(2)

As sin x and cos x are well defined in [0, 2π],

f (x) = sin x + cos x is an increasing function in the complement of interval $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

i.e., in $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$ ...(1)

12. We are given

$$(\cos x)^y = (\sin y)^x$$

Taking log of both sides, we get

$$y \log \cos x = x \log \sin y \quad ...(½)$$

Differentiating w.r.t. x, we get

$$y \cdot \frac{1}{\cos x} \cdot (-\sin x) + \log \cos x \cdot \frac{dy}{dx} = x \cdot \frac{1}{\sin y} \cdot (\cos y)\frac{dy}{dx} + \log \sin y \cdot 1 \quad ...(2)$$

or $-y \tan x + \log \cos x \frac{dy}{dx} = x \cot y \frac{dy}{dx} + \log \sin y$

$\Rightarrow \frac{dy}{dx}(\log \cos x - x \cot y) = \log \sin y + y \tan x$ ...(1)

$\therefore \frac{dy}{dx} = \frac{\log \sin y + y \tan x}{\log \cos x - x \cot y}$ ...(½)

Marks

13. For showing f is one-one ...(1½)

For showing f is onto ...(1)

As f is one-one and onto f is invertible ...(½)

For finding $f^{-1}(x) = \dfrac{4x}{4 - 3x}$ ...(1)

14. Let $I = \int \dfrac{dx}{\sqrt{5 - 4x - 2x^2}}$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{5}{2} - 2x - x^2}}$$ ...(½)

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{5}{2} - \left(2x + x^2 + 1 - 1\right)}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{7}{2} - (x + 1)^2}}$$ ...(1½)

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(\frac{7}{2}\right)^2 - (x + 1)^2}}$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x + 1}{\frac{\sqrt{7}}{\sqrt{2}}}\right) + c$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{\sqrt{2}\,(x + 1)}{\sqrt{7}}\right) + c$$ ...(2)

Marks

OR

Let $I = \int x \sin^{-1} x \; dx$

$$= \int \sin^{-1} x . x \; dx$$

$$= \sin^{-1} x \cdot \frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx \qquad ...(1)$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{-x^2}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \sqrt{1-x^2} dx - \frac{1}{2} \int \frac{dx}{\sqrt{1-x^2}} \qquad ...(1)$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \cdot \left[\frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x \right] - \frac{1}{2} \sin^{-1} x + c$$

...(1)

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{x}{4} \sqrt{1-x^2} + \frac{1}{4} \sin^{-1} x - \frac{1}{2} \sin^{-1} x + c$$

$$= \frac{x^2 \sin^{-1} x}{2} - \frac{1}{4} \sin^{-1} x + \frac{x}{4} \sqrt{1-x^2} + c \qquad ...(1)$$

15. We have

$$y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$$

Marks

$$\Rightarrow \qquad y\sqrt{1-x^2} = \sin^{-1} x$$

Differentiating w.r.t. x, we get

$$y.\frac{(-2x)}{2\sqrt{1-x^2}} + \sqrt{1-x^2}\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

or $$-xy + (1-x^2)\frac{dy}{dx} = 1$$ ...(1½)

Differentiating again,

$$-x\frac{dy}{dx} - y + (1-x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}(-2x) = 0$$

or $$(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$$ ...(2½)

which is the required result.

16. (i) P (both the students A and B pass the examination)

$= P(A \cap B)$ ...(½)

$= P(A)\ P(B)$ ...(½)

$= \frac{3}{5} \times \frac{4}{5} = \frac{12}{25}$ ...(½)

(ii) P (atleast one of the students A and B passes the examination)

= 1 – P (none of the students pass) ...(½)

$= 1 - \frac{1}{5} \times \frac{2}{5}$ ...(½)

$= 1 - \frac{2}{25}$ ...(½)

$= \frac{23}{25}$ ...(½)

Marks

When appearing in an examination, a student should have no intention of copying or cheating as it inculcates habit of dishonesty which leads to corruption and many other ills. ...(1)

17. LHS $\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$

$R_1 \rightarrow R_1 + R_2$

$R_2 \rightarrow R_2 + R_3$

$$\begin{vmatrix} a+b & a+b & -(a+b) \\ -(b+c) & b+c & b+c \\ -b & -a & a+b+c \end{vmatrix}$$ ...(2)

$$= (a+b)(b+c)\begin{vmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -b & -a & a+b+c \end{vmatrix}$$ ...(½)

$C_1 \rightarrow C_1 + C_3$

$$= (a+b)(b+c)\begin{vmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ c+a & -a & a+b+c \end{vmatrix}$$ ...(1)

$= 2(a + b)\ (b + c)\ (c + a)$ ...(½)

18. The given differential equation is

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$$

or $$\frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right)$$

Let $$y = zx$$

$\therefore$ $$\frac{dy}{dx} = z + x\frac{dz}{dx}$$ ...(1)

Marks

$\therefore \qquad z + x\dfrac{dz}{dx} = z - \tan z$

or $\qquad x\dfrac{dz}{dx} = -\tan z$ ...(1)

or $\qquad \int \cot z \, dz + \int \dfrac{dx}{x} = 0$ ...(½)

$\therefore \qquad \log \sin z + \log x = \log c$

or $\qquad \log (x \sin z) = \log c$ ...(1)

or $\qquad x \sin\left(\dfrac{y}{x}\right) = c$ ...(½)

which is the required solution.

19. The given differential equation is

$$\cos^2 x \frac{dy}{dx} + y = \tan x$$

or $\qquad \dfrac{dy}{dx} + \sec^2 x.\, y = \tan x.\, \sec^2 x$

It is a linear differential equation

Integrating factor $= e^{\int \sec^2 x \, dx} = e^{\tan x}$ ...(1)

$\therefore$ Solution of the differential equation is

$$y.e^{\tan x} = \int e^{\tan x}.\tan x \sec^2 x \, dx + c$$ ...(½)

Now, we find $\qquad I_1 = \int e^{\tan x} \cdot \tan x \sec^2 x \, dx$

Let $\qquad \tan x = t, \sec^2 x \, dx = dt$

Marks

$\therefore \quad I_1 = \int te^t \, dt$

$= t.e^t - \int e^t dt$

$= t \,.\, e^t - e^t$

$= (t - 1)e^t = (\tan x - 1)\, e^{\tan x}$ ...(2)

$\therefore$ From (i), solution is

$$y \,.\, e^{\tan x} = (\tan x - 1)\, e^{\tan x} + c$$

or $\quad y = (\tan x - 1) + ce^{-\tan x}$ ...(½)

20. Equations of the two lines are :

$$\vec{r} = (1 + \lambda)\hat{i} + (2 - \lambda)\hat{j} + (\lambda + 1)\hat{k}$$

or $\quad \vec{r} = \left(\hat{i} + 2\hat{j} + \hat{k}\right) + \lambda\left(\hat{i} - \hat{j} + \hat{k}\right)$ ...(i)

and $\quad \vec{r} = \left(2\hat{i} - \hat{j} - \hat{k}\right) + \mu\left(2\hat{i} + \hat{j} + 2\hat{k}\right)$ ...(ii)

Here $\quad \vec{a_1} = \hat{i} + 2\hat{j} + \hat{k}$ and $\vec{a_2} = 2\hat{i} - \hat{j} - \hat{k}$

and $\quad \vec{b_1} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{b_2} = 2\hat{i} + \hat{j} + 2\hat{k}$ ...(1)

$\therefore \quad \vec{a_2} - \vec{a_1} = \left(2\hat{i} - \hat{j} - \hat{k}\right) - \left(\hat{i} + 2\hat{j} + \hat{k}\right)$

$= \hat{i} - 3\hat{j} - 2\hat{k}$ ...(½)

and $\quad \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$

$= \hat{i}(-3) - \hat{j}(0) + \hat{k}(3)$

$= -3\hat{i} + 3\hat{k}$ ...(1)

Marks

$$\therefore \quad \left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right| = \sqrt{9+9} = 3\sqrt{2}$$

$$\therefore \quad \text{S.D. between the lines} = \frac{\left|\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right) \cdot \left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right)\right|}{\left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right|} \qquad \text{...(½)}$$

$$= \frac{\left|\left(\hat{i} - 3\hat{j} - 2\hat{k}\right) \cdot \left(-3\hat{i} + 3\hat{k}\right)\right|}{3\sqrt{2}}$$

$$= \frac{|-3-6|}{3\sqrt{2}}$$

$$= \frac{9}{3\sqrt{2}} = \frac{3}{\sqrt{2}} \text{ units} \qquad \text{...(1)}$$

21. $$\cot^{-1}\left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right]$$

$$= \cot^{-1}\left[\frac{\sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2} + \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}{\sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2} - \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}\right] \qquad \text{...(1)}$$

$$\text{...}\left[\because x \in \left(0, \frac{\pi}{4}\right)\right]$$

$$= \cot^{-1}\left[\frac{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) + \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) - \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}\right] \qquad \text{...(1)}$$

$$= \cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right) \qquad \text{...(1)}$$

$$= \cot^{-1}\left[\cot\left(\frac{x}{2}\right)\right]$$

$$= \frac{x}{2} \qquad \text{...(1)}$$

Marks

OR

The given equation is

$$2 \tan^{-1} (\cos x) = \tan^{-1} (2 \operatorname{cosec} x)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2 \cos x}{1 - \cos^2 x}\right) = \tan^{-1}(2 \operatorname{cosec} x) \quad \text{...(1½)}$$

$$\Rightarrow \quad \frac{2 \cos x}{\sin^2 x} = 2 \operatorname{cosec} x \quad \text{...(1)}$$

$$\Rightarrow \quad \cos x = \operatorname{cosec} x \,.\, \sin^2 x$$

$$\Rightarrow \quad \cos x = \sin x$$

$$\therefore \quad x = \frac{\pi}{4} \quad \text{...(1½)}$$

22. Unit vector along the sum of vectors

$$\overrightarrow{a} = 2\hat{i} + 4\hat{j} - 5\hat{k} \quad \text{and} \quad \overrightarrow{b} = \lambda\hat{i} + 2\hat{j} + 3\hat{k} \text{ is}$$

$$\frac{\overrightarrow{a} + \overrightarrow{b}}{\left|\overrightarrow{a} + \overrightarrow{b}\right|} = \frac{(2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2 + \lambda)^2 + 6^2 + (-2)^2}}$$

$$= \frac{(2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}} \quad \text{...(1½)}$$

We are given that dot product of above unit vector with the vector $\hat{i} + \hat{j} + \hat{k}$ is 1.

$$\therefore \quad \frac{(2 + \lambda)}{\sqrt{\lambda^2 + 4\lambda + 44}} \cdot 1 + \frac{6}{\sqrt{\lambda^2 + 4\lambda + 44}} - \frac{2}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1 \quad \text{...(1)}$$

$$\text{or} \quad 2 + \lambda + 6 - 2 = \sqrt{\lambda^2 + 4\lambda + 44}$$

$$\text{or} \quad (\lambda + 6)^2 = \lambda^2 + 4\lambda + 44$$

$$\text{or} \quad \lambda^2 + 12\lambda + 36 = \lambda^2 + 4\lambda + 44$$

Marks

or $\qquad 8\lambda = 8$

or $\qquad \lambda = 1$ ...(1½)

OR

$\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar $\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$ ...(½)

$\vec{a}+\vec{b}, \vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$ are coplanar if

$[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 0$ (½)

For showing

$$[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$$ (3)

SECTION C

23. Equation of the plane through the points A (3, – 1, 2), B (5, 2, 4) and C (–1, –1, 6) is

$$\begin{vmatrix} x-3 & y+1 & z-2 \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 0$$ ...(2½)

i.e. $\qquad 3x - 4y + 3z = 19$ ...(1½)

Distance of point (6, 5, 9) from plane $3x - 4y + 3z = 19$

$$= \frac{|18 - 20 + 27 - 19|}{\sqrt{9+16+9}}$$

$$= \frac{6}{\sqrt{34}} \text{ units}$$ ...(2)

24. The given parabola is $y^2 = x$ ....(i)

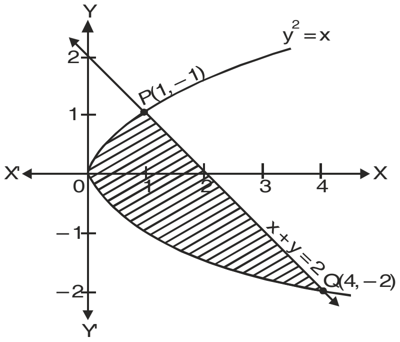
It represents a parabola with vertex at O (0, 0)

Marks

The given line is

$$x + y = 2$$

or $$x = 2 - y \qquad \text{....(ii)}$$



...(1)

Solving (i) and (ii), we get the point of intersection P (1, 1) and Q (4, – 2) ...(1)

Required area = Area of the shaded region

$$= \int_{-2}^{1} \left[(2 - y) - y^2\right] dy \qquad \text{...(2)}$$

$$= \left(2y - \frac{y^2}{2} - \frac{y^3}{3}\right)_{-2}^{1} \qquad \text{...(1)}$$

$$= \left[\left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right)\right]$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3} + 4 + 2 - \frac{8}{3}\right)$$

$$= \frac{12 - 3 - 2 + 24 + 12 - 16}{6}$$

$$= \frac{27}{6}$$

$$= \frac{9}{2} \text{ sq. units} \qquad \text{...(1)}$$

Marks

25. Let $I = \int_0^{\pi} \frac{x\ dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

or $I = \int_0^{\pi} \frac{(\pi - x)dx}{a^2 \cos^2(\pi - x) + b^2 \sin^2(\pi - x)}$

or $I = \int_0^{\pi} \frac{(\pi - x)dx}{a^2 \cos^2 x + b^2 \sin^2 x}$ ....(ii)

Adding (i) and (ii), we get

$2I = \pi\int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$ ....(iii) ...(1)

or $2I = \pi.2\int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

Using property $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$, If $f(2a - x) = f(x)$ ...(1)

or $I = \pi\int_0^{\pi/2} \frac{\sec^2 x\ dx}{a^2 + b^2 \tan^2 x}$ ...(1)

Let tan x = t then $\sec^2 x\ dx = dt$

When x = 0, t = 0 and when $x \to \frac{\pi}{2}, t \to \infty$

$\therefore \quad I = \pi\int_0^{\infty} \frac{dt}{a^2 + b^2 t^2}$ ...(1)

$= \frac{\pi}{b^2}\int_0^{\infty} \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}$

Marks

$$= \frac{\pi}{b^2} \cdot \frac{1}{a/b}\left[\tan^{-1}\frac{t}{a/b}\right]_0^{\infty}$$ ...(1)

$$= \frac{\pi}{ab}\left[\tan^{-1}\frac{bt}{a}\right]_0^{\infty}$$

$$= \frac{\pi}{ab}\left[\frac{\pi}{2}\right]$$

$$= \frac{\pi^2}{2ab}$$ ...(1)

26. Let the amount of prize for three values honesty, regularity and discipline be represented by x, y and z respectively. Then

$$5x + 4y + 3z = 11000$$

$$4x + 3y + 5z = 10700$$

$$x + y + z = 2700$$ ...(1½)

AX = B, where

$$A = \begin{bmatrix} 5 & 4 & 3 \\ 4 & 3 & 5 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 11000 \\ 10700 \\ 2700 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 5 & 4 & 3 \\ 4 & 3 & 5 \\ 1 & 1 & 1 \end{bmatrix} = -3 \neq 0$$ ...(½)

So, A^{-1} exists.

Now $$\text{adj } A = \begin{bmatrix} -2 & -1 & 11 \\ 1 & 2 & -13 \\ 1 & -1 & -1 \end{bmatrix}$$ ...(1)

Marks

$$A^{-1} = \frac{\text{adj } A}{|A|} = -\frac{1}{3}\begin{bmatrix} -2 & -1 & 11 \\ 1 & 2 & -13 \\ 1 & -1 & -1 \end{bmatrix}$$

$X = A^{-1}B$

So, $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -2 & -1 & 11 \\ 1 & 2 & -13 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 11000 \\ 10700 \\ 2700 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} -3000 \\ -2700 \\ -2400 \end{bmatrix} = \begin{bmatrix} 1000 \\ 900 \\ 800 \end{bmatrix}$$

So, $x = 1000,\ y = 900,\ z = 800$

i.e., The amount of prize for the values honesty, regularity and discipline are Rs. 1000, Rs. 900 and Rs. 800 respectively. ...(1)

(ii) I prefer honesty because corruption is the root cause of all problems for the citizens of the country. Honest persons are always disciplined and regular in approach. ...(2)

OR

26. By using elementary row transformations, we can write

$$A = IA$$

i.e., $$\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$ ...(1)

Applying $R_1 \to R_2 - R_2$, we get

$$\begin{bmatrix} 1 & -3 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$ ...(1)

Marks

Applying $R_2 \to R_2 - 2R_1$, we get

$$\begin{bmatrix} 1 & -3 & -1 \\ 0 & 9 & 2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \text{...(1)}$$

Applying $R_1 \to R_1 + R_3$, we get

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 9 & 2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \text{...(½)}$$

Applying $R_2 \to R_2 - 2R_3$, we get

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \text{...(½)}$$

Applying $R_1 \to R_1 - R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \text{...(½)}$$

Applying $R_3 \to R_3 - 4R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & 12 & 9 \end{bmatrix} A \qquad \text{...(½)}$$

$$A^{-1} = \begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & -12 & 9 \end{bmatrix} \qquad \text{...(1)}$$

Marks

27. Let the events be

E_1 : Bag I is selected

E_2 : Bag II is selected

E_3 : Bag III is selected

and A : a black and a red ball are drawn ...(1)

$$\therefore \quad P(E_1) = P(E_2) = P(E_3) = \frac{1}{3} \qquad ...(1)$$

$$P(A/E_1) = \frac{1 \times 3}{{}^6C_2} = \frac{3}{15} = \frac{1}{5}$$

$$P(A/E_2) = \frac{2 \times 1}{{}^7C_2} = \frac{2}{21}$$

$$P(A/E_3) = \frac{4 \times 3}{{}^{12}C_2} = \frac{4 \times 3}{66} = \frac{2}{11} \qquad ...(1½)$$

$$\therefore \quad P(E_1/A) = \frac{P(A/E_1).P(E_1)}{P(A/E_1)\ P/(E_1) + P(A/E_2)\ P(E_2) + P(A/E_3).P(E_3)} \qquad ...(1)$$

$$= \frac{\frac{1}{3} \times \frac{1}{5}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{2}{21} + \frac{1}{3} \times \frac{2}{11}} \qquad ...(½)$$

$$= \frac{\frac{1}{15}}{\frac{1}{15} + \frac{2}{63} + \frac{2}{33}}$$

$$= \frac{\frac{1}{15}}{\frac{551}{3465}}$$

Marks

$$= \frac{1}{15} \times \frac{3465}{551} = \frac{231}{551}$$ ...(1)

28. Let us suppose that the dealer buys x fans and y sewing machines,

Thus L.P. problem is

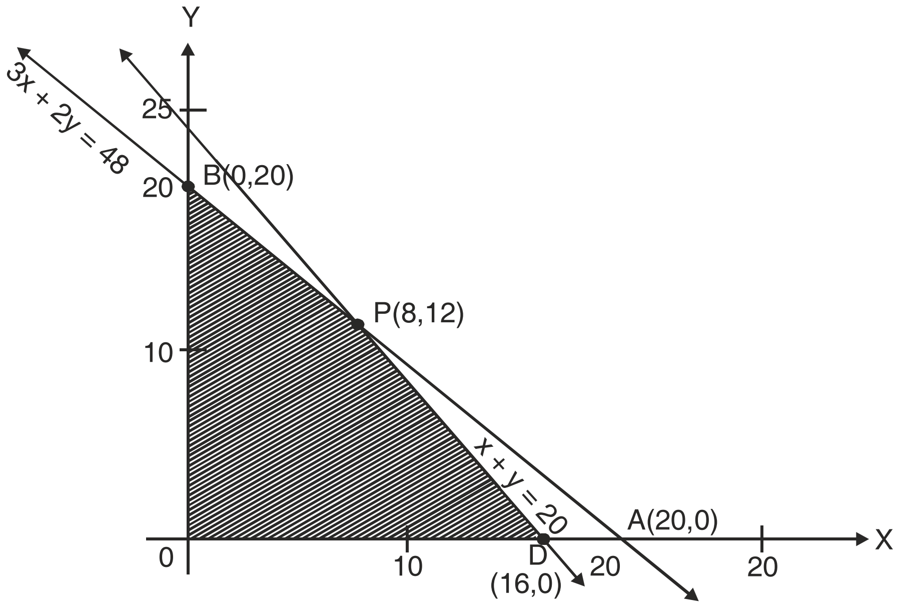
Maximise $Z = 22x + 18y$ ...(½)

subject to constraints,

$$x + y \leq 20$$

$$360x + 240y \leq 5760 \text{ or } 3x + 2y \leq 48$$

$$x \geq 0,\ y \geq 0$$ ...(1½)



For correct graph ...(1½)

The feasible region ODPB of the L.P.P. is the shaded region which has the corners O (0, 0), D (16, 0), P (8, 12) and B (0, 20)

The values of the objective function Z at O, D, P and B are :

Marks

At O, $Z = 22 \times 0 + 18 \times 0 = 0$

At D, $Z = 22 \times 16 + 18 \times 0 = 352$

At P, $Z = 22 \times 8 + 18 \times 12 = 392 \rightarrow$ Maximum

and At B, $Z = 22 \times 0 + 18 \times 20 = 360$

Thus Z is maximum at $x = 8$ and $y = 12$ and the maximum value of $z =$ Rs 392.

Hence the dealer should purchase 8 fans and 12 sewing machines to obtain maximum profit. ...(½)

Values promoted are the maximum utility of money and space of storage. ...(2)

29. Let ABC be a right angled triangle with base $BC = x$ and hypotenuse $AB = y$

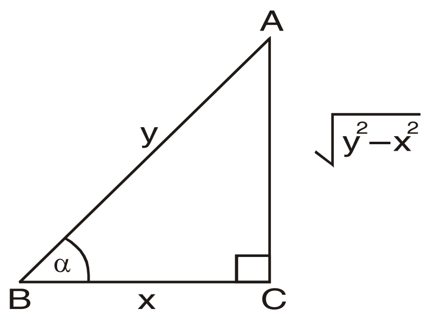
such that

$$x + y = k \text{ where } k \text{ is a constant} \quad ...(½)$$

Let α be the angle between the base and the hypotenuse.

The area of triangle, $A = \frac{1}{2} BC \times AC$

$$= \frac{1}{2} x \sqrt{y^2 - x^2}$$



...(1½)

Marks

$$\therefore \qquad A^2 = \frac{x^2}{4}(y^2 - x^2)$$

$$= \frac{x^2}{4}\left[(k - x)^2 - x^2\right]$$

or $$A^2 = \frac{x^2}{4}\left[k^2 - 2kx\right] = \frac{k^2x^2 - 2kx^3}{4} \quad ..(i)$$

Differentiating w.r.t. x we get

$$2A\frac{dA}{dx} = \frac{2k^2x - 6kx^2}{4} \qquad ....(ii)$$

or $$\frac{dA}{dx} = \frac{k^2x - 3kx^2}{4A} \qquad ...(1)$$

For maximum or minimum,

$$\frac{dA}{dx} = 0$$

$$\Rightarrow \quad \frac{k^2x - 3kx^2}{4} = 0$$

$$\Rightarrow \quad x = \frac{k}{3} \qquad ...(1)$$

Differentiating (ii) w.r.t.x. we get

$$2\left(\frac{dA}{dx}\right)^2 + 2A\frac{d^2A}{dx^2} = \frac{2k^2 - 12kx}{4}$$

Putting, $\frac{dA}{dx} = 0$ and $x = \frac{k}{3}$, we get

$$\frac{d^2A}{dx^2} = \frac{-k^2}{4A} < 0$$

Marks

$\therefore$ A is maximum when $x = \frac{k}{3}$ ...(1)

Now $x = \frac{k}{3} \Rightarrow y = k - \frac{k}{3} = \frac{2k}{3}$

$\therefore$ $\cos\alpha = \frac{x}{y} \Rightarrow \cos\alpha = \frac{k/3}{2k/3} = \frac{1}{2}$

$\alpha = \frac{\pi}{3}$ ...(1)

OR

Let the length of the tank be x metres and breadth by y metres

$\therefore$ Depth of the tank = 2 metre

$\therefore$ Volume = $x \times y \times 2 = 8$

$xy = 4$

or $y = \frac{4}{x}$ ...(1)

Area of base = xy sq m

Area of 4 walls = $2\,[2x + 2y] = 4\,(x + y)$

$\therefore$ Cost $C(x, y) = 70\,(xy) + 45\,(4x + 4y)$

or $C(x, y) = 70 \times 4 + 180\,(x + y)$ ...(1)

$\therefore$ $C(x) = 280 + 180\left(x + \frac{4}{x}\right)$ ...(½)

Now $\frac{dC}{dx} = 180\left(1 - \frac{4}{x^2}\right)$ ...(1)

For maximum or minimum, $\frac{dC}{dx} = 0$

$\therefore$ $180\left(1 - \frac{4}{x^2}\right) = 0$

Marks

or $x^2 = 4$

or $x = 2$ ...(½)

and $$\frac{d^2C}{dx^2} = 180\left(\frac{8}{x^3}\right) > 0$$

$$\left.\frac{d^2C}{dx^2}\right|_{x=2} = 180\left(\frac{8}{8}\right) > 0$$ ...(1)

$\therefore$ C is minimum at $x = 2$

Least Cost = Rs [(280 + 180 (2 + 2)]

= Rs [(280 + 720] = Rs 1000 ...(1)