Probability

Conditional Probability

Key Concepts

- If E and F are two events associated with the sample space of a random experiment, then the conditional probability of the event E given that F has occurred is denoted by P(E|F) and it is calculated by using the formula:

  $$P(E|F) = \frac{P(E \cap F)}{P(F)}, \text{ where } P(F) \neq 0$$

- Properties of conditional probability:

- If F is an event of a sample space S of an experiment, then
  P(S|F) = P(F|F) = 1

- If A and B are any two events of a sample space S and F is any event that has occurred earlier, such that P(F) ≠ 0, then
  $P((A \cup B)|F) = P(A|F) + P(B|F) - P((A \cap B)|F)$
  In particular, if A and B are disjoint events, then
  $P((A \cup B)|F) = P(A|F) + P(B|F)$

- If E and F are any two events of a sample space S and F is any event that has occurred earlier, such that P(F) ≠ 0, then
  $P(E'|F) = 1 - P(E|F)$

Solved Examples

Example 1:

Two numbers are selected at random from the integers 1 to 12. If the sum is even, then find the probability that both numbers are even.

Solution:

Out of the numbers 1 to 12, there are 6 even numbers and 6 odd numbers.

Let A = Event of choosing two even numbers from 1 to 12.

B = Event of choosing two numbers whose sum is even.

n(A) = Numbers of ways of choosing two even numbers out of 6 even numbers = $^6C_2$

n(B) = Numbers of ways of choosing two numbers out of 1 to 12 whose sum is even

= $^6C_2$ [For even pairs] + $^6C_2$ [For odd pair]
= 2 $^6C_2$

Now, n(A∩B) = Numbers of ways of choosing two even numbers out of 1 to 12 whose sum is even = $^6C_2$

Let S be the sample space.

∴ n(S) = Numbers of ways of choosing 2 numbers out of 12 numbers = $^{12}C_2$

$$\therefore P(A)=\frac{n(A)}{n(S)}=\frac{^6C_2}{^{12}C_2}=\frac{5}{22}$$
$$P(B)=\frac{n(B)}{n(S)}=2\times\frac{5}{22}=\frac{5}{11}$$
$$P(A\cap B)=\frac{n(A\cap B)}{n(S)}=\frac{^6C_2}{^{12}C_2}=\frac{5}{22}$$

Therefore, the required probability is $P(A\mid B)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{5}{22}}{\frac{5}{11}}=\frac{1}{2}$

Example 2:

Let A and B be two events such that P(A) = 0.3, P(B) = 0.5, and P(B|A) = 0.4. Find P(A∪B) and $P(A'|B')$.

Solution:

We know that,

$$P(B|A)=\frac{P(B\cap A)}{P(A)}$$
$$\Rightarrow 0.4=\frac{P(A\cap B)}{0.3}$$

⇒ P(A∩B) = 0.4 × 0.3
= 0.12

$\therefore P(A\cup B) = P(A) + P(B) - P(A\cap B)$
$= 0.3 + 0.5 - 0.12$
$= 0.68$

$$P(A'|B') = \frac{P(A'\cap B')}{P(B')} = \frac{P(A\cup B)'}{1-P(B)} = \frac{1-P(A\cup B)}{1-0.5} = \frac{1-0.68}{0.5} = 0.64$$

Multiplication Theorem on Probability

Key Concepts

- In an experiment, the conditional probability of event E, given that F has already occurred, is given as

  $$P(E|F) = \frac{P(E\cap F)}{P(F)} \quad [\text{where } P(F) \neq 0]$$

  Or, $P(E\cap F) = P(F).P(E|F)$

  Similarly, the conditional probability of event F, given that E has already occurred, is given as

  $$P(F|E) = \frac{P(F\cap E)}{P(E)} \quad [\text{where } P(E) \neq 0]$$

  $$P(F|E) = \frac{P(E\cap F)}{P(E)} \quad [\text{As } (F\cap E) = (E\cap F)]$$

  Or, $P(E\cap F) = P(E).P(F|E)$

  Hence, $P(E\cap F) = P(F).P(E|F) = P(E).P(F|E)$, where $P(E) \neq 0$, $P(F) \neq 0$

  This is known as multiplication theorem on probability for two events.

- The multiplication theorem on probability for three events E, F, and G is given by,

  $P(E\cap F\cap G) = P(E).P(F|E).P(G|E\cap F) = P(E).P(F|E).P(G|EF)$

  Similarly, multiplication rule of probability can be extended for more than 3 events.

Solved Examples

Example 1:

On 9 different cards, the numbers 1 to 9 are written. Four cards are drawn successively, without replacement, from these cards. Find the probability that the first two cards show even numbers and the last two cards show odd numbers.

Solution:

Let E denote the event that the number on the card is even and O denote the event that the number on the card is odd.

Clearly, we have to find P(EEOO).

Out of numbers 1 to 9, there are 4 even numbers.

∴ P(E) = Probability of selecting a card that shows an even number = $\frac{4}{9}$

Also, P(E|E) is the probability of selecting the second card that shows even number with the condition that first card drawn shows an even number.

Now, we are left with 9 − 1 = 8 cards, out of which 4 − 1 = 3 cards are even.

$$P(E \mid E) = \frac{3}{8}$$

Now, P(O|EE) is the probability of selecting the third card that shows an odd number with the condition that the first two cards show even numbers.

Number of cards = 8 − 1 = 7, out of which 5 cards are odd

$$P(O \mid EE) = \frac{5}{7}$$

Similarly, $P(O \mid EEO) = \frac{4}{6}$

Hence,

P(EEOO) = P(E). P(E|E). P(O|EE). P(O|EEO) $= \frac{4}{9} \times \frac{3}{8} \times \frac{5}{7} \times \frac{4}{6} = \frac{5}{63}$

Example 2:

For two events A and B, if P(AB) = 0.1, P(A|B) = 0.2, P(B|A) = 0.25, then find $P(A'B')$.

Solution:

We know that,

P(AB) = P(A).P(B|A) and P(AB) = P(B).P(A|B)

$$\Rightarrow 0.1 = P(A) \times 0.25 \text{ and } 0.1 = P(B) \times 0.2$$

$$\Rightarrow P(A) = \frac{0.1}{0.25} = 0.4 \text{ and } P(B) = \frac{0.1}{0.2} = 0.5$$

$$\begin{aligned} \text{Now, } P(A'B') &= P(A' \cap B') = P(A \cup B)' = 1 - P(A \cup B) \\ &= 1 - [P(A) + P(B) - P(AB)] \\ &= 1 - [0.5 + 0.4 - 0.1] \\ &= 1 - 0.8 \\ &= 0.2 \end{aligned}$$

Independent Events

Key Concepts:

- Two events are said to be independent events, if occurrence of one of them is not affected by the other.
- E and F are independent events, if P(E ∩ F) i.e., P(EF) = P(E) · P(F)
- If P(EF) ≠ P(E) · P(F), then event E and F are not independent events.
- Three events E, F, and G are said to be independent events, if

  P (EF) = P(E) P(F)

  P (FG) = P(F) · P(G)

  P (EG) = P(E) P(G)

  P (EFG) = P(E) · P(F) · P(G)

- **Note:**

  Independent events and mutually exclusive events are not same. If A and B are independent events, such that P(A) ≠ 0 and P (B) ≠ 0, then

$P(AB) = P(A) \cdot P(B) \neq 0$

If A and B are mutually exclusive events, then $A \cap B = \Phi$

$\therefore P(AB) = 0$

This shows that independent events and mutually exclusive events are not same.

Solved Examples

Example1:

If E and F are two events on a sample space such that $P(E'F') = 0.3$, $P(E|F) = 0.5$, and $P(F|E) = 0.4$, then show that E and F are independent events.

Solution:

We have,

$P(E|F) = 0.5$ and $P(F|E) = 0.4$

We know that,

$$P(E|F) = \frac{P(EF)}{P(F)}$$

$$\Rightarrow 0.5 = \frac{P(EF)}{P(F)}$$

$$\Rightarrow P(F) = \frac{P(EF)}{0.5} = 2P(EF)$$

We also know that,

$$P(F|E) = \frac{P(EF)}{P(E)}$$

$$\Rightarrow 0.4 = \frac{P(EF)}{P(E)}$$

$$\Rightarrow P(E) = \frac{P(EF)}{0.4} = \frac{5}{2}P(EF)$$

It is given that,

$$P(E'F') = P(E' \cap F') = 0.3$$
$$\Rightarrow P(E \cup F)' = 0.3$$
$$\Rightarrow 1 - P(E \cup F) = 0.3$$
$$\Rightarrow P(E \cup F) = 1 - 0.3 = 0.7$$
$$\Rightarrow P(E) + P(F) - P(EF) = 0.7$$
$$\Rightarrow \frac{5}{2}P(EF) + 2P(EF) - P(EF) = 0.7$$
$$\Rightarrow \frac{7}{2}P(EF) = 0.7$$
$$\Rightarrow P(EF) = 0.2$$
$$\Rightarrow P(E) = \frac{5}{2} \times 0.2 = 0.5$$
$$\text{and } P(F) = 2 \times 0.2 = 0.4$$

We have,

$P(E) \cdot P(F) = 0.5 \times 0.4 = 0.2 = P(EF)$

Hence, E and F are independent events.

Example 2:

A number is selected out of first thirty natural numbers. Let A, B, and C be denoted by the following events.

A : The number is a multiple of 2

B: The number is a multiple of 3

C: The number is a multiple of 5

Show that A, B, and C are independent events.

Solution:

The sample space S is given by,

S = {1, 2, 3, ….. 30}

$\therefore n(S) = 30$

We have,

A= {2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30}

∴ n (A) = 15

B= {3, 6, 9, 12, 15, 18, 21, 24, 27, 30}

∴ n (B) = 10

C= {5, 10, 15, 20, 25, 30}

∴ n (C) = 6

Now,

$$P(A) = \frac{15}{30} = \frac{1}{2}$$
$$P(B) = \frac{10}{30} = \frac{1}{3}$$
$$P(C) = \frac{6}{30} = \frac{1}{5}$$

Now,

A ∩ B = {6, 12, 18, 24, 30}

∴n (A ∩ B) = 5

B ∩ C = {15, 30}

∴n (B ∩ C) = 2

A ∩ C = {10, 20, 30}

∴n (A ∩ C) = 3

A ∩ B ∩ C = {30}

∴n (A ∩ B ∩ C) = 1

Therefore,

$$P(AB) = \frac{5}{30} = \frac{1}{6} = \frac{1}{2} \times \frac{1}{3} = P(A) \cdot P(B)$$
$$P(BC) = \frac{2}{30} = \frac{1}{15} = \frac{1}{3} \times \frac{1}{5} = P(B) \cdot P(C)$$
$$P(AC) = \frac{3}{30} = \frac{1}{10} = \frac{1}{2} \times \frac{1}{5} = P(A) \cdot P(C)$$
$$P(ABC) = \frac{1}{30} = \frac{1}{2} \times \frac{1}{3} \times \frac{1}{5} = P(A) \cdot P(B) \cdot P(C)$$

This shows that A, B, and C are independent events.

Bayes' Theorem

Key Concepts

- A set of events $E_1, E_2 \ldots E_n$ is said to represent a partition of the sample space S, if it satisfies the following conditions:
- $E_i \cap E_j = \phi, i \neq j;\ i, j = 1, 2, 3 \ldots n$
- $\bigcup_{i=1}^{n} E_i = S$
- $P(E_i) > 0, \forall i = 1, 2, 3 \ldots n$
- Let $\{E_1, E_2 \ldots E_n\}$ be a partition of the sample space S and suppose that each of the events $E_1, E_2 \ldots E_n$ has non-zero probability of occurrence. Let A be any event associated with S. Then,

$$P(A) = \sum_{i=1}^{n} P(E_i) \cdot P\left(\frac{A}{E_i}\right)$$

This is known as theorem of total probability.

- If $E_1, E_2 \ldots E_n$ are n non-empty events, which constitute a partition of sample space S, and A is any event of non-zero probability, then

$$P\left(\frac{E_i}{A}\right)=\frac{P(E_i)\cdot P\left(\frac{A}{E_i}\right)}{\sum_{j=1}^{n}P(E_j)P\left(\frac{A}{E_j}\right)},\ j=0,1,2\ldots n$$

This is called Bayes' theorem.

- The events E_1, E_2 … E_n are called hypothesis.
- The probability P (E_i) is called the prior probability of the hypothesis E_i.
- The conditional probability $P\left(\frac{E_i}{A}\right)$ is called the posteriori probability of the hypothesis E_i.

Solved Examples

Example 1:

There are three bags − bag **I**, bag **II**, and bag **III**. Bag **I** contains 4 silver and 6 gold coins, bag **II** contains 6 silver and 4 gold coins, and bag **III** contains 5 silver and 5 gold coins. A bag is chosen at random and a coin is drawn from it. If the drawn coin is gold coin, then find the probability that the coin drawn is from bag **III**.

Solution:

Let E_1, E_2, E_3 be the events of choosing bag **I**, bag **II**, and bag **III** respectively.

$$\therefore P(E_1)=P(E_2)=P(E_3)=\frac{1}{3}$$

Let E be the event of drawing a gold coin.

$$\therefore P\left(\frac{E}{E_1}\right)=\frac{6}{4+6}=\frac{3}{5}$$

$$P\left(\frac{E}{E_2}\right)=\frac{4}{6+4}=\frac{2}{5}$$

$$P\left(\frac{E}{E_3}\right)=\frac{5}{5+5}=\frac{1}{2}$$

Using Bayes' theorem,

$P\left(\frac{E}{E_3}\right) =$ Probability that the coin drawn is from bag **III**, given it is a gold coin

$$= \frac{P(E_3)\cdot P\left(\frac{E}{E_3}\right)}{P(E_1)\cdot P\left(\frac{E}{E_1}\right)+P(E_2)\cdot P\left(\frac{E}{E_2}\right)+P(E_3)\cdot P\left(\frac{E}{E_3}\right)}$$

$$= \frac{\frac{1}{3}\cdot\frac{1}{2}}{\frac{1}{3}\cdot\frac{3}{5}+\frac{1}{3}\cdot\frac{2}{5}+\frac{1}{3}\times\frac{1}{2}}$$

$$= \frac{\frac{1}{3}\cdot\frac{1}{2}}{\frac{1}{3}\left(\frac{3}{5}+\frac{2}{5}+\frac{1}{2}\right)}$$

$$= \frac{\frac{1}{2}}{\frac{3}{2}}$$

$$= \frac{1}{3}$$

Thus, the probability that the coin drawn is from bag **III** is $\frac{1}{3}$.

Example 2:

In a school, 200 students are girls and 300 students are boys. Out of these students, 10% of girls and 8% of boys are taller than 1.70 m. If a student is selected at random and is taller than 1.70 m, then find the probability that the selected student is a girl.

Solution:

Let E_1 and E_2 be the events of choosing a girl and a boy respectively.

Total number of students = 200 + 300 = 500

$$\therefore P(E_1) = \frac{200}{500} = \frac{2}{5}$$

$$P(E_2) = \frac{300}{500} = \frac{3}{5}$$

Let E be the event of choosing a student taller than 1.7 m.

$$\therefore \quad P\left(\frac{E}{E_1}\right)=\frac{10}{100}$$

$$P\left(\frac{E}{E_2}\right)=\frac{8}{100}$$

The probability that the selected student is a girl, given that the student is taller than 1.70 m, is given by $P\left(\frac{E_1}{E}\right)$.

Using Bayes' theorem,

$$P\left(\frac{E_1}{E}\right)=\frac{P(E_1)\cdot P\left(\frac{E}{E_1}\right)}{P(E_1)\cdot P\left(\frac{E}{E_1}\right)+P(E_2)\cdot P\left(\frac{E}{E_2}\right)}$$

$$=\frac{\frac{2}{5}\cdot\frac{10}{100}}{\frac{2}{5}\cdot\frac{10}{100}+\frac{3}{5}\cdot\frac{8}{100}}$$

$$=\frac{\frac{1}{500}(20)}{\frac{1}{500}(20+24)}$$

$$=\frac{20}{44}$$

$$=\frac{5}{11}$$

Thus, the probability that the selected student is a girl is $\frac{5}{11}$.

Random Variable and its Probability Distribution

Random Variable

- A random variable is a real-valued function whose domain is the sample space of a random experiment.

For example, if a coin is tossed thrice in succession, then the sample space *S* of the experiment is given by,

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

If *X* denotes the number of tails obtained, then *X* is a random variable for each outcome. *X* is given by:

X(TTT) = 3, *X*(HTT) = *X*(THT) = *X*(TTH) = X(TTH) = 2, *X*(HHT) =1, *X*(HHT) = *X*(HTH) = *X*(THH) = 1, *X*(HHH) = 0

Probability Distribution of a Random Variable

- The probability distribution of a random variable *X* is the system of numbers.

X	x_1	x_2	...	x_n
P*(X)*	p_1	p_2	...	p_n

Where, $p_i > 0, \sum_{i=1}^{n} p_i = 1, \ i = 1, 2, \ldots, n$

Solved Examples

Example 1:

4 cards are drawn successively (without replacement) from a well-shuffled pack of 52 cards. If *X* represents the number of red face cards drawn, then find the probability distribution of random variable *X*.

Solution:

In a deck of 52 cards, 6 cards are red face cards and remaining 46 cards are not red face cards. Since 4 cards are drawn successively (without replacement) and *X* represents the number of face cards drawn, values of *X* are 0, 1, 2, 4.

Then,

P(X = 0) = P(no red face card) = $\dfrac{{}^{6}C_0 \times {}^{46}C_4}{{}^{52}C_4}$

$$= \frac{1 \times \dfrac{46!}{42! \times 4!}}{\dfrac{52!}{48! \times 4!}}$$

$$= \frac{43 \times 44 \times 45 \times 46}{49 \times 50 \times 51 \times 52}$$

$$= \frac{32637}{54145}$$

P(X = 1) = P (1 red face card and 3 other cards)

$$= \frac{{}^{6}C_1 \times {}^{46}C_3}{{}^{52}C_4}$$

$$= \frac{6 \times \dfrac{46!}{43! \times 3!}}{\dfrac{52!}{48! \times 4!}}$$

$$= \frac{6 \times 44 \times 45 \times 46 \times 4}{49 \times 50 \times 51 \times 52}$$

$$= \frac{18216}{54145}$$

P(X = 2) = P (2 red face cards and 2 other cards)

$$= \frac{{}^{6}C_2 \times {}^{46}C_2}{{}^{52}C_4}$$

$$= \frac{15 \times \dfrac{45 \times 46}{2}}{\dfrac{52!}{48! \times 4!}}$$

$$= \frac{15 \times 45 \times 23 \times 24}{49 \times 50 \times 51 \times 52}$$

$$= \frac{624}{10829}$$

P(X = 3) = P (3 red face cards and 1 other card)

$$= \frac{{}^{6}C_{3} \times {}^{46}C_{1}}{{}^{52}C_{4}}$$

$$= \frac{20 \times 46}{\frac{52!}{48! \times 4!}}$$

$$= \frac{20 \times 46 \times 24}{49 \times 50 \times 51 \times 52}$$

$$= \frac{184}{54145}$$

P(X = 4) = P (4 red face cards)

$$= \frac{{}^{6}C_{4}}{{}^{52}C_{4}}$$

$$= \frac{15}{\frac{52!}{48! \times 4!}}$$

$$= \frac{15 \times 24}{49 \times 50 \times 51 \times 52}$$

$$= \frac{3}{54145}$$

Therefore, the probability of distribution of the random variable X is given by:

X	0	1	2	3	4
P*(X)*	$\frac{32637}{54145}$	$\frac{18216}{54145}$	$\frac{621}{10829}$	$\frac{184}{54145}$	$\frac{3}{54145}$

Example 2:

A random variable X has the following probability distribution with missing P(X = 0), P(X = 2), and P(X = 3).

X	0	1	2	3

P(*X*)		$\frac{1}{6}$		

If $P(X \le 1) = P(X \ge 2)$ and $P(X = 0) + P(X = 2) = \frac{11}{24}$, then find $P(X = 0)$, $P(X = 2)$, and $P(X = 3)$.

Solution:

We have,

$P(X \le 1) = P(X \ge 2)$

$\Rightarrow P(X = 0) + P(X = 1) = P(X = 2) + P(X = 3)$ … (i)

However, $[P(X = 0) + P(X = 1)] + [P(X = 2) + P(X = 3)] = 1$ … (ii)

Thus, from equations (i) and (ii), we obtain

$P(X = 0) + P(X = 1) = P(X = 2) + P(X = 3) = \frac{1}{2}$

Now, $P(X = 0) + P(X = 1) = \frac{1}{2}$

$\Rightarrow P(X = 0) + \frac{1}{6} = \frac{1}{2}$

$\Rightarrow P(X = 0) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

It is also given that,

$P(X = 0) + P(X = 2) = \frac{11}{24}$

$$\Rightarrow \frac{1}{3} + P(X = 2) = \frac{11}{24}$$

$$\Rightarrow P(X = 2) = \frac{11}{24} - \frac{1}{3} = \frac{1}{8}$$

However, P(X = 2) + P(X = 3) = $\frac{1}{2}$

$\Rightarrow \frac{1}{8}$ + P(X = 3) = $\frac{1}{2}$

$\Rightarrow$ P(X = 3) = $\frac{1}{2}-\frac{1}{8}=\frac{3}{8}$

Thus, P (X = 0) = $\frac{1}{3}$, P (X = 2) = $\frac{1}{8}$, P (X = 3) = $\frac{3}{8}$

Mean and Variance of Random Variables

Mean of Random Variables

- Let X be a random variable whose possible values $x_1, x_2 \ldots x_n$ occur with probabilities $p_1, p_2 \ldots p_n$ respectively. The mean of X (denoted by μ) or the expectation of X [denoted by E(X)] is the number $\sum_{i=1}^{n} x_i p_i$. This means $E(X) = \mu = \sum_{i=1}^{n} x_i p_i = x_1p_1 + x_2p_2 + \ldots + x_np_n$
- For example: Consider a random variable X with following probabilities.

***X* or x_i**	0	1	2
***P*(*X*) or p_i**	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Hence, $E(X) = \mu = x_1p_1 + x_2p_2 + x_3p_3$

$= 0\times\frac{1}{4}+1\times\frac{1}{2}+2\times\frac{1}{4}$

= 1

Variance of a Random Variable

- Let X be a random variable with possible values $x_1, x_2 \ldots. x_n$ and corresponding probabilities $p(x_1), p(x_2), \ldots. p(x_n)$. Let $\mu = E(X)$ be the mean of X. The variance of X (denoted by Var(X) or σ_x^2) is defined as:

$$\sigma_x^2 = \mathrm{Var}(X) = \sum_{i=1}^{n}(x_i - \mu)^2 p(x_i)$$

Or, $\sigma_x^2 = E(X-\mu)^2$

- Another method of finding variance of X is:

σ_x^2 = Var(X) = $E(X^2) - [E(X)]^2$, where $E(X^2) = \sum_{i=1}^{n} x_i^2 p(x_i)$ and $E(X) = \sum_{i=1}^{n} x_i p(x_i)$

- The square root of variance of X is called the standard deviation of variable X.

Solved Examples

Example 1:

The random variable X has a probability distribution $P(X)$ of the following form, where k is some number.

$$P(X) = \begin{cases} (5-x)k, & \text{if } x = 1 \text{ or } 4 \\ kx, & \text{if } x = 2 \text{ or } x = 3 \\ 0, & \text{otherwise} \end{cases}$$

Find the value of k, $E(X)$, Var(X), and Standard deviation of random variable X.

Solution:

The probability distribution of the random variable X is as follows:

X	1	2	3	4

P(*X*)	$4k$	$2k$	$3k$	k

We know that $\sum_{i=1}^{n} p_i = 1$

$\Rightarrow 4k + 2k + 3k + k = 10k = 1$

$\Rightarrow k = 0.1$

Now, $E(X) = \sum_{i=1}^{n} x_i p_i = 1 \times 4k + 2 \times 2k + 3 \times 3k + 4 \times k$

$= 21k$

$= 21 \times 0.1$

$= 2.1$

$E(X^2) = \sum_{i=1}^{n} x_i^2 p_i = (1)^2 \times 4k + (2)^2 \times 2k + (3)^2 \times 3k + (4)^2 \times k$

$= 55k$

$= 55 \times 0.1$

$= 5.5$

$\therefore \text{Var}(X) = E(X^2) - [E(X)]^2$

$= 5.5 - (2.1)^2$

$= 1.09$

S.D. $(X) = \sqrt{\text{Var}(X)} = \sqrt{1.09} = 1.04$

Example 2:

Three cards are drawn simultaneously (or successively without replacement) from a well-shuffled pack of 52 cards. Find the standard deviation of the number of red cards.

Solution:

Let X denote the number of red cards in a draw of three cards. Therefore, X is a random variable, which can assume the values 0, 1, 2, or 3.

Now, P(X = 0) = P(no red cards) = $\dfrac{{}^{26}C_3}{{}^{52}C_3} = \dfrac{\frac{26!}{23! \times 3!}}{\frac{52!}{49! \times 3!}} = \dfrac{24 \times 25 \times 26}{50 \times 51 \times 52} = \dfrac{2}{17}$

P (X = 1) = P(1 red card) $= \dfrac{{}^{26}C_1 \times {}^{26}C_2}{{}^{52}C_3}$

$$= \frac{26 \times \frac{26!}{24! \times 2!}}{\frac{52!}{49! \times 3!}}$$

$$= \frac{26 \times 25 \times 26 \times 3}{50 \times 51 \times 52}$$

$$= \frac{13}{34}$$

P(X = 2) = P(2 red cards) $= \dfrac{{}^{26}C_2 \times {}^{26}C_1}{{}^{52}C_3} = \dfrac{13}{34}$

P (X = 3) = P(3 red cards) = $\dfrac{{}^{26}C_3}{{}^{52}C_3} = \dfrac{2}{17}$

Thus, the probability distribution of X is:

X	0	1	2	3
P(*X*)	$\frac{2}{17}$	$\frac{13}{34}$	$\frac{13}{34}$	$\frac{2}{17}$

$\therefore E(X) = \sum_{i=1}^{n} x_i p(x_i)$

$= 0 \times \frac{2}{17} + 1 \times \frac{13}{34} + 2 \times \frac{13}{34} + 3 \times \frac{2}{17}$

$= \frac{51}{34} = \frac{3}{2}$

$E(X^2) = \sum_{i=1}^{n} x_i^2 p(x_i)$

$= 0^2 \times \frac{2}{17} + (1)^2 \times \frac{13}{34} + (2)^2 \times \frac{13}{34} + (3)^2 \times \frac{2}{17}$

$= \frac{91}{34}$

Now, $\text{Var}(X) = E(X^2) - [E(X)]^2$

$= \frac{91}{34} - \left(\frac{3}{2}\right)^2$

$= \frac{29}{68} = 0.43$

$\therefore$ S.D. $(X) = \sqrt{\text{Var}(X)} = \sqrt{0.43} = 0.66$ (approx)

Binomial Distribution

- Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions:
- There should be finite number of trials.
- The trials should be independent.
- Each trial has two outcomes – success or failure.
- The probability of success remains the same in each trial.
- The probability distribution of numbers of successes in an experiment consisting of n Bernoulli trials may be obtained by the binomial expansion of $(q + p)^n$ and this distribution of number of successes X can be written as:

X	0	1	2	…	*X*	…	*n*
***P*(*X*)**	${}^nC_0q^n$	${}^nC_1q^{n-1}p$	${}^nC_2q^{n-2}p^2$	…	${}^nC_xq^{n-x}p^x$	…	${}^nC_np^n$

-
- This probability distribution is known as binomial distribution with parameters *n* and *p*, where *n* = number of Bernoulli trials, *p* = probability of success in 1 trial, *q* = probability of failure in 1 trial = 1 − *p*
- The probability of *x* successes *P*(*X* = *x*) is denoted by *p*(*x*) and it is given by, $P(x) = {}^nC_xq^{n-x}p^x, x = 0, 1, 2, \ldots n$
- The binomial distribution with *n* Bernoulli trials and probability of successes in each trial as *p* is denoted as *B*(*n*, *p*).

Solved Examples:

Example 1:

A box contains 48 packets of pen, where each packet contains 5 pens. Out of these pens, 1 dozen pens are broken. From this, a dozen pens are drawn successively with replacement. Find the probability that there are at least three good pens.

Solution:

Total number of pens in the box = 48 × 5 = 240

Out of these pens, 1 dozen, i.e. 12 pens, are broken.

Therefore, *p* = Probability of getting a broken pen = $\frac{12}{240} = \frac{1}{20}$

$\therefore q = 1 - p = 1 - \frac{1}{20} = \frac{19}{20}$

Let *X* denote the number of broken pens out of 12 drawn pens. Since the drawn is done with replacement, the trials are Bernoulli trials. Clearly, *X* has the binomial distribution with *n* = 12, *p* = $\frac{1}{20}$

Now, P(at least three good pens) = P (at most 9 broken pens)

= P ($X \le 9$)

= 1 − [$P(X = 10) + P(X = 11) + P(X = 12)$]

$$=1-\left[{}^{12}C_{10}\left(\frac{19}{20}\right)^2\left(\frac{1}{20}\right)^{10}+{}^{12}C_{11}\left(\frac{19}{20}\right)\left(\frac{1}{20}\right)^{11}+{}^{12}C_{12}\left(\frac{1}{20}\right)^{12}\right]$$

$$=1-\left(66\times 361+12\times 19+1\right)\cdot\left(\frac{1}{20}\right)^{12}$$

$$=1-\frac{24055}{20^{12}}$$

Example 2:

Three dice are thrown simultaneously for 10 times. If getting a triplet is considered a success, then find the probability of trials that have less than 3 failures.

Solution:

By throwing three dice, 6 × 6 × 6 = 216 observations will be obtained, out of which there are six triplets. They are (1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 5), (6, 6, 6).

Therefore, p = Probability of success = $\frac{6}{216}=\frac{1}{36}$ (in one trial)

q = Probability of failure = $1-\frac{1}{36}=\frac{35}{36}$ (in one trial)

Let X denote the number of triplets in throwing of three dice 10 times together. These trials are Bernoulli trials. Clearly, X has the binomial distribution with n = 10 and $p = \frac{1}{36}$

Now, P (less than 3 failures)

= P (more than 7 successes)

= P ($X > 7$)

= $P(X = 8) + P(X = 9) + P(X = 10)$

$$= {}^{10}C_8\left(\frac{35}{36}\right)^2\left(\frac{1}{36}\right)^8 + {}^{10}C_9\left(\frac{35}{36}\right)\left(\frac{1}{36}\right)^9 + {}^{10}C_{10}\left(\frac{1}{36}\right)^{10}$$

$$= \left(45\times 35^2 + 10\times 35 + 1\right)\times\frac{1}{36^{10}}$$

$$= \frac{55476}{36^{10}}$$