5. Basic Algebra

5.1 Algebraic Formulae

(i) $x^n + y^n = (x + y)(x^{n-1} - x^{n-2}.y + x^{n-3}.y^2 - .. + y^{n-1})$ when n is odd. When n is odd, $x^n + y^n$ is divisible by x + y

(ii) $x^n - y^n = (x + y)(x^{n-1} - x^{n-2}.y + ... y^{n-1})$ when n is even. When n is even, $x^n - y^n$ is divisible by x + y.

(iii) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}.y + ... + y^{n-1})$ for both odd and even n. Therefore, $x^n - y^n$ is divisible by x−y.

(iv) $(a + b)^2 = a^2 + b^2 + 2ab$.

(v) $(a - b)^2 = a^2 + b^2 - 2ab$.

(vi) $(a^2 - b^2) = (a + b)(a - b)$.

(vii) $(a + b)^2 - (a - b)^2 = 4ab$.

(viii) $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$.

(ix) $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$.

(x) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$.

(xi) $(a^3 + b^3) = (a + b)(a^2 + b^2 - ab)$.

(xii) $(a^3 - b^3) = (a - b)(a^2 + b^2 + ab)$.

(xiii) $(a + b + c)^2 = [a^2 + b^2 + c^2 + 2(ab + bc + ca)]$.

(xiv) $(a^3 + b^3 + c^3 - 3abc) = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$. If $a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$.

(xv) $(x + a)(x + b) = x^2 + (a + b)x + ab$.

5.2 Linear Equations:

Consider two linear equations:-

$A_1 x + B_1 y = C_1$

$A_2 x + B_2 y = C_2$.

(i) This pair of linear equations has:- A unique solution if, $\frac{A_1}{A_2} \neq \frac{B_1}{B_2}$.

(ii) Infinite solutions if, $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$.

(iii) No solution if, $\frac{A_1}{A_2} = \frac{B_1}{B_2} \neq \frac{C_1}{C_2}$.

5.3 Quadratic Equations:

(i) General form of a quadratic equation is $ax^2 + bx + c = 0, a \neq 0$

(ii) The discriminant of a quadratic equation is D = b^2 - 4ac

(iii) The roots of the above quadratic equation are $\frac{-b+\sqrt{D}}{2a}$ and $\frac{-b-\sqrt{D}}{2a}$

(iv) Let α and β be the roots of the above quadratic equation. If D > 0, then the roots are real and unequal. The sum of the roots $\alpha + \beta$ = $\frac{-b}{a}$ and the product $\alpha\ \beta$ = $\frac{c}{a}$

(v) If D is a perfect square, then the roots are rational and unequal.

(vi) If D = 0, then the roots are real and equal and is equal to $\frac{-b}{2a}$

(vii) If D < 0, then the roots are complex and unequal. If a, b and c of the quadratic equation are rational, then the roots are conjugates of each other. Ex. if α = p + qi, then β = p - qi

(viii) If D $\geq$ 0, then, $ax^2 + bx + c = a(x - \alpha)(x - \beta)$.

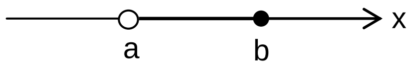
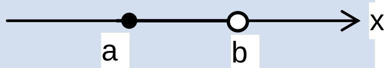
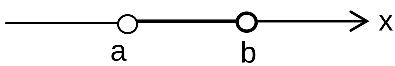
(ix) If c = a, then the roots are reciprocal.

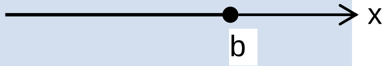
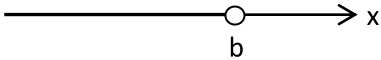
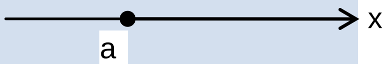
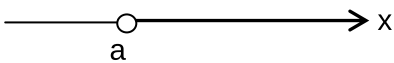
(x) If b = 0, then the roots are equal in magnitude but opposite in sign. If one of the roots of a quadratic equation with rational coefficients is irrational, then the other roots must be irrational conjugate. If $\alpha = p + \sqrt{q}$, then $\beta = p - \sqrt{q}$.

(xi) If α, β are the roots of a quadratic equation, then the equation is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

5.4 Inequalities

(i) Inequalities, Interval Notations and Graphs

Inequality	Interval Notation	Graph
a ≤ x ≤ b	[a, b]	
a < x ≤ b	(a, b]	
a ≤ x < b	[a, b)	
a < x < b	(a, b)	

$-\infty < x \leq b$, $x \leq b$	$(-\infty, b]$	
$-\infty < x < b$, $x < b$	$(-\infty, b)$	
$a \leq x < \infty$, $x \geq a$	$[a, \infty)$	
$a < x < \infty$, $x > a$	(a, ∞)	

(ii) If a > b, then b < a.

(iii) If a > b, then a − b > 0 or b − a < 0.

(iv) If a > b, then a + c > b + c.

(v) If a > b, then a − c > b − c.

(vi) If a > b and c > d, then a + c > b + d.

(vii) If a > b and c > d, then a − d > b − c.

(viii) If a > b and m > 0, then ma > mb.

(ix) If a > b and m > 0, then $\frac{a}{m} > \frac{b}{m}$.

(x) If a > b and m > 0, then ma > mb.

(xi) If a > b and m < 0, then $\frac{a}{m} < \frac{b}{m}$.

(xii) If 0 < a < b and n > 0, then an < bn.

(xiii) If 0 < a < b and n < 0, then an > bn.

(xiv) If 0 < a < b, then $\sqrt[n]{a} < \sqrt[n]{b}$.

(xv) $\sqrt{ab} \leq \frac{a+b}{2}$, where a > 0, b > 0; an equality is valid only if a = b.

(xvi) a + $\frac{1}{a} \geq 2$, where a > 0; an equality takes place only at a = 1.

(xvii) $\sqrt[n]{a_1 a_2 .... a_n} \leq \frac{a_1 + a_2 + ... + a_n}{n}$, where a_1, a_2, ..., $a^n > 0$.

(xviii) If ax + b > 0 and a > 0, then x > $-\frac{b}{a}$.

(xix) If ax + b > 0 and a > 0, then x < $-\frac{b}{a}$.

(xx) |a + b| ≤ |a| + |b|.

(xxi) If $|x| < a$, then $-a < x < a$, where $a > 0$.

(xxii) If $|x| > a$, then $x < -a$ and $x > a$, where $a > 0$.

(xxiii) If $x^2 < a$, then $|x| < \sqrt{a}$, where $a > 0$.

(xxiv) If $x^2 > a$, then $|x| > \sqrt{a}$, where $a > 0$.

5.5 Arithmetic & Geometric Progression

Arithmetic Progression:

- An arithmetic progression is a sequence of numbers in which each term is derived from the preceding term by adding or subtracting a fixed number called the common difference e.g. The sequence 9,6,3,0,-3,.... is an arithmetic progression with –3 as the common difference. The progression –3, 0, 3, 6, 9 is an Arithmetic Progression (AP) with 3 as the common difference. The general form of an Arithmetic Progression is a, a + d, a + 2d, a + 3d and so on. Thus n^{th} term of an AP series is $T_n = a + (n - 1) d$. Where T_n = n^{th} term and a = first term. Here d = common difference = $T_n - T_{n-1}$.

- Sometimes the last term is given and either 'd' is asked or 'a' is asked.

  Then formula becomes $l = a + (n - 1) d$
- There is another formula, applied to find the sum of first n terms of an AP: $S_n = n/2[2a+(n-1)d]$

- The sum of n terms is also equal to the formula $S_n = n/2(a + l)$ where is the last term.
- When three quantities are in AP, the middle one is called as the arithmetic mean of the other two. If a, b and c are three terms in AP then b = (a + c)/2.

Geometric Progression:

A geometric progression is a sequence in which each term is derived by multiplying or dividing the preceding term by a fixed number called the common ratio. The sequence 4, -2, 1, – ½,.. is a Geometric Progression (GP) for which – ½ is the common ratio.

- The general form of a GP is a, ar, ar^2, ar^3 and so on.
- Thus nth term of a GP series is $T_n = ar^{n-1}$, where a = first term and r = common ratio = T_m/T_{m-1}.
- The formula applied to calculate sum of first n terms of a GP: $S_n = a(r^n-1)/r-1$ where $\rightarrow |r| > 1$ and
  $S_n = a(1-r^n)/1-r$ where $\Rightarrow |r| < 1$.
- When three quantities are in GP, the middle one is called as the geometric mean of the other two. If a, b and c are three quantities in GP and b is the geometric mean of a and c i.e. $b = \sqrt{ac}$
- The sum of infinite terms of a GP series $S_\infty = a/1-r$