

CHAPTER

14

Inverse Trigonometric Functions

Section-A

JEE Advanced/ IIT-JEE

A

Fill in the Blanks

1. Let a, b, c be positive real numbers. Let

$$\theta = \tan^{-1} \sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1} \sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1} \sqrt{\frac{c(a+b+c)}{ab}}.$$

Then $\tan \theta =$ _____ (1981 - 2 Marks)

2. The numerical value of $\tan \left\{ 2 \tan^{-1} \left(\frac{1}{5} \right) - \frac{\pi}{4} \right\}$ is equal to _____ (1984 - 2 Marks)
3. The greater of the two angles $A = 2 \tan^{-1} (2\sqrt{2}-1)$ and $B = 3 \sin^{-1} (1/3) + \sin^{-1} (3/5)$ is _____. (1989 - 2 Marks)

C

MCQs with One Correct Answer

1. The value of $\tan \left[\cos^{-1} \left(\frac{4}{5} \right) + \tan^{-1} \left(\frac{2}{3} \right) \right]$ is _____ (1983 - 1 Mark)
- (a) $\frac{6}{17}$ (b) $\frac{7}{16}$ (c) $\frac{16}{7}$ (d) none
2. If we consider only the principle values of the inverse trigonometric functions then the value of $\tan \left(\cos^{-1} \frac{1}{5\sqrt{2}} - \sin^{-1} \frac{4}{\sqrt{17}} \right)$ is _____ (1994)
- (a) $\frac{\sqrt{29}}{3}$ (b) $\frac{29}{3}$ (c) $\frac{\sqrt{3}}{29}$ (d) $\frac{3}{29}$
3. The number of real solutions of $\tan^{-1} \sqrt{x(x+1)} + \sin^{-1} \sqrt{x^2+x+1} = \pi/2$ is _____ (1999 - 2 Marks)
- (a) zero (b) one (c) two (d) infinite

4. If $\sin^{-1} \left(x - \frac{x^2}{2} + \frac{x^3}{4} - \dots \right) + \cos^{-1} \left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \dots \right) = \frac{\pi}{2}$

for $0 < |x| < \sqrt{2}$, then x equals (2001S)

- (a) $1/2$ (b) 1 (c) $-1/2$ (d) -1
5. The value of x for which $\sin (\cot^{-1} (1+x)) = \cos (\tan^{-1} x)$ is (2004S)
- (a) $1/2$ (b) 1 (c) 0 (d) $-1/2$

6. If $0 < x < 1$, then $\sqrt{1+x^2} [\{x \cos (\cot^{-1} x) + \sin (\cot^{-1} x)\}^2 - 1]^{1/2} =$ (2008)
- (a) $\frac{x}{\sqrt{1+x^2}}$ (b) x
- (c) $x\sqrt{1+x^2}$ (d) $\sqrt{1+x^2}$

7. The value of $\cot \left(\sum_{n=1}^{23} \cot^{-1} \left(1 + \sum_{k=1}^n 2k \right) \right)$ is (JEE Adv. 2013)
- (a) $\frac{23}{25}$ (b) $\frac{25}{23}$ (c) $\frac{23}{24}$ (d) $\frac{24}{23}$

D

MCQs with One or More than One Correct

1. The principal value of $\sin^{-1} \left(\sin \frac{2\pi}{3} \right)$ is (1986 - 2 Marks)
- (a) $-\frac{2\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) none
2. If $\alpha = 3 \sin^{-1} \left(\frac{6}{11} \right)$ and $\beta = 3 \cos^{-1} \left(\frac{4}{9} \right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s) is (are) (JEE Adv. 2015)
- (a) $\cos \beta > 0$ (b) $\sin \beta < 0$
- (c) $\cos(\alpha + \beta) > 0$ (d) $\cos \alpha < 0$

E Subjective Problems

1. Find the value of : $\cos(2\cos^{-1}x + \sin^{-1}x)$ at $x = \frac{1}{5}$, where
 $0 \leq \cos^{-1}x \leq \pi$ and $-\pi/2 \leq \sin^{-1}x \leq \pi/2$.
 (1981 - 2 Marks)

2. Find all the solution of $4\cos^2 x \sin x - 2\sin^2 x = 3\sin x$
 (1983 - 2 Marks)

3. Prove that $\cos \tan^{-1} \sin \cot^{-1} x = \sqrt{\frac{x^2+1}{x^2+2}}$. (2002 - 5 Marks)

F Match the Following

DIRECTIONS (Q. 1 & 2) : Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example :

	p	q	r	s	t
A	☒	☐	☐	☒	☒
B	☐	☒	☒	☐	☐
C	☒	☒	☐	☐	☐
D	☐	☐	☐	☒	☐

If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.

1. Match the following

(2006 - 6M)

Column I**Column II**

(A) $\sum_{i=1}^{\infty} \tan^{-1} \left(\frac{1}{2i^2} \right) = t$, then $\tan t =$

(p) 1

- (B) Sides a, b, c of a triangle ABC are in AP and

(q) $\frac{\sqrt{5}}{3}$

$\cos \theta_1 = \frac{a}{b+c}, \cos \theta_2 = \frac{b}{a+c}, \cos \theta_3 = \frac{c}{a+b},$

then $\tan^2 \left(\frac{\theta_1}{2} \right) + \tan^2 \left(\frac{\theta_3}{2} \right) =$

- (C) A line is perpendicular to $x + 2y + 2z = 0$ and

(r) $\frac{2}{3}$

passes through $(0, 1, 0)$. The perpendicular distance of this line from the origin is

2. Let (x, y) be such that $\sin^{-1}(ax) + \cos^{-1}(y) + \cos^{-1}(bxy) = \frac{\pi}{2}$.

(2007)

Match the statements in Column I with statements in Column II and indicate your answer by darkening the appropriate bubbles in the 4×4 matrix given in the ORS.

Column I**Column II**

- (A) If $a = 1$ and $b = 0$, then (x, y)

- (p) lies on the circle $x^2 + y^2 = 1$

- (B) If $a = 1$ and $b = 1$, then (x, y)

- (q) lies on $(x^2 - 1)(y^2 - 1) = 0$

- (C) If $a = 1$ and $b = 2$, then (x, y)

- (r) lies on $y = x$

- (D) If $a = 2$ and $b = 2$, then (x, y)

- (s) lies on $(4x^2 - 1)(y^2 - 1) = 0$

Inverse Trigonometric Functions

DIRECTIONS (Q. 3) : Following question has matching lists. The codes for the lists have choices (a), (b), (c) and (d) out of which ONLY ONE is correct.

3. Match List I with List II and select the correct answer using the code given below the lists :

(JEE Adv. 2013)

List I**List II**

P. $\left(\frac{1}{y^2} \left(\frac{\cos(\tan^{-1} y) + y \sin(\tan^{-1} y)}{\cot(\sin^{-1} y) + \tan(\sin^{-1} y)} \right)^2 + y^4 \right)^{1/2}$ takes value

1. $\frac{1}{2}\sqrt{\frac{5}{3}}$

Q. If $\cos x + \cos y + \cos z = 0 = \sin x + \sin y + \sin z$ then

2. $\sqrt{2}$

possible value of $\cos \frac{x-y}{2}$ is

R. If $\cos\left(\frac{\pi}{4} - x\right) \cos 2x + \sin x \sin 2 \sec x = \cos x \sin 2x \sec x +$

3. $\frac{1}{2}$

$\cos\left(\frac{\pi}{4} + x\right) \cos 2x$ then possible value of $\sec x$ is

S. If $\cot\left(\sin^{-1} \sqrt{1-x^2}\right) = \sin\left(\tan^{-1}(x\sqrt{6})\right), x \neq 0,$

4. 1

then possible value of x is

Codes:

	P	Q	R	S
(a)	4	3	1	2
(b)	4	3	2	1
(c)	3	4	2	1
(d)	3	4	1	2

Section-B

JEE Main / AIEEE

1. $\cot^{-1}(\sqrt{\cos \alpha}) - \tan^{-1}(\sqrt{\cos \alpha}) = x$, then $\sin x =$
 (a) $\tan^2\left(\frac{\alpha}{2}\right)$ (b) $\cot^2\left(\frac{\alpha}{2}\right)$ [2002]
 (c) $\tan \alpha$ (d) $\cot\left(\frac{\alpha}{2}\right)$
2. The trigonometric equation $\sin^{-1} x = 2 \sin^{-1} a$ has a solution for [2003]
 (a) $|a| \geq \frac{1}{\sqrt{2}}$ (b) $\frac{1}{2} < |a| < \frac{1}{\sqrt{2}}$
 (c) all real values of a (d) $|a| < \frac{1}{2}$
3. If $\cos^{-1} x - \cos^{-1} \frac{y}{2} = \alpha$, then $4x^2 - 4xy \cos \alpha + y^2$ is equal to [2005]
 (a) $2 \sin 2\alpha$ (b) 4
 (c) $4 \sin^2 \alpha$ (d) $-4 \sin^2 \alpha$
4. If $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then the values of x is [2007]
 (a) 4 (b) 5
 (c) 1 (d) 3
5. The value of $\cot\left(\operatorname{cosec}^{-1} \frac{5}{3} + \tan^{-1} \frac{2}{3}\right)$ is
 (a) $\frac{6}{17}$ (b) $\frac{3}{17}$
 (c) $\frac{4}{17}$ (d) $\frac{5}{17}$
6. If x, y, z are in A.P. and $\tan^{-1} x, \tan^{-1} y$ and $\tan^{-1} z$ are also in A.P., then [JEE M 2013]
 (a) $x = y = z$ (b) $2x = 3y = 6z$
 (c) $6x = 3y = 2z$ (d) $6x = 4y = 3z$
7. Let $\tan^{-1} y = \tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$,
 where $|x| < \frac{1}{\sqrt{3}}$. Then a value of y is : [JEE M 2015]
 (a) $\frac{3x-x^3}{1+3x^2}$ (b) $\frac{3x+x^3}{1+3x^2}$
 (c) $\frac{3x-x^3}{1-3x^2}$ (d) $\frac{3x+x^3}{1-3x^2}$

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Inverse Trigonometric Functions

Section-A : JEE Advanced/ IIT-JEE

- A** 1. 0 2. $\frac{-7}{17}$ 3. A
- C** 1. (d) 2. (d) 3. (c) 4. (b) 5. (d) 6. (c) 7. (b)
- D** 1. (d) 2. (b, c, d)
- E** 1. $\frac{-2\sqrt{6}}{5}$ 2. $x = n\pi, n\pi + (-1)^n \frac{\pi}{10}, n\pi + (-1)^n \left(\frac{-3\pi}{10}\right)$ where $n \in \mathbb{N}$
- F** 1. (A)-(p), (B)-(r), (C)-(q) 2. (A)-p, (B)-q, (C)-p, (D)-s 3. (b)

Section-B : JEE Main/ AIEEE

1. (a) 2. None 3. (c) 4. (d) 5. (a) 6. (a) 7. (c)

Section-A JEE Advanced/ IIT-JEE

A. Fill in the Blanks

1. Let $a + b + c = u$, then

$$\theta = \tan^{-1} \sqrt{\frac{au}{bc}} + \tan^{-1} \sqrt{\frac{bu}{ca}} + \tan^{-1} \sqrt{\frac{cu}{ab}}$$

Now we know that

$$\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1} \left(\frac{x+y}{1-xy} \right) \text{ when } xy > 1$$

$$\sqrt{\frac{au}{bc}} \times \sqrt{\frac{bu}{ca}} = \frac{u}{c} = \frac{a+b+c}{c} > 1; a, b, c$$

being +ve real no's.

$$\therefore \text{ We get } \theta = \pi + \tan^{-1} \left[\frac{\sqrt{\frac{au}{bc}} + \sqrt{\frac{bu}{ca}}}{1 - \sqrt{\frac{au}{bc}} \sqrt{\frac{bu}{ca}}} \right] + \tan^{-1} \sqrt{\frac{cu}{ab}}$$

$$\theta = \pi + \tan^{-1} \left[\frac{\frac{a+b}{\sqrt{abc}} \sqrt{u}}{1 - \frac{u}{c}} \right] + \tan^{-1} \sqrt{\frac{cu}{ab}}$$

$$\theta = \pi + \tan^{-1} \left[\frac{(u-c)\sqrt{u}}{\sqrt{abc}} \times \frac{c}{-(u-c)} \right] + \tan^{-1} \sqrt{\frac{cu}{ab}}$$

$$\theta = \pi - \tan^{-1} \sqrt{\frac{uc}{ab}} + \tan^{-1} \sqrt{\frac{cu}{ab}}$$

$$[\text{Using } \tan^{-1}(-x) = -\tan^{-1} x] = \pi$$

$$\therefore \tan \theta = \tan \pi = 0$$

$$2. \quad \tan \left(2 \tan^{-1} \frac{1}{5} - \frac{\pi}{4} \right) = \tan \left[\tan^{-1} \left(\frac{2/5}{1 - (1/5)^2} \right) - \tan^{-1}(1) \right]$$

$$= \tan \left[\tan^{-1} \left(\frac{5}{12} \right) - \tan^{-1}(1) \right] = \tan \left[\tan^{-1} \left(\frac{5/12 - 1}{1 + 5/12} \right) \right]$$

$$= \tan (\tan^{-1}(-7/17)) = -7/17$$

3. We have

$$A = 2 \tan^{-1}(2\sqrt{2} - 1) = 2 \tan^{-1}(2 \times 1.414 - 1)$$

$$= 2 \tan^{-1}(1.828) > 2 \tan^{-1} \sqrt{3} = 2\pi/3$$

$$\Rightarrow A > 2\pi/3 \quad \dots (1)$$

$$\text{Also } B = 3 \sin^{-1}(1/3) + \sin^{-1}(3/5)$$

$$= \sin^{-1} \left[3 \times \frac{1}{3} - 4 \times \frac{1}{27} \right] + \sin^{-1}(3/5)$$

$$= \sin^{-1} \frac{23}{27} + \sin^{-1}(0.6) = \sin^{-1}(0.852) + \sin^{-1}(0.6)$$

$$< \sin^{-1}(\sqrt{3}/2) + \sin^{-1}(\sqrt{3}/2) = 2\pi/3$$

$$\Rightarrow B < 2\pi/3 \quad \dots (2)$$

From (1) and (2) we conclude $A > B$.

C. MCQs with ONE Correct Answer

1. (d) $\tan\left[\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right]$
 $= \tan\left[\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right]$
 $= \tan\left[\tan^{-1}\left(\frac{3/4 + 2/3}{1 - 3/4 \times 2/3}\right)\right] = \frac{17}{12} \times \frac{12}{6} = \frac{17}{6}$
2. (d) $\tan\left[\cos^{-1}\frac{1}{5\sqrt{2}} - \sin^{-1}\frac{4}{\sqrt{17}}\right]$
 $= \tan[\tan^{-1}7 - \tan^{-1}4]$
 $= \tan\left(\tan^{-1}\left(\frac{3}{29}\right)\right) = \frac{3}{29}$
3. (c) $\tan^{-1}\sqrt{x(x+1)} = \pi/2 - \sin^{-1}\sqrt{x^2+x+1}$
 $\Rightarrow \tan^{-1}\sqrt{x(x+1)} = \cos^{-1}\sqrt{x^2+x+1}$
 $\Rightarrow \cos^{-1}\frac{1}{\sqrt{x^2+x+1}} = \cos^{-1}\sqrt{x^2+x+1}$
 $\Rightarrow x^2+x+1=1 \Rightarrow x(x+1)=0$
 $\Rightarrow x=0, -1$ are the only real solutions.
4. (b) $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{3} \dots\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} \dots\right) = \frac{\pi}{2}$
 $\Rightarrow \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} \dots\right) = \frac{\pi}{2} - \sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} \dots\right)$
 $\Rightarrow \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} \dots\right) = \cos^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} \dots\right)$
 $\Rightarrow x^2 - \frac{x^4}{2} + \frac{x^6}{4} \dots = x - \frac{x^2}{2} + \frac{x^3}{4} \dots$

On both sides we have G.P. of infinite terms.

$$\therefore \frac{x^2}{1 - \left(\frac{-x^2}{2}\right)} = \frac{x}{1 - \left(\frac{-x}{2}\right)} \Rightarrow \frac{2x^2}{2+x^2} = \frac{2x}{2+x}$$

$$\Rightarrow 2x + x^3 = 2x^2 + x^3 \Rightarrow x(x-1) = 0$$

$$\Rightarrow x=0, 1 \text{ but } 0 < |x| < \sqrt{2} \Rightarrow x=1.$$

5. (d) $\sin[\cot^{-1}(1+x)] = \cos(\tan^{-1}x)$
 $\Rightarrow \sin\left[\sin^{-1}\left(\frac{1}{\sqrt{1+(1+x)^2}}\right)\right] = \cos\left[\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right]$
 $\Rightarrow \frac{1}{\sqrt{1+(1+x)^2}} = \frac{1}{\sqrt{1+x^2}}$
 $\Rightarrow 1+1+2x+x^2 = 1+x^2 \Rightarrow 2x+1=0 \Rightarrow x=-\frac{1}{2}$

6. (c) $\sqrt{1+x^2} \left[\{x \cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1 \right]^{\frac{1}{2}}$
 $= \sqrt{1+x^2} \left[\left\{ x \cos\left(\cos^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right) \right. \right.$
 $\left. \left. + \sin\left(\sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right) \right\}^2 - 1 \right]^{\frac{1}{2}}$
 $= \sqrt{1+x^2} \left[\left\{ x \cdot \frac{x}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}} \right\}^2 - 1 \right]$
 $= \sqrt{1+x^2} \left[\left(\sqrt{1+x^2} \right)^2 - 1 \right] = x\sqrt{1+x^2}$
7. (b) $\cot^{-1}\left(1 + \sum_{k=1}^n 2k\right) = \cot^{-1}[1 + n(n+1)]$
 $= \tan^{-1}\left[\frac{(n+1)-n}{1+(n+1)n}\right] = \tan^{-1}(n+1) - \tan^{-1}n$
 $\therefore \sum_{n=1}^{23} [\tan^{-1}(n+1) - \tan^{-1}n] = \tan^{-1}24 - \tan^{-1}1 = \tan^{-1}\frac{23}{25}$
 $\therefore \cot\left[\sum_{n=1}^{23} \cot^{-1}\left(1 + \sum_{k=1}^n 2k\right)\right] = \cot\left[\tan^{-1}\frac{23}{25}\right] = \frac{25}{23}$

D. MCQs with ONE or MORE THAN ONE Correct

1. (d) The principal value of
 $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left(\sin\left(\pi - \frac{\pi}{3}\right)\right)$
 $= \sin^{-1}(\sin\pi/3) = \pi/3 \therefore$ (d) is the correct answer.
2. (b, c, d)
 $\alpha = 3\sin^{-1}\frac{6}{11} > 3\sin^{-1}\frac{1}{2}$ or $\alpha > \frac{\pi}{2}$
 $\therefore \cos\alpha < 0$
 $\beta = 3\cos^{-1}\frac{4}{9} > 3\cos^{-1}\frac{1}{2}$ or $\beta > \pi$
 $\therefore \cos\beta < 0$ and $\sin\beta < 0$
 Also $\alpha + \beta > \frac{3\pi}{2} \therefore \cos(\alpha + \beta) > 0.$

E. SUBJECTIVE PROBLEMS

1. We have $\cos(2 \cos^{-1} x + \sin^{-1} x)$
 $= \cos(\cos^{-1} x + \cos^{-1} x + \sin^{-1} x)$
 $= \cos(\cos^{-1} x + \pi/2)$ {Using $\cos^{-1} x + \sin^{-1} x = \pi/2$ }
 $= -\sin(\cos^{-1} x)$
 $= -\sqrt{1 - \cos^2(\cos^{-1} x)} = -\sqrt{1 - [\cos(\cos^{-1} x)]^2}$
 $= -\sqrt{1 - x^2} = -\sqrt{1 - 1/25}$ [for $x = 1/5$]
 $= -\frac{\sqrt{24}}{5} = \frac{-2\sqrt{6}}{5}$

2. Given eq. is,
 $4 \cos^2 x \sin x - 2 \sin^2 x = 3 \sin x$
 $\Rightarrow 4 \cos^2 x \sin x - 2 \sin^2 x - 3 \sin x = 0$
 $\Rightarrow 4(1 - \sin^2 x) \sin x - 2 \sin^2 x - 3 \sin x = 0$
 $\Rightarrow \sin x [4 \sin^2 x + 2 \sin x - 1] = 0$
 $\Rightarrow$ either $\sin x = 0$ or $4 \sin^2 x + 2 \sin x - 1 = 0$
 If $\sin x = 0 \Rightarrow x = n\pi$
 $\Rightarrow$ If $4 \sin^2 x + 2 \sin x - 1 = 0 \Rightarrow \sin x = \frac{-1 \pm \sqrt{5}}{4}$
 If $\sin x = \frac{-1 \pm \sqrt{5}}{4} = \sin 18^\circ = \sin \frac{\pi}{10}$
 then $x = nx + (-1)^n \frac{\pi}{10}$
 If $\sin x = -\left(\frac{\sqrt{5}+1}{4}\right) = \sin(-54^\circ) = \sin\left(\frac{-3\pi}{10}\right)$
 then $x = n\pi + (-1)^n \left(\frac{-3\pi}{10}\right)$
 Hence, $x = n\pi, n\pi + (-1)^n \frac{\pi}{10}$ or $n\pi + (-1)^n \left(\frac{-3\pi}{10}\right)$
 where n is some integer

3. To prove that $\cos \tan^{-1} \sin \cot^{-1} x = \sqrt{\frac{x^2+1}{x^2+2}}$
 L.H.S. = $\cos [\tan^{-1} (\sin (\cot^{-1} x))]$
 $= \cos [\tan^{-1} (\sin (\sin^{-1} \frac{1}{\sqrt{1+x^2}}))] \text{ if } x > 0$
 and $\cos [\tan^{-1} (\sin (\pi - \sin^{-1} \frac{1}{\sqrt{1+x^2}}))] \text{ if } x < 0$
 In each case,
 $= \cos \left[\tan^{-1} \frac{1}{\sqrt{1+x^2}} \right] = \cos \left[\cos^{-1} \sqrt{\frac{1+x^2}{2+x^2}} \right]$
 $= \sqrt{\frac{x^2+1}{x^2+2}} = R.H.S. \quad \text{Hence Proved.}$

F. Match the Following

1. (A) $\rightarrow$ (p); (B) $\rightarrow$ (r); (C) $\rightarrow$ (q)

(A) $t = \sum_{i=1}^{\infty} \tan^{-1} \left(\frac{1}{2i^2} \right) = \sum_{i=1}^{\infty} \tan^{-1} \left[\frac{(2i+1) - (2i-1)}{1 + 4i^2 - 1} \right]$
 $= \sum_{i=1}^{\infty} [\tan^{-1}(2i+1) - \tan^{-1}(2i-1)]$
 $t = \tan^{-1} 3 - \tan^{-1} 1 + \tan^{-1} 5 - \tan^{-1} 3$
 $+ \dots + \tan^{-1}(2n+1) - \tan^{-1}(2n-1) + \dots \infty$
 $\Rightarrow t = \lim_{n \rightarrow \infty} [\tan^{-1}(2n+1) - \tan^{-1} 1]$

$= \lim_{n \rightarrow \infty} \tan^{-1} \left[\frac{2n}{1 + (2n+1)} \right] = \lim_{n \rightarrow \infty} \tan^{-1} \left[\frac{1}{1 + 1/n} \right]$

$\Rightarrow t = \tan^{-1}(1) = \frac{\pi}{4} \Rightarrow \tan t = 1, \quad (A) \rightarrow (p)$

(B) $\because a, b, c$ are in AP $\Rightarrow 2b = a + c$

$\cos \theta_1 = \frac{a}{b+c}$

$\Rightarrow \frac{1 - \tan^2 \theta_1 / 2}{1 + \tan^2 \theta_1 / 2} = \frac{a}{b+c} \Rightarrow \tan^2 \frac{\theta_1}{2} = \frac{b+c-a}{b+c+a}$

Similarly, $\tan^2 \frac{\theta_3}{2} = \frac{a+b-c}{a+b+c}$

$\Rightarrow \tan^2 \frac{\theta_1}{2} + \tan^2 \frac{\theta_3}{2} = \frac{2b}{a+b+c} = \frac{2b}{3b} = \frac{2}{3}, \quad (B) \rightarrow (r)$

(C) Equation of line through $(0, 1, 0)$ and perpendicular to

$x + 2y + 2z = 0$ is $\frac{x}{1} = \frac{y-1}{2} = \frac{z}{2} = \lambda$

For some value of λ , the foot of perpendicular from origin to line is $(\lambda, 2\lambda + 1, 2\lambda)$

Dr's of this $\perp$ from origin are $\lambda, 2\lambda + 1, 2\lambda$

$\therefore 1.\lambda + 2(2\lambda + 1) + 2.2\lambda = 0 \Rightarrow \lambda = -\frac{2}{9}$

$\therefore$ Foot of perpendicular is $\left(-\frac{2}{9}, \frac{5}{9}, -\frac{4}{9}\right)$

$\therefore$ Required distance

$= \sqrt{\frac{4}{81} + \frac{25}{81} + \frac{16}{81}} = \sqrt{\frac{45}{81}} = \frac{\sqrt{5}}{3} \quad (C) \rightarrow (q)$

2. (A) $\rightarrow$ p, (B) $\rightarrow$ q, (C) $\rightarrow$ p, (D) $\rightarrow$ s

$\sin^{-1}(ax) + \cos^{-1} y + \cos^{-1}(bxy) = \frac{\pi}{2}$

$\Rightarrow \cos^{-1} y + \cos^{-1}(bxy) = \frac{\pi}{2} - \sin^{-1}(ax) = \cos^{-1}(ax)$

Let $\cos^{-1} y = \alpha, \cos^{-1}(bxy) = \beta, \cos^{-1}(ax) = \gamma$

- $\Rightarrow y = \cos \alpha, bxy = \cos \beta, ax = \cos \gamma$
 $\therefore$ We get $\alpha + \beta = \gamma$ and $\cos \beta = bxy$
 $\Rightarrow \cos (\gamma - \alpha) = bxy$
 $\Rightarrow \cos \gamma \cos \alpha + \sin \gamma \sin \alpha = bxy$
 $\Rightarrow axy + \sin \gamma \sin \alpha = bxy \Rightarrow (a - b)xy = -\sin \alpha \sin \gamma$
 $\Rightarrow (a - b)^2 x^2 y^2 = -\sin^2 \alpha \sin^2 \gamma$
 $\quad = (1 - \cos^2 \alpha)(1 - \cos^2 \gamma)$
 $\Rightarrow (a - b)^2 x^2 y^2 = (1 - a^2 x^2)(1 - y^2) \dots (1)$
 (A) For $a = 1, b = 0$, equation (1) reduces to
 $x^2 y^2 = (1 - x^2)(1 - y^2) \Rightarrow x^2 + y^2 = 1$
 (B) For $a = 1, b = 1$ equation (1) becomes
 $(1 - x^2)(1 - y^2) = 0 \Rightarrow (x^2 - 1)(y^2 - 1) = 0$
 (C) For $a = 1, b = 2$ equation (1) reduces to
 $x^2 y^2 = (1 - x^2)(1 - y^2) \Rightarrow x^2 + y^2 = 1$
 (D) For $a = 2, b = 2$ equation (1) reduces to
 $0 = (1 - 4x^2)(1 - y^2) \Rightarrow (4x^2 - 1)(y^2 - 1) = 0$

3. (b)

$$\begin{aligned}
 \text{(P)} \quad & \left[\frac{1}{y^2} \left(\frac{\cos(\tan^{-1} y) + y \sin(\tan^{-1} y)}{\cot(\sin^{-1} y) + \tan(\sin^{-1} y)} \right)^2 + y^4 \right]^{\frac{1}{2}} \\
 &= \left[\frac{1}{y^2} \left(\frac{\cos \left(\cos^{-1} \frac{1}{\sqrt{1+y^2}} \right) + y \sin \left(\sin^{-1} \frac{y}{\sqrt{1+y^2}} \right)}{\cot \left(\cot^{-1} \frac{\sqrt{1-y^2}}{y} \right) + \tan \left(\tan^{-1} \frac{y}{\sqrt{1-y^2}} \right)} \right)^2 + y^4 \right]^{\frac{1}{2}} \\
 &= \left[\frac{1}{y^2} \left(\frac{\frac{\sqrt{1+y^2}}{y}}{\frac{1}{y(\sqrt{1-y^2})}} \right)^2 + y^4 \right]^{\frac{1}{2}} \\
 &= (1 - y^4 + y^4)^{\frac{1}{2}} = 1 \quad \therefore \text{(P)} \rightarrow (4)
 \end{aligned}$$

- (Q) We have $\cos x + \cos y = -\cos z$
 $\sin x + \sin y = -\sin z$
 Squaring and adding we get

$$(\cos x + \cos y)^2 + (\sin x + \sin y)^2 = \cos^2 z + \sin^2 z$$

$$\Rightarrow 2 + 2 \cos(x - y) = 1$$

$$\Rightarrow 4 \cos^2 \frac{x-y}{2} = 1 \quad \text{or} \quad \cos \frac{x-y}{2} = \frac{+1}{2}$$

$$\therefore Q \rightarrow (3)$$

(R) We have

$$\cos \left(\frac{\pi}{4} - x \right) \cos 2x + \sin x \sin 2x \sec x$$

$$= \cos x \sin 2x \sec x + \cos \left(\frac{\pi}{4} + x \right) \cos 2x$$

$$\Rightarrow \cos 2x \left[\cos \left(\frac{\pi}{4} - x \right) - \cos \left(\frac{\pi}{4} + x \right) \right]$$

$$= \sin 2x \sec x (\cos x - \sin x)$$

$$\Rightarrow 2 \sin \frac{\pi}{4} \sin x \cos 2x = 2 \sin x (\cos x - \sin x)$$

$$\Rightarrow 2 \sin x \left[\frac{1}{\sqrt{2}} (\cos^2 x - \sin^2 x) - (\cos x - \sin x) \right] = 0$$

$$\Rightarrow 2 \sin x (\cos x - \sin x) \left(\frac{\cos x + \sin x}{\sqrt{2}} - 1 \right) = 0$$

$$\Rightarrow \sin x = 0 \text{ or } \tan x = 1 \text{ or } \cos \left(x - \frac{\pi}{4} \right) = 1$$

$$\Rightarrow x = 0, \frac{x}{4} \Rightarrow \sec x = 1 \text{ or } \sqrt{2}$$

$$\therefore \text{(R)} \rightarrow (2)$$

$$\text{(S)} \quad \cos \left(\sin^{-1} \sqrt{1-x^2} \right) = \sin \left(\tan^{-1} x \sqrt{6} \right)$$

$$\Rightarrow \frac{x}{\sqrt{1-x^2}} = \frac{x\sqrt{6}}{\sqrt{1+6x^2}} \Rightarrow x = \pm \frac{5}{2\sqrt{3}}$$

$$\therefore \text{(S)} \rightarrow (1)$$

Hence (P) $\rightarrow$ (4), (Q) $\rightarrow$ (3), (R) $\rightarrow$ (2), (S) $\rightarrow$ (1)

Section-B

JEE Main/ AIEEE

1. (a) $\cot^{-1}(\sqrt{\cos \alpha}) - \tan^{-1}(\sqrt{\cos \alpha}) = x$

$$\tan^{-1}\left(\frac{1}{\sqrt{\cos \alpha}}\right) - \tan^{-1}(\sqrt{\cos \alpha}) = x$$

$$\Rightarrow \tan^{-1} \frac{\frac{1}{\sqrt{\cos \alpha}} - \sqrt{\cos \alpha}}{1 + \frac{1}{\sqrt{\cos \alpha}} \cdot \sqrt{\cos \alpha}} = x$$

$$\Rightarrow \tan^{-1} \frac{1 - \cos \alpha}{2\sqrt{\cos \alpha}} = x$$

$$\Rightarrow \tan x = \frac{1 - \cos \alpha}{2\sqrt{\cos \alpha}} \text{ or } \cot x = \frac{2\sqrt{\cos \alpha}}{1 - \cos \alpha}$$

$$\Rightarrow \sin x = \frac{1 - \cos \alpha}{1 + \cos \alpha} = \frac{1 - (1 - 2 \sin^2 \alpha / 2)}{1 + 2 \cos^2 \alpha / 2 - 1}$$

$$\text{or } \sin x = \tan^2 \frac{\alpha}{2}$$

2. (None) $\sin^{-1} x = 2 \sin^{-1} a$

$$-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}; \therefore -\frac{\pi}{2} \leq 2 \sin^{-1} a \leq \frac{\pi}{2}$$

$$-\frac{\pi}{4} \leq \sin^{-1} a \leq \frac{\pi}{4} \text{ or } \frac{-1}{\sqrt{2}} \leq a \leq \frac{1}{\sqrt{2}}$$

$$\therefore |a| \leq \frac{1}{\sqrt{2}}$$

Out of given four options none is absolutely correct

3. (c) $\cos^{-1} x - \cos^{-1} \frac{y}{2} = \alpha$

$$\cos^{-1} \left(\frac{xy}{2} + \sqrt{(1-x^2) \left(1 - \frac{y^2}{4} \right)} \right) = \alpha$$

$$\cos^{-1} \left(\frac{xy + \sqrt{4 - y^2 - 4x^2 + x^2 y^2}}{2} \right) = \alpha$$

$$\Rightarrow 4 - y^2 - 4x^2 + x^2 y^2$$

$$= 4 \cos^2 \alpha + x^2 y^2 - 4xy \cos \alpha$$

$$\Rightarrow 4x^2 + y^2 - 4xy \cos \alpha = 4 \sin^2 \alpha$$

4. (d) $\sin^{-1} \left(\frac{x}{5} \right) + \operatorname{cosec}^{-1} \left(\frac{5}{4} \right) = \frac{\pi}{2}$

$$\Rightarrow \sin^{-1} \left(\frac{x}{5} \right) = \frac{\pi}{2} - \operatorname{cosec}^{-1} \left(\frac{5}{4} \right)$$

$$\Rightarrow \sin^{-1} \left(\frac{x}{5} \right) = \frac{\pi}{2} - \sin^{-1} \left(\frac{4}{5} \right)$$

$$[\because \sin^{-1} x + \cos^{-1} x = \pi/2]$$

$$\Rightarrow \sin^{-1} \left(\frac{x}{5} \right) = \cos^{-1} \left(\frac{4}{5} \right)$$

$$\Rightarrow \sin^{-1} \frac{x}{5} = \sin^{-1} \frac{3}{5} \Rightarrow \frac{x}{5} = \frac{3}{5} \Rightarrow x = 3$$

5. (a) $\cot \left(\operatorname{cosec}^{-1} \frac{5}{3} + \tan^{-1} \frac{2}{3} \right) = \cot \left[\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{2}{3} \right]$

$$= \cot \left[\tan^{-1} \left(\frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4} \times \frac{2}{3}} \right) \right] = \cot \left[\tan^{-1} \frac{17}{6} \right]$$

$$= \cot \left(\cot^{-1} \frac{6}{17} \right) = \frac{6}{17}$$

6. (a) Since, x, y, z are in A.P. $\Rightarrow 2y = x + z$
Also, we have, $2 \tan^{-1} y = \tan^{-1} x + \tan^{-1} (z)$

$$\Rightarrow \tan^{-1} \left(\frac{2y}{1 - y^2} \right) = \tan^{-1} \left(\frac{x + z}{1 - xz} \right)$$

$$\Rightarrow \frac{x + z}{1 - y^2} = \frac{x + z}{1 - xz} \quad (\because 2y = x + z)$$

$$\Rightarrow y^2 = xz \text{ or } x + z = 0 \Rightarrow x = y = z$$

7. (c) $\tan^{-1} y = \tan^{-1} x + \tan^{-1} \left[\frac{2x}{1 - x^2} \right]$

$$= \tan^{-1} x + 2 \tan^{-1} x = 3 \tan^{-1} x$$

$$\tan^{-1} y = \tan^{-1} \left[\frac{3x - x^3}{1 - 3x^2} \right] \Rightarrow y = \frac{3x - x^3}{1 - 3x^2}$$