Chapter 8

OSCILLATORY MOTION AND SMALL OSCILLATIONS

End of Art 116
EXAMPLES

1. $\ddot{x} = \frac{\mu}{(a+x)^2} + \frac{\mu a}{(a+x)^3} - \frac{\mu}{(a-x)^2} - \frac{\mu a}{(a-x)^3}$, where x is small,

$$= \frac{\mu}{a^2}\left[1 - \frac{2x}{a} + 1 - \frac{3x}{a} - 1 - \frac{2x}{a} - 1 - \frac{3x}{a}\right] = -\frac{10\mu x}{a^3}$$

on neglecting squares of x. Hence the time $= 2\pi\sqrt{\frac{a^3}{10\mu}}$.

Secondly, $\ddot{y} = -2\frac{y}{r}\left(\frac{\mu}{r^2} + \frac{\mu a}{r^3}\right) = -\frac{4\mu}{a^3}y$, on neglecting squares of y.

Hence the time $= 2\pi\sqrt{\frac{a^3}{4\mu}} = \pi\sqrt{\frac{a^3}{\mu}}$.

2. $m\ddot{x} = mg - 2\lambda\frac{\sqrt{a^2+x^2}-b}{b}\cdot\frac{x}{\sqrt{a^2+x^2}}$ ....(1)

There is equilibrium when $\ddot{x} = 0$, and then $2\lambda = \frac{mgb}{a\cot\alpha - b\cos\alpha}$.

In (1) put $x = a\cot\alpha + \xi$; then

$$\ddot{\xi} = g - \frac{gb}{a\cot\alpha - b\cos\alpha}\left[\frac{a\cot\alpha+\xi}{b} - \frac{a\cot\alpha+\xi}{\sqrt{\frac{a^2}{\sin^2\alpha} + 2a\xi\cot\alpha + \xi^2}}\right]$$

$$= g - \frac{gb}{a\cot\alpha - b\cos\alpha}\left[\frac{a\cot\alpha+\xi}{b} - (a\cot\alpha+\xi)\frac{\sin\alpha}{a}\left(1 - \frac{\xi}{a}\cos\alpha\sin\alpha\right)\right]$$

$$= -\frac{g}{a}\frac{a - b\sin^3\alpha}{a\cot\alpha - b\cos\alpha}\xi = -\frac{g}{a\cot\alpha}\frac{a - b\sin^3\alpha}{a - b\sin\alpha}\xi, \text{ etc.}$$

3. If the perpendicular from the centre upon the rod makes an angle θ with the vertical, the equation of energy gives

$$2 \times \tfrac{1}{2}ma^2\dot{\theta}^2 = C - 2mg\sqrt{a^2-c^2}(1-\cos\theta).$$

Hence $a^2\ddot{\theta} = -g\sqrt{a^2-c^2}\sin\theta = -g\sqrt{a^2-c^2}\,\theta$, if θ is small, etc.

4. When the depth of the centre is x, and the inclination to the vertical of either string is θ, then

$$M\ddot{x} = Mg - \frac{4\lambda}{d}\left[\frac{x}{\cos\theta} - d\right]\cos\theta = Mg - \frac{4\lambda}{d}\left[x - \frac{dx}{\sqrt{a^2+x^2}}\right].$$

Also $\ddot{x} = 0$, when $x = c$. $\quad \therefore\ g = \frac{4\lambda c(k-d)}{Mdk}$.

Put $x = c + \xi$, where ξ is small. Then

$$\ddot{\xi} = g - \frac{4\lambda}{Md}[c+\xi]\left[1 - \frac{d}{k}\left(1 - \frac{c\xi}{k^2}\right)\right] = g - \frac{4\lambda}{Md}\left[c\frac{k-d}{k} + \xi\frac{k-d}{k} + \frac{c^2d}{k^3}\xi\right]$$

$$= -\xi\left[\frac{g}{c} + \frac{4\lambda c^2}{Mk^3}\right], \text{ etc.}$$

5. The middle point A of the rod moves, during a small oscillation, in a circle of radius equal to the length of the string, so that

$$ml\ddot{\theta} = -mg\sin\theta, \text{ and } ml\dot{\theta}^2 - 2\lambda\frac{l-a}{a} - mg\cos\theta.$$

For a small oscillation $\dot{\theta}^2$ can be neglected and θ is small, so that

$$\cos\theta = 1,\ \sin\theta = \theta,\ l = a + \frac{mga}{2\lambda},\ \text{ and } \ddot{\theta} = -\frac{g}{l}\theta.$$

Hence the period $= 2\pi\sqrt{\dfrac{l}{g}} = 2\pi\sqrt{a\left(\dfrac{1}{g} + \dfrac{m}{2\lambda}\right)}$.

6. When there is equilibrium let the string be inclined at an angle α to the horizon so that $M = 2P\sin\alpha$. When α has become $\alpha + \theta$, where θ is small, we have, on neglecting squares of small quantities,

$$Mg - 2T\sin(\alpha+\theta) = M\frac{d^2}{dt^2}[a\tan(\alpha+\theta)] = Ma\ddot{\theta}\sec^2\alpha,$$

and $$Pg - T = P\frac{d^2}{dt^2}\left[l - \frac{2a}{\cos(\alpha+\theta)}\right] = -2Pa\ddot{\theta}\sec^2\alpha\sin\alpha.$$

$$\therefore\ [M - 2P\sin(\alpha+\theta)]\,g = a\ddot{\theta}\sec^2\alpha\,[M + 4P\sin\alpha\sin(\alpha+\theta)],$$

so that $$\ddot{\theta} = -\frac{2Pg}{a}\frac{\cos^3\alpha}{M + 4P\sin^2\alpha}\theta = -\frac{g}{a}\frac{(4P^2 - M^2)^{\frac{3}{2}}}{4MP(M+P)}\theta, \text{ etc.}$$

7. $\ddot{r} - r\dot{\theta}^2 = -\dfrac{\lambda}{m}\dfrac{r-a}{a}$, $r^2\dot{\theta} = b^2\omega$, and $\lambda\dfrac{b-a}{a} = m\omega^2 b$.

$$\therefore\ \ddot{r} - \frac{b^4\omega^2}{r^3} = -\omega^2 b\,\frac{r-a}{b-a}.\ \text{ Put } r = b + \xi, \text{ where } \xi \text{ is small.}$$

$$\therefore\ \ddot{\xi} = b\omega^2\left(1 + \frac{\xi}{b}\right)^{-3} - \frac{\omega^2 b}{b-a}(b - a + \xi) = -\frac{\lambda(4b-3a)}{mab}\xi, \text{ etc.}$$

8. $m_1(\ddot{r}_1 - r_1\dot{\theta}_1^2) = -T = m_2(\ddot{r}_2 - r_2\dot{\theta}_2^2)$. ……………………………(1)

Also $\dfrac{1}{r_1}\dfrac{d}{dt}(r_1^2\dot{\theta}_1) = 0$, so that $r_1^2\dot{\theta}_1 = \text{const.} = a_1^2\omega_1$.

So $r_2^2\dot{\theta}_2 = a_2^2\omega_2$. Hence (1) gives

$$m_1\left(\ddot{r}_1 - \frac{a_1^4\omega_1^2}{r_1^3}\right) = m_2\left(\ddot{r}_2 - \frac{a_2^4\omega_2^2}{r_2^3}\right).$$

In this equation put $r_1 = a_1 + y$, and hence $r_2 = a_2 - y$, where y is small.

$$\therefore\ (m_1 + m_2)\ddot{y} = m_1 a_1\omega_1^2\left(1 - \frac{3y}{a_1}\right) - m_2 a_2\omega_2^2\left(1 + \frac{3y}{a_2}\right).$$

But, in (1), $\ddot{r}_1 = 0$ and $\ddot{r}_2 = 0$ initially, so that $m_1 a_1\omega_1^2 = m_2 a_2\omega_2^2$.

$$\therefore\ (m_1 + m_2)\ddot{y} = -3(m_1\omega_1^2 + m_2\omega_2^2)\,y.\ \text{ Hence, etc.}$$

9. $m(\ddot{r} - r\dot{\theta}^2) = -T$, and $\dfrac{1}{r}\dfrac{d}{dt}(r^2\dot{\theta}) = 0$, so that $r^2\dot{\theta} = a^2\omega$.

Also $m'(-\ddot{r}) = m'g - T$.

$\therefore\ (m + m')\ddot{r} = \dfrac{m}{r^3}a^4\omega^2 - m'g$, so that initially, when $\ddot{r}$ is zero, $ma\omega^2 = m'g$.

Put $r = a + \xi$ where ξ is small.

$$\therefore\ (m + m')\ddot{\xi} = m'g\left[\left(1 + \frac{\xi}{a}\right)^{-3} - 1\right] = -\frac{3m'g}{a}\xi, \text{ etc.}$$

10. Use polar coordinates, so that $SP=r$, $ASP=\theta$, the tangent at P is inclined at $90°-\frac{\theta}{2}$ to the vertical, and the normal at $\frac{\theta}{2}$ to SP.

Then $$r=\frac{2a}{1+\cos\theta}=\frac{2a}{2-\frac{\theta^2}{2}}=a\left(1+\frac{\theta^2}{4}\right), \text{ since } \theta \text{ is small.}$$

$$\therefore\ \dot{r}=\frac{a\theta\dot{\theta}}{2}, \text{ and } \ddot{r}=\frac{a}{2}[\theta\ddot{\theta}+\dot{\theta}^2]=\text{zero, if squares be neglected.}$$

Then $$mg\cos\theta-mg\frac{r-\frac{a}{2}}{\frac{a}{2}}-R\cos\frac{\theta}{2}=m(\ddot{r}-r\dot{\theta}^2)=0,$$

so that $R=0$, if squares of θ are neglected.

Also $$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})=-g\sin\theta+\frac{R}{m}\sin\frac{\theta}{2}=-g\theta.$$

$\therefore\ a\ddot{\theta}=-g\theta$, squares of small quantities being neglected. Hence, etc.

11. $$\ddot{y}=f(a-y)-f(a+y)-2f[\sqrt{a^2+y^2}]\frac{y}{\sqrt{a^2+y^2}}$$
$$=f(a)-yf'(a)-f(a)-yf'(a)-\frac{2y}{a}f(a), \text{ if } y \text{ be small,}$$
$$=-2\left[\frac{1}{a}f(a)+f'(a)\right]y. \text{ Hence the result.}$$

12. The centre of gravity G of the triangle ABC, at whose angular points are the particles, remains at rest, since there are no external forces acting on the system. Also always $AG=\frac{2}{3}\,.\,AB\,.\,\frac{\sqrt{3}}{2}=\frac{AB}{\sqrt{3}}$.

The resultant repulsion on each particle $=3\mu\,.\,GA$.

If the unstretched length of the string is a, then, in the position of equilibrium, $AB=2a$. Hence

$$3\mu\,.\,\frac{2a}{\sqrt{3}}=2\lambda\frac{2a-a}{a}\cos 30°=2\lambda\frac{\sqrt{3}}{2}, \text{ so that } \lambda=2\mu a.$$

When $GA=x$ we have

$$m\ddot{x}=3\mu\,.\,x-2\lambda\frac{\sqrt{3}x-a}{a}\,\frac{\sqrt{3}}{2}=-3\mu\left(x-\frac{2a}{\sqrt{3}}\right).$$

Put $x=\frac{2a}{\sqrt{3}}+\xi$, where ξ is small, and then $\ddot{\xi}=-\frac{3\mu}{m}\xi$, etc.

13. Let O be the centre, P the particle, Q any point of the ring, $\angle QOX=\theta$, and $QPX=\phi$, where OPX is a straight line. Then

$$m\ddot{x}=-2\int_0^\pi \gamma m\rho\,.\,\frac{a\,d\theta}{PQ^2}\cos\phi=-2\gamma m\rho a\int_0^\pi\frac{(a\cos\theta-x)\,d\theta}{(a^2+x^2-2ax\cos\theta)^{\frac{3}{2}}}$$
$$=-\frac{2\gamma m\rho}{a^2}\int_0^\pi(a\cos\theta-x)\left(1-\frac{2x}{a}\cos\theta\right)^{-\frac{3}{2}}, \text{ where } x \text{ is small,}$$
$$=-\frac{2\gamma m\rho}{a^2}\int_0^\pi(a\cos\theta-x+3x\cos^2\theta)\,d\theta=-\frac{2\gamma m\rho}{a^2}\left[-x\pi+\frac{3x\pi}{2}\right]=-\frac{\gamma m\rho\pi x}{a^2}.$$

Hence the time required $=2\pi\frac{a}{\sqrt{\gamma\rho\pi}}$, etc.

14. $$M\ddot{x} = -\gamma \int_{x-a}^{x+a} m \frac{M\,d\xi}{2a} \cdot \frac{1}{c^2+\xi^2} \cdot \frac{\xi}{\sqrt{c^2+\xi^2}}$$
$$= \frac{\gamma M m}{2a}\left[\frac{1}{\sqrt{c^2+(x+a)^2}} - \frac{1}{\sqrt{c^2+(x-a)^2}}\right].$$
Hence, if x is small, we have
$$\ddot{x} = -\frac{\gamma m}{a} \cdot \frac{ax}{(c^2+a^2)^{\frac{3}{2}}}, \text{ etc.}$$

15. M, m the masses of the rod and ring, $2a$ the length of the rod, and b the radius of the ring. Then
$$M\ddot{x} = -\int_{-a}^{+a} \gamma \cdot \frac{M\,d\xi}{2a} \cdot \frac{m}{(x+\xi)^2+b^2} \cdot \frac{x+\xi}{\sqrt{(x+\xi)^2+b^2}}$$
$$= \frac{\gamma M m}{2a}\left[\frac{1}{\sqrt{(x+a)^2+b^2}} - \frac{1}{\sqrt{(x-a)^2+b^2}}\right] = -\frac{\gamma M m}{(a^2+b^2)^{\frac{3}{2}}}\, x,$$
on neglecting squares of x. Hence, etc.

16. There is equilibrium when the first string is inclined at α to the vertical, where $M \sin \alpha = m \cos\left(45° - \frac{\alpha}{2}\right)$, and hence $\alpha = \frac{m}{\sqrt{2}M}$ since m, and therefore α, is small.

When this angle α becomes $\alpha+\theta$, where θ is small, then
$$mg - T = m\frac{d^2}{dt^2}\left[b - 2l \sin\left(45° - \frac{\alpha+\theta}{2}\right)\right] = ml \cos\left(45° - \frac{\alpha}{2}\right) . \ddot{\theta},$$
and
$$Ml\ddot{\theta} = T \cos\left(45° - \frac{\alpha+\theta}{2}\right) - Mg \sin(\alpha+\theta).$$
$$\therefore \ddot{\theta}\left[Ml + ml \cos\left(45° - \frac{\alpha}{2}\right) \cos\left(45° - \frac{\alpha+\theta}{2}\right)\right]$$
$$= mg \cos\left[45° - \frac{\alpha+\theta}{2}\right] - Mg \sin(\alpha+\theta).$$
$$\therefore \left(M + \frac{m}{2}\right) l\ddot{\theta} = -\theta\left[Mg - \frac{mg}{2\sqrt{2}}\right],$$
on neglecting squares of small quantities,
$$\therefore \text{ required time} = 2\pi\sqrt{\frac{2(2M+m)}{4M - m\sqrt{2}} \cdot \frac{l}{g}}, \text{ etc.}$$

17. Let O be the point of suspension and B the original position of the particle, $OB = b = \frac{4a}{3}$, where a is the unstretched length of the string. Take Bx horizontal, and By vertically downwards, as axes. Let $P(x, y)$ be any position of the particle, and $\angle BOP = \theta$. Then
$$m\ddot{x} = -\lambda \frac{OP - a}{a} \sin\theta = -\frac{3mg}{a}[x - a \sin\theta] = -\frac{3mg}{a}\left[x - \frac{ax}{b}\right] = -\frac{3mg}{4a}x,$$
and $$m\ddot{y} = mg - \lambda\frac{OP - a}{a}\cos\theta = mg - \frac{3mg}{a}[b + y - a\cos\theta] = -\frac{3mg}{a}y,$$
the squares of θ, x and y being neglected.

Hence if (α, β) is the point, P_0, to which the particle was displaced and instantaneously held at rest, we have

$$x = \alpha \cos\left[\frac{1}{2}\sqrt{\frac{3g}{a}}\,t\right], \text{ and } y = \beta \cos\left[\sqrt{\frac{3g}{a}}\,t\right] = \beta\left[\frac{2x^2}{a^2} - 1\right],$$

$\therefore x^2 = \dfrac{\alpha^2}{2\beta}(y+\beta)$, a parabola of latus-rectum $\dfrac{\alpha^2}{2\beta}$, and whose vertex, A, is at $(-\beta, 0)$.

The horizontal line through P_0 is the latus-rectum if the depth of P_0 below $A = \dfrac{1}{4}$ latus-rectum, *i.e.* if $2\beta = \dfrac{\alpha^2}{8\beta}$, *i.e.* if $\dfrac{\alpha}{\beta} = 4$, *i.e.* if the displacement makes an angle $\cot^{-1} 4$ with the horizon.

Art 117

Ex. $\mu = \dfrac{1}{4}$, and $n = \dfrac{4\pi}{3}$,

$$\therefore \text{ time of a half oscillation} = \pi \div \sqrt{n^2 - \frac{\mu^2}{4}} = \pi \div \sqrt{\frac{16\pi^2}{9} - \frac{1}{64}}$$

$$= \frac{3}{4}\left[1 + \frac{9}{32 \times 64\pi^2}\right] \text{ nearly.}$$

$$\text{Also the required factor} = e^{-\frac{\pi}{8} \div \sqrt{\frac{16\pi^2}{9} - \frac{1}{64}}} = e^{-\frac{3}{32}\left(1 + \frac{9}{32\times 64\pi^2}\right)}$$

$$= 1 - \frac{3}{32}\left(1 + \frac{9}{32 \times 64\pi^2}\right) \text{ approx.} = \frac{29}{32} \text{ approx.}$$

End of Art 124

EXAMPLES

1. Putting $x = -\dfrac{D}{C} + \lambda\cos(pt + \epsilon)$, we have $Ap^4 - Bp^2 + C = 0$, which is given to have real positive roots for p^2. Hence, etc.

2. $$\dot{x}^2 = -2\int\left(\frac{\mu x}{a} + \frac{\nu x^3}{a^3}\right)dx = \frac{\mu}{a}(a^2 - x^2) + \frac{\nu}{2a^3}(a^4 - x^4).$$

$$\therefore t\sqrt{\frac{\mu}{a}} = \int_0^x \frac{dx}{\sqrt{a^2 - x^2}\sqrt{1 + \frac{\nu}{2\mu a^2}(a^2 + x^2)}}, \text{ where } \nu \text{ is small,}$$

$$= \int_0^x \frac{dx}{\sqrt{a^2 - x^2}}\left[1 - \frac{\nu}{4\mu a^2}(a^2 + x^2)\right] \quad [\text{Put } x = a\sin\theta.]$$

$$= \int_0^{\frac{\pi}{2}}\left[1 - \frac{\nu}{4\mu} - \frac{\nu}{4\mu}\sin^2\theta\right]d\theta = \left(1 - \frac{3\nu}{8\mu}\right)\frac{\pi}{2}, \text{ etc.}$$

3. $$m\ddot{x} = mg + \lambda\left(\frac{y}{b} - \frac{x}{a}\right), \text{ and } m'(\ddot{x} + \ddot{y}) = m'g - \lambda\frac{y - b}{b}, \quad \ldots\ldots\ldots\ldots\ldots(1)$$

where a and b are the unstretched lengths of the two portions of the string. Let c, d be these lengths when there is equilibrium. On putting $x = c + \xi$ and $y = d + \eta$, we have

$$\left(D^2 + \frac{\lambda}{ma}\right)\xi - \frac{\lambda}{mb}\eta = 0, \text{ and } D^2\xi + \left(D^2 + \frac{\lambda}{m'b}\right)\eta = 0.$$

$$\therefore \left\{\left(D^2 + \frac{\lambda}{ma}\right)\left(D^2 + \frac{\lambda}{m'b}\right) + \frac{\lambda D^2}{mb}\right\}\xi = 0.$$

If we put $\xi = A\cos(pt+B)$, we have

$$p^4 - p^2\left[\frac{\lambda}{ma} + \frac{\lambda}{mb} + \frac{\lambda}{m'b}\right] + \frac{\lambda^2}{mm'ab} = 0,$$

and the required times are $\frac{2\pi}{p_1}$ and $\frac{2\pi}{p_2}$.

4. Originally, $m\ddot{x} = mg - \lambda\frac{b+x-a}{a}$, and $\lambda\frac{b-a}{a} = mg$.

$$\therefore\ m\ddot{x} = -\frac{\lambda}{a}x = -\frac{mg}{b-a}x, \text{ so that } n^2 = \frac{g}{b-a}.$$

In the second case we have

$$m\frac{d^2}{dt^2}[c\sin pt + b + x] = mg - \lambda\frac{b+x-a}{a} = -mn^2x.$$

$$\therefore\ \ddot{x} = -n^2x + cp^2\sin pt.$$

$$\therefore\ x = A\cos nt + B\sin nt + \frac{cp^2}{n^2-p^2}\sin pt = \frac{pc}{n^2-p^2}[p\sin pt - n\sin nt],$$

since when $t=0$, $x=0$ and $\frac{d}{dt}[x+c\sin pt]=0$, *i.e.* $\dot{x} = -pc$.

$$\therefore\ \text{total length required} = b + x = \text{etc.}$$

If $p=n$, the equation is $\ddot{x} = -n^2x + cn^2\sin nt$.

$$\therefore\ x = C\cos nt + D\sin nt - \frac{cnt}{2}\cos nt = -\frac{c}{2}\sin nt - \frac{cnt}{2}\cos nt, \text{ etc.}$$

5. $20g = \lambda\,.\,\frac{13\frac{1}{2}-12}{12}$, so that $\lambda = 160g$. At time t, the depth of the upper end of the spring $= \frac{1}{6}\sin\frac{10\pi t}{3}$.

Then $$20\frac{d^2}{dt^2}\left[y + \frac{1}{6}\sin\frac{10\pi t}{3}\right] = 20g - 160g(y-1).$$

$$\therefore\ \ddot{y} = \frac{100\pi^2}{54}\,.\,\sin\frac{10\pi t}{3} - 8g\left(y - \frac{9}{8}\right).$$

$$\therefore\ y - \frac{9}{8} = A\cos[\sqrt{8g}t + B] + \frac{100\pi^2}{6[72g - 100\pi^2]}\sin\frac{10\pi t}{3}.$$

The amplitude of the motion set up $= \frac{1}{6} + \frac{100\pi^2}{6(72g-100\pi^2)} = \frac{12g}{72g-100\pi^2}$

$$= \frac{384}{1304}\text{ ft. approx.} = 3{\cdot}53\text{ inches.}$$

6. $l\ddot{\theta} = -g\sin\theta - \mu l\dot{\theta}$, so that $\ddot{\theta} + \mu\dot{\theta} = -\frac{g}{l}\theta$.

$\therefore\ \theta = Ae^{-\frac{\mu t}{2}}\cos\left[\sqrt{\frac{g}{l} - \frac{\mu^2}{4}}\,t + B\right]$, and the periodic time, as in Art. 117,

$$= 2\pi \div \sqrt{\frac{g}{l} - \frac{\mu^2}{4}}.$$

7. $2\pi\sqrt{\frac{l}{g}}=2$, so that $l=g\div\pi^2$. The equation of motion is

$$\ddot{\theta}=-\frac{g}{l}\theta-\frac{1}{25}\dot{\theta}=-\pi^2\theta-\frac{1}{25}\dot{\theta}.$$

Hence, as in Art. 117,

$$\theta=Ae^{-\frac{t}{50}}\cos[nt+B]=\frac{\pi}{180}e^{-\frac{t}{50}}\left(\cos nt+\frac{1}{50n}\sin nt\right),\text{ where } n^2=\pi^2-\frac{1}{50^2}.$$

After 10 complete oscillations, when $t=10\times\frac{2\pi}{n}=20$ approx., we have

$$\theta=\frac{\pi}{180}e^{-\frac{2}{5}}=e^{-\cdot 4}\text{ degrees}=\cdot 670\text{ degree}=40'\,13''.$$

8. $\theta=A\cos\sqrt{\frac{g}{l}}t+B\sin\sqrt{\frac{g}{l}}t+\frac{1}{2}\frac{a}{l}\sqrt{\frac{g}{l}}t\sin\sqrt{\frac{g}{l}}t.$

When $t=0$, $\theta=0$ and $\dot{\theta}=\omega$, so that $A=0$ and $B=\omega\sqrt{\frac{l}{g}}$, etc.

9. At time t let ξ be the length of the spring and θ the inclination of the pendulum. Then

$$m[\ddot{\xi}\sin\theta-l\dot{\theta}^2]=mg\cos\theta-T,\quad\ldots\ldots(1)$$

and

$$\ddot{\xi}\cos\theta+l\ddot{\theta}=-g\sin\theta.\quad\ldots\ldots(2)$$

Also, since the spring is massless,

$$T\sin\theta=\lambda\frac{\xi-a}{a}.\quad\ldots\ldots(3)$$

θ and $\dot{\theta}$ being small, (1) and (3) give $\xi=a\left(1+\frac{mg}{\lambda}\theta\right)=a+\frac{al}{W}\theta.$

Then (2) gives $\qquad\ddot{\theta}\left[l+\frac{al}{W}\right]=-g\theta$, etc.

10. $ma\ddot{\theta}=-mg\sin\theta-\frac{\lambda m^2\cos\theta}{(b+2a\sin\theta)^2}$, so that, λ being small, there is equilibrium when $\theta=-\frac{\lambda m}{gb^2}$. Put $\theta=-\frac{\lambda m}{gb^2}+\phi$, where ϕ is small.

$\therefore\ a\ddot{\phi}=-g\phi+\frac{4a\lambda m}{b^3}\phi$. Hence the time $=2\pi\sqrt{\frac{a}{g}}\left[1-\frac{4a\lambda m}{b^3g}\right]^{-\frac{1}{2}}=$ etc.

11. At time t, let ϕ and θ be the inclinations to the vertical (on opposite sides) of the mast and pendulum so that $\phi=a\sin\omega t$, where $\frac{2\pi}{\omega}=8$.

Then $l\ddot{\theta}+\cos\theta\frac{d^2}{dt^2}\left[10a\cos\frac{\pi t}{4}\right]=-g\sin\theta$, where $1=2\pi\sqrt{\frac{l}{g}}$.

$$\therefore\ \ddot{\theta}+4\pi^2\theta=\frac{5\pi^4a}{2g}\sin\frac{\pi t}{4}.$$

$$\therefore\ \theta = A\cos 2\pi t + B\sin 2\pi t + \frac{40}{63}\frac{\pi^2 a}{g}\sin\frac{\pi t}{4}$$

$$= \frac{5\pi^2 a}{63g}\left[8\sin\frac{\pi t}{4} - \sin 2\pi t\right], \text{ if } \theta = 0 \text{ and } \dot{\theta} = 0 \text{ when } t = 0.$$

$$\therefore\ \frac{\theta + \phi}{\phi} = 1 + \frac{40\pi^2}{63g}\left[1 - \frac{1}{8}\frac{\sin 8\psi}{\sin\psi}\right], \text{ where } \psi = \frac{\pi t}{4}.$$

Now the maximum value of $\frac{\sin 8\psi}{\sin\psi}$ is given by

$$8\cos 8\psi \sin\psi - \sin 8\psi\cos\psi = 0, \text{ i.e. by } \tan 8\psi = 8\tan\psi.$$

On putting $\psi = \frac{7\pi}{16} + \alpha$, where α is small, we have $\alpha = -\frac{1}{316}$ radian $= -11'$ approx. and then

$$\frac{\sin 8\psi}{\sin\psi} = \frac{\sin\left(\frac{7\pi}{2} - 88'\right)}{\sin\left(\frac{7\pi}{16} - 11'\right)} = -\frac{\cos 1^\circ 28'}{\sin 78^\circ 34'} = -\frac{\cdot 9997}{\cdot 9802} = -\frac{51}{50} \text{ nearly.}$$

Hence the maximum value of $\frac{\theta + \phi}{\phi}$

$$= 1 + \frac{40\pi^2}{63g}\cdot\frac{451}{400} = 1 + \frac{451}{63 \times 32} = 1{\cdot}224 \text{ nearly.}$$

So the minimum value $= 1 + \frac{40\pi^2}{63g}\cdot\left(1 - \frac{51}{400}\right) = 1{\cdot}173$, and the mean of these $= 1{\cdot}198$, etc. $= 1\frac{1}{5}$ approx.

12. When the motion is steady there is no acceleration on the particle, so that the vertical plane through the string must pass through the centre of the horizontal circle. We thus have

$$m(a + l\sin\theta)\,\omega^2 = T\sin\theta, \text{ and } mg = T\cos\theta, \text{ etc.}$$

13. $\lambda = 3mg$. For a small oscillation, in which $r = \frac{4l}{3} + R$ where R is small, we have

$$\ddot{R} - \left(\frac{4l}{3} + R\right)\dot{\theta}^2 = g\cos\theta - \frac{\lambda}{m}\frac{\frac{l}{3} + R}{l} = -\frac{3g}{l}R,$$

and
$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = -g\sin\theta, \text{ i.e. } 2\dot{R}\dot{\theta} + \left(\frac{4l}{3} + R\right)\ddot{\theta} = -g\theta.$$

Neglecting squares of small quantities, we have

$$\ddot{R} = -\frac{3g}{l}R \text{ and } \ddot{\theta} = -\frac{3g}{4l}\theta, \text{ etc.}$$

Chapter 8

OSCILLATORY MOTION AND SMALL OSCILLATIONS

114. In the previous chapters we have had several examples of oscillatory motion. We have seen that wherever the equation of motion can be reduced to the form $\ddot{x} = n^2 x$, or $\ddot{\theta} = -n^2\theta$, the motion is simple harmonic with a period of oscillation equal to $\dfrac{2\pi}{n}$. We shall give in this chapter a few examples of a more difficult character.

115. *Small oscillations*. The general method of finding the small oscillations about a position of equilibrium is to write down the general equations of motion of the body. If there is only one variable, x say, find the value of x which makes $\dot{x}, \ddot{x} \cdots$ etc. zero, *i.e.* which gives the position of equilibrium. Let this value be a.

In the equation of motion put $x = a + \xi$, where ξ is small. For a small oscillation ξ will be small so that we may neglect its square. The equation of motion then generally reduces to the form $\ddot{\xi} = -\lambda\xi$, in which case the time of a small oscillation is $\dfrac{2\pi}{\sqrt{\lambda}}$.

For example, suppose the general equation of motion is

$$\frac{d^2x}{dt^2} + f(x)\left(\frac{dx}{dt}\right)^2 = F(x).$$

For the position of equilibrium we have $F(x) = 0,$ giving $x = a$.

Put $x = a + \xi$ and neglect ξ^2.

The equation becomes

$\dfrac{d^2\xi}{dt^2} = F(a+\xi) = F(a) + \xi F'(a) + \cdots,$ by Taylor's theorem.

Since $F(a) = 0$ this gives

$$\frac{d^2\xi}{dt^2} = \xi . F'(a).$$

If $F'(a)$ be negative, we have a small oscillation and the position of equilibrium given by $x = a$ is stable.

If $F'(a)$ be positive, the corresponding motion is not oscillatory and the position of equilibrium is unstable.

116. EX. 1. *A uniform rod, of length* $2a$*, is supported in a horizontal position by two strings attached to its ends whose other extremities are tied to a fixed point; if the unstretched length of each string be* l *and the modulus of elasticity be* n *times the weight of the rod, show that in the position of equilibrium the strings are inclined to the vertical at an angle* α *such that* $a\cot\,\alpha - l\cos\alpha = \dfrac{l}{2n}$ *and that the time of a small oscillation about the position of equilibrium is* $2\pi\sqrt{\dfrac{a}{g}\dfrac{\cot\,\alpha}{1+2n\cos^3\alpha}}$.

When the rod is at depth x below the fixed point, let θ be the inclination of each string to the vertical,

so that $x = a\cot\,\theta$ and the tension $= nmg\dfrac{\dfrac{a}{\sin\theta} - l}{l} = \dfrac{nmg}{l}\,\dfrac{a - l\,\sin\theta}{\sin\theta}$.

The equation of motion is then $\quad m\ddot{\theta} = mg - 2.\dfrac{nmg}{l}.\dfrac{a - l\sin\theta}{\sin\theta}\cos\theta,$

i.e. $$-\frac{a}{\sin^2\theta}\ddot{\theta}+\frac{2a\cos\theta}{\sin^3\theta}\dot{\theta}^2=g-2\frac{ng}{l}.\frac{a-l\sin\theta}{\sin\theta}\cos\theta,$$

i.e. $$\ddot{\theta}-2\cot\,\theta\,\dot{\theta}^2=-\frac{g}{a}\sin^2\theta+\frac{2ng}{al}\sin\theta\cos\theta(a-l\sin\theta)\ ...(1).$$

In the position of equilibrium when $\theta=\alpha$, we have $\dot{\theta}=0$ and $\ddot{\theta}=0$, and

$$\therefore\quad a\cot\,\alpha-l\cos\alpha=\frac{l}{2n}\qquad ...(2).$$

For a small oscillation put $\theta=\alpha+\Psi$, where Ψ is small, and

$$\therefore\quad \sin\theta=\sin\alpha+\Psi\cos\alpha,\text{ and }\cos\theta=\cos\alpha-\Psi\sin\alpha.$$

In this case $\dot{\theta}^2$ is the square of a small quantity and is negligible, and (1) gives

$$\ddot{\Psi}=-\frac{g}{a}(\sin\alpha+\Psi\cos\alpha)^2+\frac{2ng}{al}(\sin\alpha+\Psi\cos\alpha)\times$$

$$(\cos\alpha-\Psi\sin\alpha)[a-l(\sin\alpha+\Psi\cos\alpha)]$$

$$=-\frac{g}{a}(\sin^2\alpha+2\Psi\sin\alpha\cos\alpha)$$

$$+\frac{2ng}{al}[\sin\alpha\cos\alpha+\Psi(\cos^2\alpha-\sin^2\alpha)]$$

$$\left(\frac{l}{2n}\tan\alpha-l\Psi\cos\alpha\right)\text{ by equation (2)}$$

$$=-\Psi.\frac{g}{a}[2n\sin\alpha\cos^2\alpha+\tan\alpha]$$

$$=-\Psi.\frac{g}{a}\tan\alpha(1+2n\cos^3\alpha).$$

Hence the required time $=2\pi\sqrt{\frac{a}{g}\frac{\cot\alpha}{1+2n\cos^3\alpha}}$.

Making use of the principle of the last article, if the right-hand side of (1) be $f(\theta)$, the equation for small oscillations is

$$\ddot{\Psi} = \Psi . f'(\alpha),$$

$$\text{and} \quad f'(\alpha) = -\frac{2g}{a}\sin\alpha\cos\alpha + \frac{2ng}{al}(\cos^2\alpha - \sin^2\alpha)(a - l\sin\alpha) - \frac{2ng}{a}\sin\alpha\cos^2\alpha$$

$$= \text{ etc., as before.}$$

EX. 2. *A heavy particle is placed at the centre of a smooth circular table; n strings are attached to it and, after passing over small pulleys symmetrically arranged at the circumference of the table, each is attached to a mass equal to that of the particle on the table. If the particle be slightly displaced, show that the time of an oscillation is* $2\pi\sqrt{\frac{a}{g}\left(1+\frac{2}{n}\right)}$.

Let O be the centre of the board, $A_1, A_2, \ldots, A_n$ the pulleys, and let the particle be displaced along a line OA lying between OA_n and OA_1. When its distance $OP = x$, let $PA_r = y$, and $POA_r = \alpha_r$. Also, let a be the radius of the table and l the length of a siring.

$$\text{Then} \quad y_r = \sqrt{a^2 + x^2 - 2ax\cos\alpha_r} = a\left(1 - \frac{x}{a}\cos\alpha_r\right),$$

since x is very small.

Let T_r be the tension of the string PA_r.

$$\text{Then} \quad mg - T_r = m\frac{d^2}{dt^2}(l - y_r) = m\ddot{x}\cos\alpha_r$$

$$\therefore \quad T_r = m(g - \ddot{x}\cos\alpha_r).$$

Also $T_r \cos APA_r = m(g - \ddot{x}\cos\alpha_r).\dfrac{a\cos\alpha_r - x}{y_r}$

$$= m(g - \ddot{x}\cos\alpha_r).\frac{a\cos\alpha_r - x}{a}\left(1 + \frac{x}{a}\cos\alpha_r\right)$$

$$= \frac{m}{a^2}(g - \ddot{x}\cos\alpha_r).[a^2\cos\alpha_r - ax + ax\cos^2\alpha_r].$$

Now if $POA_1 = \alpha$, then

$$\sum\cos\alpha_r = \cos\alpha + \cos\left(\alpha + \frac{2\pi}{n}\right) + \cdots \text{to } n \text{ terms } = 0,$$

$$\sum\cos^2\alpha_r = \frac{1}{2}\left[1 + \cos 2\alpha + 1 + \cos\left(2\alpha + \frac{4\pi}{n}\right) + \cdots\right] = \frac{n}{2}, \text{ and}$$

$$\sum\cos^3\alpha_r = \frac{1}{4}\sum[3\cos\alpha_r + \cos 3\alpha_r] = 0.$$

Therefore the equation of motion of P is

$$m\ddot{x} = \sum T_r\cos APA_r = \frac{m}{a^2}\left[-agx.n + gax\frac{n}{2} - a^2\ddot{x}\frac{n}{2}\right]$$

$$\therefore \quad \ddot{x}\left(1 + \frac{n}{2}\right) = -\frac{ng}{2a}x.$$

$$\therefore \quad \ddot{x} = -\frac{ng}{a(2+n)}x,$$

and the time of a complete oscillation $= 2\pi\sqrt{\dfrac{a(2+n)}{ng}}$.

It can easily be shown that the sum of the resolved parts of the tensions perpendicular to OP vanishes if squares of x be neglected.

EX. 3. *Two particles, of masses m and m', are connected by an elastic string of natural length a and modulus of elasticity λ; m is on a smooth table and describes a circle of radius c with uniform angular velocity; the string passes through a hole in the table at the centre of* *the circle and m' hangs at rest at a distance c' below the table. Show that, if m be slightly disturbed, the periods $\dfrac{2\pi}{p}$ of small oscillations about this state of steady motion are given by the equation*

$$a^2cmm'p^4-\{mc+(4c+3c'-3a)m'\}a\lambda p^2+3(c+c'-a)\lambda^2=0.$$

At any time during the motion let x and y be the distances of m and m' from the hole and T the tension, so that the equations of motion are

$$m(\ddot{x}-x\dot{\theta}^2)=-T=-\lambda\frac{x+y-a}{a} \qquad ...(1),$$

$$\frac{1}{x}\frac{d}{dt}(x^2\dot{\theta})=0 \qquad ...(2),$$

$$\text{and} \quad m'\ddot{y}=m'g-T=m'g-\lambda\frac{x+y-a}{a} \qquad ...(3).$$

(2) gives $\quad x^2\dot{\theta}$ = const. $=h$,

so that (1) gives $\quad \ddot{x}=\dfrac{h^2}{x^3}-\dfrac{\lambda}{ma}(x+y-a) \qquad ...(4).$

When $x=c,\ y=c'$ we have equilibrium, so that $\ddot{x}=\ddot{y}=0$ then, and hence from (3) and (4)

$$m'g=m\frac{h^2}{c^3}=\frac{\lambda(c+c'-a)}{a} \qquad ...(5).$$

Hence (4) and (3) give, on putting

$x=c+\xi$ and $y=c'+\eta$ where ξ and η are small,

$$\ddot{\xi}=\frac{h^2}{c^3}\left(1-\frac{3\xi}{c}\right)-\frac{\lambda}{ma}(c+c'-a+\xi+\eta)$$

$$=-\frac{\lambda}{am}\left[\frac{4c+3c'-3a}{c}\,\xi+\eta\right],$$

and $\ddot{\eta}=-\dfrac{\lambda}{am'}(\xi+\eta).$

To solve these equations, put $\xi = A\cos(pt+\beta)$ and $\eta = B\cos(pt+\beta)$.

On substituting we have

$$A\left[-p^2+\frac{\lambda}{am}\frac{4c+3c'-3a}{c}\right]+\frac{\lambda}{am}B=0, \quad \text{and}$$

$$A.\frac{\lambda}{am'}+B\left[-p^2+\frac{\lambda}{am'}\right]=0.$$

Equating the two values of $\frac{A}{B}$ thus obtained, we have, on reduction,

$$a^2cmm'p^4-\{mc+m'(4c+3c'-3a)\}a\lambda p^2+3(c+c'-a)\lambda^2=0.$$

This equation gives two values, p_1^2 and p_2^2, for p^2, both values being positive.

The solution is thus of the form

$$\xi = A_1\cos(p_1t+\beta_1)+A_2\cos(p_2t+\beta_2)$$

with a similar expression for η.

Hence the oscillations are compounded of two simple harmonic motions whose periods are $\frac{2\pi}{p_1}$ and $\frac{2\pi}{p_2}$.

EXAMPLES

1. Two equal centres of repulsive force are at a distance $2a$, and the law of force is $\frac{\mu}{r^2}+\frac{\mu a}{r^3}$; find the time of the small oscillation of a particle on the line joining the centres.

   If the centres be attractive, instead of repulsive, find the corresponding time for a small oscillation on a straight line perpendicular to it.

2. A heavy particle is attached by two equal light extensible strings to two fixed points in the same horizontal line distant $2a$ apart; the length of each string when unstretched was b and the modulus of elasticity is λ. The particle is at rest when the strings are inclined at an angle α to the vertical, and is then slightly displaced in a vertical direction; show that the time of a complete small oscillation is

$$2\pi\sqrt{\frac{a\cot\alpha}{g}.\frac{a-b\sin\alpha}{a-b\sin^3\alpha}}.$$

3. Two heavy particles are fastened to the ends of a weightless rod, of length $2c$, and oscillate in a vertical plane in a smooth sphere of radius a; show that the time of the oscillation is the same as that of a simple pendulum of length $\frac{a^2}{\sqrt{a^2-c^2}}$.
4. A heavy rectangular board is symmetrically suspended in a horizontal position by four light elastic strings attached to the corners of the board and to a fixed point vertically above its centre. Show that the period of the small vertical oscillations is

$$2\pi\left(\frac{g}{c}+\frac{4c^2\lambda}{k^3M}\right)^{-1/2}$$

where c is the equilibrium-distance of the board below the fixed point, a is the length of a semi-diagonal, $k=\sqrt{a^2+c^2}$ and λ is the modulus.
5. A rod of mass m hangs in a horizontal position supported by two equal vertical elastic strings, each of modulus λ and natural length a. Show that, if the rod receives a small displacement parallel to itself, the period of a horizontal oscillation is

$$2\pi\sqrt{a\left(\frac{1}{g}+\frac{m}{2\lambda}\right)}.$$

6. A light string has one end attached to a fixed point A, and, after passing over a smooth peg B at the same height as A and distant $2a$ from A, carries a mass M at the other end. A ring, of mass M, can slide on the portion of the string between A and B. Show that the time of its small oscillation about its position of equilibrium is $4\pi\left[aMP(M+P)\div g(4P^2-M^2)^{3/2}\right]^{1/2}$, assuming that $2P > M$.
7. A particle, of mass m, is attached to a fixed point on a smooth horizontal table by a fine elastic string, of natural length a and modulus of elasticity λ, and revolves uniformly on the table, the string being stretched to a length b; show that the time of a small oscillation for a small additional extension of the string is $2\pi\sqrt{\dfrac{mab}{\lambda(4b-3a)}}$.
8. Two particles, of masses m_1 and m_2, are connected by a string, of length a_1+a_2, passing through a smooth ring on a horizontal table, and the particles are describing circles of radii a_1 and a_2, with angular velocities ω_1 and ω_2 respectively. Show that $m_1a_1\omega_1^2 = m_2a_2\omega_2^2$, and that the small oscillation about this state takes place in the time $2\pi\sqrt{\dfrac{m_1+m_2}{3(m_1\omega_1^2+m_2\omega_2^2)}}$.
9. A particle, of mass m, on a smooth horizontal table is attached by a fine string through a hole in the table to a particle of mass m' which hangs freely. Find the condition that the particle m may describe a circle uniformly, and show that, if m' be slightly dis-

turbed in a vertical direction, the period of the resulting oscillation is $2\pi\sqrt{\dfrac{(m+m')a}{3m'g}}$, where a is the radius of the circle.

10. On a wire in the form of a parabola, whose latus-rectum is $4a$ and whose axis is vertical and vertex downwards, is a bead attached to the focus by an elastic string of natural length $\dfrac{a}{2}$, whose modulus is equal to the weight of the bead. Show that the time of a small oscillation is $2\pi\sqrt{\dfrac{a}{g}}$.
11. At the corners of a square whose diagonal is $2a$, are the centres of four equal attractive forces equal to any function $m.f(x)$ of the distance x of the attracted particle m; the particle is placed in one of the diagonals very near the centre; show that the time of a small oscillation is $\pi\sqrt{2}\left\{\dfrac{1}{a}f(a)+f'(a)\right\}^{-1/2}$.
12. Three particles, of equal mass m, are connected by equal elastic strings and repel one another with a force n times the distance. In equilibrium each string is double its natural length; show that if the particles are symmetrically displaced (so that the three strings always form an equilateral triangle) they will oscillate in period $2\pi\sqrt{\dfrac{m}{3\mu}}$.
13. Every point of a fine uniform circular ring repels a particle with a force which varies inversely as the square of the distance; show that the time of a small oscillation of the particle about its position of equilibrium at the centre of the ring varies as the radius of the ring.
14. A uniform straight rod, of length $2a$, moves in a smooth fixed tube under the attraction of a fixed particle, of mass m, which is at a

distance c from the tube. Show that the time of a small oscillation is $2\pi\sqrt[4]{\dfrac{(a^2+c^2)^3}{\gamma^2 m^2}}$.

15. A uniform straight rod is perpendicular to the plane of a fixed uniform circular ring and passes through its centre; every particle of the ring attracts every particle of the rod with a force varying inversely as the square of the distance; find the time of a small oscillation about the position of equilibrium, the motion being perpendicular to the plane of the ring.
16. A particle, of mass M, hangs at the end of a vertical string, of length l, from a fixed point O, and attached to it is a second string which passes over a small pulley, in the same horizontal plane as and distant l from O, and is attached at its other end to a mass m, which is small compared with M. When m is allowed to drop, show that the system oscillates about a mean position with a period $2\pi\left[1+\dfrac{m}{8M}(2+\sqrt{2})\right]\sqrt{\dfrac{l}{g}}$ approximately, and find the mean position.
17. A heavy particle hangs in equilibrium suspended by an elastic string whose modulus of elasticity is three times the weight of the particle. It is then slightly displaced; show that its path is a small arc of a parabola. If the displacement be in a direction making an angle $\cot^{-1}4$ with the horizon, show that the arc is the portion of a parabola cut off by the latus-rectum.

117. *A particle of mass m moves in a straight line under a force mn^2(distance) towards a fixed point in the straight line and under a small resistance to its motion equal to $m.\mu$(velocity); to find the motion.*

The equation of motion is

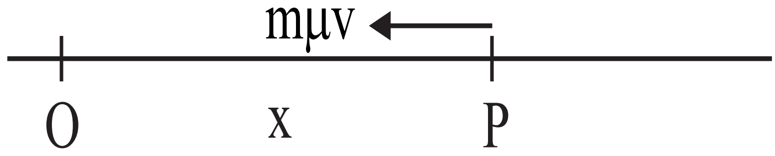
$$m\frac{d^2x}{dt^2} = -m.n^2x - m.\mu\frac{dx}{dt}, \quad i.e. \quad \frac{d^2x}{dt^2} + \mu\frac{dx}{dt} + n^2x = 0. \quad ...(1)$$

[This is clearly the equation of motion if the particle is moving so that x is increasing.

If as in the second figure the particle is moving so that x decreases, *i.e.* towards the left, the frictional resistance is towards the right, and equals $m.\mu v$. But in this case $\frac{ds}{dt}$ is negative, so that the value of v is $-\frac{dx}{dt}$; the frictional resistance is thus $m\mu\left(-\frac{dx}{dt}\right) \rightarrow$. The equation of motion is then

$$m\frac{d^2x}{dt^2} = -mn^2x + m\mu\left(-\frac{dx}{dt}\right),$$

which again becomes (1). Hence (1) gives the motion for all positions of P to the right of O, irrespective of the direction in which P is moving.



Similarly it can be shown to be the equation of motion for positions of P to the left of O, whatever be the direction in which P is moving.]

To solve (1), put $x = Le^{pt}$ and we have

$$p^2+\mu p+n^2=0 \quad \text{giving} \quad p=-\frac{\mu}{2}\pm i\sqrt{n^2-\frac{\mu^2}{4}}.$$

$$\therefore \quad x=e^{-\frac{\mu}{2}t}\left[Le^{\sqrt{n^2-\frac{\mu^2}{4}}it}+L'e^{-\sqrt{n^2-\frac{\mu^2}{4}}it}\right],$$

$$i.e. \;\; x=Ae^{-\frac{\mu}{2}t}\cos\left[\sqrt{n^2-\frac{\mu^2}{4}}t+B\right] \qquad ...(2),$$

where A and B are arbitrary constants.

If μ be small, then $Ae^{-\frac{\mu}{2}t}$ is a *slowly* varying quantity, so that (2) approximately represents a simple harmonic motion of period

$$2\pi \div \sqrt{n^2-\frac{\mu^2}{4}},$$

whose amplitude, $Ae^{-\frac{\mu}{2}t}$, is a slowly decreasing quantity. Such a motion is called a *damped* oscillation and μ measures the damping.

This period depends on the square of μ, so that, to the first order of approximation, this small frictional resistance has no effect on the period of the motion. Its effect is chiefly seen in the decreasing amplitude of the motion, which $=A\left(1-\frac{\mu}{2}t\right)$ when squares of μ are neglected, and therefore depends on the first power of μ.

Such a vibration as the above is called a free vibration. It is the vibration of a particle which moves under the action of no external periodic force.

If μ be not small compared with n, the motion cannot be so simply represented, but for all values of $\mu, < 2n$, the equation (2) gives the motion.

From (2) we have, on differentiating, that $\dot{x}=0$ when

$$\tan\left[\sqrt{n^2-\frac{\mu^2}{4}}t+B\right]=-\frac{\mu}{\sqrt{4n^2-\mu^2}}=\tan\alpha \text{ (say)} \qquad ...(3),$$

giving solutions of the form

$$\sqrt{n^2-\frac{\mu^2}{4}}t+B=\alpha,\pi+\alpha,2\pi+\alpha,\dots .$$

Hence $\dot{x}$ is zero, that is the velocity vanishes, at the ends of periods of time differing by $\pi \div \sqrt{n^2-\frac{\mu^2}{4}}$.

The times of oscillation thus still remain constant, though they are greater than when there is no frictional resistance.

If the successive values of t obtained from (3) are $t_1, t_2, t_3, \dots$ then the corresponding values of (2) are

$$Ae^{-\frac{\mu}{2}t_1}\cos\alpha, -Ae^{-\frac{\mu}{2}t_2}\cos\alpha, Ae^{-\frac{\mu}{2}t_3}\cos\alpha, \dots$$

so that the amplitudes of the oscillations form a decreasing G.P. whose common ratio $= e^{-\frac{\mu}{2}(t_2-t_1)} = e^{-\frac{\mu\pi}{2}} \div \sqrt{n^2-\frac{\mu^2}{4}}$.

If $\mu > 2n$, the form of the solution changes; for now

$$p=-\frac{\mu^2}{2}\pm\sqrt{\frac{\mu^2}{4}-n^2},$$

and the general solution is

$$x=e^{-\frac{\mu t}{2}}\left[Le^{\sqrt{\left(\frac{\mu^2}{4}-n^2\right)}.t}+L'e^{-\sqrt{\left(\frac{\mu^2}{4}-n^2\right)}.t}\right]$$

$$=e^{-\frac{\mu t}{2}}A_1\cosh\left[\sqrt{\left(\frac{\mu^2}{4}-n^2\right)}.t+B_1\right].$$

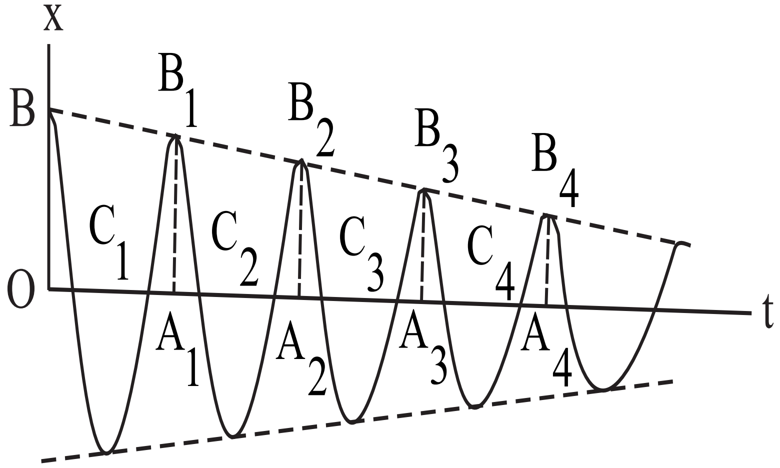
In this case the motion is no longer oscillatory.

If $\mu = n$, we have by the rules of Differential Equations

$$x = Le^{-nt} + \lim_{\gamma=0} Me^{-(n+\gamma)t} = Le^{-nt} + \lim_{\gamma=0} Me^{-nt}(1 - \gamma t + \text{ squares })$$
$$= L_1 e^{-nt} + M_1 te^{-nt} = e^{-nt}(L_1 + M_1 t).$$

EX. *The time of oscillation of a particle when there is no frictional resistance is* $1\frac{1}{2}$ *secs.; if there be a frictional resistance equal to* $\frac{1}{4} \times m \times$ *velocity, find the consequent alteration in the period and the factor which gives the ratio of successive maximum amplitudes.*

118. The motion of the last article may be represented graphically; let time t be represented by distances measured along the horizontal axis and the displacement x of the particle by the vertical ordinates. Then any displacement such as that of the last article will be represented as in the figure.



The dotted curve on which all the ends of the maximum ordinates lie is $x = \pm Ae^{-\frac{\mu}{2}t} \cos\alpha$. The times $A_1A_2, A_2A_3, A_3A_4, \ldots$, of successive periods are equal, whilst the corresponding maximum ordinates A_1B_1, $A_2B_2, A_3B_3, \ldots$, form a decreasing geometrical progression whose ratio

$$= \frac{A\cos\alpha e^{-\frac{\mu}{2}.OA_2}}{A\cos\alpha e^{-\frac{\mu}{2}.OA_1}} = e^{-\frac{\mu}{2}.\tau},$$

where τ is the time of an oscillation.

If we have a particle moving with a damped vibration of this character, and we make it automatically draw its own displacement curve as in the above figure, we can from the curve determine the forces acting on it.

For measuring the successive distances $C_1C_2, C_2C_3, \ldots$, etc, and taking their mean, we have the periodic time τ which we found in the last article to be $2\pi \div \sqrt{n^2 - \frac{\mu^2}{4}}$, so that $\frac{4\pi^2}{\tau^2} = n^2 - \frac{\mu^2}{4}$. Again, measuring the maximum ordinates $A_1B_1, A_2B_2, A_3B_3, \ldots$ finding the values of $\frac{A_2B_2}{A_1B_1}, \frac{A_3B_3}{A_2B_2}, \ldots$, and taking their mean, λ, we have the value of the quantity $e^{-\frac{\mu}{2}\tau}$, so that $-\frac{\mu}{2}\tau = \log_e \lambda$.

We thus have the values of n^2 and μ, giving the restorative force and the frictional resistance of the motion.

119. *A point is moving in a straight line with an acceleration μx towards a fixed centre in the straight line and with an additional acceleration $L\cos pt$; to find the motion.*

The equation of motion is $\quad \frac{d^2x}{dt^2} = -\mu x + L\cos pt$.

The solution of this is $x = A\cos\left(\sqrt{\mu}t + B\right) + L\frac{1}{D^2+\mu}\cos pt$

$$= A\cos\left(\sqrt{\mu}t + B\right) + L\frac{1}{\mu - p^2}\cos pt \qquad \text{...(1)}$$

If the particle starts from rest at a distance a at zero time, we have $B = 0$ and $A = a - \frac{L}{\mu - p^2}$.

$$\therefore \quad x = \left[a - \frac{L}{\mu - p^2}\right] \cos\sqrt{\mu}t + \frac{L}{\mu - p^2} \cos pt \qquad ...(2).$$

The motion of the point is thus compounded of two simple harmonic motions whose periods are $\frac{2\pi}{\sqrt{\mu}}$ and $\frac{2\pi}{p}$.

From the right-hand side of (2) it follows that, if p be nearly equal to $\sqrt{\mu}$, the coefficient $\frac{L}{\mu - p^2}$ becomes very great; in other words, the effect of the disturbing acceleration $L\cos pt$ becomes very important. It follows that the ultimate effect of a periodic disturbing force depends not only on its magnitude L, but also on its period, and that, if the period be nearly that of the free motion, its effect may be very large even though its absolute magnitude L be comparatively small.

If $p = \sqrt{\mu}$, the terms in (2) become infinite. In this case the solution no longer holds, and the second term in (1)

$$= L\frac{1}{D^2 + \mu} \cos[\sqrt{\mu}t] = L \lim_{\gamma=0} \frac{1}{D^2 + \mu} \cos(\sqrt{\mu} + \gamma)t$$

$$= L \lim_{\gamma=0} \frac{1}{\mu - (\sqrt{\mu} + \gamma)^2} \cos[\sqrt{\mu} + \gamma]t$$

$$= -L\frac{1}{2\sqrt{\mu}}[\text{ something infinite } - t\sin\sqrt{\mu}t].$$

Hence, by the ordinary theory of Differential Equations, the solution is

$$x = A_1 \cos[\sqrt{\mu}t + B_1] + \frac{L}{2\sqrt{\mu}} t \sin\sqrt{\mu}t.$$

If, as before, $x = a$ and $\dot{x} = 0$ when $t = 0$, this gives

$$x = a\cos\sqrt{\mu}t + \frac{L}{2\sqrt{\mu}} t \sin\sqrt{\mu}t,$$

and hence $\dot{x} = \left(\frac{L}{2\sqrt{\mu}} - a\sqrt{\mu}\right)\sin\sqrt{\mu}t + \frac{L}{2}t\cos\sqrt{\mu}t.$

It follows that the amplitude of the motion, and also the velocity, become very great as t gets large.

120. If, instead of a linear motion of the character of the previous article, we have an angular motion, as in the case of a simple pendulum, the equation of motion is $\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta + L\cos pt$, and the solution is similar to that of the last article.

In this case, if L be large compared with $\frac{g}{l}$ or if p be very nearly equal to $\sqrt{\frac{g}{l}}$, the free time of vibration, θ is no longer small throughout the motion and the equation of motion must be replaced by the more accurate equation $\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta + L\cos pt$.

121. As an example of the accumulative effect of a periodic force whose period coincides with the free period of the system, consider the case of a person in a swing to whom a small impulse is applied when he is at the highest point of his swing. This impulse is of the nature of a periodic force whose period is just equal to that of the swing and the effect of such an impulse is to make the swing to move through a continually increasing angle.

If however the period of the impulse is not the same as that of the swing, its effect is sometimes to help, and sometimes to oppose, the motion.

If its period is very nearly, but not quite, that of the swing its effect is for many successive applications to increase the motion, and then for many further applications to decrease the motion. In this case a great amplitude of motion is at first produced, which is then gradually destroyed, and then produced again, and so on.

122. *A particle, of mass m, is moving in a straight line under a force mn^2 (distance) towards a fixed point in the straight line, and under a frictional resistance equal to mμ (velocity) and a periodic force $mL\cos pt$; to find the motion.*

The equation of motion is

$$\frac{d^2x}{dt^2} = -n^2x - \mu\frac{dx}{dt} + L\cos pt, \quad \text{i.e.,} \quad \frac{d^2x}{dt^2} + \mu\frac{dx}{dt} + n^2x = L\cos pt.$$

The complementary function is $Ae^{-\frac{\mu}{2}t}\cos\left[\sqrt{n^2-\frac{\mu^2}{4}}t + B\right]$ ...(1), assuming $\mu < 2n$, and the particular integral

$$= \frac{1}{n^2 - p^2 + \mu D}L\cos pt$$
$$= L\frac{(n^2-p^2)\cos pt + \mu p\sin pt}{(n^2-p^2)^2 + \mu^2p^2}$$
$$= \frac{L\sin\varepsilon}{\mu p}\cos(pt-\varepsilon) \qquad ...(2),$$

where $\tan\varepsilon = \dfrac{\mu p}{n^2 - p^2}$.

The motion is thus compounded of two oscillations; the first is called the free vibration and the second the forced vibration.

Particular case. Let the period $\dfrac{2\pi}{p}$ of the disturbing force be equal to $\dfrac{2\pi}{n}$, the free period.

The solution is then, for the forced vibration, $x = \dfrac{L}{\mu n}\sin nt$.

If, as is usually the case, μ is also small, this gives a vibration whose maximum amplitude is very large. Hence we see that a small periodic force *may*, if its period is nearly equal to that of the free motion of the body, produce effects out of all proportion to its magnitude.

Hence we see why there may be danger to bridges from the accumulative effect of soldiers marching over them in step, why ships roll so heavily when the waves are of the proper period, and why a railway-carriage may oscillate considerably in a vertical direction when it is travelling at such a rate that the time it takes to go the length of a rail is equal to a period of vibration of the springs on which it rests.

Many other phenomena, of a more complicated character, are explainable on similar principles to those of the above simple case.

123. There is a very important difference between the free vibration given by (1) and the forced vibration given by (2).

Suppose for instance that the particle was initially at rest at a given finite distance from the origin. The arbitrary constants A and B are then easily determined and are found to be finite. The factor $e^{-\mu t/2}$ in (1), which gradually diminishes as time goes on, causes the expression (1) to continually decrease and ultimately to vanish. Hence the free vibration gradually dies out.

The forced vibration (2) has no such diminishing factor but is a continually repeating periodic function. Hence finally it is the only motion of any importance.

124. *Small oscillations of a simple pendulum under gravity, where the resistance = μ (velocity)^2 and μ is small.*

The equation of motion is $l\,\ddot{\theta} = -g\theta + \mu l^2\,\dot{\theta}^2$ ...(1).

[If the pendulum start from rest at an inclination α to the vertical, the same equation is found to hold until it comes to rest on the other side of the vertical.]

For a first approximation, neglect the small term $\mu l^2 \dot{\theta}^2$, and we have $\theta = A\cos\left[\sqrt{\frac{g}{l}}t + B\right]$.

For a second approximation, put this value of θ in the small terms on the right-hand side of (1), and it becomes

$$\ddot{\theta} + \frac{g}{l}\theta = \mu l.\frac{g}{l}A^2\sin^2\left[\sqrt{\frac{g}{l}}t + B\right] = \frac{A^2\mu g}{2}\left[1 - \cos\left(2\sqrt{\frac{g}{l}}t + 2B\right)\right].$$

$$\therefore\ \ \theta = A\cos\left[\sqrt{\frac{g}{l}}t + B\right] + \frac{A^2\mu l}{2} + \frac{A^2\mu l}{6}\cos\left[2\sqrt{\frac{g}{l}}t + 2B\right] \quad ...(2),$$

where $\alpha = A\cos B + \frac{A^2\mu l}{2} + \frac{A^2\mu l}{6}\cos 2B$, and

$0 = -A\sin B - \frac{A^2\mu l}{3}\sin 2B$.

$\therefore B = 0$, and $A = \alpha - \frac{2}{3}\alpha^2\mu l$, squares of μ being neglected. Hence (2) gives

$$\theta = \left(\alpha - \frac{2}{3}\alpha^2\mu l\right)\cos\left(\sqrt{\frac{g}{l}}t\right) + \frac{\alpha^2\mu l}{2} + \frac{\alpha^2\mu l}{6}\cos\left(2\sqrt{\frac{g}{l}}t\right) \quad ...(3),$$

and hence

$$\dot{\theta} = -\sqrt{\frac{g}{l}}\left(\alpha - \frac{2}{3}\alpha^2\mu l\right)\sin\left(\sqrt{\frac{g}{l}}t\right) - \frac{\alpha^2\mu l}{3} + \sqrt{\frac{g}{l}}\sin\left(2\sqrt{\frac{g}{l}}t\right) \quad ...(4).$$

$\therefore\ \dot{\theta}$ is zero when $\sin\sqrt{\frac{g}{l}}t = 0$, *i.e.* when $t = \pi\sqrt{\frac{l}{g}}$.

The time of a swing from rest to rest is therefore unaltered by the resistance, provided the square of μ be neglected. Again, when $t = \pi\sqrt{\frac{l}{g}}$.

$$\theta = -\left(\alpha - \frac{2}{3}\alpha^2\mu l\right) + \frac{\alpha^2\mu l}{2} + \frac{\alpha^2\mu l}{6} = -\left(\alpha - \frac{4}{3}\alpha^2\mu l\right).$$

Hence the amplitude of the swing is diminished by $\frac{4}{3}\alpha^2\mu l$.

Let the pendulum be passing through the lowest point of its path at time $\sqrt{\frac{l}{g}}\left(\frac{\pi}{2} + T\right)$, where T is small.

Then (3) gives

$$0 = \left(\alpha - \frac{2}{3}\alpha^2\mu l\right)(-\sin T) + \frac{\alpha^2\mu l}{2} - \frac{\alpha^2\mu l}{6}\cos 2T,$$

i.e. $\quad T\left(\alpha - \frac{2}{3}\alpha^2\mu l\right) = \frac{\alpha^2\mu l}{2} - \frac{\alpha^2\mu l}{6} = \frac{\alpha^2\mu l}{3}$, and

$$\therefore \quad T = \frac{\alpha\,\mu\, l}{3}.$$

Hence the time of swinging to the lowest point $= \sqrt{\frac{l}{g}}\left(\frac{\pi}{2} + \frac{\alpha\mu l}{3}\right)$, and of swinging up to rest again

$$= \pi\sqrt{\frac{l}{g}} - \sqrt{\frac{l}{g}}\left(\frac{\pi}{2} + \frac{\alpha\mu l}{3}\right) = \sqrt{\frac{l}{g}}\left(\frac{\pi}{2} - \frac{\alpha\mu l}{3}\right).$$

EXAMPLES

1. Investigate the rectilinear motion given by the equation

$$A\frac{d^4x}{dt^4} + B\frac{d^2x}{dt^2} + Cx + D = 0,$$

and show that it is compounded of two harmonic oscillations if the equation $Ay^2 + By + C = 0$ has real negative roots.

2. A particle is executing simple harmonic oscillations of amplitude a, under an attraction $\frac{\mu x}{a}$. If a small disturbing force $\frac{\nu x^3}{a^3}$ be introduced (the amplitude being unchanged) show that the period is, to a first approximation, decreased in the ratio $1 - \frac{3\nu}{8\mu} : 1$.
3. Two heavy particles, of masses m and m', are fixed to two points, A and B, of an elastic string OAB. The end is attached to a fixed point and the system hangs freely. A small vertical disturbance being given to it, find the times of the resultant oscillations.
4. A particle hangs at rest at the end of an elastic string whose unstretched length is a. In the position of equilibrium the length of the string is b, and $\frac{2\pi}{n}$ is the time of an oscillation about this position. At time zero, when the particle is in equilibrium, the point of suspension begins to move so that its downward displacement at time t is $\sin pt$. Show that the length of the string at time t is $b - \frac{cnp}{n^2 - p^2} \sin nt + \frac{cp^2}{n^2 - p^2} \sin pt$.
If $p = n$, show that the length of the string at time t is
$$b - \frac{c}{2} \sin nt - \frac{nct}{2} \cos nt.$$
5. A helical spring supports a weight of 20 kg.f. attached to its lower end; the natural length of the spring is 30 cm and the load causes it to extend to a length of 4 cm. The upper end of the spring is then given a vertical simple harmonic motion, the full extent of the displacement being 5 cm and 100 complete vibrations occurring in one minute. Neglecting air resistance and the inertia of the spring, investigate the motion of the suspended mass after the motion has

become steady, and show that the amplitude of the motion set up is about 9 cm.

6. If a pendulum oscillates in a medium the resistance of which varies as the velocity, show that the oscillations are isochronous.
7. The time of a complete oscillation of pendulum making small oscillations *in vacuo* is 2 seconds; if the angular retardation due to the air is .04 $\times$ (angular velocity of the pendulum and the initial amplitude is 1^o, find the inclination of the pendulum to the vertical at any subsequent time, and show that the amplitude will in 10 complete oscillation be reduced to $40'$ approximately. $[\log_{10} e = 0.4343.]$
8. The point of suspension of a simple pendulum of length l has a horizontal motion given by $x = a \cos mt$. Find the effect on the motion of the particle.
   Consider in particular the motion when m^2 is equal, or nearly equal, to $\frac{g}{l}$. In the latter case if the pendulum be passing through its vertical position with angular velocity ω at zero time, show that, so long as it is small, the inclination to the vertical at time t $= \omega \sqrt{\frac{l}{g}} \left[1 + \frac{ag}{2\omega l^2} t\right] \sin \sqrt{\frac{g}{l}} t.$
   [If O' be the position of the point of suspension at time t its acceleration is $\ddot{x}$. Hence the accelerations of P, the bob of the pendulum, are $l\ \ddot{\theta}$ perpendicular to $O'P$, $l\ \dot{\theta}^2$ along PO, and $\ddot{x}$ parallel to OO'. Hence resolving perpendicular to $O'P$,

$$l\ \ddot{\theta} + \ddot{x} \cos\theta = -g \sin\theta = -g\theta,$$

i.e. $\ddot{\theta} = -\frac{g}{l}\theta + \frac{am^2}{l} \cos mt$, since θ is small. Now solve as in Art. 119.]

9. The point of support of a simple pendulum, of weight w and length l, is attached to a massless spring which moves backwards and forwards in a horizontal line; show that the time of vibration $= 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{w}{W}\right)$, where W is the weight required to stretch the spring a distance l.
10. Two simple pendulums, each of length a, are hung from two points in the same horizontal plane at a distance b apart; the bob of each is of mass m and the mutual attraction is $\frac{\lambda m^2}{(\text{dist})^2}$, where λ is small compared with g; show that, if the pendulums be started so that they are always moving in opposite directions, the time of oscillation of each is $2\pi\sqrt{\frac{a}{g}}\left(1+\frac{2\lambda ma}{b^3 g}\right)$ nearly, about a mean position inclined at $\frac{\lambda m}{gb^2}$ radians nearly to the vertical.
11. A pendulum is suspended in a ship so that it can swing in a plane at right angles to the length of the ship, its excursions being read off on a scale fixed to the ship. The free period of oscillation of the pendulum is one second and its point of suspension is 10 metres above the centre of gravity of the ship. Show that when the ship is rolling through a small angle on each side of the vertical with a period of 8 secs., the apparent angular movement of the pendulum will be approximately 64%. greater than that of the ship.
12. The point of suspension of a simple pendulum of length l moves in a horizontal circle of radius a with constant angular velocity ω; when the motion has become steady, show that the inclination α to the vertical of the thread of the pendulum is given by the equation $\omega^2(a+l\ \sin\alpha)-g\tan\alpha=0$.

13. A pendulum consists of a light elastic string with a particle at one end and fastened at the other. In the position of equilibrium the string is stretched to $\frac{4}{3}$ of its natural length l. If the particle is slightly displaced from the position of equilibrium and is then let go, trace its subsequent path and find the times of its component oscillations.

ANSWERS WITH HINTS

Art. 116 EXAMPLES

1. $\pi\sqrt{\dfrac{a^3}{\mu}}$

9. $(m+m')\ddot{x} = -\dfrac{3m'g}{a}x$

15. $M\ddot{x} = -\dfrac{\gamma Mm}{(a^2+b^2)^{3/2}}x$

Art. 117 Ex. $\dfrac{29}{32}$

Art. 124 EXAMPLES

3. $\dfrac{2\pi}{p_1}$ and $\dfrac{2\pi}{p_2}$

7. $40'13''$

13. $\lambda = 3mg$. For a small oscillation, in which $r = \dfrac{4l}{3} + R$ where R is small, we have $\ddot{R} - \left(\dfrac{4l}{3} + R\right)\dot{\theta}^2 = g\cos\theta - \dfrac{\lambda/3 + R}{l} = -\dfrac{3g}{l}R$, and $\dfrac{1}{r}\dfrac{d}{dt}(r^2\dot{\theta}) = -g\sin\theta$, i.e., $2\dot{R}\dot{\theta} + \left(\dfrac{4l}{3} + R\right)\ddot{\theta} = -g\theta$. Neglecting squares of small quantities, we have $\ddot{R} = -\dfrac{3g}{l}R$ and $\ddot{\theta} = -\dfrac{3g}{4l}\theta$, etc.