Chapter 9

Differential Equations

Introduction to Differential Equations

An equation that involves an independent variable, dependent variable and differential coefficients of dependent variable with respect to the independent variable is called a **differential equation.**
e.g.,

(i) $x^2(d^2y / dx^2) + x^3 (dy / dx)^3 7x^2y^2$

(ii) $(x^2 + y^2) dx = (x^2 - y^2) dy$

Order and Degree of a Differential Equation

The order of a differential equation is the order of the highest derivative occurring in the equation. The order of a differential equation is always a positive integer.

The degree of a differential equation is the degree (exponent) of the derivative of the highest order in the equation, after the equation is free from negative and fractional powers of the derivatives.

Linear and Non-Linear Differential Equations

A differential equation is said to be linear, if the dependent variable and all of its derivatives occurring in the first power and there are no product of these. A linear equation of nth order can be written in the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_{n-1} \frac{dy}{dx} + P_n y = Q$$

where, $P_0, P_1, P_2, \dots, P_{n-1}$ and Q must be either constants or functions of x only.

A linear differential equation is always of the first degree but every differential equation of the first degree need not be linear.

e.g., The equations $d^2y / dx^2 + (dy / dx)^2 + xy = 0$ and $x(d^2y / dx^2) + y (dy / dx) + y = x^3$, $(dy / dx) d^2y / dx^2 + y = 0$ are not linear.

Solution of Differential Equations

A solution of a differential equation is a relation between the variables, not involving the differential coefficients, such that this relation and the derivative obtained from it satisfy the given differential equation.

e.g., Let $d^2y / dx^2 + y = 0$

Integrating above equation twicely, we get $y = A \cos x + B \sin x$.

General Solution

If the solution of the differential equation contains as many independent arbitrary constants as the order of the differential equation, then it is called the general solution or the complete integral of the differential equation.

e.g., The general solution of $d^2y / dx^2 + y = 0$ is $y = A \cos x + B \sin x$ because it contains two arbitrary constants A and B, which is equal to the order of the equation.

Particular Solution

Solution obtained by giving particular values to the arbitrary constants in the general solution is called a particular solution. e.g., In the

previous example, if $A = B = 1$, then $y = \cos x + \sin x$ is a particular solution of the differential equation $d^2y / dx^2 + y = 0$.

Solution of a differential equation is also called its primitive.

Formation of Differential Equation

Suppose, we have a given equation with n arbitrary constants $f(x, y, c_1, c_2, \ldots, c_n) = 0$.

Differentiate the equation successively n times to get n equations.

Eliminating the arbitrary constants from these $n + 1$ equations leads to the required differential equations.

Solutions of Differential Equations of the First Order and First Degree

A differential equation of first degree and first order can be solved by following method.

1. **Inspection Method**

If the differential equation' can be written as f [f_1(x, y) d {f_1(x, y)}] + φ [f_2(x, y) d {f_2(x, y)}] +... = 0] then each term can be integrated separately.

For this, remember the following results

1. $xdy + ydx = d(xy)$
2. $d(x+y) = dx + dy$
3. $d\left(\frac{y}{x}\right) = \frac{x\,dy - ydx}{x^2}$
4. $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$
5. $d\left(\frac{x^2}{y}\right) = \frac{2xy\,dx - x^2\,dy}{y^2}$
6. $d\left(\frac{y^2}{x}\right) = \frac{2xy\,dy - y^2\,dx}{x^2}$
7. $d\left(\frac{x^2}{y^2}\right) = \frac{2xy^2\,dx - 2x^2y\,dy}{y^4}$
8. $d\left(\frac{y^2}{x^2}\right) = \frac{2x^2y\,dy - 2xy^2\,dx}{x^4}$
9. $\frac{xdy + ydx}{xy} = d(\log xy)$
10. $\frac{ydx - xdy}{xy} = d\left(\log \frac{x}{y}\right)$
11. $\frac{xdy - ydx}{xy} = d\left(\log \frac{y}{x}\right)$
12. $\frac{dx + dy}{x + y} = d\log(x + y)$
13. $\frac{xdx + ydy}{x^2 + y^2} = d\left(\log\sqrt{x^2 + y^2}\right)$
14. $\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$
15. $\frac{ydx - xdy}{x^2 + y^2} = d\left(\tan^{-1}\frac{x}{y}\right)$
16. $d\left(\frac{-1}{xy}\right) = \frac{xdy + ydx}{x^2y^2}$
17. $d\left(\frac{e^x}{y}\right) = \frac{ye^x dy - e^x dy}{y^2}$
18. $d\left(\frac{e^y}{x}\right) = \frac{xe^y dy - e^y dx}{x^2}$
19. $d(\sqrt{x^2 + y^2}) = \frac{xdx + ydy}{\sqrt{x^2 + y^2}}$
20. $d(x^m y^n) = x^{m-1} \cdot y^{n-1}(mydx + nx\,dy)$
21. $d\left(\frac{1}{2}\log\frac{x+y}{x-y}\right) = \frac{xdy - ydx}{x^2 - y^2}$
22. $\frac{d[f(x,y)]^{1-n}}{1-n} = \frac{f'(x,y)}{[f(x,y)]^n}$
23. $d\left(\frac{1}{y} - \frac{1}{x}\right) = d\left(\frac{1}{y}\right) - d\left(\frac{1}{x}\right) = \frac{dx}{x^2} - \frac{dy}{y^2}$

2. **Variable Separable Method**

If the equation can be reduced into the form $f(x)\,dx + g(y)\,dy = 0$, we say that the variable have been separated. On integrating this reduced, form, we get $\int f(x)\,dx + \int g(y)\,dy = C, = C$, where C is any arbitrary constant.

3. **Differential Equation Reducible to Variables Separable Method**

A differential equation of the form $dy / dx = f(ax + by + c)$ can be reduced to variables separable form by substituting

$ax + by + c = z \Rightarrow a + b\, dy / dx = dz / dx$

The given equation becomes

$1 / b\,(dz / dx - a)\, f(z) \Rightarrow dz / dx = a + b\, f(z) \Rightarrow dz / a + bf(z) = dx$

Hence, the variables are separated in terms of z and x.

4. **Homogeneous Differential Equation**

A function $f(x, y)$ is said to be homogeneous of degree n, if
$f(\lambda x, \lambda y) = \lambda^n f(x, y)$

Suppose a differential equation can be expressed in the form
$dy / dx = f(x, y) / g(x, y) = F\,(y / x)$

where, $f(x, y)$ and $g(x, y)$ are homogeneous function of same degree. To solve such types of equations, we put $y = vx$

$\Rightarrow dy / dx = v + x\, dv / dx$.

The given equation, reduces to

$v + x\, dv / dx = F(v)$

$\Rightarrow x\, dv / dx = F(v) - v$

$\therefore dv / F(v) - v = dx / x$

Hence, the variables are separated in terms of v and x.

5. **Differential Equations Reducible to Homogeneous Equation**

The differential equation of the form

$dy / dx = a_1x + b_1y + c_1 / a_2x + b_2y + c_2$ ……(i)

put $X = X + h$ and $y = Y + k$

$\therefore dY / dX = a_1 X + b_1 Y + (a_1h + b_1k + c_1) / a_2X + b_2 Y + (a_2h + b_2k + c_2)$ ……(ii)

We choose h and k, so as to satisfy $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$.

On solving, we get

$h / b_1c_2 - b_2c_1 = k / c_1a_2 - c_2a_1 = 1 / a_1b_2 - a_2b_1$

$\therefore h = b_1c_2 - b_2c_1 / a_1b_2 - a_2b_1$ and $k = c_1a_2 - c_2a_1 / a_1b_2 - a_2b_1$

Provided $a_1b_2 - a_2b_1 \neq 0$, $a_1 / a_2 \neq b_a / b_2$

Then, Eq, (ii) reduces to $dY / dX = (a_1 X + b_1 Y) / (a_2X + b_2 Y)$, which is a homogeneous form and will be solved easily.

6. Exact Differential Equation

Differential equation $M(x,y)\ dy + N(x,y)\ dy = 0$ is called an exact ifferential equation.

If a function $u\ (x, y)$ exist such that, $du = Mdx + Ndy$.

Necessary and Sufficient Condition for an Equation to be an Exact Differential Equation

Differential equation $Mdx + Ndy = 0$ where, M and N are the functions of x and y, will be an exact differential equation, if
$\partial N / \partial y = \partial N / \partial x$

Solution of Exact Differential Equation

$$\underset{(y\text{Constant})}{\int} M\,dx + \underset{(\text{Only those terms of } N \text{ which independent from } x)}{\int} N\,dy = C$$

7. **Linear Differential Equation**

A linear differential equation of the first order can be either of the following forms

(i) $dy / dx + Py = Q$, where P and Q are functions of x or constants.

(ii) $dx / dy + Rx = S$, where Rand S are functions of y or constants.
Consider the differential Eq. (i) i.e., $dy / dx + Py = Q$

$$\therefore \qquad \text{IF} = e^{\int P\,dx}$$

Multiply both the sides by $e^{\int P\,dx}$, we get

$$e^{\int P\,dx} \frac{dy}{dx} + Pye^{\int P\,dx} = Qe^{\int P\,dx}$$

$$\Rightarrow \qquad \frac{d}{dx}\left(ye^{\int P\,dx}\right) = Qe^{\int P\,dx}$$

Integrating, we get

$$ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + C$$

i.e.,

$$y\,(\text{IF}) = \int Q\,(\text{IF})\,dx + C$$

where, $\text{IF} = \text{Integrating Factor} = e^{\int P dx}$

Similarly, for the second differential equation $dx / dy + Rx = S$, the integrating factor, $\text{IF} = e^{\int R\,dy}$ and the general solution is
$x\,(\text{IF}) = \int S\,(\text{IF})\,dy + C$

8. **Differential Equation Reducible to Linear Form**

Bernoulli's Equation An equation of the form $dy / dx + Py = Qy^n$, where P and Q are functions of x along or constants, is called Bernoulli's equation.

Divide both the sides by y^n, we get

$y^{-n}\,dy / dx + Py^{-n+1} = Q$

Put $y^{-n+1} = z$

$\Rightarrow (-n + 1)y^{-n}\,dy / dx = dz / dx$

The equation reduces to

$1 / 1 - n\ dz / dx + Pz = Q \Rightarrow dz / dx + (1 - n)\ Pz = Q\ (1 - n)$

which is linear in z and can be solved in the usual manner.

9. **Clairaut Form for Differential Equation**

Differential equation $y = Px + f(p)$, where $P = dy / dx$ ... (i)

is called clairaut form of differential equation. In which, get its general solution by replacing P from C.

Now, differential on both sides of Eq, (i) with respect to x and put $dy / dx = P$.

$P = P + x\ dp / dx + f'(P)\ dp / dx = 0$

$\Rightarrow [x + f'(p)]\ dp / dx = 0$

$\Rightarrow dp / dx = 0 \Rightarrow p = C$

10. **Orthogonal Trajectory**

Any curve, which cuts every member of a given family of curves at right angle, is called an orthogonal trajectory of the family.

Procedure for finding the Orthogonal Trajectory

(i) Let $f(x,y,c) = 0$ be the equation of the given family of curves, where 'c' is an arbitrary parameter.

(ii) Differentiate $f = 0$, with respect to 'x' and eliminate 0, i.e., from a differential equation.

(iii) Substitute $(- dx / dy)$ for (dy / dx) in the above differential equation. This will give the differential equation of the orthogonal trajectories.

(iv) By solving this differential equation, we get the required orthogonal trajectories.

Important Formulas: Differential Equations

Differential Equations Of First Order And First Degree

Definitions :

1. An equation that involves independent and dependent variables and the derivatives of the dependent variables is called a **Differential Equations**.

2. A differential equation is said to be ordinary , if the differential coefficients have reference to a single independent variable only and it is said to be PARTIAL if there are two or more independent variables . We are concerned with ordinary differential equations only. eg. $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$

$= 0$ is a partial differential equation.

3. Finding the unknown function is called solving or integrating the differential equation . The solution of the differential equation is also called its Primitive, because the differential equation can be regarded as a relation derived from it.

4. The order of a differential equation is the order of the highest differential coefficient occuring in it.

5. The degree of a differential equation which can be written as a polynomial in the derivatives is the degree of the derivative of the highest order occuring in it , after it has been expressed in a form free from radicals & fractions so far as derivatives are concerned, thus the differential equation : $f(x, y)\left[\frac{d^m y}{dx^m}\right]^p + \phi(x, y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q$

$= 0$ is order m & degree p. Note that in the differential equation $ey''' - xy'' + y = 0$ order is three but degree doesn't apply.

6. Formation of a differential equation : If an equation in independent and dependent variables having some arbitrary constant is given , then a differential equation is obtained as follows: Differentiate the given equation w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it . Eliminate the arbitrary constants .

The eliminant is the required differential equation . Consider forming a differential equation for $y^2 = 4a(x + b)$ where a and b are arbitary constant .

Note : A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

7. General And Particular Solutions : The solution of a differential equation which contains a number of independent arbitrary constants equal to the order of the differential equation is called the general solution (or complete integral or complete primitive) . A solution obtainable from the general solution by giving particular values to the constants is called a **Particular solution**.

Note that the general solution of a differential equation of the nth order contains 'n' & only 'n' independent arbitrary constants. The arbitrary constants in the solution of a differential equation are said to be independent, when it is impossible to deduce from the solution an equivalent relation containing fewer arbitrary constants. Thus the two arbitrary constants A, B in the equation $y = A\, ex + B$ are not independent since the equation can be written as $y = A\, e^{B}.\, ex = C\, ^{ex}$. Similarly the solution $y = A \sin x + B \cos (x + C)$ appears to contain three arbitrary constants, but they are really equivalent to two only.

8. Elementary Types Of First Order & First Degree Differential Equations.

TYPE————1. Variables separable : If the differential equation can be expressed as ; $f(x)dx + g(y)dy = 0$ then this is said to be variable − separable type.

A general solution of this is given by $\int f(x)\, dx + \int g(y)\, dy = c$

where c is the arbitrary constant . consider the example $(dy/dx) = e^{x-y} + x^2.\, e^{-y}$.

Note : Sometimes transformation to the polar co−ordinates facilitates separation of variables.

In this connection it is convenient to remember the following differentials. If $x = r \cos\theta$; $y = r \sin\theta$ then,

(i) $x\, dx + y\, dy = r\, dr$

(ii) $dx_2 + dy_2 = dr_2 + r2\, d\theta_2$

(iii) $x\, dy - y\, dx = r_2\, d\theta$ If $x = r \sec\theta$ & $y = r \tan\theta$ then $x\, dx - y\, dy = r\, dr$ and $x\, dy - y\, dx = r_2 \sec\theta\, d\theta$.

TYPE– 2 : $\frac{dy}{dx} = f(ax + by + c)$, $b \neq 0$.

To solve this , substitute $t = ax + by + c$. Then the equation reduces to separable type in the variable t and x which can be solved. Consider the example $(x + y)^2 \frac{dy}{dx} = a^2$.

TYPE– 3. Homogeneous equations

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$, where $f(x, y)$ & $\varphi(x, y)$ are homogeneous functions of x & y , and of the same degree , is called **Homogeneous** .This equation may also be reduced to the form $\frac{dy}{dx} = g\left(\frac{x}{y}\right)$ & is solved by putting $y = vx$ so that the dependent variable y is changed to another variable v, where v is some unknown function, the differential equation is transformed to an equation with variables separable. Consider $\frac{dy}{dx} + \frac{y(x+y)}{x^2} = 0$.

TYPE– 4. Equations reducible to the homogeneous form:

If $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ where $a_1 b_2 - a_2 b_1 \neq 0$, i.e. $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$

then the substitution $x = u + h$, $y = v + k$ transform this equation to a homogeneous type in the new variables u and v where h and k are arbitrary constants to be chosen so as to make the given equation homogeneous which can be solved by the method as given in Type – 3.

If (i) $a_1 b_2 - a_2 b_1 = 0$, then a substitution $u = a_1 x + b_1 y$ transforms the differential equation to an equation with variables separable. and

(ii) $b_1 + a_2 = 0$, then a simple cross multiplication and substituting $d(xy)$ for $x\,dy + y\,dx$ & integrating term by term yields the result easily.

Consider $\frac{dy}{dx} = \frac{x - 2y + 5}{2x + y - 1}$; $\frac{dy}{dx} = \frac{2x + 3y - 1}{4x + 6y - 5}$ & $\frac{dy}{dx} = \frac{2x - y + 1}{6x - 5y + 4}$

(iii) In an equation of the form : $yf(xy)\,dx + xg(xy)dy = 0$ the variables can be separated by the substitution $xy = v$.

Important note :

(a) The function $f(x, y)$ is said to be a homogeneous function of degree n if for any real number $t\ (\neq 0)$, we have $f(tx, ty) = tn\ f(x, y)$.

For e.g. $f(x, y) = ax^{2/3} + hx^{1/3} . y^{1/3} + by^{2/3}$ is a homogeneous function of degree 2/3

(b) A differential equation of the form $\frac{dy}{dx} = f(x, y)$ is homogeneous if $f(x, y)$ is a homogeneous function of degree zero i.e. $f(tx, ty) = t^{\circ} f(x, y) = f(x, y)$. The function f does not depend on x & y separately but only on their ratio $\frac{y}{x}$ or $\frac{x}{y}$

Linear diferential equations: A differential equation is said to be linear if the dependent variable & its differential coefficients occur in the first degree only and are not multiplied together The nth order linear differential equation is of the form $a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + ... + a_n(x) . y = \phi(x)$. Where $a_0(x)$, $a_1(x)$ ..... $a_n(x)$ a

coefficients of the differential equation. Note that a linear differential equation is always of the first degree but every differental equation of the first degree need not be linear. e.g. the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y^2 = 0$ is not linear, though its degree is 1.

TYPE ———— 5. Linear differential equations of first order :

The most general form of a linear differential equations of first order is $\frac{dy}{dx} + Py = Q$,

where P& Q are functions of x . To solve such an equation multiply both sides by $e^{\int P dx}$

Note : (1) The factor $e^{\int P dx}$ on multiplying by which the left hand side of the differential equation becomes the differential coefficient of some function of x & y , is called integrating factor of the differential equation popularly abbreviated as I. F.

(2) It is very important to remember that on multiplying by the integrating factor , the left hand side becomes the derivative of the product of y and the I. F. (3) Some times a given differential equation becomes linear if we take y as the independent variable and x as the dependent variable. e.g. the equation ;

$(x + y + 1)\frac{dy}{dx} = y^2 + 3$ can be written as $(y^2 + 3)\frac{dx}{dy} = x + y + 1$ which is a linear differential

equation.

TYPE————6. Equations reducible to linear form :

The equation $\frac{dy}{dx} + py = Q$. yn where P & Q functions of x , is reducible to the linear form by dividing it by y^n & then substituting $y^{-n+1} = Z$. Its solution can be obtained as in Type————5. Consider the example $(x^3 y^2 + xy)\, dx = dy$.

The equation $\frac{dy}{dx} + Py = Q$. y^n is called Bernouli's equation.

9. Trajectories: Suppose we are given the family of plane curves. Φ (x, y, a) = 0 depending on a single parameter a. A curve making at each of its points a fixed angle α with the curve of the family passing through that point is called an isogonal trajectory of that family ; if in particular α = π/2, then it is called an orthogonal trajectory.

Orthogonal trajectories : We set up the differential equation of the given family of curves. Let it be of the form F (x, y, y') = 0 The differential equation of the orthogonal trajectories is of the form $F\left(x, y, -\frac{1}{y'}\right) = 0$ The general integral of this equation Φ1 (x, y, C) = 0 gives the family of orthogonal trajectories.

Note : Following exact differentials must be remembered :

(i) $x\,dy + y\,dx = d(xy)$

(ii) $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$

(iii) $\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$

(iv) $\dfrac{x\,dy + y\,dx}{xy} = d(\ln xy)$

(v) $\dfrac{dx + dy}{x + y} = d(\ln(x+y))$

(vi) $\dfrac{x\,dy - y\,dx}{xy} = d\left(\ln\dfrac{y}{x}\right)$

(vii) $\dfrac{y\,dx - x\,dy}{xy} = d\left(\ln\dfrac{x}{y}\right)$

(viii) $\dfrac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$

(ix) $\dfrac{y\,dx - x\,dy}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{x}{y}\right)$

(x) $\dfrac{x\,dx + y\,dy}{x^2 + y^2} = d\left[\ln\sqrt{x^2 + y^2}\right]$

(xi) $d\left(-\dfrac{1}{xy}\right) = \dfrac{x\,dy + y\,dx}{x^2y^2}$

(xii) $d\left(\dfrac{e^x}{y}\right) = \dfrac{y e^x\,dx - e^x\,dy}{y^2}$

(xiii) $d\left(\dfrac{e^y}{x}\right) = \dfrac{x e^y\,dy - e^y\,dx}{x^2}$,