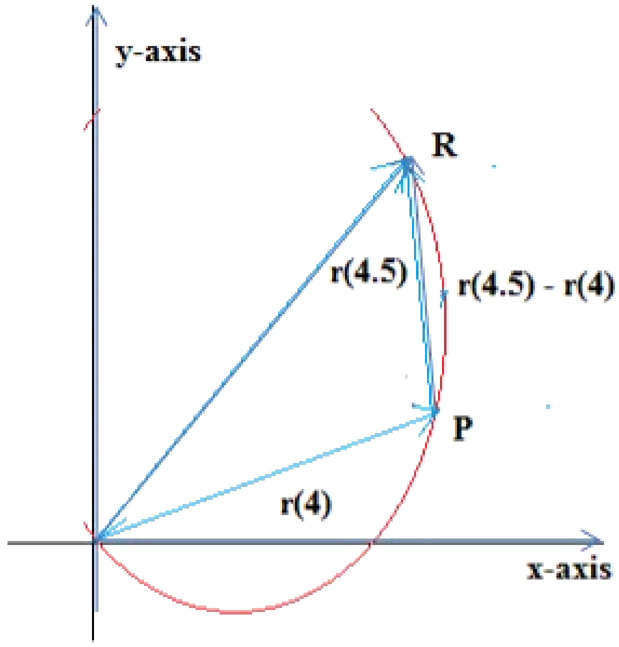
# Exercise 13.2

## Answer 1E.

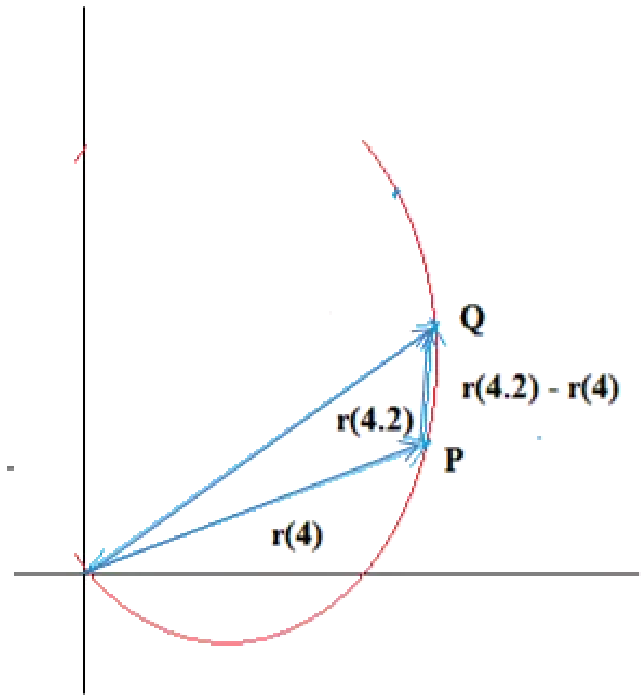
$(a)$

The objective is to draw the vectors $r(4.5)-r(4)$ and $r(4.2)-r(4)$.

Draw the graph of the vector, $r(4.5)-r(4)$

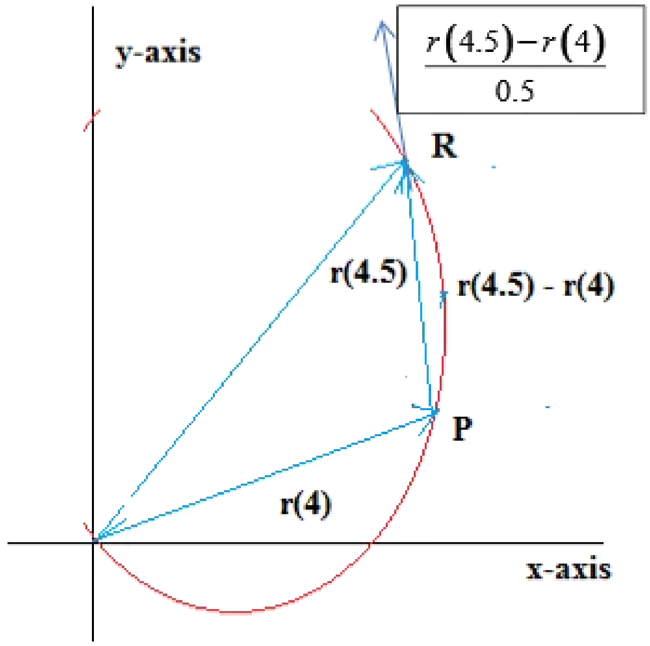



Draw the graph of the vector, $r(4.2)-r(4)$

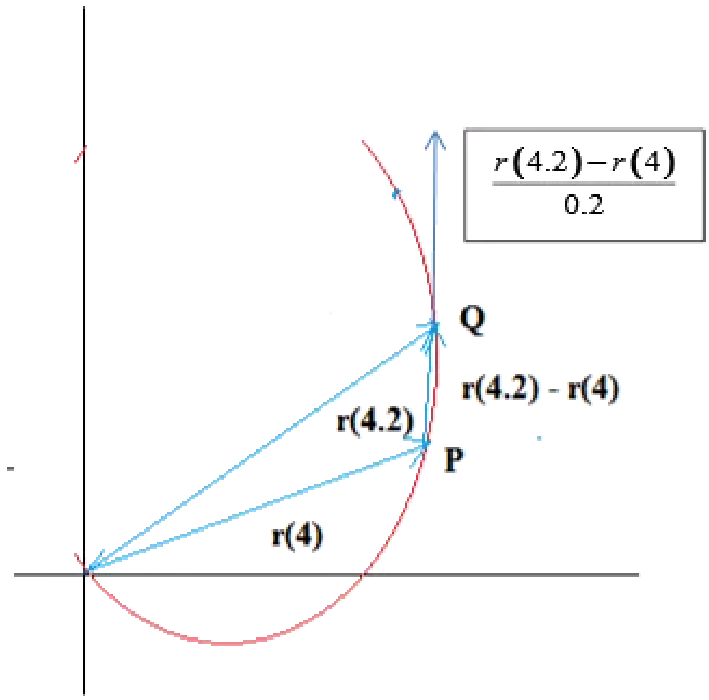


$(b)$

Draw the vector $\frac{r(4.5)-r(4)}{0.5}$




Draw the vector $\frac{r(4.2)-r(4)}{0.2}$



$(c)$

Write the expression for $r'(4)$ and the unit tangent vector $T(4)$

Expression for $r'(4) = \lim_{h \to 0} \frac{r(4+h) - r(4)}{h}$

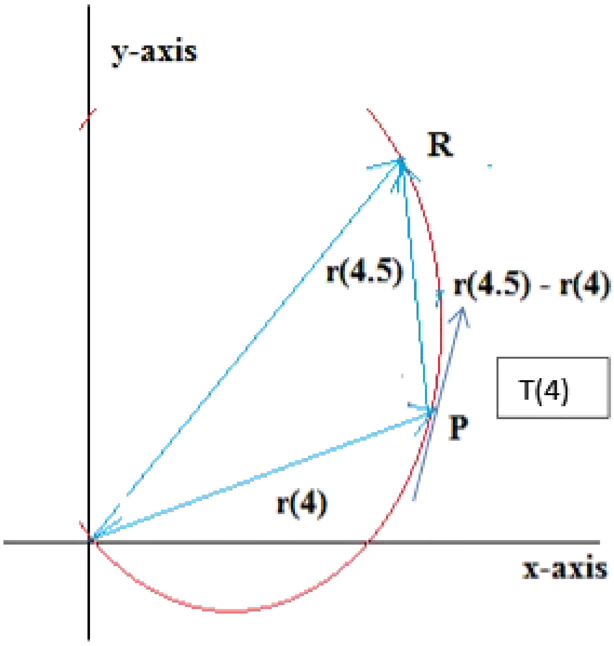
Unit tangent vector, $T(t) = \frac{r'(t)}{|r'(t)|}$

$$T(4) = \frac{r'(4)}{|r'(4)|}$$

$(d)$

Draw the vector $T(4)$

Draw the tangent line at $P$



**Answer 4E.**

(a)

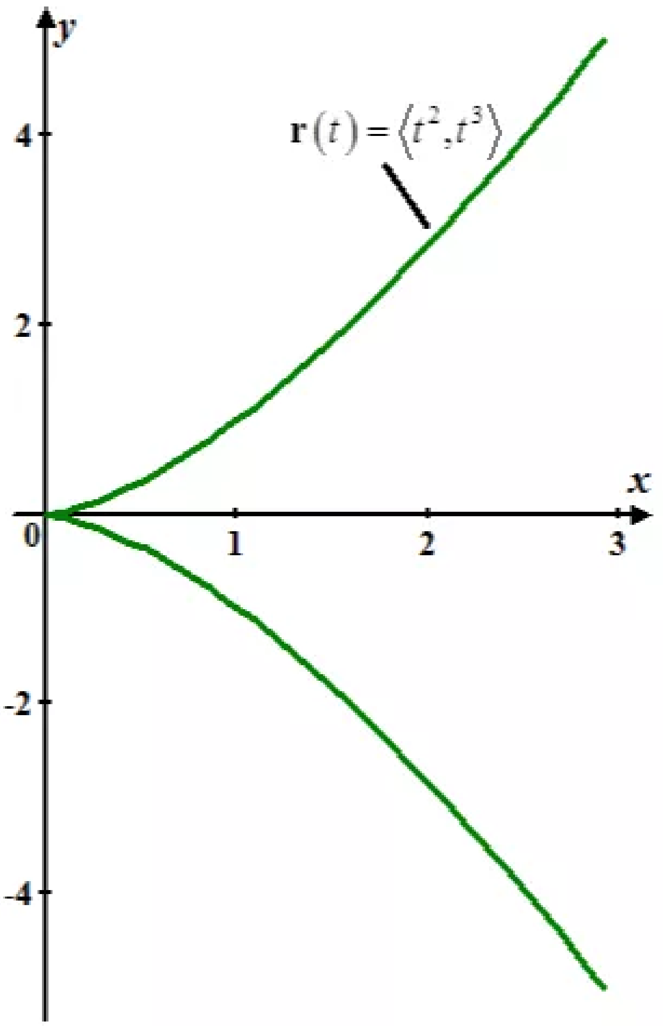
Consider the following vector equation:

$\mathbf{r}(t)=\langle t^2,t^3\rangle$ ...... (1)

The parametric equations are as follows:

$x=t^2,\ y=t^3$

Sketch the plane curve for the vector equation $\mathbf{r}(t)=\langle t^2,t^3\rangle$



**Answer 5E.**

Consider the vector equation,

$$\mathbf{r}(t) = \sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$$

Rewrite the vector equation as,

$$\mathbf{r}(t) = \langle \sin t, 2\cos t \rangle$$

(a)

The objective is to sketch the plane curve with the given vector equation.

Comparing it with $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, we get $x = \sin t, y = 2\cos t$

$$y = 2\cos t$$

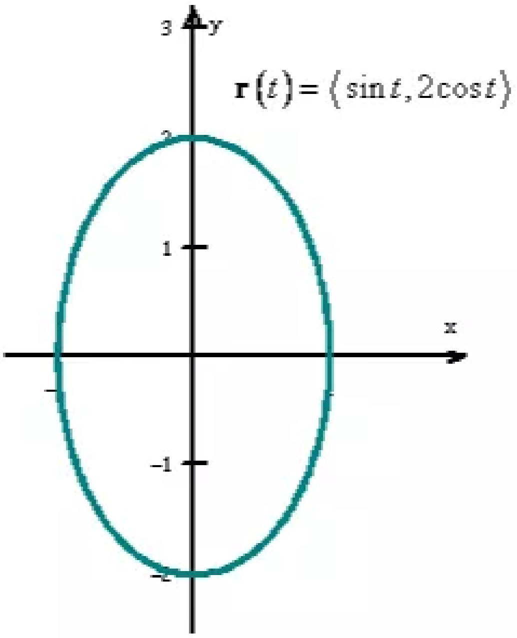
$$\begin{aligned} y &= 2\sqrt{1-\sin^2 t} \\ &= 2\sqrt{1-x^2} \end{aligned}$$

$$\left(\frac{y}{2}\right)^2 = 1 - x^2$$

$$\frac{x^2}{1^2} + \frac{y^2}{2^2} = 1$$

It represents the ellipse with major axis is *y*-axis.

The sketch of the plane curve is,



(b)

Consider $\mathbf{r}(t)=\langle \sin t, 2\cos t\rangle$

The objective is to find $\mathbf{r}'(t)$.

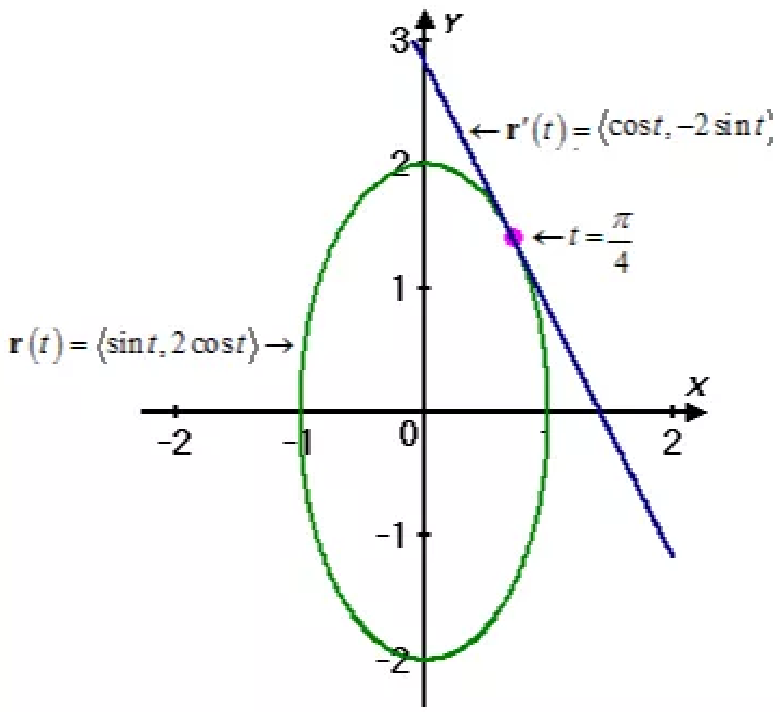
Differentiate $\mathbf{r}(t)$ with respect to '$t$'.

$$\mathbf{r}'(t)=\left\langle \frac{d}{dt}(\sin t), \frac{d}{dt}(2\cos t)\right\rangle$$
$$=\langle \cos t, -2\sin t\rangle$$

Hence, the derivative of $\mathbf{r}(t)$ is $\boxed{\mathbf{r}'(t)=\langle \cos t, -2\sin t\rangle}$

(c)

The objective is to sketch the position vector of $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ at $t=\frac{\pi}{4}$




**Answer 6E.**

The given vector equation is

$$\vec{r}(t)=e^{t}\hat{i}+e^{-t}\hat{j}$$

From the parametric equations

$$x=e^{t},\ y=e^{-t}$$

Then $\quad x=\frac{1}{y}$

Or $\quad xy=1$

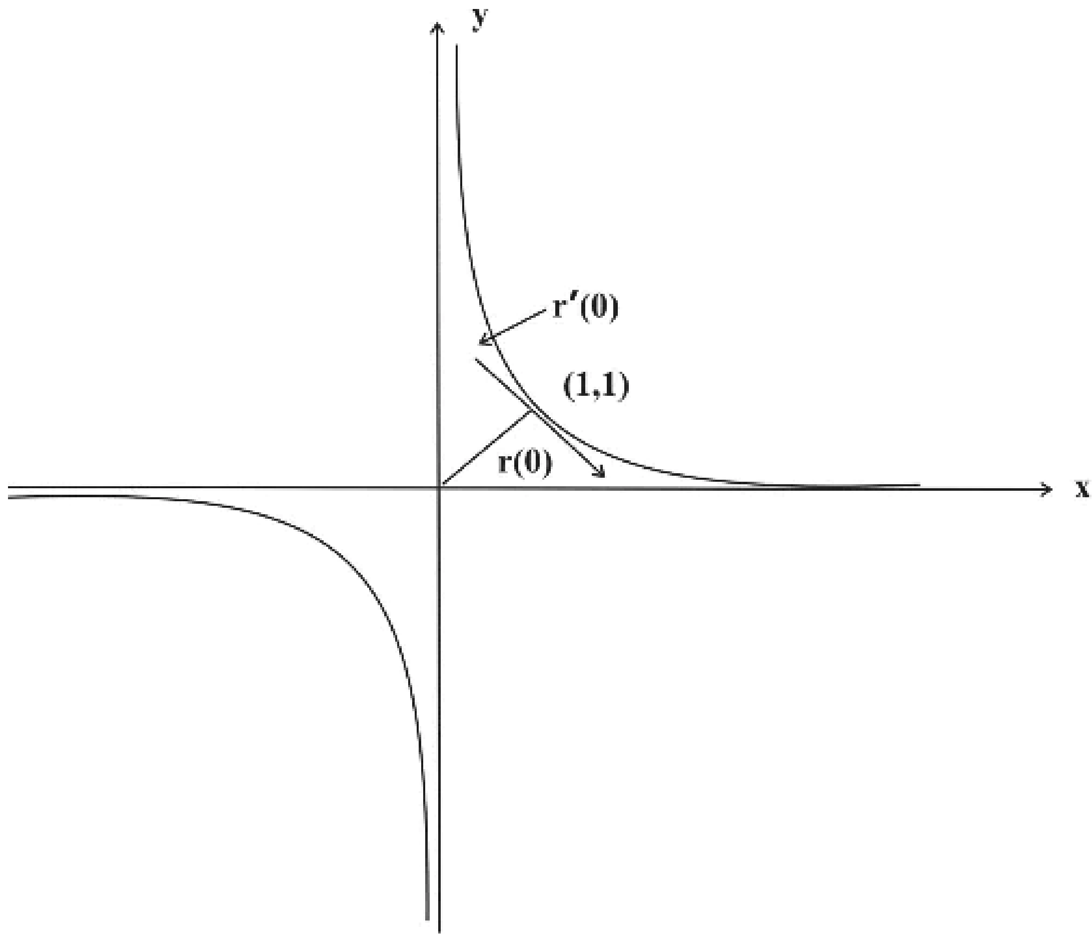
Which is an equilateral hyperbola rotated through 45°

(B)

Now $\vec{r}\,'(t) = \left(\frac{d}{dt}e^{t}\right)\hat{i} + \left(\frac{d}{dt}e^{-t}\right)\hat{j}$

i.e. $\vec{r}\,'(t) = e^{t}\hat{i} - e^{-t}\hat{j}$

(A), (C)



(a)

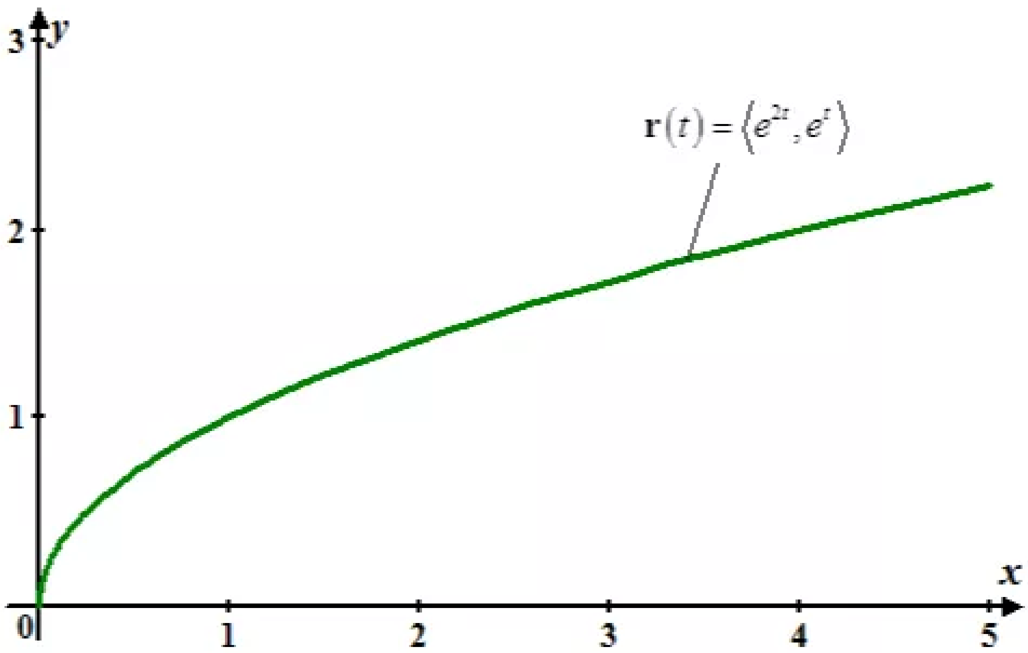
Consider the vector equation

$\mathbf{r}(t) = e^{2t}\mathbf{i} + e^{t}\mathbf{j}$ ...... (1)

The parametric equations are

$x = e^{2t},\ y = e^{t}$

Sketch the plane curve for the vector equation $\mathbf{r}(t) = e^{2t}\mathbf{i} + e^{t}\mathbf{j}$




(b)

To find the derivative of the vector equation $\mathbf{r}(t) = e^{2t}\mathbf{i} + e^{t}\mathbf{j}$

Recall the theorem,

If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, where $f, g,$ and $h$ are differentiable functions, then

$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$ ...... (2)

Now differentiate $\mathbf{r}(t) = e^{2t}\mathbf{i} + e^{t}\mathbf{j}$.

$\mathbf{r}'(t) = \left(\frac{d}{dt}\left(e^{2t}\right)\right)\mathbf{i} + \left(\frac{d}{dt}\left(e^{t}\right)\right)\mathbf{j}$ Using theorem (2)

$\mathbf{r}'(t) = 2e^{2t}\mathbf{i} + e^{t}\mathbf{j}$

Therefore the derivative of the vector equation $\mathbf{r}(t) = e^{2t}\mathbf{i} + e^{t}\mathbf{j}$ is

$\boxed{\mathbf{r}'(t) = 2e^{2t}\mathbf{i} + e^{t}\mathbf{j}}$ ...... (3)

(c)

Now find the position vector at $t=0$

Substitute $t=0$ in vector equation (1)

$$\mathbf{r}(0)=e^{2(0)}\mathbf{i}+e^{(0)}\mathbf{j}$$
$$\mathbf{r}(0)=\mathbf{i}+\mathbf{j}$$

Therefore the position vector at $t=0$ is

$$\boxed{\mathbf{r}(0)=\mathbf{i}+\mathbf{j}}$$

Now find the tangent vector at $t=0$

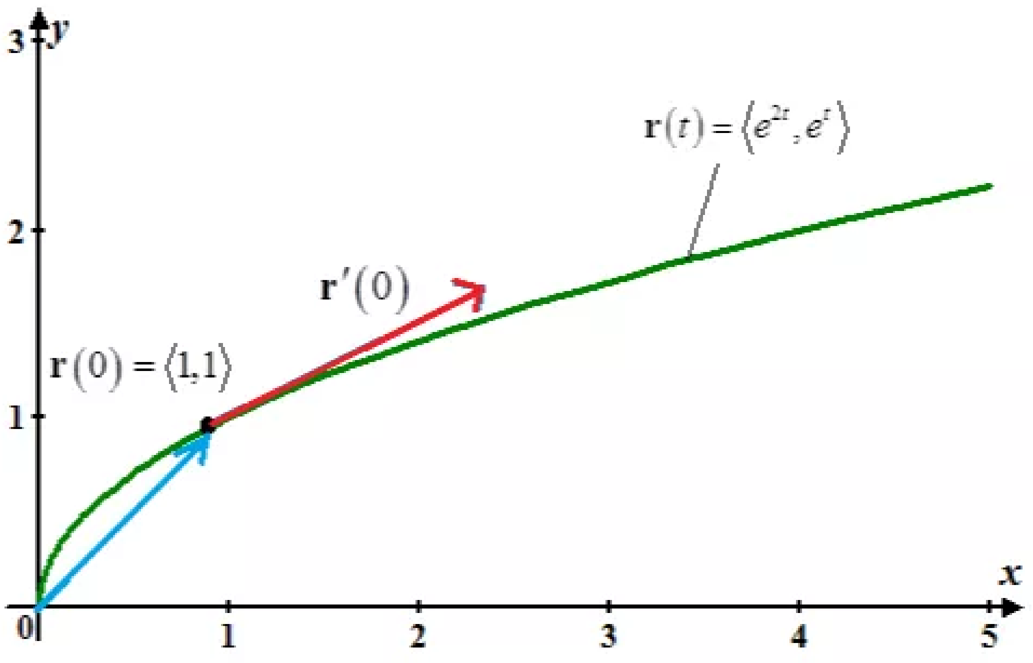
Substitute $t=0$ in vector equation (3)

$$\mathbf{r}'(0)=2e^{2(0)}\mathbf{i}+e^{(0)}\mathbf{j}$$
$$\mathbf{r}(0)=2\mathbf{i}+\mathbf{j}$$

Therefore the position vector at $t=0$ is

$$\boxed{\mathbf{r}(0)=2\mathbf{i}+\mathbf{j}}$$

Sketch of the position vector $\mathbf{r}(0)=\langle 1,1\rangle$ and $\mathbf{r}'(0)=\langle 2,1\rangle$



## Answer 9E.

Consider the following vector function:

$$\mathbf{r}(t) = \langle t\sin t, t^2, t\cos 2t \rangle$$

Find the derivative of the vector function $\mathbf{r}(t)$.

Let, $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = \langle t\sin t, t^2, t\cos 2t \rangle$.

Here, $f(t) = t\sin t, g(t) = t^2$ and $h(t) = t\cos 2t$.

Differentiate each component of $\mathbf{r}(t)$.

$$\begin{aligned} f'(t) &= \frac{d}{dt}(t\sin t) \\ &= t\frac{d}{dt}(\sin t) + \sin t\frac{d}{dt}(t) \\ &= t\cos t + \sin t(1) \\ &= t\cos t + \sin t \end{aligned}$$

$$\begin{aligned} g'(t) &= \frac{d}{dt}(t^2) \\ &= 2t \end{aligned}$$

$$\begin{aligned} h'(t) &= \frac{d}{dt}(t\cos 2t) \\ &= t\frac{d}{dt}(\cos 2t) + \cos 2t\frac{d}{dt}(t) \\ &= t(-\sin 2t)\frac{d}{dt}(2t) + \cos 2t(1) \\ &= t(-\sin 2t)(2) + \cos 2t \\ &= -2t\sin 2t + \cos 2t \end{aligned}$$

Clearly, the functions $f(t), g(t)$ and $h(t)$ are differentiable, then the derivative of the vector function $\mathbf{r}(t)$ is given as follows:

$$\begin{aligned} \mathbf{r}'(t) &= \langle f'(t), g'(t), h'(t) \rangle \\ &= \langle (t\cos t + \sin t), 2t, (-2t\sin 2t + \cos 2t) \rangle \end{aligned}$$

Therefore, the derivative of the vector function $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = \boxed{\langle (t\cos t + \sin t), 2t, (-2t\sin 2t + \cos 2t) \rangle}.$$

## Answer 10E.

We are suppose to find the derivative of the vector function.

Given $\vec{r}\left(t\right) = \ < \tan t,\ \sec t,\ 1/t^2 >$

The derivative is $\vec{r}'\left(t\right) = \ < \sec^2\left(t\right),\ \sec\left(t\right)\tan\left(t\right),\ \frac{-2}{t^3} >$

## Answer 11E.

We know that the derivative of any vector valued function of the form $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is given by $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$.

Evaluate $f'(t)$.

$$\begin{aligned} f'(t) &= \frac{d}{dt}(t) \\ &= 1 \end{aligned}$$

Now, find $g'(t)$.

$$\begin{aligned} g'(t) &= \frac{d}{dt}(1) \\ &= 0 \end{aligned}$$

Determine $h'(t)$.

$$\begin{aligned} h'(t) &= \frac{d}{dt}\left(2\sqrt{t}\right) \\ &= \frac{1}{\sqrt{t}} \end{aligned}$$

Replace the obtained values in $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$.

Thus, we get the derivative as $\boxed{\mathbf{r}'(t) = \mathbf{i} + \frac{1}{\sqrt{t}}\mathbf{k}}$.

## Answer 12E.

We know that the derivative of any vector valued function of the form $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is given by $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$.

Evaluate $f'(t)$.

$$f'(t) = \frac{d}{dt}\left(\frac{1}{1+t}\right) = -\frac{1}{(1+t)^2}$$

Now, find $g'(t)$.

$$g'(t) = \frac{d}{dt}\left(\frac{t}{1+t}\right) = \frac{1}{(1+t)^2}$$

Determine $h'(t)$.

$$h'(t) = \frac{d}{dt}\left(\frac{t^2}{1+t}\right) = \frac{t(2+t)}{(1+t)^2}$$

Replace the obtained values in $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$.

$$\mathbf{r}'(t) = -\frac{1}{(1+t)^2}\mathbf{i} + \frac{1}{(1+t)^2}\mathbf{j} + \frac{t(2+t)}{(1+t)^2}\mathbf{k}$$

Thus, we get the derivative as $\boxed{\mathbf{r}'(t) = -\frac{1}{(1+t)^2}\mathbf{i} + \frac{1}{(1+t)^2}\mathbf{j} + \frac{t(2+t)}{(1+t)^2}\mathbf{k}}$.

## Answer 13E.

The given vector function is

$$\vec{r}(t) = e^{t^2}\hat{i} - \hat{j} + \ln(1+3t)\hat{k}$$

Then the derivative is

$$\vec{r}\,'(t) = \frac{d}{dt}\left(e^{t^2}\right)\hat{i} - \frac{d}{dt}(1)\,\hat{j} + \frac{d}{dt}\left(\ln(1+3t)\right)\hat{k}$$

i.e. $$\vec{r}\,'(t) = 2te^{t^2}\hat{i} - (0)\,\hat{j} + \frac{1}{1+3t}\times(3)\,\hat{k}$$

i.e. $$\vec{r}\,'(t) = 2te^{t^2}\hat{i} + \frac{3}{1+3t}\hat{k}$$

## Answer 14E.

The given vector function is

$$\vec{r}(t) = at\cos 3t\,\hat{i} + b\sin^3 t\,\hat{j} + c\cos^3 t\,\hat{k}$$

Then the derivative is

$$\vec{r}'(t) = \left[a\frac{d}{dt}(t\cos 3t)\right]\hat{i} + b\frac{d}{dt}\left(\sin^3 t\right)\hat{j} + c\frac{d}{dt}\left(\cos^3 t\right)\hat{k}$$

i.e. $\vec{r}'(t) = a\left[\cos 3t - 3t\sin 3t\right]\hat{i} + 3b\sin^2 t\cos t\,\hat{j} - 3c\cos^2 t\sin t\,\hat{k}$

The given vector function is

$$\vec{r}(t) = a + tb + t^2c$$

Then the derivative is

$$\vec{r}'(t) = \frac{d}{dt}(1)\,a + \frac{d}{dt}(t)\,b + \frac{d}{dt}\left(t^2\right)c$$

i.e. $\vec{r}'(t) = (0)a + (1)b + (2t)c$

i.e. $\vec{r}'(t) = b + 2tc$

## Answer 16E.

Consider the vector function,

$$\mathbf{r}(t) = t\,\mathbf{a}\times(\mathbf{b} + t\,\mathbf{c})$$

Here, $\mathbf{r}(t)$ is the cross product of $t\mathbf{a}$ and $\mathbf{b} + t\mathbf{c}$.

Assume that $\mathbf{u}(t) = t\mathbf{a}$ and $\mathbf{v}(t) = \mathbf{b} + t\mathbf{c}$

Recall that,

If $\mathbf{u}(t), \mathbf{v}(t)$ are differentiable functions, then

$$\frac{d}{dt}(\mathbf{u}(t)\times\mathbf{v}(t)) = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t) \quad \text{......} \quad (1)$$

Then the derivative of given vector function is

$\mathbf{r}'(t) = \frac{d}{dt}(t\mathbf{a})\times(\mathbf{b} + t\mathbf{c}) + t\mathbf{a}\times\frac{d}{dt}(\mathbf{b} + t\mathbf{c})$ Apply (1)

$= \mathbf{a}\frac{d}{dt}(t)\times(\mathbf{b} + t\mathbf{c}) + t\mathbf{a}\times\left(\frac{d}{dt}(\mathbf{b}) + \frac{d}{dt}(t\mathbf{c})\right)$

$= \mathbf{a}\times(\mathbf{b} + t\mathbf{c}) + t\,\mathbf{a}\times\left(0 + \mathbf{c}\frac{d}{dt}(t)\right)$ Use $\frac{d}{dx}(c) = 0$

$= \mathbf{a}\times(\mathbf{b} + t\mathbf{c}) + t\,\mathbf{a}\times\mathbf{c}$ Simplify

$= \mathbf{a}\times(\mathbf{b} + t\mathbf{c}) + t\,\mathbf{a}\times\mathbf{c}$ Apply the rule $\mathbf{a}\times(\mathbf{b} + \mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$

$= \mathbf{a}\times\mathbf{b} + \mathbf{a}\times t\mathbf{c} + t\mathbf{a}\times\mathbf{c}$

$= \mathbf{a}\times\mathbf{b} + t(\mathbf{a}\times\mathbf{c}) + t(\mathbf{a}\times\mathbf{c})$ Use the rule $(c\mathbf{a})\times\mathbf{b} = c(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(c\mathbf{b})$

$= \mathbf{a}\times\mathbf{b} + 2t(\mathbf{a}\times\mathbf{c})$ Simplify

Thus, the derivative of the vector function $\mathbf{r}(t)$ is $\boxed{\mathbf{a}\times\mathbf{b}+2t(\mathbf{a}\times\mathbf{c})}$.

## Answer 17E.

Given vector function is $r(t)=\langle te^{-t},2\arctan t,2e^{t}\rangle$

$$=te^{-t}i+2\arctan tj+2e^{t}k$$

Differentiate each component of r with respect to $t$, we have

$$r'(t)=\frac{d}{dt}\left(te^{-t}i\right)+\frac{d}{dt}\left(2\arctan tj\right)+\frac{d}{dt}\left(2e^{t}k\right)$$

$$=\left(t\frac{d}{dt}\left(e^{-t}\right)+e^{-t}\frac{d}{dt}(t)\right)i+2\frac{d}{dt}(\arctan t)j+2\frac{d}{dt}\left(e^{t}\right)k$$

$$=\left(-te^{-t}+e^{-t}\right)i+\frac{2}{1+t^{2}}j+2e^{t}k$$

We now find $r(t), r'(t)$ at $t=0$

$$\left[r(t)\right]_{t=0}=0e^{0}i+2\arctan 0j+2e^{0}k=2k \quad \left(\text{since } e^{0}=1, \tan 0=0\right)$$

$$\left[r'(t)\right]_{t=0}=\left(-0e^{0}+e^{0}\right)i+\frac{2}{1+0^{2}}j+2e^{0}k=i+2j+2k$$

The unit tangent vector at the point $(0,0,2)$ is

$$T(0)=\frac{r'(0)}{|r'(0)|}$$

$$=\frac{i+2j+2k}{\sqrt{1^{2}+2^{2}+2^{2}}}$$

$$=\frac{i+2j+2k}{\sqrt{1+4+4}}$$

$$=\frac{i+2j+2k}{\sqrt{9}}$$

Therefore the unit tangent vector at the point with the given value of the parameter $t$ is, $T(t)=\dfrac{i+2j+2k}{3}$

**Answer 18E.**

Let $C$ be a smooth curve represented by $\mathbf{r}$ on an interval $I$. If $\mathbf{T}'(t) \neq 0$, then the principal unit normal vector at $t$ is defined as $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$.

Let us start by evaluating $\mathbf{T}(t)$ given by $\dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$.

We have $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

Evaluate $x'(t)$, $y'(t)$, and $z'(t)$.

$$x'(t) = \frac{d}{dt}(t^3 + 3t) = 3t^2 + 3 \qquad y'(t) = \frac{d}{dt}(t^2 + 1) = 2t \qquad z'(t) = \frac{d}{dt}(3t + 4) = 3$$

We get $\mathbf{r}'(t) = (3t^2 + 3)\mathbf{i} + 2t\mathbf{j} + 3\mathbf{k}$.

We know that $\|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$.

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{(3t^2 + 3)^2 + (2t)^2 + 3^2} \\ &= \sqrt{9t^4 + 18t^2 + 9 + 4t^2 + 9} \\ &= \sqrt{9t^4 + 22t^2 + 18}\end{aligned}$$

Substitute the obtained values in $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$.

$$\mathbf{T}(t) = \frac{(3t^2 + 3)\mathbf{i} + 2t\mathbf{j} + 3\mathbf{k}}{\sqrt{9t^4 + 22t^2 + 18}}$$

Substitute 1 for $t$.

$$\begin{aligned}\mathbf{T}(1) &= \frac{(3(1)^2 + 3)\mathbf{i} + 2(1)\mathbf{j} + 3\mathbf{k}}{\sqrt{9(1)^4 + 22(1)^2 + 18}} \\ &= \frac{6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}}{\sqrt{49}} \\ &= \frac{1}{7}(6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})\end{aligned}$$

Thus, we get $\boxed{\mathbf{T}(1) = \frac{1}{7}(6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})}$

**Answer 19E.**

The given vector function is

$$\vec{r}(t) = \cos t\,\hat{i} + 3t\,\hat{j} + 2\sin 2t\,\hat{k}$$

Then $\vec{r}'(t) = \frac{d}{dt}(\cos t)\hat{i} + \frac{d}{dt}(3t)\hat{j} + \frac{d}{dt}(2\sin 2t)\hat{k}$

$= -\sin t\hat{i} + 3\hat{j} + 4\cos 2t\hat{k}$

At t = 0,

$\vec{r}'(0) = -\sin 0\hat{i} + 3\hat{j} + 4\cos 0\hat{k}$

i.e. $\vec{r}'(0) = 3\hat{j} + 4\hat{k}$

And $\left|\vec{r}'(0)\right| = \sqrt{3^2 + 4^2}$

$= \sqrt{9+16} = \sqrt{25} = 5$

As we know $\vec{T}(t) = \frac{\vec{r}'(t)}{\left|\vec{r}'(t)\right|}$

Then $\vec{T}(0) = \frac{\vec{r}'(0)}{\left|\vec{r}'(0)\right|}$

i.e. $\vec{T}(0) = \frac{1}{5}\left(3\hat{j} + 4\hat{k}\right)$

i.e. $\vec{T}(0) = \frac{3}{5}\hat{j} + \frac{4}{5}\hat{k}$

## Answer 20E.

Consider the vector function $\mathbf{r}(t) = 2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \tan t\,\mathbf{k}$.

The objective is to find the unit tangent vector $\mathbf{T}(t)$ at the point where $t = \frac{\pi}{4}$.

The tangent vector to the curve defined by $\mathbf{r}(t)$ at a point $P$ is given by $\mathbf{r}'(t)$.

So the unit tangent vector is $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\left|\mathbf{r}'(t)\right|}$.

Differentiate $\mathbf{r}(t) = 2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \tan t\,\mathbf{k}$ with respect to '$t$'.

$$\mathbf{r}'(t) = \frac{d}{dt}(2\sin t)\mathbf{i} + \frac{d}{dt}(2\cos t)\mathbf{j} + \frac{d}{dt}(\tan t)\mathbf{k}$$

$$= 2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j} + \sec^2 t\,\mathbf{k}$$

The point where $t=\frac{\pi}{4}$ corresponds to

$$\mathbf{r}\left(\frac{\pi}{4}\right)=2\sin\left(\frac{\pi}{4}\right)\mathbf{i}+2\cos\left(\frac{\pi}{4}\right)\mathbf{j}+\tan\left(\frac{\pi}{4}\right)\mathbf{k}$$
$$=2\left(\frac{1}{\sqrt{2}}\right)\mathbf{i}+2\left(\frac{1}{\sqrt{2}}\right)\mathbf{j}+1\mathbf{k}$$
$$=\sqrt{2}\,\mathbf{i}+\sqrt{2}\mathbf{j}+\mathbf{k}$$
$$=\left(\sqrt{2},\sqrt{2},1\right)$$

and

$$\mathbf{r}'\left(\frac{\pi}{4}\right)=2\cos\left(\frac{\pi}{4}\right)\mathbf{i}-2\sin\left(\frac{\pi}{4}\right)\mathbf{j}+\sec^2\left(\frac{\pi}{4}\right)\mathbf{k}$$
$$=2\left(\frac{1}{\sqrt{2}}\right)\mathbf{i}-2\left(\frac{1}{\sqrt{2}}\right)\mathbf{j}+\left(\sqrt{2}\right)^2\mathbf{k}$$
$$=\sqrt{2}\mathbf{i}-\sqrt{2}\,\mathbf{j}+2\mathbf{k}$$

$$\left|\mathbf{r}'\left(\frac{\pi}{4}\right)\right|=\sqrt{\left(\sqrt{2}\right)^2+\left(-\sqrt{2}\right)^2+(2)^2}$$
$$=\sqrt{2+2+4}$$
$$=\sqrt{8}$$
$$=2\sqrt{2}$$

Therefore, the unit tangent vector $\mathbf{T}(t)$ at the point $\left(\sqrt{2},\sqrt{2},1\right)$ is,

$$\mathbf{T}\left(\frac{\pi}{4}\right)=\frac{\mathbf{r}'\left(\frac{\pi}{4}\right)}{\left|\mathbf{r}'\left(\frac{\pi}{4}\right)\right|}$$
$$=\frac{\sqrt{2}\mathbf{i}-\sqrt{2}\,\mathbf{j}+2\mathbf{k}}{2\sqrt{2}}$$
$$=\boxed{\left(\frac{1}{2}\mathbf{i}-\frac{1}{2}\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k}\right)}$$

**Answer 21E.**

$$\vec{r}(t)= <t,t^2,t^3>$$
$$\vec{r}\,'(t)=<1,2t,3t^2>$$
$$\vec{r}\,''(t)= <0,2,6t>$$

$$\vec{T}(1) = \frac{\vec{r}\,'(1)}{|\vec{r}\,'(1)|}$$
$$= \frac{<1,2,3>}{\sqrt{1^2+2^2+3^2}}$$
$$= <\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}>$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix}$$
$$= (12t^2 - 6t^2)\hat{i} - (6t - 0)\hat{j} + (2 - 0)\hat{k}$$
$$= 6t^2\hat{i} - 6t\,\hat{j} + 2\hat{k}$$

Therefore $\vec{r}\,'(t) \times \vec{r}\,''(t) = <6t^2, -6t, 2>$

## Answer 22E.

Consider the vector function

$$\mathbf{r}(t) = \langle e^{2t}, e^{-2t}, \; t e^{2t} \rangle$$

Then

$$\mathbf{r}'(t) = \langle 2e^{2t}, -2e^{-2t}, t \cdot 2e^{2t} + e^{2t} \cdot 1 \rangle$$
$$= \langle 2e^{2t}, -2e^{-2t}, 2te^{2t} + e^{2t} \rangle$$

And

$$\mathbf{r}''(t) = \langle 4e^{2t}, 4e^{-2t}, 2t \cdot 2e^{2t} + e^{2t} \cdot 2 + 2e^{2t} \rangle$$
$$= \langle 4e^{2t}, 4e^{-2t}, 4te^{2t} + 4e^{2t} \rangle$$

Now evaluate the vector functions

$$\mathbf{r}'(0) = \langle 2e^{2\cdot 0}, -2e^{-2\cdot 0}, 2(0)e^{2\cdot 0} + e^{2\cdot 0} \rangle$$
$$= \langle 2, -2, 1 \rangle$$

And

$$|\mathbf{r}'(t)| = \sqrt{4e^{4t} + 4e^{4t} + (2te^{2t} + e^{2t})^2}$$
$$|\mathbf{r}'(0)| = \sqrt{4+4+1}$$
$$= 3$$

Therefore, unit tangent vector is

$$\mathbf{T}(0)=\frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|}$$
$$=\frac{\langle 2,-2,1\rangle}{3}$$
$$=\boxed{\left\langle \frac{2}{3},-\frac{2}{3},\frac{1}{3}\right\rangle}$$

Now

$$\mathbf{r}''(t)=\langle 4e^{2t},4e^{-2t},4te^{2t}+4e^{2t}\rangle$$
$$\mathbf{r}''(0)=\boxed{\langle 4,4,4\rangle}$$

Note that, if $\mathbf{a}=\langle a_1,a_2,a_3\rangle$ and $\mathbf{b}=\langle b_1,b_2,b_3\rangle$, then the **dot product** of **a** and **b** is the number $\mathbf{a}\cdot\mathbf{b}$ given by

$$\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$$

Consider

$$\mathbf{r}'(t)\cdot\mathbf{r}''(t)=\langle 2e^{2t},-2e^{-2t},2te^{2t}+e^{2t}\rangle\cdot\langle 4e^{2t},4e^{-2t},4te^{2t}+4e^{2t}\rangle$$
$$=(2e^{2t})(4e^{2t})+(-2e^{-2t})(4e^{-2t})+(2te^{2t}+e^{2t})(4te^{2t}+4e^{2t})$$
$$=8e^{4t}-8e^{-4t}+8t^2e^{4t}+8te^{4t}+4te^{4t}+4e^{4t}$$
$$=\boxed{(8t^2+12t+12)e^{4t}-8e^{-4t}}$$

## Answer 23E.

Consider the parametric equations of a curve:

$$x=1+2\sqrt{t},\quad y=t^3-t,\quad z=t^3+t$$

The vector equation of the curve is $r(t)=\langle 1+2\sqrt{t},t^3-t,t^3+t\rangle$.

Differentiate $r(t)$ with respect to $t$.

$$r'(t)=\left\langle \frac{d}{dt}(1+2\sqrt{t}),\frac{d}{dt}(t^3-t),\frac{d}{dt}(t^3+t)\right\rangle$$
$$=\left\langle \frac{2}{2\sqrt{t}},3t^2-1,3t^2+1\right\rangle$$
$$=\left\langle \frac{1}{\sqrt{t}},3t^2-1,3t^2+1\right\rangle$$

The value of $t$ corresponding to the point $(3,0,2)$ is $t=1$.

Find the equation of the tangent line to the given curve at $t=1$.

At $t=1$:

$$r(1)=\left\langle 1+2\sqrt{1},1^3-1,1^3+1\right\rangle=\langle 3,0,2\rangle.$$

$$r'(1)=\left\langle \frac{1}{\sqrt{1}},3(1)^2-1,3(1)^2+1\right\rangle$$
$$=\langle 1,2,4\rangle$$

The vector equation of the tangent line is

$$\begin{aligned} v(t)&=r(1)+t\cdot r'(1)\\ &=\langle 3,0,2\rangle+t\cdot\langle 1,2,4\rangle\\ &=\langle 3,0,2\rangle+\langle t,2t,4t\rangle\\ &=\langle 3+t,2t,2+4t\rangle \end{aligned}$$

The standard parameterization of the tangent line to the given curve at the point $(3,0,2)$ is

$\boxed{x=3+t,\ \ y=2t,\ \ z=2+4t}$.

**Answer 24E.**

We are suppose to find parametric equations for the tangent line to the curve with the given parametric equations at the specified point.

Given $x=e^t, y=te^t, z=te^{t^2}$ ; $(1, 0, 0)$

The curve will pass through (1,0,0) when t = 0

$$\vec{r}(t)= <e^t,\ e^t+te^t,\ e^{t^2}+2t^2e^{t^2}>$$

$$\vec{r}(0)= <1,\ 1+0,\ 1+0> = <1,\ 1,\ 1>$$

so the line is given by

$$<1,\ 0,\ 0> +t<1,\ 1,\ 1>$$

so parametric equations are

x=1+t, y=0+t, z=0+t

i.e. x = 1+ t , y = t and z = t

## Answer 25E.

The parametric equations of the curve are:

$$x = e^{-t}\cos t,\ \ y = e^{-t}\sin t,\ z = e^{-t}$$

The given point is (1, 0, 1)

By taking $1 = e^{-t}\cos t,\ \ 0 = e^{-t}\sin t,\ 1 = e^{-t}$

The parametric value of t corresponding to the point (1, 0, 1) is $t = 0$

The vector equation of the curve is

$$\vec{r}(t) = \langle e^{-t}\cos t,\ e^{-t}\sin t,\ e^{-t} \rangle$$

Then $\vec{r}\,'(t) = \langle -e^{-t}(\sin t + \cos t),\ e^{-t}(\cos t - \sin t),\ -e^{-t} \rangle$

The tangent vector at $t = 0$ is

$$\vec{r}\,'(0) = \langle -1, 1, -1 \rangle$$

Now the tangent line is the line through (1, 0, 1) and parallel to vector <-1, 1, -1> then the parametric equations of tangent are:

$$x = 1 + (-1)t$$

$$y = 0 + (1)t$$

$$z = 1 + (-1)t$$

i.e. $\boxed{x = 1 - t,\ \ y = t,\ \ z = 1 - t}$

## Answer 26E.

Let $C$ be a smooth curve represented by $\mathbf{r}$ on an interval $I$. The unit tangent vector $\mathbf{T}(t)$ at defined as $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$, $\mathbf{r}'(t) \neq \mathbf{0}$.

We have $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

Evaluate $x'(t)$.

$$x'(t) = \frac{d}{dt}\left(\sqrt{t^2+3}\right)$$
$$= \frac{t}{\sqrt{t^2+3}}$$

Now, find $y'(t)$.

$$y'(t) = \frac{d}{dt}\left[\ln\left(t^2+3\right)\right]$$
$$= \frac{2t}{t^2+3}$$

Determine $z'(t)$

$$z'(t) = \frac{d}{dt}(t)$$
$$= 1$$

We get $\mathbf{r}'(t) = \frac{t}{\sqrt{t^2+3}}\mathbf{i} + \frac{2t}{t^2+3}\mathbf{j} + \mathbf{k}$.

Replace $t$ with 1 in $\mathbf{r}'(t) = \frac{t}{\sqrt{t^2+3}}\mathbf{i} + \frac{2t}{t^2+3}\mathbf{j} + \mathbf{k}$.

$$\mathbf{r}'(1) = \frac{1}{\sqrt{1^2+3}}\mathbf{i} + \frac{2(1)}{1^2+3}\mathbf{j} + \mathbf{k}$$
$$= \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \mathbf{k}$$

We thus get the tangent vector as $\left\langle \frac{1}{2}, \frac{1}{2}, 1 \right\rangle$.

The parametric equations of a line in space parallel to a nonzero vector $\mathbf{v} = \langle a, b, c \rangle$ and passing through the point $P(x_1, y_1, z_1)$ are $x = x_1 + at$, $y = y_1 + bt$, and $z = z_1 + ct$. The numbers $a$, $b$, and $c$ are called direction numbers.

We have $x_1 = 2$, $y_1 = \ln 4$, and $z_1 = 1$. The direction numbers are $a = \frac{1}{2}$,

$b = \frac{1}{2}$, and $c = 1$.

Thus, the parametric equations are $\boxed{x = 2 + \frac{1}{2}t,\ y = \ln 4 + \frac{1}{2}t,\ \text{and } z = 1 + t.}$

## Answer 27E.

We have $x^2 + y^2 = 25$ and $y^2 + z^2 = 20$ at $(3, 4, 2)$.

From $x^2 + y^2 = 25$, we get the parametric equations for $x$ as $5\cos t$ and $y$ as $5\sin t$, where $0 \le t \le 2\pi$.

Substitute the known values in $y^2 + z^2 = 20$ and solve for $y$.

$$(5\sin t)^2 + z^2 = 20$$
$$z^2 = 20 - 25\sin^2 t$$
$$z = \pm\sqrt{20 - 25\sin^2 t}$$

Since the point of intersection is (3, 4, 2), we take $z = \sqrt{20 - 25\sin^2 t}$.

We get the vector function as $\mathbf{r}(t) = 5\cos t\,\mathbf{i} + 5\sin t\,\mathbf{j} + \sqrt{20 - 25\sin^2 t}\,\mathbf{k}$.

Also, we get $\mathbf{r}'(t) = -5\sin t\,\mathbf{i} + 5\cos t\,\mathbf{j} - \dfrac{25\sin t\cos t}{\sqrt{20 - 25\sin^2 t}}\mathbf{k}$

At (3, 4, 2) we have $\cos t = \dfrac{3}{5}$ and $\sin t = \dfrac{4}{5}$.

We have to find $\mathbf{r}'(t)$ at the point $(3,4,2)$.

$$\mathbf{r}'(t) = -5\left(\frac{4}{5}\right)\mathbf{i} + 5\left(\frac{3}{5}\right)\mathbf{j} - \frac{25\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)}{\sqrt{20 - 25\left(\frac{4}{5}\right)^2}}\mathbf{k}$$

$$= -4\mathbf{i} + 3\mathbf{j} - \frac{12}{2}\mathbf{k}$$

$$= -4\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$$

We thus get the tangent vector as $\langle -4, 3, -6\rangle$.

The parametric equations of a line in space parallel to a nonzero vector $\mathbf{v} = \langle a, b, c\rangle$ and passing through the point $P(x_1, y_1, z_1)$ are $x = x_1 + at, y = y_1 + bt$ and $z = z_1 + ct$.
The numbers $a$, $b$, and $c$ are called direction numbers.

We have $x_1 = 3, y_1 = 4$ and $z_1 = 2$.
The direction numbers are $a = -4, b = 3,$ and $c = -6$.
Thus, the parametric equations are $x = 3 - 4t, y = 4 + 3t, z = 2 - 6t$

Therefore the vector equation is $\boxed{\mathbf{r}(t) = (3 - 4t)\mathbf{i} + (4 + 3t)\mathbf{j} + (2 - 6t)\mathbf{k}}$

**Answer 28E.**

Given $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, e^t\rangle$, where $0 \le t \le \pi$.

Differentiating we get

$\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, e^t\rangle$.

Since the tangent line is parallel to the plane $\sqrt{3}x + y = 1$, we have to find $t$ such that $\langle -2\sin t, 2\cos t, e^t\rangle$ is perpendicular to $\mathbf{P}\langle\sqrt{3}, 1, 0\rangle$.

Then, we get $\mathbf{r}'(t)\cdot\mathbf{P} = 0$.

Substituting the known values in $\mathbf{r}'(t)\cdot\mathbf{P} = 0$.

$$\langle -2\sin t, 2\cos t, e^t\rangle \cdot \langle\sqrt{3}, 1, 0\rangle = 0$$

$$-2\sqrt{3}\sin t + 2\cos t = 0$$

$$\sqrt{3}\sin t = \cos t$$

$$\tan t = \frac{1}{\sqrt{3}}$$

$$\Rightarrow t = \frac{\pi}{6}$$

Plug in $t$ with $\frac{\pi}{6}$ in $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, e^t \rangle$.

$$\mathbf{r}(t) = \left\langle 2\cos\frac{\pi}{6}, 2\sin\frac{\pi}{6}, e^{\frac{\pi}{6}} \right\rangle$$

$$= \left\langle \sqrt{3}, 1, e^{\frac{\pi}{6}} \right\rangle$$

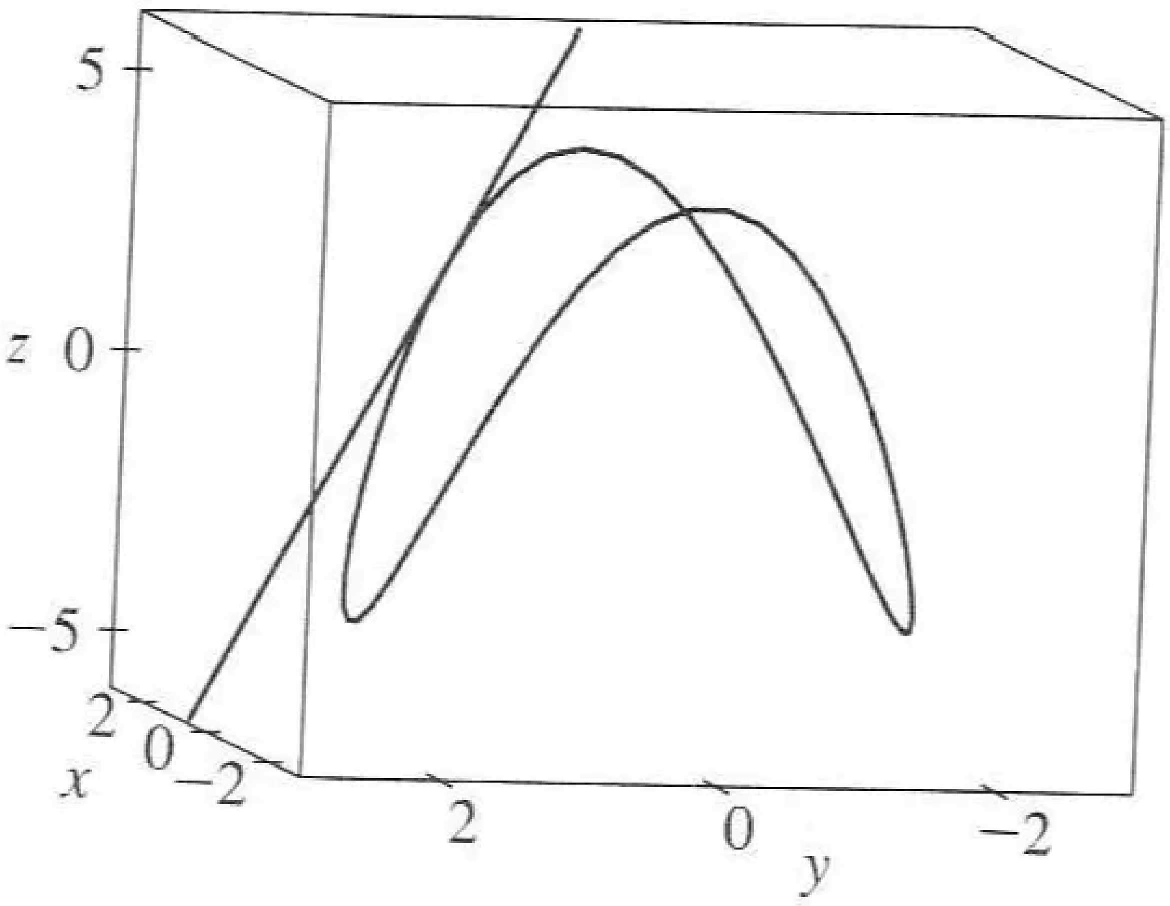
Therefore, tangent line is parallel to the plane $\sqrt{3}x + y = 1$ at

$$\boxed{\left\langle \sqrt{3}, 1, e^{\frac{\pi}{6}} \right\rangle}$$

**Answer 30E.**

We are given that $r(t) = \langle 2\cos t, 2\sin t, 4\cos 2t \rangle$. So from this we find it's derivative to be $r'(t) = \langle -2\sin t, 2\cos t, -8\sin 2t \rangle$. At (√3, 1, 2), t = π/6 and **r**′(π/6) = $\langle -1, \sqrt{3}, -4\sqrt{3} \rangle$. Therefore, the parametric equations of the tangent line are

$$x = \sqrt{3} - t,\ y = 1 + \sqrt{3}\,t,\ z = 2 - 4\sqrt{3}\,t.$$



## Answer 31E.

Consider the parametric equations, $x = t\cos t, y = t, z = t\sin t : (-\pi, \pi, 0)$ and the point $(-\pi, \pi, 0)$.

The object is to find the parametric equations for the tangent line at point $(-\pi, \pi, 0)$ and illustrate by graphing both the curve and the tangent line on common screen.

The parametric equaion is,

$$r(t) = \langle x(t), y(t), z(t) \rangle$$
$$= \langle t\cos t, t, t\sin t \rangle$$

The derivative of the parametric equation is,

$$r'(t) = \langle x'(t), y'(t), z'(t) \rangle$$
$$= \langle -t\sin t + \cos t, 1, t\cos t + \sin t \rangle$$

The parameter value corresponding to the point $(-\pi, \pi, 0)$ is $t = \pi$, so

$$r'(t) = \langle -t\sin t + \cos t, 1, t\cos t + \sin t \rangle$$
$$r'(\pi) = \langle -\pi\sin(\pi) + \cos\pi, 1, \pi\cos(\pi) + \sin(\pi) \rangle$$
$$= \langle -1, 1, -\pi \rangle$$

The parametric equations for the tangent line to the curve with parametric equations at the specified point $(-\pi, \pi, 0)$ are,

$$x = -\pi + t(-1),\ y = \pi + t(1) \text{ and } z = 0 + t(-\pi)$$
$$x = -\pi - t,\quad y = \pi + t \text{ and } z = -\pi t$$

Hence, the parametric equations for the tangent line to the curve is,

$$\boxed{x = -\pi - t, y = \pi + t, \text{ and} z = -\pi t}$$

To draw the graphing both the curve and the tangent line on a common screen, use the maple software as follows:

Type the below commands on maple worksheet.

$$with(plots):$$

$$A := spacecurve([t\cos t, t, t\sin t], t = -5..5, axes = normal, color = blue):$$

$$B := spacecurve([-t\sin t + \cos t, 1, t\cos t + \sin t], t = -5..5, axes = normal, color = green):$$

$$display(A, B);$$

The input and outputs are as shown below.

Input:

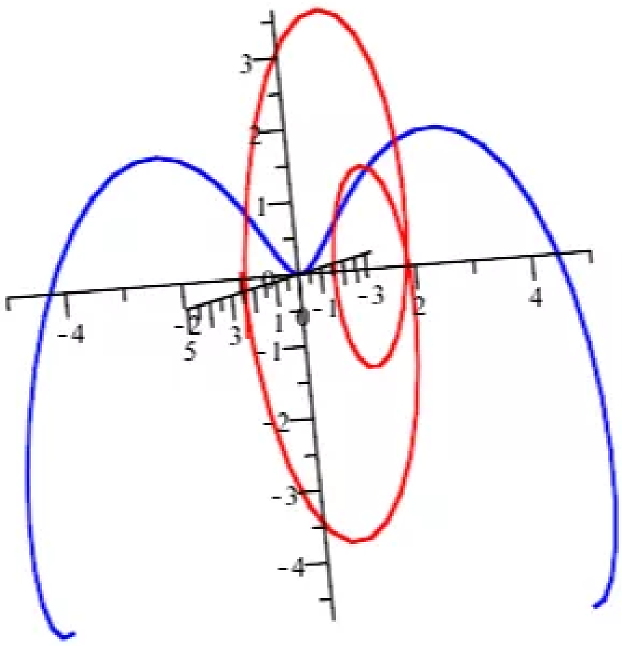
> $with(plots):$

> $A := spacecurve([t\cdot\cos(t), t, t\cdot\sin(t)], t = -5..5, axes = normal, color = blue);$

> $B := spacecurve([-t\cdot\sin(t) + \cos(t), 1, t\cdot\cos(t) + \sin(t)], t = -5..5, axes = normal, color = red);$

> $display(A, B);$

Output:




Q32E.

(A)

The vector equation of the curve is

$$\vec{r}(t) = <\sin \pi t, 2\sin \pi t, \cos \pi t>$$

The point corresponding to $t = 0$ is

$$\left(\sin 0, 2\sin 0, \cos 0\right)$$

i.e. $\left(0, 0, 1\right)$

And the point corresponding to $t = 1/2$ is:

$$\left(\sin\frac{\pi}{2}, 2\sin\frac{\pi}{2}, \cos\frac{\pi}{2}\right)$$

i.e. $\left(1, 2, 0\right)$

Now $\vec{r}\,'(t) = <\pi\cos\pi t,\ 2\pi\cos\pi t,\ -\pi\sin\pi t>$

Then $\vec{r}\,'(0) = <\pi, 2\pi, 0>$

And $\vec{r}\,'\left(\frac{1}{2}\right) = <0, 0, -\pi>$

The equation of tangent line at (0, 0, 1) is

$$\vec{r}(0) + u\vec{r}\,'(0)$$

i.e. $<0, 0, 1> + u<\pi, 2\pi, 0>$

i.e. $<u\pi, u2\pi, 1>$

And the equation of tangent at (1, 2, 0) is:

$$\vec{r}\left(\frac{1}{2}\right) + v\vec{r}\,'\left(\frac{1}{2}\right)$$

i.e. $<1, 2, 0> + v<0, 0, -\pi>$

i.e. $<1, 2, -v\pi>$

The tangent lines will intersect if for some $u$ and $v$,

$$<u\pi, 2u\pi, 1> = <1, 2, -v\pi>$$

$$\Rightarrow u\pi = 1,\ 2u\pi = 2,\ 1 = -v\pi$$

$$\Rightarrow u = \frac{1}{\pi},\ v = -\frac{1}{\pi}$$

Thus the point of intersection is: $\boxed{(1, 2, 1)}$

**Answer 33E.**

The given curves are:

$$\vec{r}_1(t) = <t, t^2, t^3>$$

And $\vec{r}_2(t) = <\sin t, \sin 2t, t>$

We know the angle between two curves is equal to the angle between their tangents at the point of intersection.

Consider $\vec{r}_1(t) = <t, t^2, t^3>$

Then $\vec{r}_1\,'(t) = <1, 2t, 3t^2>$

And then $\vec{r}_1\,'(0) = <1, 0, 0>$

And $\left|\vec{r}_1\,'(0)\right| = \sqrt{1^2 + 0 + 0} = 1$

Thus tangent vector at origin is

$$\vec{T_1}(t) = \frac{\vec{r_1}(0)}{\left|\vec{r_1}(0)\right|}$$

i.e. $\vec{T_1}(t) = <1,0,0>$ ----------------- (1)

Then $\left|\vec{T_1}(t)\right| = 1$

Now consider $\vec{r_2}(t) = <\sin t, \sin 2t, t>$

Then $\vec{r_2'}(t) = <\cos t, 2\cos 2t, 1>$

Then $\vec{r_2'}(0) = <1,2,1>$

And $\left|\vec{r_2'}(0)\right| = \sqrt{1^2+2^2+1^2}$

$= \sqrt{6}$

Thus the tangent vector at origin is

$$\vec{T_2}(t) = \frac{\vec{r_2'}(0)}{\left|\vec{r_2'}(0)\right|} \quad \text{-------------- (2)}$$

i.e. $\vec{T_2}(t) = <\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}>$

Then $\left|\vec{T_2}(t)\right| = \sqrt{\frac{1}{6}+\frac{4}{6}+\frac{1}{6}}$

$= \sqrt{\frac{6}{6}} = 1$

Let $\theta$ be the angle between tangents (1) and (2)

Then $\cos\theta = \frac{\vec{T_1}(t).\vec{T_2}(t)}{\left|\vec{T_1}(t)\right|\left|\vec{T_2}(t)\right|}$

$= \frac{<1,0,0>.<\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}>}{(1)(1)}$

$= \frac{1}{\sqrt{6}}$

Then $\theta = \cos^{-1}\left(1/\sqrt{6}\right) \approx 66^\circ$

Hence the angle between two given curves is $\boxed{66^\circ}$

**Answer 34E.**

Consider the vectors

$$\mathbf{r}_1(t) = <t, 1-t, 3+t^2>$$

And $\mathbf{r}_2(s) = <3-s, s-2, s^2>$

Then the parametric equations are

$$x = t,\ y = 1-t,\ z = 3+t^2$$

And $x = 3-s,\ y = s-2,\ z = s^2$

If the two curves intersect then they have a common point that is for some values of s and t,

$$\mathbf{r}_1(t) = \mathbf{r}_2(t)$$

Then $<t, 1-t, 3+t^2> = <3-s, s-2, s^2>$

Equate the corresponding coordinates

$t = 3-s$ ...... (1)

$1-t = s-2$ ...... (2)

$3+t^2 = s^2$ ...... (3)

Substitute $t = 3-s$ from equation (1) in equation (3), $3+t^2 = s^2$

That is

$$\begin{aligned} 3+(3-s)^2 &= s^2 \\ 3+9-6s+s^2 &= s^2 \\ 12 &= 6s \\ s &= 2 \end{aligned}$$

Substitute $s = 2$ in equation (1), $t = 3-s$

So,

$$\begin{aligned} t &= 3-s \\ &= 3-2 \\ &= 1 \end{aligned}$$

These values satisfy equations (2) ,because $1-2 = 1-2$

Then the two curves intersect and then point of intersection will be obtained by substituting $t = 1$ in $<t, 1-t, 3+t^2>$ or $s = 2$ in $<3-s, s-2, s^2>$

Therefore, the point of intersection is $\boxed{(1,0,4)}$ .

To find the angle of intersection of curves,

Consider $\mathbf{r}_1(t) = < t, 1-t, 3+t^2 >$

Then $\mathbf{r}_1'(t) = < 1, -1, 2t >$

Therefore $\mathbf{r}_1'(1) = < 1, -1, 2 >$ Since $t = 1$

And $\left|\mathbf{r}_1'(1)\right| = \sqrt{1+1+4} = \sqrt{6}$

So, tangent vector at (1, 0, 4) is

$$\mathbf{T}_1(1) = \frac{\mathbf{r}_1'(1)}{\left|\mathbf{r}_1'(1)\right|} = < \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}} > \quad \ldots\ldots (4)$$

Now consider $\mathbf{r}_2(s) = < 3-s, s-2, s^2 >$

Then $\mathbf{r}_2'(s) = < -1, 1, 2s >$

Therefore $\vec{r}_2'(2) = < -1, 1, 4 >$ Since $s = 2$

And $\left|\vec{r}_2'(2)\right| = \sqrt{1+1+16} = \sqrt{18}$

The tangent vector at (1, 0, 4) is

$$\mathbf{T}_2(2) = \frac{\mathbf{r}_2'(2)}{\left|\mathbf{r}_2'(2)\right|}$$

$$= < \frac{-1}{\sqrt{18}}, \frac{1}{\sqrt{18}}, \frac{4}{\sqrt{18}} > \quad \ldots\ldots (5)$$

Let $\theta$ be the angle between tangent (4) and (5)

Then $\cos\theta = \dfrac{\mathbf{T}_1(1).\mathbf{T}_2(2)}{\left|\mathbf{T}_1(1)\right|\left|\mathbf{T}_2(2)\right|}$

$$= \frac{< \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}} > . < -\frac{1}{\sqrt{18}}, \frac{1}{\sqrt{18}}, \frac{4}{\sqrt{18}} >}{(1)(1)}$$ Since $\mathbf{T}_1(1), \mathbf{T}_2(2)$ are unit vectors

$$= -\frac{1}{6\sqrt{3}} - \frac{1}{6\sqrt{3}} + \frac{8}{6\sqrt{3}} = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}}$$

So $\theta = \cos^{-1}\left( 1/\sqrt{3} \right) = 54.7^{\circ}$

Since the angle between curves is equal to the angle between tangents at the point of intersection, then the angle between given curves is $54.7^{\circ}$.

## Answer 35E.

Consider the integral,

$$\int_0^2 \left(t\mathbf{i} - t^3\mathbf{j} + 3t^5\mathbf{k}\right)dt$$

Evaluate the integral,

$$\int_0^2 \left(t\mathbf{i} - t^3\mathbf{j} + 3t^5\mathbf{k}\right)dt$$

$$\begin{aligned}\int_0^2 \left(t\mathbf{i} - t^3\mathbf{j} + 3t^5\mathbf{k}\right)dt &= \left(\int_0^2 t\,dt\right)\mathbf{i} - \left(\int_0^2 t^3\,dt\right)\mathbf{j} + \left(\int_0^2 3t^5\,dt\right)\mathbf{k} \\ &= \left(\frac{1}{2}\left[t^2\right]_0^2\right)\mathbf{i} - \left(\frac{1}{4}\left[t^4\right]_0^2\right)\mathbf{j} + \left(\frac{3}{6}\left[t^6\right]_0^2\right)\mathbf{k} \\ &= \frac{1}{2}(4-0)\mathbf{i} - \frac{1}{4}(16-0)\mathbf{j} + \frac{1}{2}(64-0)\mathbf{k} \\ &= 2\mathbf{i} - 4\mathbf{j} + 32\mathbf{k}\end{aligned}$$

Therefore,

$$\int_0^2 \left(t\mathbf{i} - t^3\mathbf{j} + 3t^5\mathbf{k}\right)dt = \boxed{2\mathbf{i} - 4\mathbf{j} + 32\mathbf{k}}$$

## Answer 36E.

$$\int_0^1 \left(\frac{4}{1+t^2}\hat{j} + \frac{2t}{1+t^2}\hat{k}\right)dt$$

$$= \left(\int_0^1 \frac{4}{1+t^2}dt\right)\hat{j} + \left(\int_0^1 \frac{2t}{1+t^2}dt\right)\hat{k}$$

$$= 4\tan^{-1}t\Big]_0^1\,\hat{j} + \ln\left(1+t^2\right)\Big]_0^1\,\hat{k}$$

$$= 4\left[\tan^{-1}1 - \tan^{-1}0\right]\hat{j} + \left[\ln(2) - \ln(1)\right]\hat{k}$$

$$= 4\left(\pi/4\right)\hat{j} + \ln(2)\hat{k}$$

$$= \pi\hat{j} + \ln 2\,\hat{k}$$

## Answer 37E.

Consider the definite integral

$$\int_0^{\frac{\pi}{2}} \left(3\sin^2 t\cos t\,\mathbf{i} + 3\sin t\cos^2\,\mathbf{j} + 2\sin t\cos t\,\mathbf{k}\right)dt$$

This means that we can evaluate an integral of a vector function by integrating each component function.

We can extend the Fundamental theorem of calculus to continuous vector functions as follows:

$$\int_a^b r(t)\,dt = \mathbf{R}(t)\big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

Where **R** is an antiderivative of $r$, that is, $\mathbf{R}'(t) = \mathbf{r}(t)$. We use the notation $\int \mathbf{r}(t)\,dt$ for indefinite integrals (anti derivatives).

Rewrite the definite integral as

$$\int_0^{\frac{\pi}{2}} \left(\left(3\sin^2 t\cos t\right)\mathbf{i} + \left(3\sin t\cos^2\right)\mathbf{j} + \left(2\sin t\cos t\right)\mathbf{k}\right)dt$$

$$= \left(\int_0^{\frac{\pi}{2}} 3\sin^2 t\cos t\,dt\right)\mathbf{i} + \left(\int_0^{\frac{\pi}{2}} 3\sin t\cos^2\,dt\right)\mathbf{j} + \left(\int_0^{\frac{\pi}{2}} 2\sin t\cos t\,dt\right)\mathbf{k} \quad \text{...... (1)}$$

To solve

$$\int_0^{\frac{\pi}{2}} 3\sin^2 t\cos t\,dt$$

Let $\sin t = u$

$$\Rightarrow \cos t\,dt = du$$

As $t=0$, $u=1$ and $t=\frac{\pi}{2}$, $u=0$

$$\int_0^{\frac{\pi}{2}} 3\sin t\cos^2 dt = -\int_1^0 3u^2 du$$
$$= -u^3\Big]_1^0$$
$$= -\left(0^3-1^3\right)$$

$$\int_0^{\frac{\pi}{2}} 3\sin^2 t\cos t\, dt = 1 \quad \cdots\cdots (3)$$

To solve

$$\int_0^{\frac{\pi}{2}} 2\sin t\cos t\, dt$$

$$= \int_0^{\frac{\pi}{2}} \sin 2t\, dt \quad \text{Since } 2\sin t\cos t = \sin 2t$$

$$= -\frac{\cos 2t}{2}\Big]_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{2}[\cos\pi - \cos 0]$$

$$= -\frac{1}{2}[-1-1]$$

$$= 1$$

$$\int_0^{\frac{\pi}{2}} 2\sin t\cos t\, dt = 1 \quad \cdots\cdots (4)$$

Therefore, substitute the values of the equations $(2)$, $(3)$, $(4)$ in equation $(1)$ we get

$$\int_0^{\frac{\pi}{2}} \left(\left(3\sin^2 t\cos t\right)\mathbf{i} + \left(3\sin t\cos^2\right)\mathbf{j} + \left(2\sin t\cos t\right)\mathbf{k}\right)dt = \boxed{\mathbf{i}+\mathbf{j}+\mathbf{k}}$$

**Answer 38E.**

To evaluate $\int_1^2 \left(t^2 i + t\sqrt{t-1}\, j + t\sin\pi t k\right)dt = I\ (\text{let})$

$$I = i\int_1^2 \left(t^2\right)dt + j\int_1^2 \left(t\sqrt{t-1}\right)dt + k\int_1^2 \left(t\sin\pi t\right)dt = iI_1 + jI_2 + kI_3\ (\text{let}) \quad \cdots\cdots\cdots (1)$$

$$I_1 = \int_1^2 \left(t^2\right)dt = \left(\frac{t^3}{3}\right)_1^2$$

$$= \left(\frac{2^3}{3} - \frac{1}{3}\right)$$

$$= \left(\frac{8}{3} - \frac{1}{3}\right)$$

$$= \frac{7}{3}$$

$$I_2 = \int_1^2 \left(t\sqrt{t-1}\right) dt$$

put $t - 1 = u^2$ implies $t = u^2 + 1$ and $dt = 2udu$

limits vary from 0 to 1, using this in $I_1$, we have

$$I_2 = \int_0^1 \left(u^2 + 1\right)\sqrt{u^2}\, 2udu = 2\int_0^1 \left(u^3 + u\right) udu$$

$$= 2\int_0^1 \left(u^4 + u^2\right) du$$

$$= 2\left(\frac{u^5}{5} + \frac{u^3}{3}\right)_0^1$$

$$= 2\left(\frac{1}{5} + \frac{1}{3}\right)$$

$$= 2\left(\frac{8}{15}\right)$$

$$= \frac{16}{15}$$

$$I_3 = \int_1^2 (t \sin \pi t)\, dt$$

$$= \left[t\int (\sin \pi t)\, dt - \int \left(\frac{d}{dt}(t)(t \sin \pi t)\, dt\right) dt\right]_1^2$$

$$= \left(\frac{-t \cos \pi t}{\pi}\right)_1^2 - \left[\int \left(\frac{-\cos \pi t}{\pi}\right) dt\right]_1^2$$

$$= \frac{-2\cos 2\pi}{\pi} + \frac{\cos \pi}{\pi} + \left[\frac{\sin \pi t}{\pi^2}\right]_1^2$$

$$= \frac{-2}{\pi} + \frac{(-1)}{\pi} + \frac{\sin 2\pi}{\pi^2} - \frac{\sin \pi}{\pi^2} \quad (\text{since } \cos \pi = -1, \cos 2\pi = 1)$$

$$= \frac{-3}{\pi} \quad (\text{since } \sin \pi = 0 = \sin 2\pi)$$

now put $I_1, I_2$ and $I_3$ in $(1)$, we have

$$I = \frac{7}{3}i + \frac{16}{15}j - \frac{3}{\pi}k$$

$$\quad 3 \quad 15 \quad \pi$$

Therefore $\int_1^2 \left(t^2 i + t\sqrt{t-1} j + t \sin \pi t k\right) dt = \frac{7}{3} i + \frac{16}{15} j - \frac{3}{\pi} k$

**Answer 39.**

Consider the integral

$$\int \left(\sec^2 t \ \mathbf{i} + t\left(t^2+1\right)^3 \mathbf{j} + t^2 \ln t \mathbf{k}\right) \ \ldots\ldots \ (1)$$

The objective is to solve the above integral.

Recall the result:

The integral of vector valued function of the form $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\int \mathbf{r}(t) dt = \left(\int f(t)\, dt\right)\mathbf{i} + \left(\int g(t)\, dt\right)\mathbf{j} + \left(\int h(t)\, dt\right)\mathbf{k} \ \ldots\ldots \ (2)$$

Use the result in (1).

$$\int \left(\sec^2 t \ \mathbf{i} + t\left(t^2+1\right)^3 \mathbf{j} + t^2 \ln t \mathbf{k}\right) dt = \int \sec^2 t dt \ \mathbf{i} + \int t\left(t^2+1\right)^3 dt \mathbf{j} + \int t^2 \ln t dt \ \mathbf{k}$$

Evaluate the integrals of the components of the vector function.

Consider the integral,

$\int \sec^2 t dt = \tan t + c_1$ Since $\int \sec^2 \theta d\theta = \tan \theta + c$

Therefore the integral is,

$$\int \sec^2 t dt = \tan t + c_1 \ \ldots\ldots \ (3)$$

Consider the integral $\int t\left(t^2+1\right)^3 dt$

Put $t^2+1=u$ then $2tdt=du$

The integral becomes,

$$\int t\left(t^2+1\right)^3 dt=\int u^3\frac{du}{2}$$
$$=\frac{1}{2}\int u^3 du$$

$=\frac{1}{2}\left(\frac{u^4}{4}\right)+c_2$ Since $\int x^n dx=\frac{x^{n+1}}{n+1}$

$$=\frac{u^4}{8}+c_2$$

$=\frac{\left(t^2+1\right)^4}{8}+c_2$ Substitute $u=t^2+1$

Therefore the integral is:

$$\int t\left(t^2+1\right)^3 dt=\frac{\left(t^2+1\right)^4}{8}+c_2 \quad \cdots\cdots \quad (4)$$

Consider the integral,

$$\int t^2 \ln t dt$$

Using integration by parts $\int u\,dv=u\,v-\int u'v$

Take $u=\ln t, dv=t^2dt$.

Then $du=\frac{1}{t}, v=\frac{t^3}{3}$

We obtain,

$$\int t^2 \ln t\,dt=\int\left(\underbrace{\ln t}_{u}\right)\left(\underbrace{t^2dt}_{dv}\right)$$
$$=(\ln t)\left(\frac{t^3}{3}\right)-\int\left[\frac{d}{dt}(\ln t)\right]\left(\frac{t^3}{3}\right)dt$$
$$=\frac{1}{3}t^3\ln t-\int\left(\frac{1}{t}\right)\left(\frac{t^3}{3}\right)dt$$

Therefore the integral

$$\int t^2 \ln tdt=\frac{1}{3}t^3\ln t-\frac{1}{9}t^3+c_3 \quad \cdots\cdots \quad (5)$$

Finally find the integral of the vector function (1) by substituting the limits of the components from (3), (4), and (5).

$$\int\left(\sec^2 t\ \mathbf{i}+t\left(t^2+1\right)\mathbf{j}+t^2\ln t\mathbf{k}\right)dt$$

$$=\int\sec^2 tdt\ \mathbf{i}+\int t\left(t^2+1\right)dt\mathbf{j}+\int t^2\ln tdt\ \mathbf{k}$$

$$=\left(\tan t+c_1\right)\ \mathbf{i}+\left(\frac{\left(t^2+1\right)^4}{4}+c_2\right)\mathbf{j}+\left(\frac{1}{3}t^3\ln t-\frac{1}{9}t^3+c_3\right)\ \mathbf{k}$$

$$=\tan t\ \mathbf{i}+\frac{1}{8}\left(t^2+1\right)^4\ \mathbf{j}+\left(\frac{1}{3}t^3\ln t-\frac{1}{9}t^3\right)\ \mathbf{k}+\mathbf{C}$$

Constant vector, $\mathbf{C}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$

Therefore the integral of the vector function (1) is

$$\boxed{\int\left(\sec^2 t\ \mathbf{i}+t\left(t^2+1\right)\mathbf{j}+t^2\ln t\mathbf{k}\right)dt=\tan t\ \mathbf{i}+\frac{1}{8}\left(t^2+1\right)^4\ \mathbf{j}+\left(\frac{1}{3}t^3\ln t-\frac{1}{9}t^3\right)\ \mathbf{k}+\mathbf{C}}$$

**Answer 41E.**

Given that

$$\overline{r}\,'\left(t\right)=2t\,\overline{i}\ +3t^2\overline{j}\ +\sqrt{t}\ \overline{k}$$

$$\overline{r}\left(t\right)=\ \int\left(2t\overline{i}\ +3t^2\overline{j}\ +\sqrt{t}\ \overline{k}\right)\mathrm{d}\,t$$

$$\overline{r}\left(t\right)=\ t^2\overline{i}\ +t^3\overline{j}\ +2/3t^{3/2}\overline{k}\ +c$$

$$\overline{r}\left(1\right)=\ \overline{i}\ +\overline{j}\ +2/3\overline{k}\ +c=\ \overline{i}\ +\overline{j}$$

Therefore, $c$ must equal $-2/3\overline{k}$

And, $\overline{r}\left(t\right)=\ t^2\overline{i}\ +t^3\overline{j}\ +\left(2/3t^{3/2}\ -2/3\right)\overline{k}$

## Answer 42E.

Given $r'(t) = ti + e^t j + te^t k$ and $r(0) = i + j + k$

Integrate $r'(t)$ with respect to $t$, we have

$$r(t) = \int \left(ti + e^t j + te^t k\right) dt$$

$$= i\int t dt + j\int e^t dt + k\int te^t dt$$

$$= i\frac{t^2}{2} + je^t + k\left(t\int e^t dt - \int\left(\frac{d}{dt}(t)\int e^t dt\right)dt\right)$$

$$= \frac{t^2}{2}i + e^t j + k\left(te^t - \int (1)e^t dt\right)$$

$$= \frac{t^2}{2}i + e^t j + k\left(te^t - e^t\right)$$

$$= \frac{t^2}{2}i + e^t j + e^t(t-1)k + C, \text{ here } C \text{ is constant of integration}$$

now $r(0) = 0i + e^0 j + e^0(0-1)k + C$

$= j - k + C$

implies $i + j + k = j - k + C$ (from given)

implies $C = i + 2k$

put this in $r(t)$, we have

$$r(t) = \frac{t^2}{2}i + e^t j + e^t(t-1)k + i + 2k$$

$$= \left(\frac{t^2}{2} + 1\right)i + e^t j + \left[e^t(t-1) + 2\right]k$$

## Answer 43E.

Let $\vec{u}(t) = < f_1(t), f_2(t), f_3(t) >$

And $\vec{v}(t) = < g_1(t), g_2(t), g_3(t) >$

$\vec{u}(t) + \vec{v}(t) = < f_1(t) + g_1(t), f_2(t) + g_2(t), f_3(t) + g_3(t) >$

Then $\frac{d}{dt}\left[\vec{u}(t)+\vec{v}(t)\right] = <\frac{d}{dt}\left(f_1(t)+g_1(t)\right), \frac{d}{dt}\left(f_2(t)+g_2(t)\right),$

$$\frac{d}{dt}\left(f_3(t)+g_3(t)\right)>$$

$$= <\frac{d}{dt}f_1(t)+\frac{d}{dt}\left(g_1(t)\right), \frac{d}{dt}\left(f_2(t)\right)+\frac{d}{dt}g_2(t)$$

$$\frac{d}{dt}\left(f_3(t)\right)+\frac{d}{dt}g_3(t)>$$

$$= <\frac{d}{dt}f_1(t), \frac{d}{dt}f_2(t), \frac{d}{dt}, f_3(t)>$$

$$+ <\frac{d}{dt}g_1(t), \frac{d}{dt}g_2(t), \frac{d}{dt}g_3(t)>$$

$$= \frac{d}{dt}\vec{u}(t)+\frac{d}{dt}\vec{v}(t)$$

Therefore $\frac{d}{dt}\left[\vec{u}(t)+\vec{v}(t)\right] = \vec{u}\,'(t)+\vec{v}\,'(t)$

## Answer 44E.

Let $\vec{u}(t) = <f_1(t), f_2(t), f_3(t)>$

Then $f(t)\vec{u}(t) = <f(t)f_1(t), f(t)f_2(t), f(t)f_3(t)>$

$$\frac{d}{dt}\left[f(t)u(t)\right] = <\frac{d}{dt}\left(f(t)f_1(t)\right), \frac{d}{dt}\left(f(t)f_2(t)\right), \frac{d}{dt}\left(f(t)f_3(t)\right)>$$

$$= <f'(t)f_1(t)+f(t)f_1'(t), f'(t)f_2(t)+f(t)f_2'(t),$$

$$f'(t)f_3(t)+f(t)f_3'(t)>$$

$$= <f'(t)f_1(t), f'(t)f_2(t), f'(t)f_3(t)>$$

$$+ <f(t)f_1'(t), f(t)f_2'(t), f(t)f_3'(t)>$$

$$= f'(t)<f_1(t), f_2(t), f_3(t)>$$

$$+f(t)<f_1'(t), f_2'(t), f_3'(t)>$$

$$= f'(t)\vec{u}(t)+f(t)\vec{u}\,'(t)$$

Therefore $\frac{d}{dt}\left[f(t)\vec{u}(t)\right] = f'(t)\vec{u}(t)+f(t)\vec{u}\,'(t)$

**Answer 45E.**

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are differentiable functions

Need to show that, $\frac{d}{dt}\left[\mathbf{u}(t)\times\mathbf{v}(t)\right]=\mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$

Let $\mathbf{u}(t)=\langle f_1(t),f_2(t),f_3(t)\rangle$, $\mathbf{v}(t)=\langle g_1(t),g_2(t),g_3(t)\rangle$

Find the derivatives $\mathbf{u}'(t),\mathbf{v}'(t)$

$$\mathbf{u}'(t)=\left\langle f_1'(t),f_2'(t),f_3'(t)\right\rangle$$
$$\mathbf{v}'(t)=\left\langle g_1'(t),g_2'(t),g_3'(t)\right\rangle$$

Use the definition of cross product to evaluate $\mathbf{u}(t)\times\mathbf{v}(t)$

$$\mathbf{u}(t)\times\mathbf{v}(t)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ f_1(t) & f_2(t) & f_3(t)\\ g_1(t) & g_2(t) & g_3(t)\end{vmatrix}$$
$$=\mathbf{i}\left(f_2(t)g_3(t)-g_2(t)f_3(t)\right)+\mathbf{j}\left(g_1(t)f_3(t)-f_1(t)g_3(t)\right)$$
$$+\mathbf{k}\left(f_1(t)g_2(t)-g_1(t)f_2(t)\right)$$

Then

$$\mathbf{u}(t)\times\mathbf{v}(t)=\left\langle \begin{matrix}f_2(t)g_3(t)-f_3(t)g_2(t),f_3(t)g_1(t)-f_1(t)g_3(t),\\ f_1(t)g_2(t)-f_2(t)g_1(t)\end{matrix}\right\rangle \text{ ...... (1)}$$

Recall that,

If $\mathbf{r}(t)=\langle f(t),g(t),h(t)\rangle$ where $f,g,h$ are differentiable functions, then

$$\mathbf{r}(t)=\langle f'(t),g'(t),h'(t)\rangle \text{ ...... (2)}$$

Differentiate (1) with respect to $t$.

$$\frac{d}{dt}(\mathbf{u}(t)\times\mathbf{v}(t))=\left\langle \begin{matrix}\frac{d}{dt}(f_2(t)g_3(t)-f_3(t)g_2(t)),\frac{d}{dt}(f_3(t)g_1(t)-f_1(t)g_3(t)),\\ \frac{d}{dt}(f_1(t)g_2(t)-f_2(t)g_1(t))\end{matrix}\right\rangle$$

Use (2)

$$=\left\langle \begin{matrix}\left(f_2'(t)g_3(t)+f_2(t)g_3'(t)-f_3'(t)g_2(t)-f_3(t)g_2'(t)\right),\\ \left(f_3'(t)g_1(t)+f_3(t)g_1'(t)-f_1'(t)g_3(t)-f_1(t)g_3'(t)\right),\\ \left(f_1'(t)g_2(t)+f_1(t)g_2'(t)-f_2'(t)g_1(t)-f_2(t)g_1'(t)\right)\end{matrix}\right\rangle$$

Use product rule of differentiation

Now, find $\mathbf{u}'(t)\times\mathbf{v}(t),\mathbf{u}(t)\times\mathbf{v}'(t)$

$$\mathbf{u}'(t)\times\mathbf{v}(t)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ f_1'(t) & f_2'(t) & f_3'(t)\\ g_1(t) & g_2(t) & g_3(t)\end{vmatrix}$$

$$=\mathbf{i}\left(f_2'(t)g_3(y)-g_2(t)f_3'(t)\right)+\mathbf{j}\left(g_1(t)f_3'(t)-f_1'(t)g_3(t)\right)$$
$$+\mathbf{k}\left(f_1'(t)g_2(t)-g_1(t)f_2'(t)\right)$$

$$=\left\langle \begin{matrix} f_2'(t)g_3(y)-g_2(t)f_3'(t),g_1(t)f_3'(t)-f_1'(t)g_3(t),\\ f_1'(t)g_2(t)-g_1(t)f_2'(t)\end{matrix}\right\rangle$$

$$\mathbf{u}(t)\times\mathbf{v}'(t)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ f_1(t) & f_2(t) & f_3(t)\\ g_1'(t) & g_2'(t) & g_3'(t)\end{vmatrix}$$

$$=\mathbf{i}\left(f_2(t)g_3'(y)-g_2'(t)f_3(t)\right)+\mathbf{j}\left(g_1'(t)f_3(t)-f_1(t)g_3'(t)\right)$$
$$+\mathbf{k}\left(f_1(t)g_2'(t)-g_1'(t)f_2(t)\right)$$

$$=\left\langle \begin{matrix} f_2(t)g_3'(y)-g_2'(t)f_3(t),g_1'(t)f_3(t)-f_1(t)g_3'(t),\\ f_1(t)g_2'(t)-g_1'(t)f_2(t)\end{matrix}\right\rangle$$

Find $\mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$

$$\mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)=\left\langle \begin{matrix} f_2'(t)g_3(y)-g_2(t)f_3'(t),g_1(t)f_3'(t)-f_1'(t)g_3(t),\\ f_1'(t)g_2(t)-g_1(t)f_2'(t)\end{matrix}\right\rangle$$
$$+\left\langle \begin{matrix} f_2(t)g_3'(y)-g_2'(t)f_3(t),g_1'(t)f_3(t)-f_1(t)g_3'(t),\\ f_1(t)g_2'(t)-g_1'(t)f_2(t)\end{matrix}\right\rangle$$

$$=\left\langle \begin{matrix} f_2'(t)g_3(y)-g_2(t)f_3'(t)+f_2(t)g_3'(y)-g_2'(t)f_3(t),\\ g_1(t)f_3'(t)-f_1'(t)g_3(t)+g_1'(t)f_3(t)-f_1(t)g_3'(t),\\ f_1'(t)g_2(t)-g_1(t)f_2'(t)+f_1(t)g_2'(t)-g_1'(t)f_2(t)\end{matrix}\right\rangle$$

From these computations, it is clear that, $\boxed{\frac{d}{dt}\left[\mathbf{u}(t)\times\mathbf{v}(t)\right]=\mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)}$

Thus, it is proved.

**Answer 46E.**

Let $\vec{u}(t) = < f_1(t), f_2(t), f_3(t) >$

$\vec{u}(f(t)) = < f_1(f(t)), f_2(f(t)), f_3(f(t)) >$

$$\frac{d}{dt}\left[\vec{u}(f(t))\right] = < \frac{d}{dt}(f_1(f(t))), \frac{d}{dt}(f_2(f(t)))\frac{d}{dt}(f_3(f(t))) >$$

$$= < f_1'(f(t)).f'(t), f_2'(f(t))f'(t), f_3'(f(t))f'(t) >$$

$$= f'(t) < f_1'(f(t)), f_2'(f(t)), f_3'(f(t)) >$$

$$= f'(t)u'(f(t))$$

Therefore $\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t).\vec{u}'(f(t))$

**Answer 47E.**

Given vector functions are $u(t) = \langle \sin t, \cos t, t \rangle = \sin ti + \cos tj + tk$

and $v(t) = \langle t, \cos t, \sin t \rangle = ti + \cos tj + \sin tk$

From the formula 4 of theorem 3, we have

$$\frac{d}{dt}(u(t) \cdot v(t)) = u'(t) \cdot v(t) + u(t) \cdot v'(t) \quad \text{.........} \quad (1)$$

consider $u'(t) = \frac{d}{dt}(\sin ti + \cos tj + tk)$

$$= \frac{d}{dt}(\sin t)i + \frac{d}{dt}(\cos t)j + \frac{d}{dt}(t)k$$

$$= \cos ti - \sin tj + k$$

$v'(t) = \frac{d}{dt}(ti + \cos tj + \sin tk)$

$$= \frac{d}{dt}(t)i + \frac{d}{dt}(\cos t)j + \frac{d}{dt}(\sin t)k$$

$$= i - \sin tj + \cos tk$$

put $u(t), u'(t), v(t), v'(t)$ in (1), we have

$$\frac{d}{dt}(u(t) \cdot v(t)) = (\cos ti - \sin tj + k) \cdot (ti + \cos tj + \sin tk) + (\sin ti + \cos tj + tk) \cdot (i - \sin tj + \cos tk)$$

$$= t\cos t - \sin t\cos t + \sin t + \sin t - \cos t\sin t + t\cos t$$

Therefore $\frac{d}{dt}(u(t) \cdot v(t)) = 2t\cos t + 2\sin t - 2\sin t\cos t$

**Answer 48E.**

If $\vec{u}(t) = \sin t\hat{i} + \cos t\hat{j} + t\hat{k}$

$\vec{v}(t) = t\hat{i} + \cos t\hat{j} + \sin t\hat{k}$

Then $\vec{u}'(t) = \cos t\hat{i} - \sin t\hat{j} + \hat{k}$

$\vec{v}'(t) = \hat{i} - \sin t\hat{j} + \cos t\hat{k}$

Then using the theorem

$$\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right]=\vec{u}'(t)\times\vec{v}(t)+\vec{u}(t)\times\vec{v}'(t)$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \cos t & -\sin t & 1\\ t & \cos t & \sin t\end{vmatrix}+\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \sin t & \cos t & t\\ 1 & -\sin t & \cos t\end{vmatrix}$$

$$=\left(-\sin^2 t-\cos t\right)\hat{i}+\left(t-\cos t\sin t\right)\hat{j}+\left(\cos^2 t+t\sin t\right)\hat{k}$$

$$+\left(\cos^2 t+t\sin t\right)\hat{i}+\left(t-\sin t\cos t\right)\hat{j}+\left(-\sin^2 t-\cos t\right)\hat{k}$$

$$=\left(-\sin^2 t-\cos t+\cos^2 t+t\sin t\right)\hat{i}+\left(t-\sin t\cos t+t-\sin t\cos t\right)\hat{j}$$

$$+\left(\cos^2 t+t\sin t-\sin^2 t-\cos t\right)\hat{k}$$

$$=\left(\cos 2t-\cos t+t\sin t\right)\hat{i}+2\left(t-\sin t\cos t\right)\hat{j}+\left(\cos 2t+t\sin t-\cos t\right)\hat{k}$$

**Answer 49E.**

On applying the product rule of differentiation, we get $f'(t) = \mathbf{u}'(t)\mathbf{v}(t)+\mathbf{u}(t)\mathbf{v}'(t)$ or $f'(2) = \mathbf{u}'(2)\mathbf{v}(2)+\mathbf{u}(2)\mathbf{v}'(2)$.

Substitute 2 for $t$ in $\mathbf{v}(t) = \langle t, t^2, t^3\rangle$.

$\mathbf{v}(2) = \langle 2, 4, 8\rangle$

Let $\mathbf{v}(t) = x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$. Then, $\mathbf{v}'(t) = x'(t)\mathbf{i}+y'(t)\mathbf{j}+z'(t)\mathbf{k}$.

$$x'(t) = \frac{d}{dt}(t) = 1 \qquad y'(t) = \frac{d}{dt}(t^2) = 2t \qquad z'(t) = \frac{d}{dt}(t^3) = 3t^2$$

We get $\mathbf{v}'(t) = \mathbf{i}+2t\mathbf{j}+3t^2\mathbf{k}$.

Replace $t$ with 2.

$\mathbf{v}'(2) = \mathbf{i}+2(2)\mathbf{j}+3(2)^2\mathbf{k}$

$\mathbf{v}'(2) = \mathbf{i}+4\mathbf{j}+12\mathbf{k}$

Substitute the known values in $f'(2) = \mathbf{u}'(2)\mathbf{v}(2)+\mathbf{u}(2)\mathbf{v}'(2)$.

$$\begin{aligned} f'(2) &= \langle 3, 0, 4\rangle\langle 2, 4, 8\rangle+\langle 1, 2, -1\rangle\langle 1, 4, 12\rangle \\ &= \langle (3)(2)+(0)(4)+(4)(8)\rangle+\langle (1)(1)+(2)(4)+(-1)(12)\rangle \\ &= 6+32+1+8-12 \\ &= 35 \end{aligned}$$

Therefore, we get the $f'(2)$ as $\boxed{35}$.

## Answer 50E.

On applying the product rule of differentiation, we get

$\mathbf{r}'(t) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ or $\mathbf{r}'(2) = \mathbf{u}'(2) \times \mathbf{v}(2) + \mathbf{u}(2) \times \mathbf{v}'(2)$.

Substitute 2 for $t$ in $\mathbf{v}(t) = \langle t, t^2, t^3 \rangle$.

$$\mathbf{v}(2) = \langle 2, 4, 8 \rangle$$

Let $\mathbf{v}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then, $\mathbf{v}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

$$x'(t) = \frac{d}{dt}(t) = 1 \qquad y'(t) = \frac{d}{dt}(t^2) = 2t \qquad z'(t) = \frac{d}{dt}(t^3) = 3t^2$$

We get $\mathbf{v}'(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$.

Replace $t$ with 2.

$$\mathbf{v}'(2) = \mathbf{i} + 2(2)\mathbf{j} + 3(2)^2\mathbf{k}$$
$$\mathbf{v}'(2) = \mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$$

Substitute the known values in $\mathbf{r}'(2) = \mathbf{u}'(2) \times \mathbf{v}(2) + \mathbf{u}(2) \times \mathbf{v}'(2)$.

$$\begin{aligned}\mathbf{r}'(2) &= \langle 3, 0, 4 \rangle \times \langle 2, 4, 8 \rangle + \langle 1, 2, -1 \rangle \times \langle 1, 4, 12 \rangle \\ &= \langle -16, -16, 12 \rangle + \langle 28, -13, 2 \rangle \\ &= \langle 12, -29, 14 \rangle\end{aligned}$$

Therefore, we get the $\mathbf{r}'(2)$ as $\boxed{\langle 12, -29, 14 \rangle}$.

## Answer 51E.

We know that

$$\frac{d}{dt}\left[\vec{r}(t) \times \vec{r}\,'(t)\right] = \vec{r}\,'(t) \times \vec{r}\,'(t) + \vec{r}(t) \times \vec{r}\,''(t)$$

But $\vec{r}\,'(t) \times \vec{r}\,'(t) = 0$

Therefore $\frac{d}{dt}\left[\vec{r}(t) \times \vec{r}\,'(t)\right] = \vec{r}(t) \times \vec{r}\,''(t)$

## Answer 52E.

$$\frac{d}{dt}\left[\vec{u}(t).\vec{v}(t) \times \vec{w}(t)\right]$$
$$= \frac{d}{dt}\left[\vec{u}(t)\right].\left(\vec{v}(t) \times \vec{w}(t)\right) + \vec{u}(t).\frac{d}{dt}\left(\vec{v}(t) \times \vec{w}(t)\right)$$
$$= \vec{u}\,'(t).\left(\vec{v}(t) \times \vec{w}(t)\right) + \vec{u}(t).\left[\vec{v}\,'(t) \times \vec{w}(t) + \vec{v}(t) \times \vec{w}\,'(t)\right]$$

## Answer 53E.

We know that

$$\left|\vec{r}(t)\right|^2 = \vec{r}(t)\cdot\vec{r}(t) \qquad \ldots (1)$$

$$\frac{d}{dt}\left|\vec{r}(t)\right|^2 = 2\left|\vec{r}(t)\right|\frac{d}{dt}\left|\vec{r}(t)\right| \qquad \ldots (2)$$

$$\frac{d}{dt}\left[u(t)\cdot v(t)\right] = u'(t)\cdot v(t) + u(t)\cdot v'(t)$$

Now from (1) we have

$$\left|\vec{r}(t)\right|^2 = \vec{r}(t)\cdot\vec{r}(t)$$

Then differentiating both sides with respect to t,

$$\frac{d}{dt}\left|\vec{r}(t)\right|^2 = \frac{d}{dt}\left(\vec{r}(t)\cdot\vec{r}(t)\right)$$

$$2\left|\vec{r}(t)\right|\frac{d}{dt}\left|\vec{r}(t)\right| = \vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t) \qquad \text{(Using (1), (2))}$$

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i.e. $2\left|\vec{r}(t)\right|\frac{d}{dt}\left|\vec{r}(t)\right| = \vec{r}(t)\cdot\vec{r}\,'(t) + \vec{r}(t)\cdot\vec{r}\,'(t)$ (As $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$)

i.e. $2\left|\vec{r}(t)\right|\frac{d}{dt}\left|\vec{r}(t)\right| = 2\vec{r}(t)\cdot\vec{r}\,'(t)$

Since $\left|\vec{r}(t)\right| \neq 0$ (given)

$$\frac{2\left|\vec{r}(t)\right|\frac{d}{dt}\left|\vec{r}(t)\right|}{2\left|\vec{r}(t)\right|} = \frac{2\vec{r}(t)\cdot\vec{r}\,'(t)}{2\left|\vec{r}(t)\right|} \qquad \text{(Dividing with } 2\left|\vec{r}(t)\right|\text{)}$$

Then $\frac{d}{dt}\left|\vec{r}(t)\right| = \frac{\vec{r}(t).\vec{r}\,'(t)}{\left|\vec{r}(t)\right|}$

Therefore $\boxed{\frac{d}{dt}\left|\vec{r}(t)\right| = \frac{\vec{r}(t).\vec{r}\,'(t)}{\left|\vec{r}(t)\right|}}$

## Answer 54E.

Let the position vector $\vec{r}(t)$ be

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

Then the parametric equations are

$$x = f(t), \quad y = g(t), \quad z = h(t) \qquad \text{---------------- (1)}$$

Also $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$

It is given that $\vec{r}(t)$ and $\vec{r}\,'(t)$ are perpendicular then $\vec{r}(t).\vec{r}\,'(t) = 0$

i.e. $\langle f(t), g(t), h(t) \rangle . \langle f'(t), g'(t), h'(t) \rangle = 0$

i.e. $f(t)f'(t) + g(t)g'(t) + h(t)h'(t) = 0$

Then using equation (1)

$$x\frac{dx}{dt}+y\frac{dy}{dt}+z\frac{dz}{dt}=0$$

Or $$2x\frac{dx}{dt}+2y\frac{dy}{dt}+2z\frac{dz}{dt}=0$$

Or $$\frac{d}{dt}\left(x^2+y^2+z^2\right)=0$$

Or $$x^2+y^2+z^2= \text{ constant } =k^2\,(\text{say})$$

Which is the equation of a sphere with centre at origin

Hence we say that if the position vector $\vec{r}(t)$ is always perpendicular to tangent vector $\vec{r}'(t)$, then the curve is a sphere with centre at origin.

**Answer 55E.**

Consider the following vector function:

$$\mathbf{u}(t)=\mathbf{r}(t)\cdot\left[\mathbf{r}'(t)\times\mathbf{r}''(t)\right]$$

Differentiate the above function:

$$\frac{d}{dt}(\mathbf{u}(t))=\frac{d}{dt}\left[\mathbf{r}(t)\cdot\left[\mathbf{r}'(t)\times\mathbf{r}''(t)\right]\right]$$

Use product rule to find the derivative:

$$\frac{d}{dt}(\mathbf{u}(t))=\mathbf{r}'(t).(\mathbf{r}'(t)\times\mathbf{r}''(t))+\mathbf{r}(t).\left[\mathbf{r}''(t)\times\mathbf{r}''(t)+\mathbf{r}'(t)\times\mathbf{r}'''(t)\right] \text{......} \quad (1)$$

It is known that:

$$\mathbf{r}'(t).(\mathbf{r}'(t)\times\mathbf{r}''(t))=0$$

Also

$$\mathbf{r}''(t)\times\mathbf{r}''(t)=0$$

Use these facts in (1):

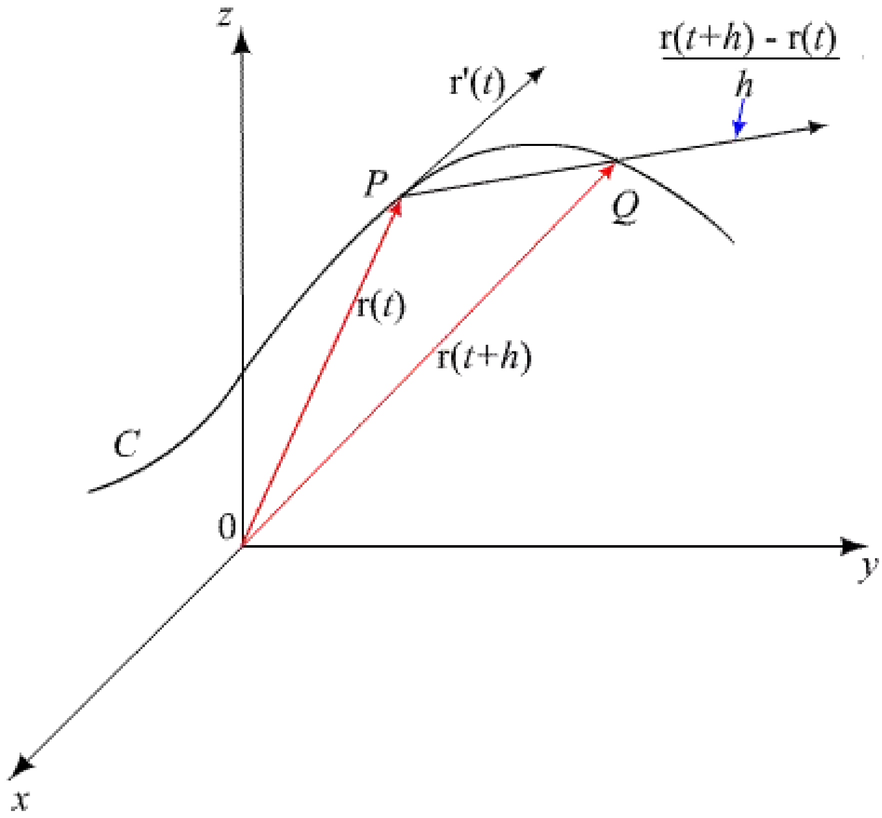
$$\frac{d}{dt}\left[\mathbf{r}(t).(\mathbf{r}'(t)\times\mathbf{r}''(t))\right]=\mathbf{r}(t).\left[\mathbf{r}'(t)\times\mathbf{r}'''(t)\right]$$

Therefore, it has been proved that:

$$\boxed{\mathbf{u}'(t)=\mathbf{r}(t).\left[\mathbf{r}'(t)\times\mathbf{r}'''(t)\right]}$$

**Answer 56.**

Let us start by sketching the tangent vector of a curve.




From the above figure, we have $\overrightarrow{OP} = \mathbf{r}(t)$ and $\overrightarrow{OQ} = \mathbf{r}(t+h)$.

Now, we know that $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$ or $\overrightarrow{PQ} = \mathbf{r}(t+h) - \mathbf{r}(t)$ and is in the increasing direction of $t$.

Consider the case when $h > 0$.

We can say that $\frac{1}{h}\left[\mathbf{r}(t+h) - \mathbf{r}(t)\right]$ has the same direction as $\overrightarrow{PQ}$.

Thus, the tangent vector defined by $\mathbf{r}(t)$ points in the direction of $t$.

Now, let $h < 0$ and say $h = -h'$ with $h' > 0$.
We have

$\overrightarrow{OP} = \mathbf{r}(t)$ and $\overrightarrow{OQ}' = \mathbf{r}(t - h')$.

Then, we get

$$\begin{aligned}\mathbf{r}(t - h') - \mathbf{r}(t) &= \overrightarrow{OQ}' - \overrightarrow{OP} \\ &= \overrightarrow{PQ}'\end{aligned}$$

We can thus say that $\overrightarrow{PQ}'$ is in the direction of decreasing $t$.
Now,

$$\begin{aligned}\frac{1}{h}\left[\mathbf{r}(t - h') - \mathbf{r}(t)\right] &= -\frac{1}{h'}\left[\mathbf{r}(t - h') - \mathbf{r}(t)\right] \\ &= \frac{\mathbf{r}(t) - \mathbf{r}(t - h')}{h'}\end{aligned}$$

has the same direction as that of $-\overrightarrow{PQ}'$.
Therefore, the tangent vector points in the direction of increasing $t$.