Chapter 3

Pair of Straight Lines

Contents

Gabriel Lame

The general equation of second degree

$$ax^2 + by^2 + 2gx + 2fy + c = 0$$

represents pair of straight line; if $\Delta = 0$ and $ab - h^2 \leq 0$

Clairaut (1729 A.D.) was the first to gave the distance formulae although in clumsy form. He also gave the intercept form of the linear equation.

In 1818, Gabriel Lame a civil engineer gave mE + mE' = 0 as the curve passing through the point of intersection of two loci E = 0 and E' = 0.

Pair of Straight Lines

3.1 Equation of Pair of Straight lines

Let the equation of two lines be

$a'x + b'y + c' = 0$ .....(i) and $a''x + b''y + c'' = 0$ .....(ii)

Hence $(a'x + b'y + c')(a''x + b''y + c'') = 0$ is called the joint equation of lines (i) and (ii) and conversely, if joint equation of two lines be $(a'x + b'y + c')$ $(a''x + b''y + c'') = 0$ then their separate equation will be $a'x + b'y + c' = 0$ and $a''x + b''y + c'' = 0$.

(1) **Equation of a pair of straight lines passing through origin :** The equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight line passing through the origin where a, h, b are constants.

Let the lines represented by $ax^2 + 2hxy + by^2 = 0$ be $y - m_1 x = 0$ and $y - m_2 x = 0$

where, $m_1 = \dfrac{-h + \sqrt{h^2 - ab}}{b}$ and $m_2 = \dfrac{-h - \sqrt{h^2 - ab}}{b}$ then, $m_1 + m_2 = -\dfrac{2h}{b}$ and $m_1 m_2 = \dfrac{a}{b}$

Then, two straight lines represented by $ax^2 + 2hxy + by^2 = 0$ are $ax + hy + y\sqrt{h^2 - ab} = 0$ and $ax + hy - y\sqrt{h^2 - ab} = 0$.

Note :

- The lines are real and distinct if $h^2 - ab > 0$
- The lines are real and coincident if $h^2 - ab = 0$
- The lines are imaginary if $h^2 - ab < 0$
- If the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ and $a'x^2 + 2h'xy + b'y^2 = 0$ should have one line common, then $(ab' - a'b)^2 = 4(ah' - a'h)(hb' - h'b)$.
- The equation of the pair of straight lines passing through origin and perpendicular to the pair of straight lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by $bx^2 - 2hxy + ay^2 = 0$
- If the slope of one of the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ be the square of the other, then $a^2 b + ab^2 - 6abh + 8h^3 = 0$.
- If the slope of one of the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ be λ times that of the other, then $4\lambda h^2 = ab(1 + \lambda)^2$.

(2) **General equation of a pair of straight lines :** An equation of the form,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

where a, b, c, f, g, h are constants, is said to be a general equation of second degree in x and y.

The necessary and sufficient condition for $ax^2+2hxy+by^2+2gx+2fy+c=0$ to represent a pair of straight lines is that $abc+2fgh-af^2-bg^2-ch^2=0$ or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$

(3) **Separate equations from joint equation:** The general equation of second degree be $ax^2+2hxy+by^2+2gx+2fy+c=0$. To find the lines represented by this equation we proceed as follows :

Step I : Factorize the homogeneous part $ax^2+2hxy+by^2$ into two linear factors. Let the linear factors be $a'x+b'y$ and $a''x+b''y$.

Step II: Add constants c' and c'' in the factors obtained in step I to obtain $a'x+b'y+c'$ and $a''x+b''y+c''$. Let the lines be $a'x+b'y+c'=0$ and $a''x+b''y+c''=0$.

Step III : Obtain the joint equation of the lines in step II and compare the coefficients of x, y and constant terms to obtain equations in c' and c''.

Step IV : Solve the equations in c' and c'' to obtain the values of c' and c''.

Step V : Substitute the values of c' and c'' in lines in step II to obtain the required lines.

Example: 1 If the sum of the slopes of the lines given by $x^2-2cxy-7y^2=0$ is four times their product. Then c has the value

[AIEEE 2004]

(a) – 2 (b) – 1 (c) 2 (d) 1

Solution: (c) We know that, $m_1+m_2=\frac{-2h}{b}$ and $m_1m_2=\frac{a}{b}$.

Given, $m_1+m_2=4m_1m_2 \Rightarrow \frac{-2c}{7}=4\left(\frac{1}{-7}\right) \Rightarrow c=2$

Example: 2 If one of the lines represented by the equation $ax^2+2hxy+by^2=0$ be $y=mx$, then

(a) $bm^2+2hm+a=0$ (b) $bm^2+2hm-a=0$ (c) $am^2+2hm+b=0$ (d) $bm^2-2hm+a=0$

Solution: (a) Substituting the value of y in the equation $ax^2+2hxy+by^2=0$

$\Rightarrow ax^2+2hx(mx)+b(mx)^2=0 \Rightarrow a+2hm+bm^2=0$

Example: 3 If the equation $12x^2-10xy+2y^2+11x-5y+K=0$ represent two straight lines, then the value of K is **[MP PET 20**

(a) 1 (b) 2 (c) 0 (d) 3

Solution: (b) Condition for pair of lines, $abc+2fgh-af^2-bg^2-ch^2=0$, Here $a=12,\ h=-5, b=2,\ g=11/2,\ f=-5/2,\ c=K$

Then, $12\times 2\times K+2\times\frac{-5}{2}\times\frac{11}{2}-12\times\left(\frac{-5}{2}\right)^2-2\times\left(\frac{11}{2}\right)^2-K(-5)^2=0$. On solving, we get K= 2.

3.2 Angle between the Pair of Lines

(1) The angle θ between the pair of lines represented by $ax^2+2hxy+by^2=0$ is given by $\tan\theta=\left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$

(i) The lines are coincident if the angle between them is zero.

$\therefore$ Lines are coincident *i.e.*, $\theta = 0 \Rightarrow \tan\theta = 0 \Rightarrow \frac{2\sqrt{h^2 - ab}}{a+b} = 0 \Rightarrow h^2 - ab = 0 \Rightarrow h^2 = ab$

Hence, the lines represented by $ax^2 + 2hxy + by^2 = 0$ are coincident, $iff\ h^2 = ab$

(ii) The lines are perpendicular if the angle between them is $\pi/2$.

$\therefore \theta = \frac{\pi}{2} \Rightarrow \cot\theta = \cot\frac{\pi}{2} \Rightarrow \cot\theta = 0 \Rightarrow \frac{a+b}{2\sqrt{h^2 - ab}} = 0 \Rightarrow a + b = 0 \Rightarrow$ coeff. of x^2 + coeff. of $y^2 = 0$

Thus, the lines represented by $ax^2 + 2hxy + by^2 = 0$ are perpendicular *iff* $a + b = 0$ *i.e.,* coeff. of x^2 + coeff. of $y^2 = 0$.

(2) The angle between the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is given by

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| \Rightarrow \theta = \tan^{-1}\left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

(i) The lines are parallel if the angle between them is zero. Thus, the lines are parallel *iff*

$\theta = 0 \Rightarrow \tan\theta = 0 \Rightarrow \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = 0 \Rightarrow h^2 = ab$.

Hence, the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are parallel *iff* $h^2 = ab$ and $af^2 = bg^2$ or $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$.

(ii) The lines are perpendicular if the angle between them is $\pi/2$.

Thus, the lines are perpendicular *i.e.,* $\theta = \pi/2 \Rightarrow \cot\theta = 0 \Rightarrow \frac{a+b}{2\sqrt{h^2 - ab}} = 0$

$\Rightarrow a + b = 0 \Rightarrow$ coeff. of x^2 + coeff. of $y^2 = 0$

Hence, the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are perpendicular *iff* $a + b = 0$

i.e., coeff. of x^2 + coeff. of $y^2 = 0$.

(iii) The lines are coincident, if $g^2 = ac$.

Example: 4 The angle between the lines $x^2 - xy - 6y^2 - 7x + 31y - 18 = 0$ is **[Karnataka CET 2003]**

(a) 45^o (b) 60^o (c) 90^o (d) 30^o

Solution: (b) Angle between the lines is $\theta = \tan^{-1}\left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \tan^{-1}\left|\frac{2\sqrt{\left(\frac{-1}{2}\right)^2 - 1\times(-6)}}{1+(-6)}\right| = \tan^{-1}\left|\frac{2\sqrt{\frac{1}{4}+6}}{1+(-6)}\right| = \tan^{-1}|-1| = \tan^{-1}(1) = \frac{\pi}{4}$, 45^o

Example: 5 If the angle between the pair of straight lines represented by the equation $x^2 - 3xy + \lambda y^2 + 3x - 5y + 2 = 0$ is $\tan^{-1}\left(\frac{1}{3}\right)$, where λ is a non- negative real number, then λ is

(a) 2 (b) 0 (c) 3 (d) 1

Solution: (a) Given that $\theta = \tan^{-1}\left(\frac{1}{3}\right) \Rightarrow \tan\theta = \frac{1}{3}$

Now, since $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| \Rightarrow \frac{1}{3} = \left|\frac{2\sqrt{\left(\frac{-3}{2}\right)^2 - \lambda}}{\lambda + 1}\right| \Rightarrow (\lambda+1)^2 = 9(9-4\lambda) \Rightarrow \lambda^2 + 38\lambda - 80 = 0$

$\Rightarrow \lambda^2 + 40\lambda - 2\lambda - 80 = 0 \Rightarrow \lambda(\lambda+40) - 2(\lambda+40) = 0 \Rightarrow (\lambda-2)(\lambda+40) = 0 \Rightarrow \lambda = 2$ or – 40, but λ is a non-negative real number. Hence $\lambda = 2$.

Example: 6 The angle between the pair of straight lines represented by $2x^2 - 7xy + 3y^2 = 0$ is **[Kurukshetra CEE 2002]**

(a) 60^o (b) 45^o (c) $\tan^{-1}(7/6)$ (d) 30^o

Solution: (b) Angle between the lines is , $\theta = \tan^{-1}\left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \tan^{-1}\left|\frac{2\sqrt{\left(-\frac{7}{2}\right)^2 - (2)(3)}}{2+3}\right| \Rightarrow \theta = \tan^{-1}\left(\frac{2}{5}.\frac{5}{2}\right) = \tan^{-1}(1) \Rightarrow$

$\theta = 45^o$

3.3 Bisectors of the Angles between the Lines

(1) The joint equation of the bisectors of the angles between the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2 - y^2}{a-b} = \frac{xy}{h}$ .....(i)

$\Rightarrow \quad hx^2 - (a-b)xy - hy^2 = 0$

Here, coefficient of x^2 + coefficient of $y^2 = 0$. Hence, the bisectors of the angles between the lines are perpendicular to each other. The bisector lines will pass through origin also.

Note : ❑ If $a = b$, the bisectors are $x^2 - y^2 = 0$ *i.e.,* $x - y = 0, x + y = 0$

❑ If $h = 0$, the bisectors are $xy = 0$ *i.e.,* $x = 0, y = 0$.

❑ If bisectors of the angles between lines represented by $ax^2 + 2hxy + by^2 = 0$ and $a'x^2 + 2h'xy + b'y^2 = 0$ are same, then $\frac{h'}{h} = \frac{a'-b'}{a-b}$.

❑ If the equation $ax^2 + 2hxy + by^2 = 0$ has one line as the bisector of the angle between the coordinate axes, then $4h^2 = (a+b)^2$.

(2) The equation of the bisectors of the angles between the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are given by $\frac{(x-\alpha)^2 - (y-\beta)^2}{a-b} = \frac{(x-\alpha)(y-\beta)}{h}$, where α, β is the point of intersection of the lines represented by the given equation.

Example: 7 The equation of the bisectors of the angles between the lines represented by $x^2 + 2xy\cot\theta + y^2 = 0$ is

(a) $x^2 - y^2 = 0$ (b) $x^2 - y^2 = xy$ (c) $(x^2 - y^2)\cot\theta = 2xy$ (d) None of these

Solution: (a) Equation of bisectors is given by $\frac{x^2 - y^2}{a-b} = \frac{xy}{h}$ or $\frac{x^2 - y^2}{0} = \frac{xy}{\cot\theta} \Rightarrow x^2 - y^2 = 0$

Example: 8 If the bisectors of the lines $x^2 - 2pxy - y^2 = 0$ be $x^2 - 2qxy - y^2 = 0$, then

[MP PET 1993; DCE 1999; Rajasthan PET 2003; AIEEE 2003]

(a) $pq + 1 = 0$ (b) $pq - 1 = 0$ (c) $p + q = 0$ (d) $p - q = 0$

Solution: (a) Bisectors of the angle between the lines $x^2 - 2pxy - y^2 = 0$ is $\frac{x^2 - y^2}{xy} = \frac{1-(-1)}{-p} \Rightarrow px^2 + 2xy - py^2 = 0$

But it is represented by $x^2 - 2qxy - y^2 = 0$. Therefore $\frac{p}{1} = \frac{2}{-2q} \Rightarrow pq = -1 \Rightarrow pq + 1 = 0$

3.4 Point of Intersection of Lines represented by $ax^2+2hxy+by^2+2gx+2fy+c = 0$

Let $\phi \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$$\frac{\partial \phi}{\partial x} = 2ax + 2hy + 2g = 0 \qquad \text{(Keeping } y \text{ as constant)}$$

and $$\frac{\partial \phi}{\partial y} = 2hx + 2by + 2f = 0 \qquad \text{(Keeping } x \text{ as constant)}$$

For point of intersection $\frac{\partial \phi}{\partial x} = 0$ and $\frac{\partial \phi}{\partial y} = 0$

We obtain, $ax + hy + g = 0$ and $hx + by + f = 0$

On solving these equations, we get $\frac{x}{fh - bg} = \frac{y}{gh - af} = \frac{1}{ab - h^2}$ *i.e.* $(x, y) = \left(\frac{bg - fh}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right)$

Also, since $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, from first two rows

$a \quad h \quad g \Rightarrow ax + hy + g = 0$ and

$h \quad b \quad f \Rightarrow hx + by + f = 0$ and then solve, we get the point of intersection.

Note : ❑ The point of intersection of lines represented by $ax^2 + 2hxy + by^2 = 0$ is (0, 0).

Example: 9 The point of intersection of the lines represented by the equation $2x^2 + 3y^2 + 7xy + 8x + 14y + 8 = 0$ is

(a) $(0,2)$ (b) $(1,2)$ (c) $(-2,0)$ (d) $(-2,1)$

Solution: (c) Let $\phi \equiv 2x^2 + 3y^2 + 7xy + 8x + 14y + 8 = 0$

$\frac{\partial \phi}{\partial x} = 4x + 7y + 8 = 0$ and $\frac{\partial \phi}{\partial y} = 6y + 7x + 14 = 0$

On solving these equations, we get $x = -2, \ y = 0$

Trick : If the equation is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

The points of intersection are given by $\left\{\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right\}$. Hence point is (– 2, 0)

Example: 10 If the pair of straight lines $xy - x - y + 1 = 0$ and line $ax + 2y - 3 = 0$ are concurrent, then a =

(a) – 1 (b) 0 (c) 3 (d) 1

Solution: (d) Given that equation of pair of straight lines $xy - x - y + 1 = 0$

$\Rightarrow \quad (x-1)(y-1)=0 \quad \Rightarrow \quad x-1=0$ or $y-1=0$

The intersection point of $x-1=0, y-1=0$ is (1,1)

$\therefore$ Lines $x-1=0, y-1=0$ and $ax+2y-3=0$ are concurrent.

$\therefore$ The intersecting points of first two lines satisfy the third line.

Hence, $a+2-3=0 \Rightarrow a=1$

3.5 Equation of the Lines joining the Origin to the Points of Intersection of a given Line and a given Curve

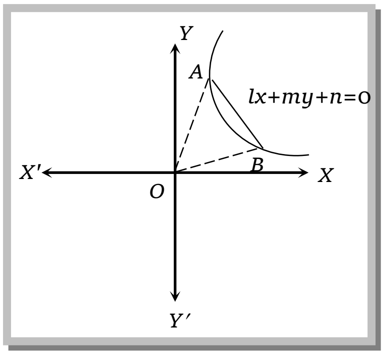
The equation of the lines which joins origin to the point of intersection of the line $lx+my+n=0$ and curve $ax^2+2hxy+by^2+2gx+2fy+c=0$, can be obtained by making the curve homogeneous with the help of line $lx+my+n=0$, which is

$$ax^2+2hxy+by^2+2(gx+fy)\left(\frac{lx+my}{-n}\right)+c\left(\frac{lx+my}{-n}\right)^2=0$$

We have $ax^2+2hxy+by^2+2gx+2fy+c=0$ ......(i)

and $lx+my+n=0$ .....(ii)

Suppose the line (ii) intersects the curve (i) at two points *A* and *B*. We wish to find the combined equation of the straight lines *OA* and *OB*. Clearly *OA* and *OB* pass through the origin, so their joint equation is a homogeneous equation of second degree in *x* and *y*.



From equation (ii), $lx+my=-n \Rightarrow \dfrac{lx+my}{-n}=1$ ......(iii)

Now, consider the equation

$$ax^2+2hxy+by^2+2gx\left(\frac{lx+my}{-n}\right)+2fy\left(\frac{lx+my}{-n}\right)+c\left(\frac{lx+my}{-n}\right)^2=0 \quad .....(iv)$$

Clearly, this equation is a homogeneous equation of second degree. So, it represents a pair of straight lines passing through the origin. Moreover, it is satisfied by the points *A* and *B*.

Hence (iv) represents a pair of straight lines *OA* and *OB* through the origin *O* and the points *A* and *B* which are points of intersection of (i) and (ii).

Example: 11 The lines joining the origin to the point of intersection of the circle $x^2+y^2=3$ and the line $x+y=2$ are

(a) $y-(3+2\sqrt{2})x=0$ (b) $x-(3+2\sqrt{2})y=0$ (c) $x-(3-2\sqrt{2})y=0$ (d) $y-(3-2\sqrt{2})x=0$

Solution: (a,b,c,d) Make homogenous the equation of circle, we get $x^2-6xy+y^2=0$

$$\Rightarrow \quad x=\frac{6y\pm\sqrt{(36-4)y^2}}{2}=\frac{6y\pm 4\sqrt{2}y}{2}=3y\pm 2\sqrt{2}y$$

Hence, the equation are $x=(3+2\sqrt{2})y$ and $x=(3-2\sqrt{2})y$

Also after rationalizing these equations becomes $y-(3+2\sqrt{2})x=0$ and $y-(3-2\sqrt{2})x=0$.

Example: 12 The pair of straight lines joining the origin to the points of intersection of the line $y=2\sqrt{2}x+c$ and the circle $x^2+y^2=2$ are at right angles, if

[MP PET 1996]

(a) $c^2-4=0$ (b) $c^2-8=0$ (c) $c^2-9=0$ (d) $c^2-10=0$

Solution: (c) Pair of straight lines joining the origin to the points of intersection of the line $y=2\sqrt{2}x+c$ and the circle $x^2+y^2=2$ are

$$\Rightarrow x^2+y^2+(-2)\left(\frac{2\sqrt{2}x-y}{-c}\right)^2=0 \Rightarrow x^2+y^2-\frac{2}{c^2}\left(8x^2+y^2-4\sqrt{2}xy\right)=0 \Rightarrow x^2\left(1-\frac{16}{c^2}\right)+y^2\left(1-\frac{2}{c^2}\right)+\frac{8\sqrt{2}xy}{c^2}=0$$

If these lines are perpendicular, $1-\frac{16}{c^2}+1-\frac{2}{c^2}=0$

$$\Rightarrow \frac{2c^2-18}{c^2}=0 \Rightarrow c^2-9=0.$$

3.6 Removal of First degree Terms

Let point of intersection of lines represented by $ax^2+2hxy+by^2+2gx+2fy+c=0$ ......(i) is (α,β).

Here $(\alpha,\beta)=\left(\frac{bg-fh}{h^2-ab},\frac{af-gh}{h^2-ab}\right)$

For removal of first degree terms, shift the origin to (α,β) *i.e.*, replacing *x* by $(X+\alpha)$ and *y* be $(Y+\beta)$ in (i).

Alternative Method : Direct equation after removal of first degree terms is

$$aX^2+2hXY+bY^2+(g\alpha+f\beta+c)=0$$

Where $\alpha=\frac{bg-fh}{h^2-ab}$ and $\beta=\frac{af-gh}{h^2-ab}$

3.7 Removal of the Term *xy* from *f* (*x*, *y*) = *ax*² + 2*hxy* +*by*² without changing the Origin

Clearly, $h\neq 0$. Rotating the axes through an angle θ, we have,

$x=X\cos\theta-Y\sin\theta$ and $y=X\sin\theta+Y\cos\theta$

$\therefore \quad f(x,y)=ax^2+2hxy+by^2$

After rotation, new equation is $F(X,Y)=(a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta)X^2$

$$+2\{(b-a)\cos\theta\sin\theta+h(\cos^2\theta-\sin^2\theta)XY$$

$$+(a\sin^2\theta-2h\cos\theta\sin\theta+b\cos^2\theta)Y^2$$

Now coefficient of XY = 0. Then we get $\cot 2\theta=\frac{a-b}{2h}$

Note : ❑ Usually, we use the formula, $\tan 2\theta=\frac{2h}{a-b}$ for finding the angle of rotation, θ. However, if $a=b$, we use $\cot 2\theta=\frac{a-b}{2h}$ as in this case $\tan 2\theta$ is not defined.

Example: 13 The new equation of curve $12x^2+7xy-12y^2-17x-31y-7=0$ after removing the first degree terms

(a) $12X^2-7XY-12Y^2=0$ (b) $12X^2+7XY+12Y^2=0$

(c) $12X^2+7XY-12Y^2=0$ (d) None of these

Solution: (c) Let $\phi \equiv 12x^2+7xy-12y^2-17x-31y-7=0$ .....(i)

$\therefore \frac{\partial \phi}{\partial x} \equiv 24x+7y-17=0$ and $\frac{\partial \phi}{\partial y} \equiv 7x-24y-31=0$

Their point of intersection is $(x,y) \equiv (1,-1)$

Here $\alpha = 1, \beta = -1$

Shift the origin to (1, –1) then replacing $x = X+1$ and $y = Y-1$ in (i), the required equation is

$12(X+1)^2+7(X+1)(Y-1)-12(Y-1)^2-17(X+1)-31(Y-1)-7=0$ *i.e.*, $12X^2+7XY-12Y^2=0$

Alternative Method : Here $\alpha = 1$ and $\beta = -1$ and $g = -17/2, f = -31/2, c = -7$

$\therefore g\alpha + f\beta + c = -\frac{17}{2}\times 1 - \frac{31}{2}\times -1 - 7 = 0$

$\therefore$ Removed equation is $aX^2+2hXY+bY^2+(g\alpha+f\beta+c)=0$

i.e., $12X^2+7XY-12Y^2+0=0 \Rightarrow 12X^2+7XY-12Y^2=0$.

Example: 14 Mixed term xy is to be removed from the general equation $ax^2+by^2+2hxy+2gx+2fy+c=0$, one should rotate the axes through an angle θ given by $\tan 2\theta =$

(a) $\frac{a-b}{2h}$ (b) $\frac{2h}{a+b}$ (c) $\frac{a+b}{2h}$ (d) $\frac{2h}{a-b}$

Solution: (d) Let (x', y') be the coordinates on new axes, then put $x = x'\cos\theta - y'\sin\theta,\ y = x'\sin\theta + y'\cos\theta$ in the equation, then the coefficient of xy in the transformed equation is 0.

So, $2(b-a)\sin\theta.\cos\theta + 2h\cos 2\theta = 0 \Rightarrow \tan 2\theta = \frac{2h}{a-b}$

3.8 Distance between the Pair of parallel Straight lines

If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represent a pair of parallel straight lines, then the distance between them is given by $2\sqrt{\frac{g^2-ac}{a(a+b)}}$ or $2\sqrt{\frac{f^2-bc}{b(a+b)}}$

Example: 15 Distance between the pair of lines represented by the equation $x^2-6xy+9y^2+3x-9y-4=0$ **[Kerala (Engg.) 20**

(a) $\frac{15}{\sqrt{10}}$ (b) $\frac{1}{2}$ (c) $\sqrt{\frac{5}{2}}$ (d) $\frac{1}{\sqrt{10}}$

Solution: (c) The distance between the pair of straight lines given by

$ax^2+2hxy+by^2+2gx+2fy+c=0$ is $2\sqrt{\frac{g^2-ac}{a(a+b)}}$, Here $a=1, b=9, c=4,\ g=\frac{3}{2} = 2\times\sqrt{\frac{\frac{9}{4}-(-4)}{1(1+9)}} = 2\times\sqrt{\frac{\frac{25}{4}}{10}} = \sqrt{\frac{5}{2}}$

Example: 16 Distance between the lines represented by the equation $x^2+2\sqrt{3}xy+3y^2-3x-3\sqrt{3}y-4=0$ is **[Roorkee 1989]**

(a) 5/2 (b) 5/4 (c) 5 (d) 0

Solution: (a) First check for parallel lines *i.e.*, $\frac{a}{h}=\frac{h}{b}=\frac{g}{f} \Rightarrow \frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}=\frac{\frac{-3}{2}}{\frac{-3\sqrt{3}}{2}}$

which is true, hence lines are parallel. $\therefore$ Distance between them is $2\sqrt{\frac{g^2-ac}{a(a+b)}} = 2\sqrt{\frac{(-3/2)^2-1(-4)}{1(1+3)}}$

$=5/2$

3.9 Some Important Results

(1) The lines joining the origin to the points of intersection of the curves $ax^2 + 2hxy + by^2 + 2gx = 0$ and $a'x^2 + 2h'xy + b'y^2 + 2g'x = 0$ will be mutually perpendicular, if $g(a'+b') = g'(a+b)$.

(2) If the equation $hxy + gx + fy + c = 0$ represents a pair of straight lines, then $fg = ch$.

(3) The pair of lines $(a^2 - 3b^2)x^2 + 8abxy + (b^2 - 3a^2)\ y^2 = 0$ with the line $ax + by + c = 0$ form an equilateral triangle.

(4) The area of a triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ is given by $\frac{n^2\sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2}$

(5) The lines joining the origin to the points of intersection of line $y = mx + c$ and the circle $x^2 + y^2 = a^2$ will be mutually perpendicular, if $a^2(m^2 + 1) = 2c^2$.

(6) If the distance of two lines passing through origin from the point (x_1, y_1) is *d*, then the equation of lines is $(xy_1 - yx_1)^2 = d^2(x^2 + y^2)$

(7) The lines represented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will be equidistant from the origin, if $f^4 - g^4 = c(bf^2 - ag^2)$

(8) The product of the perpendiculars drawn from (x_1, y_1) on the lines $ax^2 + 2hxy + by^2 = 0$ is given by

$$\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}$$

(9) The product of the perpendiculars drawn from origin on the lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

$$\frac{c}{\sqrt{(a-b)^2 + 4h^2}}$$

(10) If the lines represented by the general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are perpendicular, then the square of distance between the point of intersection and origin is $\frac{f^2 + g^2}{h^2 + b^2}$

(11) The square of distance between the point of intersection of the lines represented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and origin is $\frac{c(a+b) - f^2 - g^2}{ab - h^2}$

Example: 17 The area of the triangle formed by the lines $4x^2 - 9xy - 9y^2 = 0$ and $x = 2$ is **[Roorkee 2000]**

(a) 2 (b) 3 (c) $\frac{10}{3}$ (d) $\frac{20}{3}$

Solution: (c) The area of triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ is given by $\left|\frac{n^2\sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2}\right|$

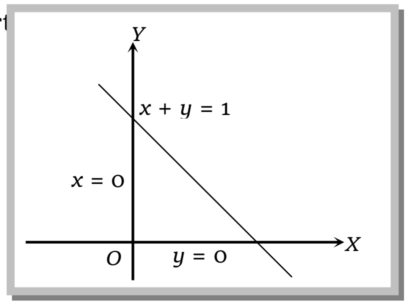
Here $a = 4, b = -9, h = -\frac{9}{2}, l = 1, m = 0, n = -2$, then area of triangle

$$= \left| \frac{(-2)^2 \sqrt{\left(\frac{-9}{2}\right)^2 - 4 \times \frac{-9}{2}}}{-9 \times (1)^2} \right| = \left| \frac{4\sqrt{\frac{81}{4} + \frac{36}{2}}}{-9} \right| = \left| \frac{-30}{9} \right| = \frac{10}{3}$$

Example: 18 The orthocentre of the triangle formed by the lines $xy = 0$ and $x + y = 1$ is **[IIT 1995]**

(a) (0, 0) (b) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (c) $\left(\frac{1}{3}, \frac{1}{3}\right)$ (d) $\left(\frac{1}{4}, \frac{1}{4}\right)$

Solution: (a) Lines represented by $xy = 0$ is $x = 0$, $y = 0$. Then the triangle formed is right angled triangle at $O(0, 0)$, therefore $O(0, 0)$ is its ort



Example: 19 If the pair of straight lines given by $Ax^2 + 2Hxy + By^2 = 0, (H^2 > AB)$ forms an equilateral triangle with line $ax + by + c = 0$ then $(A + 3B)(3A + B)$ is **[EAMCET 2003]**

(a) H^2 (b) $-H$ (c) $2H^2$ (d) $4H^2$

Solution: (d) We know that the pair of lines $(a^2 - 3b^2)x^2 + 8abxy + (b^2 - 3a^2)y^2 = 0$ with the line $ax + by + c = 0$ form an equilateral triangle. Hence comparing with $Ax^2 + 2Hxy + By^2 = 0$ then $A = a^2 - 3b^2$, $B = b^2 - 3a^2$, $2H = 8ab$

Now $(A + 3B)(3A + B) = (-8a^2)(-8b^2) \Rightarrow (8ab)^2 = (2H)^2 = 4H^2$.

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