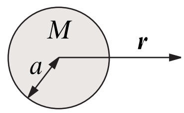
3.3 Gravitation

Newtonian gravitation

Newton's law of gravitation	$\boldsymbol{F}_1 = \frac{Gm_1m_2}{r_{12}^2}\hat{\boldsymbol{r}}_{12}$	(3.40)
Newtonian field equations[a]	$\boldsymbol{g} = -\nabla\phi$	(3.41)
	$\nabla^2\phi = -\nabla\cdot\boldsymbol{g} = 4\pi G\rho$	(3.42)
Fields from an isolated uniform sphere, mass M, $\boldsymbol{r}$ from the centre	$\boldsymbol{g}(\boldsymbol{r}) = \begin{cases} -\frac{GM}{r^2}\hat{\boldsymbol{r}} & (r > a) \\ -\frac{GMr}{a^3}\hat{\boldsymbol{r}} & (r < a) \end{cases}$	(3.43)
	$\phi(\boldsymbol{r}) = \begin{cases} -\frac{GM}{r} & (r > a) \\ \frac{GM}{2a^3}(r^2 - 3a^2) & (r < a) \end{cases}$	(3.44)

$m_{1,2}$	masses
$\boldsymbol{F}_1$	force on m_1 $(= -\boldsymbol{F}_2)$
$\boldsymbol{r}_{12}$	vector from m_1 to m_2
$\hat{}$	unit vector
G	constant of gravitation
$\boldsymbol{g}$	gravitational field strength
ϕ	gravitational potential
ρ	mass density
$\boldsymbol{r}$	vector from sphere centre
M	mass of sphere
a	radius of sphere



[a]The gravitational force on a mass m is $m\boldsymbol{g}$.

General relativity[a]

Quantity	Equation	No.	Symbol	Meaning
Line element	$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}\tau^2$	(3.45)	$\mathrm{d}s$	invariant interval
			$\mathrm{d}\tau$	proper time interval
			$g_{\mu\nu}$	metric tensor
			$\mathrm{d}x^\mu$	differential of x^μ
Christoffel symbols and covariant differentiation	$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}(g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta})$	(3.46)	$\Gamma^\alpha{}_{\beta\gamma}$	Christoffel symbols
	$\phi_{;\gamma} = \phi_{,\gamma} \equiv \partial\phi/\partial x^\gamma$	(3.47)	$,\alpha$	partial diff. w.r.t. x^α
	$A^\alpha_{;\gamma} = A^\alpha_{,\gamma} + \Gamma^\alpha{}_{\beta\gamma} A^\beta$	(3.48)	$;\alpha$	covariant diff. w.r.t. x^α
	$B_{\alpha;\gamma} = B_{\alpha,\gamma} - \Gamma^\beta{}_{\alpha\gamma} B_\beta$	(3.49)	ϕ	scalar
			A^α	contravariant vector
			B_α	covariant vector
Riemann tensor	$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma} + \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta}$	(3.50)	$R^\alpha{}_{\beta\gamma\delta}$	Riemann tensor
	$B_{\mu;\alpha;\beta} - B_{\mu;\beta;\alpha} = R^\gamma{}_{\mu\alpha\beta} B_\gamma$	(3.51)		
	$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}\ ; \quad R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\gamma\delta}$	(3.52)		
	$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$	(3.53)		
Geodesic equation	$\frac{\mathrm{D}v^\mu}{\mathrm{D}\lambda} = 0$	(3.54)	v^μ	tangent vector $(= \mathrm{d}x^\mu/\mathrm{d}\lambda)$
	where $\frac{\mathrm{D}A^\mu}{\mathrm{D}\lambda} \equiv \frac{\mathrm{d}A^\mu}{\mathrm{d}\lambda} + \Gamma^\mu{}_{\alpha\beta} A^\alpha v^\beta$	(3.55)	λ	affine parameter (e.g., τ for material particles)
Geodesic deviation	$\frac{\mathrm{D}^2\xi^\mu}{\mathrm{D}\lambda^2} = -R^\mu{}_{\alpha\beta\gamma} v^\alpha \xi^\beta v^\gamma$	(3.56)	ξ^μ	geodesic deviation
Ricci tensor	$R_{\alpha\beta} \equiv R^\sigma{}_{\alpha\sigma\beta} = g^{\sigma\delta} R_{\delta\alpha\sigma\beta} = R_{\beta\alpha}$	(3.57)	$R_{\alpha\beta}$	Ricci tensor
Einstein tensor	$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$	(3.58)	$G^{\mu\nu}$	Einstein tensor
			R	Ricci scalar $(= g^{\mu\nu} R_{\mu\nu})$
Einstein's field equations	$G^{\mu\nu} = 8\pi T^{\mu\nu}$	(3.59)	$T^{\mu\nu}$	stress-energy tensor
			p	pressure (in rest frame)
Perfect fluid	$T^{\mu\nu} = (p+\rho) u^\mu u^\nu + p g^{\mu\nu}$	(3.60)	ρ	density (in rest frame)
			u^ν	fluid four-velocity
Schwarzschild solution (exterior)	$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$	(3.61)	M	spherically symmetric mass (see page 183)
			(r,θ,ϕ)	spherical polar coords.
			t	time
Kerr solution (outside a spinning black hole)	$\mathrm{d}s^2 = -\frac{\Delta - a^2\sin^2\theta}{\varrho^2}\mathrm{d}t^2 - 2a\frac{2Mr\sin^2\theta}{\varrho^2}\mathrm{d}t\,\mathrm{d}\phi + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\varrho^2}\sin^2\theta\,\mathrm{d}\phi^2 + \frac{\varrho^2}{\Delta}\mathrm{d}r^2 + \varrho^2\,\mathrm{d}\theta^2$	(3.62)	J	angular momentum (along z)
			a	$\equiv J/M$
			Δ	$\equiv r^2 - 2Mr + a^2$
			ϱ^2	$\equiv r^2 + a^2\cos^2\theta$

[a]General relativity conventionally uses the $(-1,1,1,1)$ metric signature and "geometrized units" in which $G = 1$ and $c = 1$. Thus, $1\,\mathrm{kg} = 7.425 \times 10^{-28}\,\mathrm{m}$ etc. Contravariant indices are written as superscripts and covariant indices as subscripts. Note also that $\mathrm{d}s^2$ means $(\mathrm{d}s)^2$ etc.