Long Answer Questions-I-D (PYQ)

[4 Marks]

Q.1. Verify Lagrange's mean value theorem for the following function:

$f(x) = x^2 + 2x + 3$, for [4, 6].

Ans.

$f(x) = x^2 + 2x + 3$ for $[4, 6]$

i. Given function is a polynomial hence it is continuous.

ii. $f'(x) = 2x + 2$ which is differentiable.

$f(4) = 16 + 8 + 3 = 27$ and $f(6) = 36 + 12 + 3 = 51$

$\Rightarrow$ $f(4) \neq f(6)$. All conditions of mean value theorem are satisfied.

$\therefore$ There exist at least one real value $c \in (4, 6)$

such that $f'(c) = \frac{f(6)-f(4)}{6-4} = \frac{24}{2} = 12$

$\Rightarrow$ $2c + 2 = 12$ or $c = 5 \in (4, 6)$

Hence, Lagrange' mean value theorem is verified.

Q.2. Verify Mean Value theorem for the function $f(x) = 2 \sin x + \sin 2x$ on [0, π].

Ans.

We have,

$f(x) = 2 \sin x + \sin 2x$

$f(x)$ is continuous in [0, π] being trigonometric function.

Also $f(x)$ is differentiable on (0, π).

Hence, condition of Mean Value theorem is satisfied.

Therefore, mean value theorem is applicable.

So, ∃ a real number *c* such that

$$f'(c) = \frac{f(\pi)-f(0)}{\pi-0} \qquad \ldots(i)$$

Now $f(0) = 2 \sin 0 + \sin 0 = 0$

$f(p) = 2 \sin \pi + \sin 2\pi = 0$

and $f'(x) = 2 \cos x + 2 \cos 2x$

∴ $f'(c) = 2 \cos c + 2 \cos 2c$

From (*i*)

$$2 \cos c + 2 \cos 2c = \frac{0-0}{\pi}$$

⇒ $2 \cos c + 2 \cos 2c = 0$

⇒ $2 \cos c + 2(2 \cos^2 c - 1) = 0$

⇒ $\cos c + 2 \cos^2 c - 1 = 0$

⇒ $2 \cos^2 c + \cos c - 1 = 0$

⇒ $2 \cos^2 c + 2 \cos c - \cos c - 1 = 0$

⇒ $2 \cos c (\cos c + 1) - 1 (\cos c + 1) = 0$

$\Rightarrow \quad (\cos c + 1)(2\cos c - 1) = 0$

$\Rightarrow \quad \cos c = -1$ and $\cos c = \frac{1}{2}$

$\Rightarrow \quad c = \pi$ and $c = \frac{\pi}{3}$

$\because \quad c = \frac{\pi}{3} \in (0, \pi)$

Hence Mean Value theorem is verified.

Long Answer Questions-I-D (OIQ)

[4 Marks]

Q.1. Verify Lagrange's mean value theorem for the function $f(x) = x + \frac{1}{2}$ in $[1, 3]$.

Ans.

Given, $f(x) = x + \frac{1}{x}$ or $f(x) = \frac{x^2+1}{x}$

i. Since $f(x)$ is a rational function such that the denominator is not zero for any value in $[1, 3]$, it is a continuous function.

ii. $f'(x) = 1 - \frac{1}{x^2}$ which exist in $(1, 3)$ $\quad \therefore \quad f(x)$ is differentiable in $(1, 3)$

Thus, all the conditions of Lagrange's Mean Value theorem are satisfied. Hence, there exist at least one real value c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad ...(i)$$

where $f'(c) = 1 - \frac{1}{c^2}$; $f(b) = f(3) = \frac{10}{3}$ and $f(a) = f(1) = 2$ $\qquad ...(ii)$

From (i) and (ii), we get

$$\Rightarrow \quad 1-\frac{1}{c^2}=\frac{\frac{10}{3}-2}{3-1}$$

$$\Rightarrow \quad \frac{c^2-1}{c^2}=\frac{2}{3}$$

$$\Rightarrow \quad 3c^2-3=2c^2$$

$$\Rightarrow \quad c^2=3$$

$$\Rightarrow \quad c=\pm\sqrt{3}$$

Neglecting $c=-\sqrt{3}$ as $-\sqrt{3}\notin(1,3)$ $\therefore$ $c=\sqrt{3}\in(1,3)$

Hence, Lagrange's mean value theorem is verified.

Q.2. Using Rolle's theorem, find the points on the curve $y = x^2$, where $x \in [-2, 2]$ and the tangent is parallel to x-axis.

Ans.

$f(x) = x^2$

i. $f(x)$ is a polynomial, hence continuous in $[-2, 2]$

ii. $f'(x) = 2x$ which exist in $[-2, 2]$

$\therefore$ $f(x)$ is differentiable in $[-2, 2]$

iii. $f(-2) = (-2)^2 = 4$

$f(2) = (2)^2 = 4$

$\therefore \quad f(2) = f(-2)$

Thus, all the conditions of Rolle's theorem are applicable, then there exist at least one real value c, such that

$f'(c) = 0$

$\Rightarrow \quad 2c = 0$

$\Rightarrow \quad c = 0$

when $x = 0, y = (0)^2 = 0$

$\therefore \quad (0, 0)$ is the required point.