Mathematics & Statistics

Academic Year: 2016-2017 Marks: 70
Date & Time: 6th March 2017, 11:00 am
Duration: 3h

Question 1: [12]

Question 1: Select and write the appropriate answer from the given alternatives in each of the following sub-questions: [6]

Question 1.1.1: If the points A(2, 1, 1), B(0, -1, 4) and C(k, 3, -2) are collinear, then k [2]
(A) 0
(B) 1
(C) 4
(D) -4

Solution: (c) 4
solution:
Direction Ratio of bar(AB)=(-2, -2, 3)
Direction Ratio of bar(BC)=(k, 4, -6)

If point A,B and C are collinear then $\frac{\text{DR of } \overline{AB}}{\text{DR of } \overline{BC}}$=constatnt

$$-\frac{2}{k}=-\frac{2}{4}$$

$$k=4$$

Question 1.1.2: [2]

The inverse of the matrix $\begin{bmatrix}-1 & 5\\-3 & 2\end{bmatrix}$ is ______

1/13 $\begin{bmatrix}2 & -5\\3 & -1\end{bmatrix}$

1/13 $\begin{bmatrix}-1 & 5\\-3 & 2\end{bmatrix}$

1/13 $\begin{bmatrix}-1 & -3\\5 & 2\end{bmatrix}$

1/13 $\begin{bmatrix}1 & 5\\3 & -2\end{bmatrix}$

Solution:

(A) 1/13 $\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

|A| = - 2 + 15 = 13

$$A^{-1} = \frac{\text{adj}A}{|A|} = \frac{\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}}{13}$$

$$A^{-1} = \frac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$

Question 1.1.3: In Δ ABC, if a = 13, b = 14 and c = 15, then sin (A/2) = _______. **[2]**

(A) $\frac{1}{5}$

(B) $\sqrt{\frac{1}{5}}$

(C) $\frac{4}{5}$

(D) $\frac{2}{5}$

Solution:

(B) $\sqrt{\frac{1}{5}}$

$$s = \frac{a+b+c}{2} = \frac{13+14+15}{2} = 21$$

$$\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sqrt{\frac{(21-14)(21-15)}{14 \times 15}} = \sqrt{\frac{1}{5}}$$

Question 1.2: Attempt any THREE of the following: **[12]**

Question 1.2.1: Find the volume of the parallelepiped whose coterminous edges are given by vectors **[2]**

$2\hat{i} + 3\hat{j} - 4\hat{k}, 5\hat{i} + 7\hat{j} + 5\hat{k}$ and $4\hat{i} + 5\hat{j} - 2\hat{k}$

Solution: If $\vec{a}\ \vec{b}$ and $\vec{c}$ are conterminous edges of parallelepiped then the volume of the parallelepiped = $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

where $\vec{a} = 2\hat{i} + 3\hat{j} - 4\hat{k}$

$\vec{b} = 5\hat{i} + 7\hat{j} + 5\hat{k}$

$\vec{c} = 4\hat{i} + 5\hat{j} - 2\hat{k}$

$\therefore V = \left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{bmatrix} 2 & 3 & -4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}$

$= 2(-14 - 25) - 3(-10 - 20) - 4(25 - 28)$

$= 2(-39) - 3(-30) - 4(-3)$

$= -78 + 90 + 12$

$= 24\text{cube Unit}$

Question 1.2.2: In Δ ABC, prove that, a (b cos C - c cos B) = b^2 - c^2. **[2]**

Solution:

Taking LHS

$a(b\cos C - c\cos B)$

$= ab\cos C - ac\cos B$

$= ab\left(\frac{a^2 + b^2 - c^2}{2ab}\right) - ac\left(\frac{a^2 + c^2 - b^2}{2ac}\right)$

$\frac{a^2 + b^2 - c^2 - a^2 - c^2 + b^2}{2}$

$= \frac{2b^2 - 2c^2}{2}$

$= b^2 - c^2(\text{RHS})$

Question 1.2.3: If from a point Q (a, b, c) perpendiculars QA and QB are drawn to the YZ and ZX planes respectively, then find the vector equation of the plane QAB. **[2]**

Solution: QA and QB are the perpendiculars from the point Q(a, b, c) to YZ and ZX planes.

∴ A = (0, b, c) and B = (a, 0, c)

The required plane is passing through O(0, 0, 0), A(0, b, c) and B(a, 0, c)

The vector equation of the plane passing through the O,A,B is

$\bar{r}.\left(\overline{OA} \times \overline{OB}\right) = \bar{0}.\left(\overline{OA} \times \overline{OB}\right)$

i.e; $\bar{r}.\left(\vec{a} \times \vec{b}\right) = 0$

Now, $\overline{OA} = \bar{a} = 0.\,\hat{i} + b\hat{j} + c\hat{k}$

and $\overline{OB} = \bar{b} = a\hat{i} + 0.\,\hat{j} + c\hat{k}$

$$\therefore \overline{OA} \times \overline{OB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & b & c \\ a & 0 & c \end{vmatrix}$$

$= (bc - 0)\hat{i} - (0 - ac)\hat{j} + (0 - ab)\hat{k}$

$= bc\hat{i} + ac\hat{j} - ab\hat{k}$

$\therefore$ from (1), the vector equation of the required plane is

$$\bar{r}.\left(bc\hat{i} + ac\hat{j} - ab\hat{k}\right) = 0$$

Question 1.2.4: Find the cartesian equation of the line passing through the points A(3, 4, -7) and B(6,-1, 1). **[2]**

Solution:

Equation of line passing through the point $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Equation of line passing through the point A(3, 4,-7) and B(6,-1,1) is

$$\frac{x-3}{6-3} = \frac{y-4}{-1-4} = \frac{z-(-7)}{1-(-7)}$$

$$\frac{x-3}{3} = \frac{y-4}{-5} = \frac{z+7}{8}$$

Question 1.2.5: Write the following statement in symbolic form and find its truth value: ∀ n ∈ N, n^2 + n is an even number and n^2 - n is an odd number. **[2]**

Solution: Let p ≡ ∀ n ∈ N, n^2 + n is an even number

Let q ≡ ∀ n ∈ N, n^2 - n is an odd number

The symbolic form of given statement is
(p ∧ q)

Truth value of given statement is

p ≡ ∀ n ∈ N, n^2 + n is an even number (T)

q ≡ ∀ n ∈ N, n^2 - n is an odd number (F)

(∵ from n = 1, n^2 - n = 0, which is not an odd number)

∴ (p ∧ q) ≡ T ∧ F ≡ F

∴ the given statement is false

Question 2: [14]

Question 2.1 | Attempt any TWO of the following: [6]

Question 2.1.1: Using truth tables, examine whether the statement pattern $(p \wedge q) \vee (p \wedge r)$ is a tautology, contradiction or contingency. [3]

Solution: No of rows = $2^n = 2^3 = 8$
No. of columns = m+ n=3+3= 6

p	q	r	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

In the last column, the truth values of the statement is neither all T nor all F. Hence, it is neither a tautology nor a contradiction i.e. it is a contingency.

Question 2.1.2: [3]

Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$

Solution:

The lines are

$\frac{X-1}{2} = \frac{Y-2}{3} = \frac{Z-3}{4}$ ....(1)

$\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$ ....(2)

Here

$$x_1 = 1, y_1 = 2, z_1 = 3 \text{ and } a_1 = 2, b_1 = 3, c_1 = 4$$

$$x_2 = 2, y_2 = 4, z_2 = 5 \text{ and } a_2 = 3, b_2 = 4, c_2 = 5$$

Shortest distance between the lines is

$$d=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

Now $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=\begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

$=1(15-16)-2(10-12)+2(8-9)=-1+4-2=1$

and $(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2=(15-16)^2+(12-10)^2+(8-9)^2$

$=1+4+1=6$

Hence, the shortest distance between the lines (i) and (ii) is $=\left|\frac{1}{\sqrt{6}}\right|=\frac{1}{\sqrt{6}}\,units$

Question 2.1.3: Find the general solution of the equation sin 2x + sin 4x + sin 6x = 0 `[3]`

Solution:

$(\sin 2x+\sin 6x)+\sin 4x=0$

$2\sin 4x.\cos 2x+\sin 4x=0$

$\sin 4x(2\cos 2x+1)=0$

$\sin 4x=0 \text{ or } 2\cos 2x+1=0$

$\sin 4x=0 \text{ or } \cos 2x=-\frac{1}{2}=-\cos\frac{\pi}{3}=\cos\left(\pi-\frac{\pi}{3}\right)$

Using $\sin x=0 \Rightarrow x=n\pi$

$\sin 4x=0$

$4x=n\pi$

The genral solution is x

$x=\frac{n\pi}{4}$

using $\cos x=\cos\alpha \Rightarrow x=2mx\pm\alpha$

$\cos 2x=\cos\left(\frac{2\pi}{3}\right)$

$2x=2m\pi\pm\frac{2\pi}{3}$

The genral solution is x

$x = m\pi \pm \frac{\pi}{3}$ where $m, n \in z$

Question 2.2: Attempt any TWO of the following: [8]

Question 2.2.1: Solve the following equations by method of reduction: [4]
x-y + z = 4,
2x + y - 3z = 0,
x + y + z = 2

Solution: x-y + z = 4,

2x + y - 3z = 0,

x + y + z = 2

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$R_2 - 2R_1$ and $R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -5 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ -2 \end{bmatrix}$$

$x - y + z = 4$ ....(1)

$3y - 5z = -8$ ....(2)

$2y = -2$ ....(3)

y=-1

By equation (2)

$-3 - 5z = -8$
$-5z = -5$
$z = 1$
By equation (1)

$x + 1 + 1 = 4$
$x = 2$
Ans : $x = 2, y = -1, z = 1$

Question 2.2.2: [4]

If θ is the measure of acute angle between the pair of lines given by $ax^2 + 2hxy + by^2 = 0$, then prove that $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|, a+b \neq 0$

Solution:

Let m_1 and m_2 be the slopes of the lines represented by the equation

$ax^2 + 2hxy + by^2 = 0$ ....(1)

Then their separate equation are

$y = m_1 x$ and $y = m_2 x$

therefore their combined equation is

$(y - m_1 x)(y - m_2 x) = 0$

i.e $m_1 m_2 x^2 - (m_1 + m_2)xy + y^2 = 0$ ....(2)

Since (1) and (2) represent the same two lines, comparing the coefficients, we get

$$\frac{m_1 m_2}{a} = \frac{1}{b} = \frac{m_1 + m_2}{2h}$$

$$\therefore m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1 m_2 = \frac{a}{b}$$

Thus $(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2$

$$(m_1 - m_2)^2 = \left(-\frac{2h}{b}\right)^2 - 4\left(\frac{a}{b}\right)$$

$$= (m_1 - m_2)^2 = \frac{4(h^2 - ab)}{b^2}$$

Let the angle between y = m_1x and y =m_2x be θ .

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|,$$

$$= \left|\frac{\sqrt{m_1 - m_2}^2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{\frac{4\sqrt{h^2 - ab}}{b^2}}{1 + \frac{a}{b}}\right|,$$

$$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|, \quad \text{if} \quad a + b \neq 0$$

Hence to proved.

Question 2.2.3: Using vector method, find in centre of the triangle whose vertices are P(0, 4, 0), Q(0, 0, 3) and R(0, 4, 3). **[4]**

Solution:

Let $\bar{p}, \bar{q}, \bar{r}$ be the position vectors of vertices P, Q, R of Δ PQR respectively

$$\bar{p} = 4\hat{j}, \bar{q} = 3\hat{k}, \bar{r} = 4\hat{j} + 3\hat{k}$$

$$\overline{PQ} = \bar{q} - \bar{p} = 3\hat{k} - 4\hat{j} = -4\hat{j} + 3\hat{k}$$

$$\overline{QR} = \bar{r} - \bar{q} = 4\hat{j} + 3\hat{k} - 3\hat{k} = 4\hat{j}$$

$$\overline{RP} = \bar{p} - \bar{r} = 4\hat{j} - 4\hat{j} - 3\hat{k} = -3\hat{k}$$

Let x, y, z be the lengths of opposites of vertices P,Q,R respectively.

$$x = \left|\overline{QR}\right| = 4$$

$$y = \left|\overline{RP}\right| = 3$$

$$z = \left|\overline{PQ}\right| = \sqrt{16 + 9} = \sqrt{25} = 5$$

If H($\bar{h}$)is the incentre of ΔPQR then

$$\bar{h}=\frac{x\bar{p}+y\bar{q}+z\bar{r}}{x+y+z}$$

$$=\frac{4\left(4\hat{j}\right)+3\left(3\hat{k}\right)+5\left(4\hat{j}+3\hat{k}\right)}{4+3+5}$$

$$=\frac{16\hat{j}+9\hat{k}+20\hat{j}+15\hat{k}}{12}$$

$$=\frac{36\hat{j}+24\hat{k}}{12}=3\hat{j}+2\hat{k}$$

Question 3: [14]

Question 3.1 | Attempt any TWO of the following: [6]

Question 3.1.1: Construct the switching circuit for the statement (p ∧ q) ∨ (~ p) ∨ (p ∧ ~ q). [3]

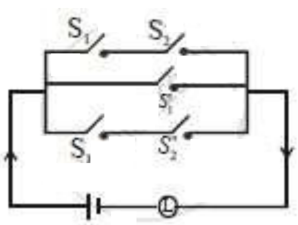
Solution:

Let

p = Switch S_1 is closed

q = Switch S_2 is closed

$\sim p = switch \quad S_1'$ and $\sim q \equiv S_2'$



Question 3.1.2: Find the joint equation of the pair of lines passing through the origin which are perpendicular respectively to the lines represented by $5x^2 + 2xy - 3y^2 = 0$. [3]

Solution 1:

Comparing the equation $5x^2+2xy-3y^2=0$, we get,

$a=5, 2h=+2, b=-3$

Let m_1 and m_2 be the slopes of the lines represented by $5x^2+2xy-3y^2=0$

$$m_1 + m_2 = -\frac{2h}{b} = -\frac{2}{-3} = \frac{2}{3} \qquad ......(1)$$

$$m_1 m_2 = \frac{a}{b} = \frac{5}{-3}$$

Now required lines are perpendicular to these lines

their slopes are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$

Since these lines are passing through the origin, their separate equations are

$$y = -\frac{1}{m_1}x \text{ and } y = -\frac{1}{m_2}x$$

$$\therefore m_1 y = -x \text{ and } m_2 y = -x$$

$$x + m_1 y = 0 \text{ and } x + m_2 y = 0$$

their combined equation is

$$(x + m_1 y)(x + m_2 y) = 0$$

$$x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 = 0$$

$$x^2 + \frac{2}{3}xy + \frac{-5}{3}y^2 = 0$$

$$3x^2 + 2xy - 5y^2 = 0$$

Solution 2: Given homogeneous equation is

$5x^2 + 2xy - 3y^2 = 0$

Which is factorisable

$5x^2 + 5xy - 3xy - 3y^2 = 0$

$5x(x + y) - 3y(x + y) = 0$

$(x + y)(5x - 3y) = 0$

$\therefore$ $x + y = 0$ and $5x - 3y = 0$ are the two lines represented by the given equation

$\Rightarrow$Their slopes are -1 and 5/3

Required two lines are respectively perpendicular to these lines.

$\therefore$ Slopes of required lines are 1 and 3/5 and the lines pass thought origin

$\therefore$ Their individual equations are y = 1.x and $y = -\frac{3}{5}x$

i.e x - y = 0 and 3x + 5y = 0

∴ Their joint equation is

(x - y) (3x + 5y) = 0

$3x^2 - 3xy + 5xy - 5y^2 = 0$

$3x^2 + 2xy - 5y^2 = 0$

Question 3.1.3: [3]

Show that:

$\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$

Solution:

Let a = $\cos^{-1}\left(\frac{4}{5}\right)$ and b = $\cos^{-1}\left(\frac{12}{13}\right)$

Let a = $\cos^{-1}\left(\frac{4}{5}\right)$

cos a = $\frac{4}{5}$

We know that

$\sin^2 a = 1 - \cos^2 a$

sin a = $\sqrt{1 - \cos^2 a}$

$= \sqrt{1 - \left(\frac{4}{5}\right)^2} = \sqrt{1 - \frac{16}{25}}$

$= \sqrt{\frac{25 - 16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

Let b = $\cos^{-1}\left(\frac{12}{13}\right)$

cos b = $\frac{12}{13}$

W know that

$\sin^2 b$ = 1 - $\cos^2 b$

sin b = $\sqrt{1-\cos^2 b}$

$= \sqrt{1-\left(\frac{12}{13}\right)^2} = \sqrt{1-\frac{144}{169}}$

$= \sqrt{\frac{169-144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$

We know that

cos (a+b) = cos a cos b - sin a sin b

Putting values

cos a = $\frac{4}{5}$, sin a = $\frac{3}{5}$

& cos b = $\frac{12}{13}$, sin b = $\frac{5}{13}$

cos (a+b) = $\frac{4}{5} \times \frac{12}{13} \times \frac{3}{5} \times \frac{5}{13}$

$= \frac{48}{65} - \frac{3}{13}$

$= \frac{48-15}{65}$

$= \frac{33}{65}$

$\therefore \cos(a+b) = \frac{33}{65}$

$a + b = \cos^{-1}\left(\frac{33}{65}\right)$

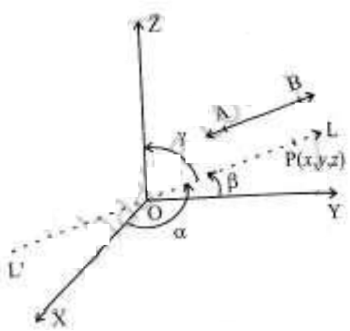
$\cos^{-1}\frac{4}{5} + \cos^{-1}\left(\frac{12}{15}\right) = \cos^{-1}\left(\frac{33}{65}\right)$

Hence LH.S = R.H.S

Question 3.2: Attempt any TWO of the following **[8]**

Question 3.2.1: If l, m, n are the direction cosines of a line, then prove that $l^2 + m^2 + n^2 = 1$. Hence find the direction angle of the line with the X axis which makes direction angles of 135° and 45° with Y and Z axes respectively. **[4]**

Solution: Let α, β, γ be the angles made by the line with X-, Y-, Z- axes respectively.



$l = \cos\alpha, m = \cos\beta$ and $n = \cos\gamma$

Let $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ be any non-zero vector along the line.

Since $\hat{i}$ is the unit vector along X-axis,

$\bar{a}.\hat{i} = |\bar{a}|.\left|\hat{i}\right|\cos\alpha = a\cos\alpha$

Also, $\bar{a}.\hat{i} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right).\hat{i}$

$= a_1 \times 1 + a_2 \times 0 + a_3 \times 0 = a_1$

$a\cos\alpha = a_1$ ....(1)

Since $\hat{j}$ is the unit vector along Y-axis,

$\bar{a}.\hat{j} = |\bar{a}|.\left|\hat{j}\right|\cos\beta = a\cos\beta$

$\bar{a}.\hat{j} = |\bar{a}|.\left|\hat{j}\right| \cos\beta = a\cos\beta$

$\bar{a}.\hat{j} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right).\hat{j}$

$= a_1 \times 0 + a_2 \times 1 + a_3 \times 0 = a_2$

$a\cos\beta = a_2$ ....(2)

similarly $a\cos\gamma = a_3$ ....(3)

from equations (1), (2) and (3),

$a^2\cos^2\alpha + a^2\cos^2\beta + a^2\cos^2\gamma = a_1^2 + a_2^2 + a_3^2$

$a^2\left(\cos^2\alpha + \cos^2\beta + \cos^2\gamma\right) = a^2 \qquad \left[a = |\bar{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}\right]$

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$i.el^2 + m^2 + n^2 = 1$

also

$\alpha = ?, \beta = 135^\circ, \gamma = 45^\circ$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 45^\circ$

$\cos^2\alpha + \cos^2 135^\circ + \cos^2 45^\circ = 1$

$\cos^2\alpha + \frac{1}{2} + \frac{1}{2} = 1$

$\cos^\alpha = 0$

$\therefore \alpha = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$

Question 3.2.2: Find the vector and cartesian equations of the plane passing through the points A(1, 1, -2), B(1, 2, 1) and C(2, -1, 1). **[4]**

Solution:

The vector equation of the plane passing through the points $A(\bar{a}), B(\bar{b})$ and $C(\bar{c})$

$\bar{r}.\left(\overline{AB} \times \overline{AC}\right) = \bar{a}\left(\overline{AB} \times \overline{AC}\right)$............(1)

Let $\bar{a} = \hat{i} + \hat{j} - 2\hat{k}, \bar{b} = \hat{i} + 2\hat{j} + \hat{k}, \bar{c} = 2\hat{i} - \hat{j} + \hat{k}$

$\therefore \overline{AB} = \bar{b} - \bar{a} = \left(\hat{i} + 2\hat{j} + \hat{k}\right) - \left(\hat{i} + \hat{j} - 2\hat{k}\right) = \hat{j} + 3\hat{k}$

and $\overline{AC} = \bar{c} - \bar{a} = \left(2\hat{i} - \hat{j} + \hat{k}\right) - \left(\hat{i} - \hat{j} - 2\hat{k}\right) = \hat{i} - 2\hat{j} + 3\hat{k}$

$$\therefore \overline{AB} \times \overline{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 3 \\ 1 & -2 & 3 \end{vmatrix}$$

$= (3+6)\hat{i} - (0-3)\hat{j} + (0-1)\hat{k}$

$= 9\hat{i} + 3\hat{j} - \hat{k}$

$\bar{a}.\left(\overline{AB} \times \overline{AC}\right) = \left(\hat{i} + \hat{j} - 2\hat{k}\right).\left(9\hat{i} + 3\hat{j} - \hat{k}\right)$

$= 1(9) + 1(3) + (-2)(-1)$

$= 9 + 3 + 2 = 14$

from (1), the vector equation of the required plane is

$\bar{r}.\left(9\hat{i} + 3\hat{j} - \hat{k}\right) = 14$

$\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\left(9\hat{i} + 3\hat{j} - \hat{k}\right) = 14$

$\therefore$ the cartesian equation of the plane is

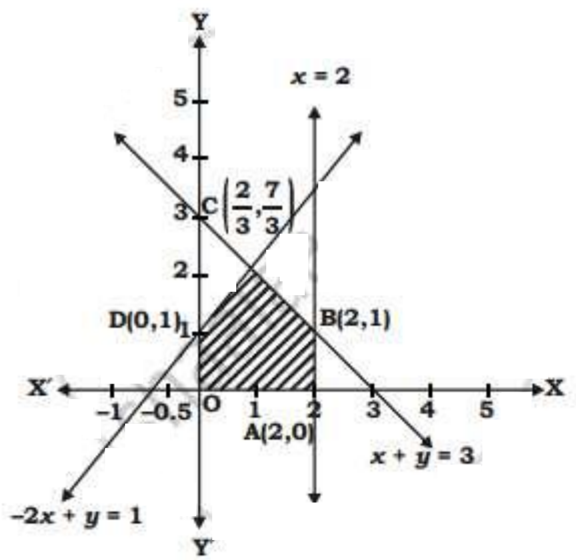
$9x + 3y - z = 14$

Question 3.2.3: Solve the following LPP by using graphical method. **[4]**
Maximize : Z = 6x + 4y
Subject to x ≤ 2, x + y ≤ 3, -2x + y ≤ 1, x ≥ 0, y ≥ 0.
Also find maximum value of Z.

Solution:

Inequalities	x ≤ 2	x+y≤3	-2x+y≤1
Equalities	x=2	x+y=3	-2x+y=1
Intercept form	$\frac{x}{2} = 1$	$\frac{x}{3} + \frac{y}{3} = 1$	$\frac{x}{-\frac{1}{2}} + \frac{y}{1} = 1$
Origin Test	0≤2	0+0≤3	-2(0)+0≤1

	True Origin Side	True Origin Side	True Origin Side



Shaded portion OABC is the feasible region, Where O(0,0) A(2, 0) D(0, 1), B(2, 1)

For C :

x + y = 3

− 2x + y = 1

− − −

3x = 2

∴ x = 2/3

2/3+y=3 i.e y=7/3

$\therefore c\left(\frac{2}{3},\frac{7}{3}\right)$

Z = 6x + 4y

Z at O(0, 0) = 6(0) + 4(0) = 0

Z at A(2, 0) = 6(2) + 4(0) = 12

Z at B(2, 1) = 6(2) + 4(1) = 16

Z at $c\left(\frac{2}{3},\frac{7}{3}\right)=6\left(\frac{2}{3}\right)+\left(\frac{7}{3}\right)4=\frac{40}{3}$

Z at D(0,1) = 6(0) + 4(1) = 4 Thus, Z is maximized at B(2, 1) and its maximum value is 16.

Question 4: [12]

Question 4.1 | Select and write the appropriate answer from the given alternatives in each of the following sub-questions: [6]

Question 4.1.1: Derivatives of $\tan^3\theta$ with respect to $\sec^3\theta$ at $\theta=\pi/3$ is [2]

(A) $\frac{3}{2}$

(B) $\frac{\sqrt{3}}{2}$

(C) $\frac{1}{2}$

(D) $-\frac{\sqrt{3}}{2}$

Solution:

(B) $\frac{\sqrt{3}}{2}$

Let $y = \tan^3\theta, \text{ and } x = \sec^3\theta$

$$\frac{dy}{d\theta} = 3\tan^2\theta.\sec^2\theta, \frac{dx}{d\theta} = 3\sec^2\theta.\sec\theta\tan\theta$$

$$\frac{dy}{dx} = \sin\theta = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

Question 4.1.2: The equation of tangent to the curve $y = 3x^2 - x + 1$ at the point (1, 3) is [2]
(a) y=5x+2
(b)y=5x-2
(c)y=1/5x+2
(d)y=1/5x-2

Solution: y = 5x – 2

$$\frac{dy}{dx} = 6x - 1 \text{ at } (1,3)$$

Slope of the tangent at (1, 3) = (6 - 1) = 5

Equation of tangent is $y - y_1 = m(x - x_1)$

y - 3 = 5(x - 1)

5x - y - 2 = 0
y = 5x - 2

Question 4.1.3: The expected value of the number of heads obtained when three fair coins are tossed simultaneously is **[2]**
(A) 1
(B) 1.5
(C) 0
(D) -1

Solution: (B) 1.5

p=1/2

q=1/2 [∵q=1-(p)]

n=3

Expected value E(X) = np $=3 \times \frac{1}{2} = 1.5$

Question 4.2 | Attempt any THREE of the following: **[6]**

Question 4.2.1: Find dy/dx if x sin y + y sin x = 0. **[2]**

Solution: x sin y + y sin x = 0
Differentiate w.r.t. x both side

$$\left[x\cos y\frac{dy}{dx} + \sin y\right] + \left[y\cos x + \sin x\frac{dy}{dx}\right] = 0$$

$$\therefore \sin y + y\cos x = \frac{dy}{dx}(-\sin x - x\cos y)$$

$$\therefore \frac{dy}{dx} = -\left(\frac{\sin y + y\cos x}{\sin x + x\cos y}\right)$$

Question 4.2.2: Test whether the function is increasing or decreasing. **[2]**

f(x) = $\mathbf{x} - \frac{1}{\mathbf{x}}$, x ∈ R, x ≠ 0,

Solution:

f(x) $= \mathbf{x} - \frac{1}{\mathbf{x}}, \mathbf{x} \in \mathrm{R}$

$\therefore \mathbf{f}\prime(\mathbf{x}) = 1 - \left(-\frac{1}{\mathbf{x}^2}\right) = 1 + \frac{1}{\mathbf{x}^2}$

$\because \mathbf{x} \neq 0,$ for all values of x, $\mathbf{x}^2 > 0$

$\therefore \frac{1}{\mathbf{x}^2} > 0, 1 + \frac{1}{\mathbf{x}^2}$ is always positive

thus f'(x)>o , for all x ∈ R

Hence f(x) is increasing function.

Question 4.2.3: [2]

Evaluate: $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$

Solution:

$Let I = \int \left(\frac{\sin \sqrt{x}}{\sqrt{x}}\right) dx$

$Let \sqrt{x} = t$

$\frac{1}{\sqrt{x}} = \frac{dt}{dx}$

$\frac{1}{\sqrt{x}} dx = 2dt$

$\therefore I = 2 \int \sin t dt$

$= -2 \cos t + C$

$= -2 \cos(\sqrt{x}) + C$

Question 4.2.4: Form the differential equation by eliminating arbitrary constants from the relation $y = Ae^{5x} + Be^{-5x}$ [2]

Solution:

$y = Ae^{5x} + Be^{-5x}$

Differentitating w.r.t. x

$\frac{dy}{dx} = A.\,e^{5x}.(5) + Be^{-5x}(-5)$

$\therefore \frac{dy}{dx} = 5A.\,e^{5x} - 5Be^{-5x}$

Again differentitating w.r.t. x

$\frac{d^2y}{dx^2} = 5Ae^{5x}.(5) - 5Be^{-5x}.(-5)$

$\frac{d^2y}{dx^2} = 25Ae^{5x} + 25Be^{-5x}$

$\frac{d^2y}{dx^2} = 25\left(Ae^{5x} + Be^{-5x}\right)$

$\frac{d^2y}{dx^2} = 25y$

$\frac{d^2y}{dx^2} - 25y = 0$ is the required differential equation

Question 4.2.5: The probability that a bomb will hit a target is 0.8. Find the probability that out of 10 bombs dropped, exactly 4 will hit the target. [2]

Solution: Let r = no of bombs hit the target

p=0.8

q=0.2 (1-p=q)

n=10 r=4

$$p(r=4) = {}^{n}C_{r}p^{r}q^{n-r} \quad r=0,1,2\ldots\ldots\ldots,n$$

$$= {}^{10}C_{4}(0.8)^{4}(0.2)^{6}$$

$$= {}^{10}C_{4}\left(\frac{8}{10}\right)^{4}\left(\frac{2}{10}\right)^{6}$$

$$= \frac{10!}{4!6!} \times (2)^{18}\left(\frac{1}{10}\right)^{10}$$

$$= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2} \times (2)^{18} \times \left(\frac{1}{10}\right)^{10}$$

$$= 210 \times (2)^{18} \times \left(\frac{1}{10}\right)^{10}$$

$$= \frac{262144 \times 210}{(10)^{10}} = \frac{55050240}{(10)^{10}}$$

$$= Anti[\log 210 + 18\log 2 - 10]$$

$$= Anti[2.3222 + 18\log(0.3010) - 10]$$

$$= Anti(3.7402)$$

$$= 0.0055$$

Question 5: [14]

Question 5.1: Attempt any TWO of the following: [6]

Question 5.1.1: If u and v are two functions of x then prove that [3]

$$\int uvdx = u\int vdx - \int\left[d\frac{u}{dx}\int vdx\right]dx$$

Hence evaluate, $\int xe^{x}dx$

Solution:

Let $\int v dx = w \ldots\ldots (1)$

$then \quad \frac{dw}{dx} = v \ldots\ldots (2)$

$Now \frac{d}{dx}(u, w) = u.\frac{d}{dx}(w) + w\frac{d}{dx}(u)$

$= u.v + w\frac{du}{dx} \ldots\ldots\ \text{from}(2)$

By definition of integration.

$u.w = \int \left[u.v + w\frac{du}{dx}\right] dx$

$= \int u.v dx + \int w.\frac{du}{dx} dx$

$\int u.v dx = u.w - \int w\frac{du}{dx} dx$

$= u\int v dx - \int \left[\frac{du}{dx}\int v.dx\right] dx$

[next section only required for question 2]

Hence, $\int xe^x dx = x.\int e^x dx - \int \left[\frac{d}{dx}x.\int e^x dx\right] dx$

$= xe^x - \int 1 \times e^x dx$

$= xe^x - e^x + c$

Question 5.1.2: Solve: dy/dx = cos(x + y) [3]

Solution: Given,

$$\frac{dy}{dx} = \cos(x + y) \text{ ...(i)}$$

Put $x + y = v$ ...(ii)

$\therefore\ y = v - x$

$$\therefore\ \frac{dy}{dx} = \frac{dv}{dx} - 1 \text{ ...(iii)}$$

Substituting (ii) and (iii) in (i), we get

$$\frac{dv}{dx} - 1 = \cos v$$

$$\therefore\ \frac{dv}{dx} = 1 + \cos v$$

$$\therefore\ \frac{dv}{dx} = 2\cos^2\left(\frac{v}{2}\right)$$

$$\therefore\ \frac{1}{\cos^2\left(\frac{v}{2}\right)} dv = 2dx$$

$$\therefore \sec^2\left(\frac{v}{2}\right) dv = 2dx$$

Integrating on both sides, we get

$$\int \sec^2\left(\frac{v}{2}\right) dv = 2\int dx$$

$$\therefore 2\tan\left(\frac{v}{2}\right) = 2x + c\prime$$

$$\therefore \tan\left(\frac{v}{2}\right) = x + \frac{c\prime}{2}$$

$$\therefore \tan\left(\frac{x+y}{2}\right) = x + c, \text{ where } c = \frac{c\prime}{2}$$

Question 5.1.3: [3]

If $f(x) = \dfrac{e^{x^2} - \cos x}{x^2}$, for x= 0, is continuous at x = 0, find f(0).

Solution: f(x) is continuous at x = 0

$$\lim_{x\to 0} f(x) = f(0)$$

$$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2} = \lim_{x\to 0} \frac{\left(e^{x^2} - 1\right) + (1 - \cos x)}{x^2}$$

$$= \lim_{x\to 0} \left(\frac{e^{x^2} - 1}{x^2} + \frac{1 - \cos x}{x^2} \right)$$

$$= \lim_{x\to 0} \left(\frac{e^{x^2} - 1}{x^2} + \frac{2\sin^2\left(\frac{x}{2}\right)}{x^2} \right)$$

$$= \lim_{x\to 0} \left(\frac{e^{x^2} - 1}{x^2} + 2\left(\frac{\sin\left(\frac{x}{2}\right)}{x} \right)^2 \right)$$

$$= \lim_{x\to 0} \left(\frac{e^{x^2} - 1}{x^2} + 2\left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} \times \frac{1}{2} \right)^2 \right)$$

$$= \lim_{x\to 0} \frac{e^{x^2} - 1}{x^2} + \frac{1}{2}\left(\lim_{x\to 0} \frac{\sin\left(\frac{x}{2}\right)}{x} \right)^2$$

$$= 1 + \frac{1}{2}(1)^2$$

$$= \frac{3}{2}$$

Thus, f(0)=3/2

Question 5.2: Attempt any TWO of the following: [8]

Question 5.2.1: If y = f(x) is a differentiable function of x such that inverse function x = f^{-1} (y) exists, then prove that x is a differentiable function of y and [4]

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} \text{ where } \frac{dy}{dx} \neq 0$$

Solution 1: Let δy be the increment in y corresponding to an increment δx in x.

as $\delta x \to 0, \delta y \to 0$

Now y is a differentiable function of x.

$$\therefore \lim_{\delta x\to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

Now $\frac{\delta y}{\delta x} \times \frac{\delta x}{\delta y} = 1$

$\therefore \frac{\delta x}{\delta y} = \frac{1}{\frac{\delta y}{\delta x}}$

Taking limits on both sides as $\delta x \to 0, we \geq t$

$\lim_{\delta x \to 0} \frac{\delta x}{\delta y} = \lim_{\delta x \to 0} \left[\frac{1}{\frac{\delta y}{\delta x}}\right] = \frac{1}{\lim_{dx \to 0} \frac{\delta y}{\delta x}}$

$\lim_{\delta x \to 0} \frac{\delta x}{\delta y} = \frac{1}{\lim_{dx \to 0} \frac{\delta y}{\delta x}}$ ....[as $\delta x \to 0, \delta y \to 0$]

Since limit in R.H.S. exists

limit in L.H.S. also exists and we have,

$\lim_{\delta y \to 0} \frac{\delta x}{\delta y} = \frac{dx}{dy}$

$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$, where $\frac{dy}{dx} \neq 0$

Let $y = \tan^{-1} x$

$x = \tan y \Rightarrow \cos y = \frac{1}{\sqrt{1+\tan^2 y}} = \frac{1}{\sqrt{1+x^2}}$

$\therefore \sec^y . \frac{dy}{dx} = 1 \Rightarrow \frac{dx}{dy} = \sec^2 y$

$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sec^2 y} = \cos^2 y \Rightarrow \frac{dy}{dx} = \cos^y$

$\frac{d(\tan^{-1} x)}{dx} = \cos^2 y = (\cos y)^2 = \left(\frac{1}{\sqrt{1+x^2}}\right)^2$

$\therefore \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

Solution 2: 'y' is a differentiable function of 'x'.

Let there be a small change δx in the value of 'x'.

Correspondingly, there should be a small change δy in the value of 'y'.

As δx → 0, δy → 0

Consider, $\frac{\delta x}{\delta y} \times \frac{\delta y}{\delta x} = 1$

$\therefore \quad \frac{\delta x}{\delta y} = \frac{1}{\frac{\delta y}{\delta x}}, \frac{\delta y}{\delta x} \neq 0$

Taking $\lim_{\delta x \to 0}$ on both sides, we get

$$\lim_{\delta x \to 0}\left(\frac{\delta x}{\delta y}\right) = \frac{1}{\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)}$$

Since 'y' is a differentiable function of 'x'

$\therefore \quad \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{dy}{dx}$

As $\delta x \to 0$, $\delta y \to 0$

$$\therefore \quad \lim_{\delta y \to 0}\left(\frac{\delta x}{\delta y}\right) = \frac{1}{\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)} \quad \text{....(i)}$$

$\therefore$ limits on R.H.S. of (i) exist and are finite.

Hence, limits on L.H.S. of (i) also should exist and be finite.

$\therefore \quad \lim_{\delta y \to 0}\left(\frac{\delta x}{\delta y}\right) = \frac{dx}{dy}$ exists and is finite.

$$\therefore \quad \frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}, \frac{dy}{dx} \neq 0$$

Question 5.2.2: A telephone company in a town has 5000 subscribers on its list and collects fixed rent charges of Rs.3, 000 per year from each subscriber. The company proposes to increase annual rent and it is believed that for every increase of one rupee in the rent, one subscriber will be discontinued. Find what increased annual rent will bring the maximum annual income to the company. **[4]**

Solution: Here, the number of subscribers = 5000 and annual rental charges per subscriber = Rs.3000.
For every increase of 1 rupee in the rent, one subscriber will be discontinued.
Let the rent be increased by Rs. x.
New rental charges per year = 3000 + x
and number of subscribers after the increase in rental charges = 5000 – x. [1 M]
Let R be the annual income of the company.

Then, R = (3000+x)(5000-x)

$=15000000-3000x+5000x-x^2$

$$= 15000000 + 2000x - x^2 \text{ and } \frac{d^2R}{dx^2} = -2$$

R is maximum if $\frac{\mathrm{dR}}{dx} = 0 i.e, 2000 - 2x = 0$

$i.e$ if $x = 1000$

$$\left(\frac{d^2R}{dx^2}\right)_{x=1000} = -2 < 0$$

By the second derivative test, R is maximum when x =1000.
Thus, the annual income of the company is maximum when the annual rental charges are increased by Rs.1000.

Question 5.2.3: **[4]**

Evaluate: $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}} dx$

Solution:

$$\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}} dx$$

Let $I = \int_{-a}^{a} \sqrt{\frac{a-x}{a+x}} dx$

$$= \int_{-a}^{a} \sqrt{\frac{(a-x)(a-x)}{(a+x)(a-x)}} dx$$

$$= \int_{-a}^{a} \frac{a-x}{\sqrt{a^2-x^2}} dx$$

$$= \int_{-a}^{a} \frac{a}{\sqrt{a^2-x^2}} dx - \int_{-a}^{a} \frac{x}{\sqrt{a^2-x^2}} dx$$

[but $\frac{a}{\sqrt{a^2-x^2}}$ is an is an even function and $\frac{x}{\sqrt{a^2-x^2}}$ is an odd function]

$$= 2a.\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 2a.\left[\sin^{-1} 1 - \sin^{-1} 0\right]$$

$$= 2a\left[\frac{\pi}{2} - 0\right]$$

$$\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}.dx = \pi a$$

Question 6: [14]

Question 6.1: Attempt any TWO of the following: [6]

Question 6.1.1: Discuss the continuity of the following function, at x = 0. **[3]**

$f(x) = \frac{x}{|x|}, f \text{ or } x \neq 0$

$= 1$,for $x = 0$

Solution: f(0)=1............(given)............(1)

for$x > 0, |x| = x$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{|x|}$

$= \lim_{x \to 0^+} \frac{x}{x}$

$= \lim_{x \to 0^+} (1)$

$= 1$

for x<0,|x|=-x

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} \frac{x}{|x|}$

$= \lim_{x \to 0^-} -\frac{x}{x}$

$= \lim_{x \to 0^-} (-1)$

$= -1$

$\therefore \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$

f is discontinuous at x = 0

here $\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(X)$

$\therefore \lim_{x \to 0} f(x)$does not exist

hence, it is discontinuous at x = 0

Question 6.1.2: If the population of a country doubles in 60 years, in how many years will it be triple under the assumption that the rate of increase in proportional to the number of inhabitants? **[3]**
[Given : log 2 = 0.6912 and log 3 = 1.0986.]

Solution: Let P be the population of the country at time t.

Given $\frac{\mathrm{dP}}{dt} \propto P$

$\therefore \frac{\mathrm{dP}}{dt} = kP$ (where k is a constant)

$\therefore \frac{1}{P} dP = kdt$

Integrating both the side w.r.t x

$\int \frac{1}{P} dp = k \int 1dt + c$

$\log P = kt + c$

$P = e^{kt+c} = e^{kt}.e^{c}$

Let $e^{c} = \alpha$

$\therefore P = \alpha.e^{kt}$

Let initial population at t = 0

$\therefore N = \alpha.e^{0} \quad \therefore N = \alpha$

$P = N.e^{kt}$

Given P = 2N when t = 60 years,

$\therefore 2N = Ne^{60k}$

$\therefore 2 = e^{60k} \Rightarrow k = \frac{1}{60} \log 2$

$\therefore P = N.e^{60k}$

Required t when P = 3N

$3 = e^{kt} \Rightarrow \log 3 = kt$

$\log 3 = \left(\frac{1}{60} \log 2\right).t$

$t = \frac{60 \log 3}{\log 2}$

$= \frac{60 \times 1.0986}{0.6912}$

$= 95.4 years (\approx.)$

The population of the counter will triple approximately in 95.4 years.

Question 6.1.3: A fair coin is tossed 8 times. Find the probability that it shows heads exactly 5 times. **[3]**

Solution: Let X = Number of heads
p = probability of getting head in one toss

p=1/2

$$q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$$

Given n=8

$$x \sim B\left(8, \frac{1}{2}\right)$$

The p.m.f. of X is given as

$$P(X = x) = {}^nC_x p^x q^{n-x}$$

$$i.e P(x) = {}^8C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{)} (8 - x), x = 0, 1, 2, 3, \ldots.., 8$$

P(exactly 5 heads) = p[X=5]

$$= P(5) = {}^8C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{8-5}$$

$$= {}^8C_3 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^3 [\because {}^nC_x = {}^nC_{n-x}]$$

$$= \frac{8 \times 7 \times 6}{1 \times 2 \times 3} \times \frac{1}{256} = \frac{7}{32}$$

∴ P[X = 5] = 0.21875

Hence, the probability of getting exactly 5 heads is 0.21875

Question 6.1.3: A fair coin is tossed 8 times. Find the probability that it shows heads at least once **[3]**

Solution: Let X = Number of heads
p = probability of getting head in one toss

p=1/2

$$q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$$

Given n=8

$x \sim B\left(8, \frac{1}{2}\right)$

The p.m.f. of X is given as

$$P(X = x) = {}^{n}C_{x} p^{x} q^{n-x}$$

$$i.e P(x) = {}^{8}C_{x}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{)}(8-x), x = 0, 1, 2, 3, \ldots.., 8$$

P (getting heads at least once)

$$P[X \geq 1] = 1 - P[X = 0]$$

$$= 1 - P(0) = 1 - {}^{8}C_{0}\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{8-0}$$

$$= 1 - \left(\frac{1}{2}\right)^{8} = 1 - \frac{1}{256} = \frac{255}{256}$$

$$P[X \geq 1] = 0.996$$

Question 6.2: Attempt any TWO of the following: **[8]**

Question 6.2.1: **[4]**

Find: $I = \int \frac{dx}{\sin x + \sin 2x}$

Solution:

$$I = \int \frac{dx}{\sin x + \sin 2x}$$

$$= \int \frac{1}{\sin x + 2\sin x \cos x} dx$$

$$= \int \frac{1}{\sin x(1 + 2\cos x)} dx$$

$$= \int \frac{\sin x}{\sin^{2} x(1 + 2\cos x)} dx$$

Let u=cosx

⇒du=−sinxdx

Also,

$$\sin^2 x = 1 - \cos^2 x = 1 - u^2$$

$$\therefore\ I = \int -\frac{1}{(1-u^2)(1+2u)}du$$

$$= \int \frac{1}{(1+u)(1-u)(1+2u)}du$$

Using partial fractions, we get

$$\frac{1}{(1+u)(1-u)(1+2u)} = \frac{A}{1+u} + \frac{B}{1-u} + \frac{C}{1+2u}$$

$$\Rightarrow -1 = A(1-u)(1+2u) + B(1+u)(1+2u) + C(1+u)(1-u)$$

$$\Rightarrow -1 = A(1+u-2u^2) + B(1+3u+2u^2) + C(1-u^2)$$

$$\Rightarrow -1 = (-2A+2B-C)u^2 + (A+3B)u + (A+B+C)$$

Equating the respective coefficients on the LHS and the RHS, we get

−2A+2B−C=0 .....(1)

A+3B=0 .....(2)

A+B+C=−1 .....(3)

Adding (1), (2) and (3), we get

6B=−1

⇒B=−1/6

From (2), we get

A=−3B

⇒A=1/2

from (3), we get

C=−1−A−B

⇒C=−4/3

So,

$$\frac{1}{(1+u)(1-u)(1+2u)}=\frac{1}{2(1+u)}-\frac{1}{6(1-u)}-\frac{4}{3(1+2u)}$$

$$\Rightarrow I=\int\left[\frac{1}{2(1+u)}-\frac{1}{6(1-u)}-\frac{4}{3(1+2u)}\right]du$$

$$=\frac{1}{2}\log(1+u)+\frac{1}{6}\log(1-u)-\frac{4}{\times 2}\log(1+2u)+C$$

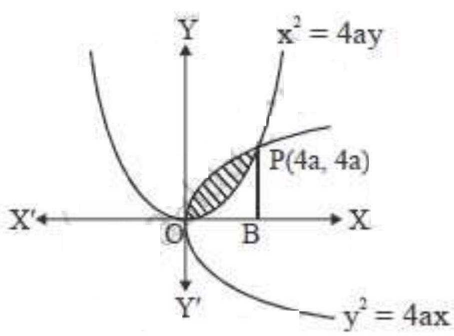
$$=\frac{1}{2}\log(1+\cos x)+\frac{1}{6}\log(1-\cos x)-\frac{2}{3}\log(1+2\cos x)+C$$

Question 6.2.2: Find the area of the region lying between the parabolas y^2 = 4ax and x^2 = 4ay. **[4]**

Solution: The equations of the parabolas are

$y^2=4ax$.............(1)

$x^2=4ay$.............(2)



$$\left[\frac{x^2}{4a}\right]^2=4ax by (2)$$

$$x^4=64a^3x$$

$$x\left[x^3-(4a)^3\right]=0$$

x=0 and x=4a

y=0 and y=4a

The points of intersection of curves are O(0, 0), P(4a, 4a)

The required areas is, A = (Area under parabola y2 = 4ax) – (Area under parabola x2 = ay)

$$= \int_0^{4a} \sqrt{4ax}dx - \int_0^{4a} \frac{x^2}{4a}dx$$

$$= \sqrt{4a}.\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^{4a} - \frac{1}{4a}\frac{1}{3}\left[x^3\right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3} \times 4a\sqrt{4a} - \frac{1}{12a} \times 64a^3$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2$$

$$= \frac{16}{3}a^2 sq. units$$

Question 6.2.3: Given the p. d. f. (probability density function) of a continuous random variable x as: **[4]**

$$f(x) = \frac{x^2}{3}, -1$$

= 0 , otherwise

Determine the c. d. f. (cumulative distribution function) of x and hence find P(x < 1), P(x ≤ -2), P(x > 0), P(1 < x < 2)

Solution: c.d.f. of the continuous random variable is given by

$$F(x) = \int_{-1}^{x} \frac{y^2}{3}dx$$

$$= \left[\frac{y^3}{9}\right]_{-1}^{x}$$

$$= \frac{x^3+1}{9}, x \in R$$

Consider $P(X<1)=F(1)=(1^3+1)/9=2/9$

$$P(x \leq -2) = 0$$

$$P(X > 0) = 1 - P(X \leq 0)$$

$P(X > 0) = 1 - P(X \leq 0)$

$= 1 - F(0)$

$= 1 - \left(\frac{0}{9} + \frac{1}{9}\right)$

$= \frac{8}{9}$

$P(1 < x < 2) = F(2) - F(1)$

$= 1 - \left(\frac{1}{9} + \frac{1}{9}\right)$

$= \frac{7}{9}$