Long Answer Type Questions

[4 MARKS]

Que 1. In ΔPQR, right angled at Q, PR + QR = 25 cm and PQ = 5 cm. Determine the values of sin P, cos P and tan P.

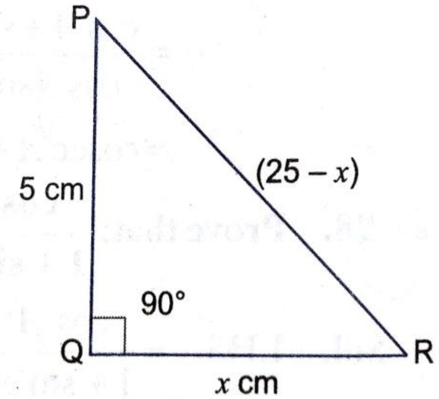

Fig. 10.6

Sol. We have a right-angled ΔPQR in which $\angle Q = 90°$.

Let QR = x cm

Therefore, PR = (25 – x) cm

By Pythagoras Theorem, we have

$PR^2 = PQ^2 + QR^2$

$\Rightarrow \quad (25 - x)^2 = 5^2 + x^2 \qquad \Rightarrow (25 - x)^2 - x^2 = 5^2$

$\Rightarrow \quad (25 - x - x)(25 - x + x) = 25$

$\Rightarrow \quad (25 - 2x)\, 25 = 25 \qquad \Rightarrow 25 - 2x = 1$

$\Rightarrow \quad 25 - 1 = 2x \qquad \Rightarrow 24 = 2x$

$\therefore \quad x = 12\ cm.$

Hence, QR =12 cm

PR (25 – x) cm = 25 – 12 = 13 cm

PQ = 5 cm

$\therefore \quad \sin P = \frac{QR}{PR} = \frac{12}{13}$, Cos P = $\frac{PQ}{PR} = \frac{5}{13}$; $\quad \tan P = \frac{QR}{PQ} = \frac{12}{5}$

Que 2. In triangle ABC, right-angled at B, if $\tan A = \frac{1}{\sqrt{3}}$, find the value of:

(i) sin A cos C + cos A sin C **(ii) cos A cos C – sin A sin C.**

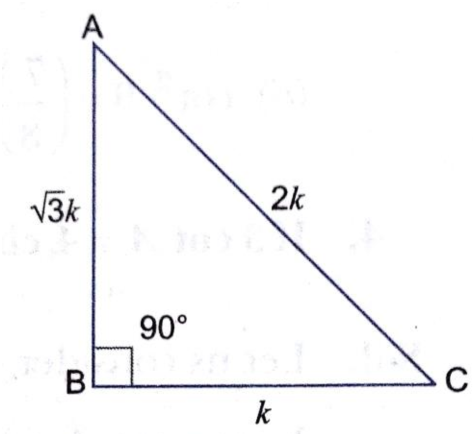

Fig. 10.7

Sol. We have a right-angled ΔABC in which $\angle B = 90°$.

And, tan A = $\frac{1}{\sqrt{3}}$

Now, $\tan A = \frac{1}{\sqrt{3}} = \frac{BC}{AB}$

Let BC = k and AB = $\sqrt{3}\,k$

∴ By Pythagoras Theorem, we have

$$AC^2 = AB^2 + BC^2$$

$\Rightarrow \quad AC^2 = \left(\sqrt{3}k\right)^2 + (k)^2 = 3k^2 + k^2$

$\Rightarrow \quad AC^2 = 4k^2 \qquad \therefore \quad AC = 2k$

Now, $\sin A = \frac{Perpendicular}{Hypotenuse} = \frac{k}{2k} = \frac{1}{2}$; $\qquad \cos A = \frac{Base}{Hypotenuse} = \frac{\sqrt{3}k}{2k} = \frac{\sqrt{3}}{2}$

$sin\ C = \frac{Perpendicular}{Hypotenuse} = \frac{\sqrt{3}k}{2k} = \frac{\sqrt{3}}{2}$, $\qquad \cos C = \frac{Base}{Hypotenuse} = \frac{k}{2k} = \frac{1}{2}$

(i) $\sin A \,.\cos C + \cos A.\sin C = \frac{1}{2} \times \frac{1}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} = \frac{1}{4} + \frac{3}{4} = \frac{4}{4} = 1.$

(ii) $\cos A.\cos C - \sin A.\sin C = \frac{\sqrt{3}}{2} \times \frac{1}{2} - \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0.$

Que 3. If $\cot\theta = \frac{7}{8}$, evaluate: (i) $\frac{(1+\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(1-\cos\theta)}$, (ii) $\cot^2\theta$.

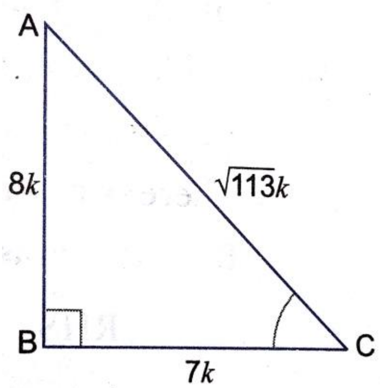

Fig. 10.8

Sol. Let us draw a right triangle ABC in which $\angle B = 90°$ and $\angle c = \theta$.

We have, $\cot \theta = \frac{7}{8} = \frac{Base}{Perpendicular} = \frac{BC}{AB}$ (given)

Let BC = 7k and AB = 8k
Therefore, by Pythagoras Theorem

$$AC^2 = AB^2 + BC^2 = (8k)^2 + (7k)^2 = 64k^2 + 49k^2$$

$$AC^2 = 113k^2 \qquad \therefore \quad AC = \sqrt{113}k$$

$$\therefore \quad \sin\theta = \frac{Perpendicular}{Hypotenuse} = \frac{AB}{AC} = \frac{8k}{\sqrt{113}k} = \frac{8}{\sqrt{113}}$$

And $\cos\theta = \frac{Base}{Hypotenuse} = \frac{BC}{AC} = \frac{7}{\sqrt{113}k} = \frac{7}{\sqrt{113}}$

(i) $$\frac{(1+\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(1-\cos\theta)} = \frac{1-\sin^2\theta}{1-\cos^2\theta} = \frac{1-\left(\frac{8}{\sqrt{113}}\right)^2}{1-\left(\frac{7}{\sqrt{113}}\right)^2}$$

$$= \frac{1-\frac{64}{113}}{1-\frac{49}{113}} = \frac{\frac{113-64}{113}}{\frac{113-49}{113}} = \frac{49}{64}.$$

Alternate method:

$$\frac{(1+\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(1-\cos\theta)} = \frac{1-\sin^2\theta}{1-\cos^2\theta} = cot^2\,\theta = \left(\frac{7}{8}\right)^2 = \frac{49}{64}$$

(ii) $$cot^2\,\theta = \left(\frac{7}{8}\right)^2 = \frac{49}{64}.$$

Que 4. If 3 cot A = 4, check whether $\frac{1-\tan^2 A}{1+\tan^2 A} = \cos^2 A - \sin^2 A$ or not.

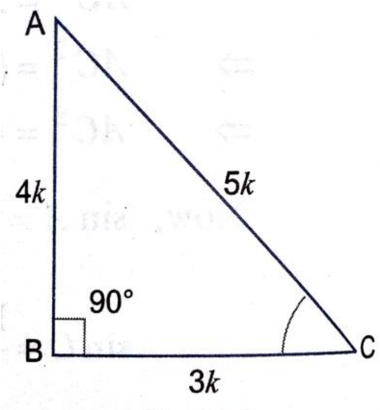

Fig. 10.9

Sol. Let us consider a right triangle ABC in which $\angle B = 90°$.

Now, $cot\,A = \frac{Base}{Perpendicular} = \frac{AB}{BC} = \frac{4}{3}$

Let AB = 4k and BC = 3k
$\therefore$ By Pythagoras Theorem

$$AC^2 = AB^2 + BC^2$$

$\Rightarrow \quad AC^2 = (4k)^2 + (3k)^2 = 16k^2 + 9k^2$

$AC^2 = 25k^2$

$\therefore \; AC = 5k$

Therefore, $\tan A = \frac{Perpendicular}{Base} = \frac{BC}{AB} = \frac{3k}{4k} = \frac{3}{4}$

And, $\sin A = \frac{Perpendicular}{Hypotenuse} = \frac{BC}{AC} = \frac{3k}{5k} = \frac{3}{5}$

$\cos A = \frac{Base}{Hypotenuse} = \frac{AB}{AC} = \frac{4k}{5k} = \frac{4}{5}$

Now, LHS $= \frac{1-\tan^2 A}{1+\tan^2 A}$

$$= \frac{1-\left(\frac{3}{4}\right)^2}{1+\left(\frac{3}{4}\right)^2} = \frac{1-\frac{9}{16}}{1+\frac{9}{16}} = \frac{16-9}{16+9} = \frac{7}{25}$$

$RHS = \cos^2 A - \sin^2 A = \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{16}{25} - \frac{9}{25} - \frac{7}{25}$

Hence, $\frac{1-\tan^2 A}{1+\tan^2 A} = \cos^2 A - \sin^2 A.$

Que 5. Write all the other trigonometric ratios of $\angle A$ in terms of sec A.

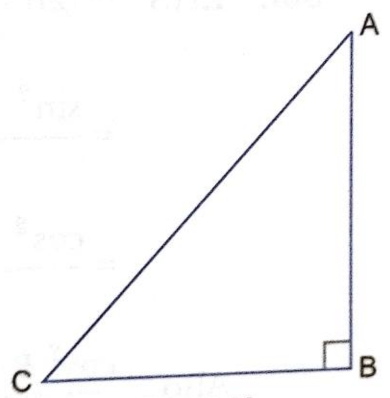

Fig. 10.10

Sol. Let us consider a right-angled ΔABC, in which $\angle B = 90°$.

For $\angle A$, we have

Base = AB, Perpendicular = BC and Hypotenuse = AC

$\therefore \quad sec\ A = \frac{Hypotenuse}{Base} = \frac{AC}{AB}$

$\Rightarrow \frac{sec\ A}{1} = \frac{AC}{AB} \qquad \Rightarrow \qquad$ AC = AB sec A

Let AB = K and AC = k sec A

$\therefore$ By Pythagoras Theorem, we have

$$AC^2 = AB^2 + BC^2 \quad \Rightarrow \quad k^2\, sec^2\, A = k^2 + BC^2$$

$$\therefore \quad BC^2 = k^2\, sec^2\, A - k^2 \quad \Rightarrow \quad BC = k\sqrt{sec^2\, A - 1}$$

$$\therefore \quad \sin A = \frac{BC}{AC} = \frac{k\sqrt{sec^2\, A - 1}}{k\, sec\, A} = \frac{\sqrt{sec^2\, A - 1}}{sec\, A}$$

$$\cos A = \frac{AB}{AC} = \frac{k}{k\, sec\, A} = \frac{1}{sec\, A}$$

$$\tan A = \frac{BC}{AB} = \frac{k\sqrt{sec^2\, A-1}}{k} = \sqrt{sec^2\, A - 1}$$

$$cot\, A = \frac{1}{\tan A} = \frac{1}{\sqrt{sec^2\, A-1}}$$

$$\mathrm{cosec}\, A = \frac{AC}{BC} = \frac{k\, sec\, A}{k\sqrt{sec^2\, A-1}} = \frac{sec\, A}{\sqrt{sec^2\, A-1}}.$$

Que 6. Prove that: $\left(\frac{1+\tan^2 A}{1+cot^2 A}\right) = \left(\frac{1-\tan A}{1-cot\, A}\right)^2 = \tan^2 A.$

Sol. LHS = $\left(\frac{1+\tan^2 A}{1+cot^2 A}\right) = \frac{sec^2\, A}{\mathrm{cosec}^2\, A}$

$$= \frac{\frac{1}{\cos^2 A}}{\frac{1}{\sin^2 A}} = \frac{\sin^2 A}{\cos^2 A} = \tan^2 A$$

$$RHS = \left(\frac{1-\tan A}{1-cot\, A}\right)^2 = \left(\frac{1-\tan A}{1-\frac{1}{\tan A}}\right)^2$$

$$= \left(\frac{1-\tan A}{\frac{\tan A-1}{\tan A}}\right)^2 = \left(\frac{1-\tan A}{\tan A-1}\times \tan A\right)^2 = (-\tan A)^2 = \tan^2 A$$

LHS = RHS.

Que 7. Prove that: $\tan^2 A - \tan^2 B = \frac{\cos^2 B-\cos^2 A}{\cos^2 B\cos^2 A} = \frac{\sin^2 A-\sin^2 B}{\cos^2 A\cos^2 B}.$

Sol. LHS $= \tan^2 A - \tan^2 B = \frac{\sin^2 A}{\cos^2} - \frac{\sin^2 B}{\cos^2 B}$

$$= \frac{\sin^2 A\cos^2 B-\cos^2 A\sin^2 B}{\cos^2 A\cos^2 B} = \frac{(1-\cos^2 A)\cos^2 A\,(1-\cos^2 B)}{\cos^2 A\cos^2 B}$$

$$= \frac{\cos^2 B-\cos^2 A\cos^2 B-\cos^2 A+\cos^2 A\cos^2 B}{\cos^2 A\cos^2 B} = \frac{\cos^2 B-\cos^2 A}{\cos^2 A\cos^2 B}$$

Also $\frac{\cos^2 B-\cos^2 A}{\cos^2 A\cos^2 B} = \frac{(1-\sin^2 B)-(1-\sin^2 A)}{\cos^2 A\cos^2 B}$

$$= \frac{\sin^2 A-\sin^2 B}{\cos^2 A\cos^2 B} = RHS.$$