DEFINITE INTEGRALS

	Properties of Definite Integrals :-
P-I	$\int_a^b f(x)dx = \int_a^b f(t)dt$ e.g $\int_0^{\frac{\pi}{2}} sin\ t\ dt = \int_a^{\frac{\pi}{2}} sin\ x\ dx$
P-II	$\int_a^b f(x)dx = \int_{-b}^a f(x)dx$
P-III	$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ Where $a < c < b$ e.g $\int_0^a f(x)dx = \int_a^{2a} f(x)dx + \int_0^{2a} f(x)dx$
P-IV	$\int_0^a f(x)dx = \int_0^a f(a-x)dx$ Proof: Taking RHS $\int_0^a f(a-x)dx$ Put $a - x = t$ when $x = 0 \Rightarrow t = a$ $-dx = dt \Rightarrow dx = -dt$ when $x = a \Rightarrow t = a$ ∴ RHS = $-\int_a^0 f(t)dt$ = $\int_0^a f(t)dt$ ......(by P–II) = $\int_0^a f(x)dx$ ......(by P–I) = LHS ∴ $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ proved
P-V	$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ Proof: Do yourself by put $a + b - x = t$
P-VI	$\int_b^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & ; \quad f(2a-x) = f(x) \\ 0 & ; \quad if\ f(2a-x) = -f(x) \end{cases}$ Mainly $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$
P-VII	**Even - function property** $\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx & ; \quad if\ f(x) \rightarrow \text{even} \\ 0 & ; \quad if\ f(x) \rightarrow \text{odd} \end{cases}$ If $f(-x) = f(x)$ then $f(x)$ is an even function If $f(-x) = -f(x)$ then $f(x)$ is an odd function
Q.1)	Evaluate $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\ dx.$
Sol.1)	$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\ dx$ ....(1)

	$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)} + \sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\, dx$ .... $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx$ ....(2) Adding (1) & (2) $2\,I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$ $= \int_0^{\frac{\pi}{2}} 1\,.\,dx$ $= (x)_0^{\frac{\pi}{2}}$ $2\,I = \frac{\pi}{2}$ $I = \frac{\pi}{4}$ ans.
Q.2)	Evaluate $I = \int_0^{\frac{\pi}{2}} \sin(2x)\log(\tan x)\ dx$
Sol.2)	$I = \int_0^{\frac{\pi}{2}} \sin(2x)\log(\tan x)\ dx$ ....(1) $I = \int_0^{\frac{\pi}{2}} \sin\left[2\left(\frac{\pi}{2}-x\right)\right].\log\left[\tan\left(\frac{\pi}{2}-x\right)\right]\ dx$ ....$\left[\int_0^a f(x)dx = \int_0^a f(a-x)\right]$ $I = \int_0^{\frac{\pi}{2}} \sin(\pi-2x).\log(\cot x)dx$ $I = \int_0^{\frac{\pi}{2}} \sin(2x).\log(\cot x)dx$ ....(2) $[\because sin(\pi-2x) = \sin(2x)]$ Adding $eq.$(1) & (2) $2\,I = \int_0^{\frac{\pi}{2}} \sin(2x)[\log(\tan x) + \log(cot\ x)]\,dx$ $2\,I = \int_0^{\frac{\pi}{2}} \sin(2x)\log(\tan x\ .cot\ x)\ dx$ $2\,I = \int_0^{\frac{\pi}{2}} \sin(2x)\,.\log(1)\ dx$ $2\,I = \int_0^{\frac{\pi}{2}} 0\,dx$ ... ...$\{\because\ \log 1 = 0\}$ $\therefore\ I = 0$ ans......
Q.3)	$I = \int_0^{\frac{\pi}{4}} \log(1+\tan x)\ dx$
Sol.3)	$I = \int_0^{\frac{\pi}{4}} \log(1+\tan x)\ dx$ ....(1) $I = \int_0^{\frac{\pi}{2}} log\left[1 + tan\left(\frac{\pi}{4}-x\right)\right]dx$ .....$\left[\int_0^a f(x)dx = \int_0^a f(a-x)\ dx\right]$ $I = \int_0^{\frac{\pi}{4}} log\left[1+\frac{1-tan\,x}{1+tan\,x}\right]\ dx$ .....$\{\tan(A-B)\ formula\}$ $I = \int_0^{\frac{\pi}{4}} log\left[\frac{1+tanx+1-\ tan\,x}{1+\ tan\,x}\right]\ dx$

	$I = \int_0^{\frac{\pi}{4}} log\left(\frac{2}{1+\ tanx}\right)\ dx$ .....(2) $Eq.(1)\ +\ (2)$ $2\,I = \int_0^{\frac{\pi}{4}} log\left(1 + tan\,x\ .\frac{2}{1+tanx}\right)\ dx$ $2\,I = \int_0^{\frac{\pi}{4}} log\ (2)\ dx$ $2\,I = log\,2\,(x)_0^{\frac{\pi}{4}}$ $2\,I = log\,2\left[\frac{\pi}{4} - 0\right]$ $\therefore\ I = \frac{\pi}{8} log\,2\quad ans.$
Q.4)	$I = \int_0^{\frac{\pi}{2}} 2\log(\cos x) - \log(\sin(2x))dx$
Sol.4)	$I = \int_0^{\frac{\pi}{2}} 2\ log(cos\ x) - log(sin(2x))dx$ $I = \int_0^{\frac{\pi}{2}} log(cos^2x) - log\big(sin(2x)\big)\ dx$ $I = \int_0^{\frac{\pi}{2}} log\left(\frac{cos^2x}{sin(2x)}\right)\ dx$ $I = \int_0^{\frac{\pi}{2}} log\left(\frac{cos^2x}{sin(2x)}\right)\ dx$ $I = \int_0^{\frac{\pi}{2}} log\left[\frac{cot\ x}{2}\right] dx\quad ......(1)$ $I = \int_0^{\frac{\pi}{2}} \log\left[\frac{\cot\left(\frac{\pi}{2}-x\right)}{2}\right] dx\quad ......\left[\int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$ $I = \int_0^{\frac{\pi}{2}} log\ \left(\frac{tan\,x}{2}\right)\quad ......(2)$ $Eq.(1)\ +\ (2)$ $2\,I = \int_0^{\frac{\pi}{2}} log\left(\frac{cot\ x}{2}\ .\frac{tan\ x}{2}\right)\ dx$ $2\,I = \int_0^{\frac{\pi}{2}} log\left(\frac{1}{4}\right) dx\quad ......\{\tan x\,.\cot x\ =\ 1\}$ $2\,I = \int_0^{\frac{\pi}{2}} log(1) - log(4)dx$ $2\,I = \int_0^{\frac{\pi}{2}} - log\ 4\ dx\quad ......\{\log 1\ =\ 0\}$ $2\,I = -log\ 4\ [x]_0^{\frac{\pi}{2}}$ $2\,I = -log\ 4\left[\frac{\pi}{2}\right]$ $I = -\frac{\pi}{4}\ log\ 4\quad ans.$ (or) $I = \frac{-\pi}{4}\log(2)^2$ $I = \frac{-\pi}{2}\log 2\quad ans.$

Q.5)	$I = \int_0^1 \log\left(\frac{1}{x} - 1\right) dx$
Sol.5)	$I = \int_0^1 log\left(\frac{1-x}{x}\right) dx$ ....(1) $= \int_0^1 log\left[\frac{1-(1-x)}{1-x}\right] dx$ ....$\left[\int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$ $= \int_0^1 log\left[\frac{x}{1-x}\right] dx$ ....(2) (1) + (2) $2I = \int_0^1 \log\left(\frac{1-x}{x} . \frac{x}{1-x}\right) dx$ $= \int_0^1 \log(1)\, dx$ $2I = 0$ ……..$\{\because \log 1 = 0\}$ $I = 0$ ans.
Q.6)	$I = \int_0^5 \frac{\sqrt[3]{x}}{\sqrt[3]{x} + \sqrt[3]{5-x}} dx$
Sol.6)	$I = \int_0^5 \frac{\sqrt[3]{x}}{\sqrt[3]{x} + \sqrt[3]{5-x}} dx$ … … (1) $I = \int_0^5 \frac{\sqrt[3]{5-x}}{\sqrt[3]{5-x} + \sqrt[3]{5-(5-x)}} dx$ … … $\left[\int_0^a f(x)dx = \int_0^a f(a-x)\right]$ $I = \int_0^5 \frac{\sqrt[3]{5-x}}{\sqrt[3]{5-x} + \sqrt[3]{x}} dx$ ....(2) (1) + (2) $2I = \int_0^5 \frac{\sqrt[3]{x} + \sqrt[3]{5-x}}{\sqrt[3]{x} + \sqrt[3]{5-x}} dx$ $= \int_0^5 1. dx$ $= (x)_0^5$ $2I = 5$ $I = \frac{5}{2}$ ans.
Q.7)	Show that $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$
Sol.7)	R.H.S $\int_0^a f(x)dx + \int_0^a f(2a-x)dx$ Put $2a - x = t$ when $x = 0$; $t = 2a$ $-dx = dt$ $dx = -dt$ when $x = a$; $t = a$ $\therefore R.H.S = \int_0^a f(x)dx - \int_{2a}^a f(t)dt$ $= \int_0^a f(x)dx + \int_a^{2a} f(t)\, dt$ ....$\left[\int_0^b f(x)dx = -\int_b^a f(x)\right]$ $= \int_0^a f(x)dx + \int_a^{2a} f(x)dx$ ....$\left[\int_a^b f(x)dx = \int_a^b f(t)dt\right]$

	$= \int_0^{2a} f(x)dx$ ....$\left[\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx\right]$ $= LHS$ Proved
Q.8)	Show that $I = \int_0^1 x(1-x)^n\, dx$
Sol.8)	$I = \int_0^1 (1-x)[1-(1-x)]^n dx$ ....$[\int_0^a f(x)dx = \int_0^a f(a-x)dx]$ $I = \int_0^1 (1-x)(x)^n dx$ $I = \int_0^1 x^n - x^{n+1} dx$ $I = \left[\frac{x^n}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$ $I = \left[\frac{1}{n+1} - \frac{1}{n+2}\right] - [0-0]$ $I = \frac{n+2-n-1}{(n+1)\,(n+2)}$ $I = \frac{1}{(n+1)\,(n+2)}$ ans.
Q.9)	$I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx$
Sol.9)	$I = \int_0^{\frac{\pi}{2}} \frac{sin^2 x}{sin\, x + cos\, x} dx$ ....(1) $I = \int_0^{\frac{\pi}{2}} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)} dx$ ....$(P-IV)$ $I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos x + \sin x} dx$ ....(2) $(1) + (2)$ $2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x} dx$ $(Type:- single\ sin\ x\ \&\ cos\ x)$ $2I = \int_0^{\frac{\pi}{2}} \frac{1}{\frac{2\tan x}{1+\tan^2\frac{x}{2}} + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}} dx$ $2I = \int_0^{\frac{\pi}{2}} \frac{1+tan^2\left(\frac{x}{2}\right)}{2\,tan\frac{x}{2} + 1 - tan^2\frac{x}{2}} dx$ $2I = \int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{2\tan\frac{x}{2} + 1 - \tan^2\left(\frac{x}{2}\right)} dx$ Put $\tan\left(\frac{x}{2}\right) = t$ when $x = 0$; $\tan(0) = t$ $\sec^2\left(\frac{x}{2}\right) . \frac{1}{2}\, dx = dt$ $t = 0$ $Sec^2\left(\frac{x}{2}\right) dx = 2dt$ when $x = \frac{\pi}{2}$; $\tan\left(\frac{\pi}{4}\right) = t$, $t = 1$

	$\therefore\ 2I = 2\int_0^1 \frac{dt}{-t^2+2t+1}$ $I = -\int_0^1 \frac{1}{t^2-2t-1}\,dt$ $= -\int_0^1 \frac{1}{(t-1)^2-1-1}\,dt$ $= -\int_0^1 \frac{1}{(t-1)^2-(\sqrt{2})^2}\,dt$ $= \int_0^1 \frac{1}{(\sqrt{2})^2-(t-1)^2}\,dt$ $= \frac{1}{2\sqrt{2}}\left[\log\left\|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right\|\right]_0^1$ $= \frac{1}{2\sqrt{2}}\left[\log\left\|\frac{\sqrt{2}-0}{\sqrt{2}+0}\right\| - \log\left\|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right\|\right]$ $= \frac{1}{2\sqrt{2}}\left[\log(1) - \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$ $I = \frac{-1}{2\sqrt{2}}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$ *ans.* (Or) $I = \frac{-1}{2\sqrt{2}}\log\left[\frac{(\sqrt{2}-1)(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}\right]$ ……{*Rationalize*} $= \frac{-1}{2\sqrt{2}}\log\left[\frac{(\sqrt{2}-1)^2}{2-1}\right]$ $= -\frac{2}{2\sqrt{2}}\log(\sqrt{2}-1)$ $I = \frac{-1}{\sqrt{2}}\log(\sqrt{2}-1)$ ans.
Q.10)	$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{1+\sin x.\cos x}\,dx$
Sol.10)	$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{1+\sin x.\cos x}\,dx$ …….(1) $I = \int_0^{\frac{\pi}{2}} \frac{\cos^2\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right).\cos\left(\frac{\pi}{2}-x\right)}\,dx$ $I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\cos x.\ \sin x}\,dx$ ……..(2) $(1) + (2)$ $2I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x.\cos x}\,dx$ ………$\{\sin^2 x + \cos^2 x = 1\}$ *Type: Divide by* $\cos^2 x$ *Divide N & D by* $\cos^2 x$ $2I = \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{\sec^2 x + \tan x}\,dx$ $2I = \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{1+\tan^2 + \tan x}\,dx$

Put $tan\,x = t$ when $x = 0$; $t = 0$

$\therefore$ $\sec^2 x.dx = dt$ when $x = \frac{\pi}{2}$; $t = \infty$

$\therefore$ $2I = \int_0^{\infty} \frac{dt}{t^2+t+1}$

Perfect square

$$2I = \int_0^{\infty} \frac{1}{\left(t+\frac{1}{2}\right)^2 - \frac{1}{4} + 1} dt$$

$$2I = \int_0^{\infty} \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$2I = \frac{2}{\sqrt{3}} \left[\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_0^{\infty}$$

$$2I = \frac{2}{\sqrt{3}} \left[\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^{\infty}$$

$$2I = \frac{2}{\sqrt{3}} \left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$2I = \frac{2}{\sqrt{2}} \left[\frac{\pi}{2} - \frac{\pi}{6}\right]$$

$$2I = \frac{2}{\sqrt{2}} \left[\frac{\pi}{3}\right]$$

$I = \frac{\pi}{3\sqrt{3}}$ ans.