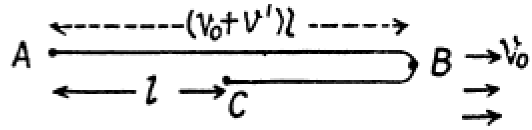
Kinematics (Part - 1)

Q. 1. A motorboat going downstream overcame a raft at a point A; τ = 60 min later it turned back and after some time passed the raft at a distance l = 6.0 km from the point A. Find the flow velocity assuming the duty of the engine to be constant.

Ans. 1. Let v_0 be the stream velocity and v' the velocity of motorboat with respect to water. The motorboat reached point B while going downstream with velocity $(v_0 + v')$ and then returned with velocity $(v' - v_0)$ and passed the raft at point C. Let t be the time for the raft (which flows with stream with velocity v_0) to move from point A to C, during which the motorboat moves from A to B and then from B to C.
Therefore

$$\frac{l}{v_0} = \tau + \frac{(v_0 + v')\tau - l}{(v' - v_0)}$$

On solving we get $v_0 = \frac{l}{2\tau}$



Q. 2. A point traversed half the distance with a velocity v_0. The remaining part of the distance was covered with velocity v_1 for half the time, and with velocity v_2 for the other half of the time. Find the mean velocity of the point averaged over the whole time of motion.

Ans. Let s be the total distance traversed by the point and tx the time taken to cover half the distance. Further let 2t be the time to cover the rest half of the distance.
Therefore

$$\frac{s}{2} = v_0 t_1 \quad \text{or} \quad t_1 = \frac{s}{2 v_0} \qquad (1)$$

and $$\frac{s}{2} = (v_1 + v_2) t \quad \text{or} \quad 2t = \frac{s}{v_1 + v_2} \qquad (2)$$

Hence the sought average velocity

$$\langle v \rangle = \frac{s}{t_1 + 2t} = \frac{s}{[s/2 v_0] + [s/(v_1 + v_2)]} = \frac{2 v_0 (v_1 + v_2)}{v_1 + v_2 + 2 v_0}$$

Q. 3. A car starts moving rectilinearly, first with acceleration ω = 5.0 m/s² (the initial velocity is equal to zero), then uniformly, and finally, decelerating at the same rate ω, comes to a stop. The total time of motion equals τ = 25 s. The average velocity during that time is equal to (v) = 72 km per hour. How long does the car move uniformly?

Ans. As the car starts from rest and finally comes to a stop, and the rate of acceleration and deceleration are equal, the distances as well as the times taken are same in these phases of motion.
Let Δf be the time for which the car moves uniformly. Then the acceleration / deceleration time is $\frac{\tau - \Delta t}{2}$ each. So,

$$\langle v \rangle \tau = 2\left\{\frac{1}{2} w \frac{(\tau - \Delta t)^2}{4}\right\} + w \frac{(\tau - \Delta t)}{2} \Delta t$$

or $\Delta t^2 = \tau^2 - \frac{4 \langle v \rangle \tau}{w}$

Hence $\Delta t = \tau \sqrt{1 - \frac{4 \langle v \rangle}{w \tau}} = 15 \text{ s.}$

Q. 4. A point moves rectilinearly in one direction. Fig. 1.1 shows the distance s traversed by the point as a function of the time t. Using the plot find:

(a) the average velocity of the point during the time of motion;
(b) the maximum velocity;
(c) the time moment t_0 at which the instantaneous velocity is equal to the mean velocity averaged over the first t_0 seconds.

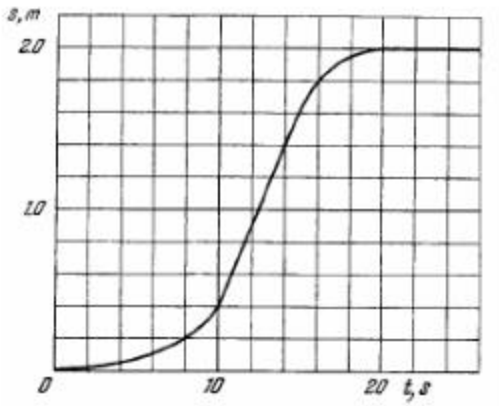
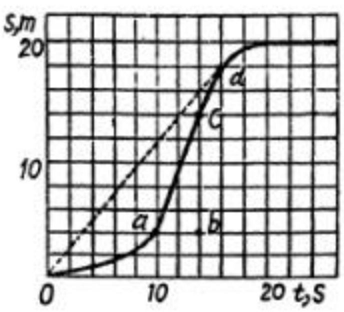


Fig. 1.1.

Ans.



(a) Sought average velocity

$$\langle v\rangle = \frac{s}{t} = \frac{200\text{ cm}}{20\text{s}} = 10\text{ cm/s}$$

(b) For the maximum velocity, ds/dt should be maximum. From the figure ds/dt is maximum for all points on the line ac, thus the sought maximum velocity becomes average velocity for the line ac and is equal to :

$$\frac{bc}{ab} = \frac{100\text{ cm}}{4\text{s}} = 25\text{ cm/s}$$

(c) Time t_0 should be such that corresponding to it the slope ds/dt should pass through the ds s point O (origin), to satisfy the relationship $\frac{ds}{dt} = \frac{s}{t_0}$. From figure the tangent at point d passes through the origin and thus corresponding time $t = t_0 = 16$ s.

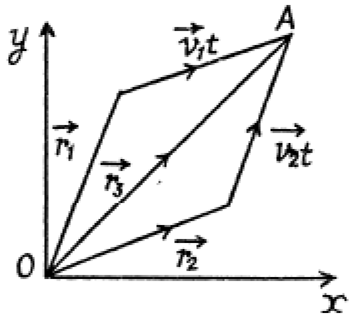
Q. 5. wo particles, 1 and 2, move with constant velocities v_1 and v_2. At the initial moment their radius vectors are equal to r_1 a nd r_2. How must these four vectors be interrelated for the particles to collide?

Ans. Xet the particles collide at the point A (Fig.), whose position vector is $\vec{r_3}$ (say). If t be the time taken by each particle to reach at point A, from triangle law of vector addition :

$$\vec{r}_3 = \vec{r}_1 + \vec{v}_1 t = \vec{r}_2 + \vec{v}_2 t$$

so, $\vec{r}_1 - \vec{r}_2 = (\vec{v}_2 - \vec{v}_1)\,t$ (1)

therefore, $t = \dfrac{|\vec{r}_1 - \vec{r}_2|}{|\vec{v}_2 - \vec{v}_1|}$ (2)



From Eqs. (1) and (2)

$$\vec{r}_1 = \vec{r}_2 - (\vec{v}_2 - \vec{v}_1)\frac{|\vec{r}_1 - \vec{r}_2|}{|\vec{v}_2 - \vec{v}_1|}$$

or, $\dfrac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|} = \dfrac{\vec{v}_2 - \vec{v}_1}{|\vec{v}_2 - \vec{v}_1|}$, , which is the sought relationship.

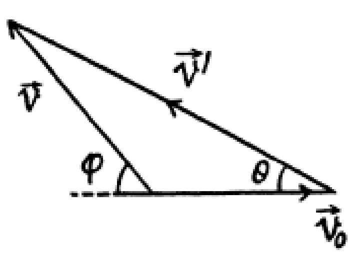
Q. 6. A ship moves along the equator to the east with velocity v_0 = 30 km/hour. The southeastern wind blows at an angle φ = 60° to the equator with velocity v = 15 km/hour. Find the wind velocity v' relative to the ship and the angle φ' between the equator and the wind direction in the reference frame fixed to the ship.

Ans. We have $\vec{v}' = \vec{v} - \vec{v}_0$ (1)

$$v' = \sqrt{v_0^2 + v^2 + 2\,v_0\,v\cos\varphi} = 39.7 \text{ km/hr} \quad (2)$$

and $\dfrac{v'}{\sin(\pi - \varphi)} = \dfrac{v}{\sin\theta}$ or, $\sin\theta = \dfrac{v\sin\varphi}{v'}$

or $\theta = \sin^{-1}\left(\dfrac{v\sin\varphi}{v'}\right)$



Using (2) and putting the values of v and d θ - 19.1'

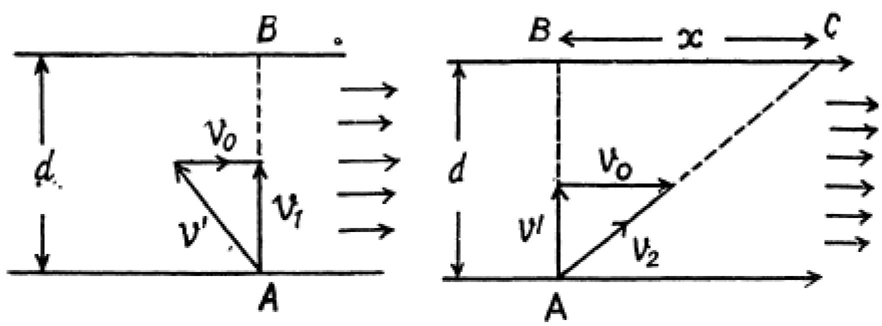
Q. 7. Two swimmers leave point A on one bank of the river to reach point B lying right across on the other bank. One of them crosses the river along the straight line AB while the other swims at right angles to the stream and then walks the distance that he has been carried away by the stream to get to point B. What was the velocity u of his walking if both swimmers reached the destination simultaneously? The stream velocity v_0 = 2.0 km/hour and the velocity if of each swimmer with respect to water equals 2.5 km per hour.

Ans. Let one of the swimmer (say 1) cross the river along AB, which is obviously the shortest path. Time taken to cross the river by the swimmer 1.

$$t_1 = \frac{d}{\sqrt{v'^2 - v_0^2}}, \quad \text{(where AB = d is the width of the river)} \qquad (1)$$

For the other swimmer (say 2), which follows the quickest-path, the time taken to cross the river.

$$t_2 = \frac{d}{v'} \qquad (2)$$



In the time t_2, drifting of the swimmer 2, becomes

$$x = v_0 t_2 = \frac{v_0}{v'} d, \quad \text{(using Eq. 2)} \qquad (3)$$

If t_3 be the time for swimmer 2 to walk the distance x to come from C to B (Fig.), the

$$t_3 = \frac{x}{u} = \frac{v_0 d}{v' u} \quad \text{(using Eq. 3)} \qquad (4)$$

According to the problem $t_1 = t_2 + 1_3$

$$\text{or } \frac{d}{\sqrt{v'^2 - v_0^2}} = \frac{d}{v'} + \frac{v_0 d}{v' u}$$

On solving we get

$$u = \frac{v_0}{\left(\frac{1 - v_0^2}{v'^2}\right)^{-\frac{1}{2}} - 1} = 3 \text{ km/hr.}$$

Q. 8. Two boats, A and B, move away from a buoy anchored at the middle of a river along the mutually perpendicular straight lines: the boat A along the river, and the boat B across thg river. Having moved off an equal distance from the buoy the boats returned. Find the ratio of times of motion of boats τ_A/τ_B if the velocity of each boat with respect to water is i1 = 1.2 times greater than the stream velocity.

Ans. Let / be the distance covered by the boat A along the river as well as by the boat 3 acre the river. Let v_0 be the stream velocity and v' the velocity of each boat with respect water. Therefore time taken by the boat A in its journey

$$t_A = \frac{l}{v' + v_0} + \frac{l}{v' - v_0}$$

and for the boat B $t_B = \frac{l}{\sqrt{v'^2 - v_0^2}} + \frac{l}{\sqrt{v'^2 - v_0^2}} = \frac{2l}{\sqrt{v'^2 - v_0^2}}$

Hence, $\frac{t_A}{t_B} = \frac{v'}{\sqrt{v'^2 - v_0^2}} = \frac{\eta}{\sqrt{\eta^2 - 1}}$ $\left(\text{where } \eta = \frac{v'}{v}\right)$

On substitution $t_A/t_B = 1{\cdot}8$

Q. 9. A boat moves relative to water with a velocity which is n = 2.0 times less than the river flow velocity. At what angle to the stream direction must the boat move to minimize drifting?

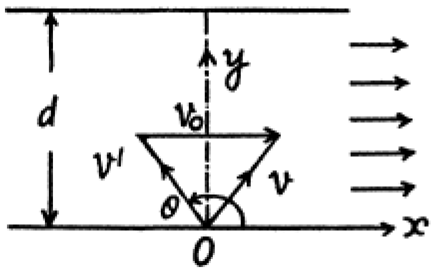
Ans. Let v0 be the stream velocity and v' the velocity of boat with respect to water. A $\frac{v_0}{v'} = \eta = 2 > 0,$ some drifting of boat is inevitable.

Let $\vec{v'}$ make an angle 0 with flow direction. (Fig.), then the time taken to cross the rive $t = \frac{d}{v' \sin\theta}$ (where d is the width of the river)

In this time interval, the drifting of the boat

$x = (v' \cos\theta + v_0)\, t$

$$= (v' \cos\theta + v_0) \frac{d}{v' \sin\theta} = (\cot\theta + \eta \operatorname{cosec}\theta)\, d$$



For x_{min} (minimum drifting)

$$\frac{d}{d\theta}(\cot\theta + \eta \operatorname{cosec}\theta) = 0, \text{ which yields}$$

$$\cos\theta = -\frac{1}{\eta} = -\frac{1}{2}$$

Hence, $\theta = 120°$

Q. 10. Two bodies were thrown simultaneously from the same point: one, straight up, and the other, at an angle of $\theta = 60°$ to the horizontal. The initial velocity of each body is equal to $v_0 = 25$ m/s. Neglecting the air drag, find the distance between the bodies $t = 1.70$ s later.

Ans. The solution of this problem becomes simple in the frame attached with one of the bodies.

Let the body thrown straight up be 1 and the other body be 2, then for the body 1 in the frame of 2 from the kinematic equation for constant acceleration :

$$\vec{r}_{12} = \vec{r}_{0(12)} + \vec{v}_{0(12)}\, t + \frac{1}{2}\vec{w}_{12}\, t^2$$

So, $\vec{r}_{12} = \vec{v}_{0(12)}\, t$, (because $\vec{w}_{12} = 0$ and $\vec{r}_{0(12)} = 0$)

or, $|\vec{r}_{12}| = |\vec{v}_{0(12)}|\, t \qquad (1)$

But $|\vec{v}_{01}| = |\vec{v}_{02}| = v_0$

So, from properties of triangle

$$v_{0(12)} = \sqrt{v_0^2 + v_0^2 - 2\, v_0\, v_0 \cos(\pi/2 - \theta_0)}$$

Hence, the sought distance

$$|\vec{r}_{12}| = v_0 \sqrt{2(1 - \sin\theta)}\; t = 22 \text{ m.}$$

Q. 11. Two particles move in a uniform gravitational field with an acceleration g. At the initial moment the particles were located at one point and moved with velocities v_1 = 3.0 m/s and v_2 = 4.0 m/s horizontally in opposite directions. Find the distance between the particles at the moment when their velocity vectors become mutually perpendicular.

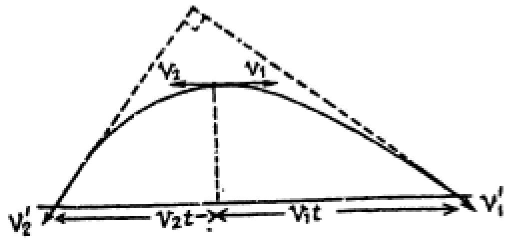
Ans. 1 Let the velocities of the pariti (say $\vec{v}_1{}'$ and $\vec{v}_2{}'$) be comes mutually perpendicular after time t. Then their velocitis become

$$\vec{v}_1{}' = \vec{v}_1 + \vec{g}\, t;\quad \vec{v}_2{}' = \vec{v}_2 + \vec{g}\, t \qquad (1)$$

As $\vec{v}_1{}' \perp \vec{v}_2{}'$ so, $\vec{v}_1{}' \cdot \vec{v}_2{}' = 0$

or, $(\vec{v}_1 + \vec{g}\, t) \cdot (\vec{v}_2 + \vec{g}\, t) = 0$

or $\quad -v_1 v_2 + g^2 t^2 = 0$



Hence, $t = \frac{\sqrt{v_1 v_2}}{g}$ (3)

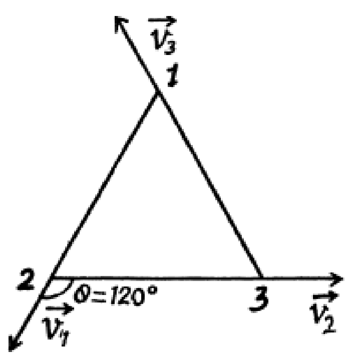
Now form the Eq. $\vec{r}_{12} = \vec{r}_{0(12)} + \vec{v}_{0(12)} t + \frac{1}{2} \vec{w}_{12} t^2$

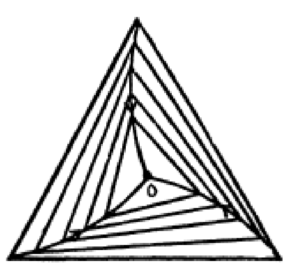
$|\vec{r}_{12}| = |\vec{v}_{0(12)}| t$, (because here $\vec{w}_{12} = 0$ and $\vec{r}_{0(12)} = 0$)

$|\vec{r}_{12}| = \frac{v_1 + v_2}{g} \sqrt{v_1 v_2}$ (as $|\vec{v}_{0(12)}| = v_1 + v_2$)

Q. 12. Three points are located at the vertices of an equilateral triangle whose side equals a. They all start moving simultaneously with velocity v constant in modulus, with the first point heading continually for the second, the second for the third, and the third for the first. How soon will the points converge?

Ans. From the symmetry of the problem all the three points are always located at the vertices of equilateral trian gles of varying side length and finally meet at the centriod of the initial equilateral triangle whose side length is a, in the sought time interval (say t).





Let us consider an arbitrary equilateral triangle of edge length / (say).

Then the rate by which 1 approaches 2, 2 approches 3, and 3 approches 1, becomes

$$\frac{-dl}{dt} = v - v\cos\left(\frac{2\pi}{3}\right)$$

On integrating : $-\int_a^0 dl = \frac{3v}{2}\int_0^t dt$

$$a = \frac{3}{2}vt \quad \text{so} \quad t = \frac{2a}{3v}$$

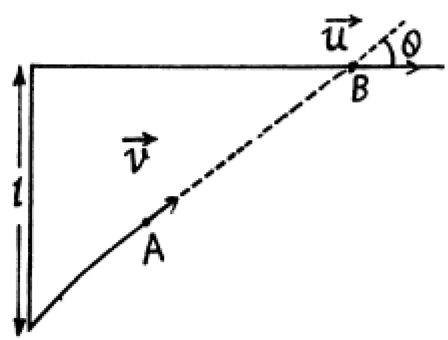
Q. 13. Point A moves uniformly with velocity v so that the vector v is continually "aimed" at point B which in its turn moves rectilinearly and uniformly with velocity u< v. At the initial moment of time v ⊥ u and the points are separated by a distance 1. How soon will the points converge?

Ans. Let us locate the points A and B at an arbitrary instant of time (Fig.).
If A and B are separated by the distance s at this moment, then the points converge or point A approaches B with velocity -ds/dt = v - u cos α where angle a varies with time.
On intergating,

$$-\int_l^0 ds = \int_0^T (v - u\cos\alpha)\,dt,$$

(where T is the sought time.)

$$\text{or} \qquad l = \int_0^T (v - u\cos\alpha)\,dt \qquad (1)$$



As both A and B cover the same distance in jc-direction during the sought time interval, so the other condition which is required, can be obtained by the equation

$$\Delta x = \int v_x \, dt$$

So, $uT = \int_0^T v \cos \alpha \, dt$ (2)

Solving (1) and (2), w e get $T = \frac{ul}{v^2 - u^2}$

One can see that if w = v, or u < v, point A cannot catch B.

Q. 14. A train of length l = 350 m starts moving rectilinearly with constant acceleration w = 3.0.10^{-2} m/s^2; t = 30 s after the start the locomotive headlight is switched on (event 1) , and τ = 60 s after that event the tail signal light is switched on (event 2) . Find the distance between these events in the reference frames fixed to the train and to the Earth. How and at what constant velocity V relative to the Earth must a certain reference frame K move for the two events to occur in it at the same point?

Ans. In the reference frame fixed to the train, the distance between the two events is obviously equal to l. Suppose the train starts moving at time t = 0 in the positive x direction and take the origin (x = 0) at the head-light o f the train at t = 0. Then the coordinate o f first event in the earth's frame is

$$x_1 = \frac{1}{2} w t^2$$

and similarly the coordinate of the second event is

$$x_2 = \frac{1}{2} w (t + \tau)^2 - l$$

The distance between the two events is obviously.

$$x_1 - x_2 = l - w\tau \, (t + \tau/2) = 0{\cdot}242 \text{ km}$$

in the reference frame fixed on the earth..

For the two events to occur at the same point in the reference frame K, moving with constant velocity V relative to the earth, the distance travelled by the frame in the time interval T must be equal to the above distance.

Thus $V\tau = l - w\tau(t + \tau/2)$

So, $V = \frac{l}{\tau} - w(t + \tau/2) = 4{\cdot}03$ m/s

The frame K must clearly be moving in a direction opposite to the train so that if (for example) the origin of the frame coincides with the point x_1 on the earth at time t, it coincides with the point x_2 at time t + τ.

Q. 15. An elevator car whose floor-to-ceiling distance is equal to 2.7 m starts ascending with constant acceleration 1.2 m/s^2; 2.0 s after the start a bolt begins falling from the ceiling of the car. Find:

(a) the bolt's free fall time;
(b) the displacement and the distance covered by the bolt during the free fall in the reference frame fixed to the elevator shaft.

Ans. (a) One good way to solve the problem is to work in the elevator's frame having the observer at its bottom (Fig.).

Let us denote the separation between floor and celing by h = 2-7 m. and the acceleration of the elevator by w = 1.2 m / s^2

From the kinematical formula

$$y = y_0 + v_{0y} t + \frac{1}{2} w_y t^2 \qquad (1)$$

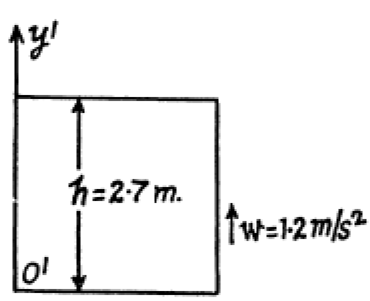
Here $y = 0, y_0 = +h, v_{0y} = 0$

and $w_y = w_{\text{bolt}(y)} - w_{\text{ele}(y)}$

$$= (-g) - (w) = -(g + w)$$

So, $0 = h + \frac{1}{2}\{-(g + w)\} t^2$

or, $t = \sqrt{\frac{2h}{g + w}} = 0{\cdot}7$ s.

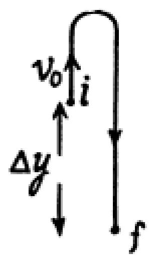


$$v_0 = (1 \cdot 2)(2) = 2 \cdot 4 \text{ m/s}$$

In the reference frame attached with the elevator shaft (ground) and pointing the y-axis upward, we have for the displacement of the bolt^

$$\Delta y = v_{0_y} t + \frac{1}{2} w_y t^2$$

$$= v_0 t + \frac{1}{2}(-g) t^2$$



or $\Delta y = (2 \cdot 4)(0 \cdot 7) + \frac{1}{2}(-9 \cdot 8)(0 \cdot 7)^2 = -0 \cdot 7 \text{ m.}$

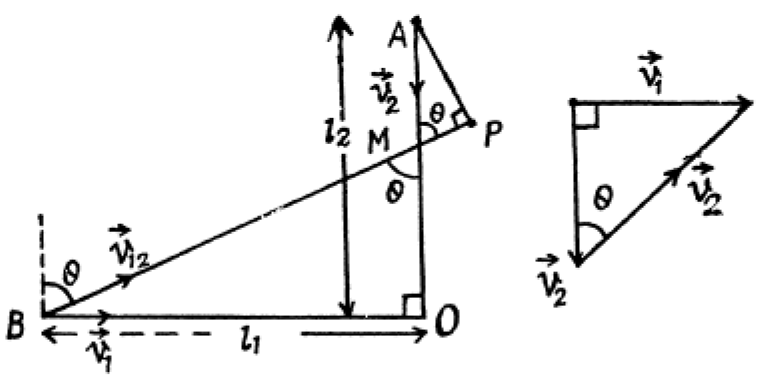
Hence the bolt comes down or displaces downward relative to the point, when it loses contact with the elevator by the amount 0.7 m (Fig.).
Obviously the total distance covered by the bolt during its free fall time

$$s = |\Delta y| + 2\left(\frac{v_0^2}{2g}\right) = 0 \cdot 7 \text{ m} + \frac{(2 \cdot 4)^2}{(9 \cdot 8)} \text{ m} = 1 \cdot 3 \text{ m.}$$

Kinematics (Part - 2)

Q. 16. Two particles, 1 and 2, move with constant velocities v_1 and v_2 along two mutually perpendicular straight lines toward the intersection point 0. At the moment t = 0 the particles were located at the distances l_1 a nd l_2 from the point 0. How soon will the distance between the particles become the smallest? What is it equal to?

Ans. Let the particle 1 and 2 be at points B and A at t = 0 at the distances l_1 and l_2 from intersection point O .
Let us fix the inertial frame with the particle 2. Now the particle 1 moves in relative to this reference frame with a relative velocity $\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$ and its trajectory is the straight line BP. Obviously, the minimum distance between the particles is equal to the length of the perpendicular AP dropped from point A on to the straight line BP (Fig.).



From Fig. (b), $v_{12} = \sqrt{v_1^2 + v_2^2}$, and $\tan\theta = \frac{v_1}{v_2}$ (1)

The shortest distaiice

$$AP = AM \sin\theta = (OA - OM)\sin\theta = (l_2 - l_1 \cot\theta)\sin\theta$$

or $AP = \left(l_2 - l_1 \frac{v_2}{v_1}\right) \frac{v_1}{\sqrt{v_1^2 + v_2^2}} = \frac{v_1 l_2 - v_2 l_1}{\sqrt{v_1^2 + v_2^2}}$ (using 1)

The sought time can be obtained directly from the condition that $(l_1 - v_1 t)^2 + (l_2 - v_2 t)^2$ is minimum. This gives $t = \frac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}$.

Q. 17. From point A located on a highway (Fig. 1.2) one has to get by car as soon as possible to point B located in the field at a distance 1 from the highway. It is known that the car moves in the field ri times slower than on the highway. At what distance from point D one must turn off the highway?

Ans. Let the car turn off the highway at a distance x from the point D.
So, CD = x, and if the speed of the car in the field is v, then the time taken by the car to cover the distance AC = AD - x on the highway

$$t_1 = \frac{AD - x}{\eta v} \quad (1)$$

and the time taken to travel the distance CB in the field

$$t_2 = \frac{\sqrt{l^2 + x^2}}{v} \quad (2)$$

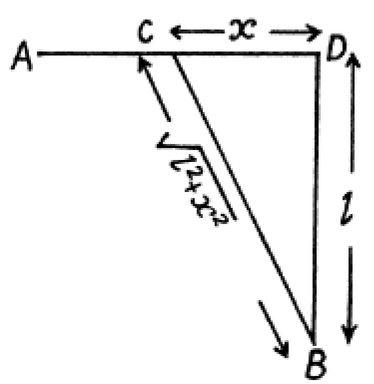
So, the total time elapsed to move the car from point A to B

$$t = t_1 + t_2 = \frac{AD - x}{\eta v} + \frac{\sqrt{l^2 + x^2}}{v}$$

For t to be minimum

$$\frac{dt}{dx} = 0 \quad \text{or} \quad \frac{1}{v}\left[-\frac{1}{\eta} + \frac{x}{\sqrt{l^2 + x^2}}\right] = 0$$

or $\eta^2 x^2 = l^2 + x^2$ or $x = \dfrac{l}{\sqrt{\eta^2 - 1}}$



Q. 18. A point travels along the x axis with a velocity whose projection vx is presented as a function of time by the plot in Fig. 1.3.

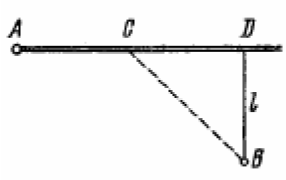


Fig. 1.2.

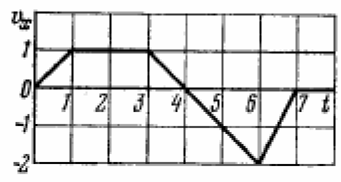


Fig. 1.3.

Assuming the coordinate of the point x = 0 at the moment t = 0, draw the approximate time dependence plots for the acceleration w_x, the x coordinate, and the distance covered s.

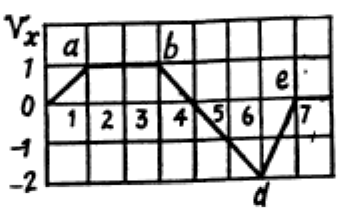
Ans. To plot x (f), s (t) and w_x (t) let us partion the given plot v_x (t) into five segments (for detailed analysis) as shown in the figure.
For the part oa : w_x =1 and v_x = t = v

Thus, $\Delta x_1(t) = \int v_x\,dt = \int_0^t dt = \frac{t^2}{2} = s_1(t)$

Putting $t = 1$, we get, $\Delta x_1 = s = \frac{1}{2}$ unit

For the part ab :

$w_x = 0$ and $v_x = v =$ constant $= 1$



Thus $\Delta x_2(t) = \int v_x\,dt = \int_1^t dt = (t-1) = s_2(t)$

Putting $t = 3$, $\Delta x_2 = s_2 = 2$ unit

For the part $b4$: $w_x = 1$ and $v_x = 1-(t-3) = 4-t) = v$

Thus $\Delta x_3(t) = \int_3^t (4-t)\,dt = 4t - \frac{t^2}{2} - \frac{15}{2} = s_3(t)$

Putting $t = 4$, $\Delta x_3 = x_3 = \frac{1}{2}$ unit

For the part $4d$: $v_x = -1$ and $v_x = -(1-4) = 4-$

So, $v = |v_x| = t - 4 \quad \text{for} \quad t > 4$

Thus $\Delta x_4(t) = \int_4^t (1 - t)\, dt = 4t - \frac{t^2}{2} - 8$

Putting $t = 6,\ \Delta x_4 = -1$

Similarly $s_4(t) = \int |v_x|\, dt = \int_4^t (t - 4)\, dt = \frac{t^2}{2} - 4t + 8$

Putting $t = 6,\ s_4 = 2$ unit

For the part d 7 : $w_x = 2$ and $v_x = -2 + 2(t - 6) = 2(t - 7)$

$v = |v_x| = 2(7 - t)$ for $t \leftarrow 7$

Now, $\Delta x(t) = \int_t^6 2\ (t - 7)\, dt = t^2 - 14t + 48$

Putting $t = 4,\ \Delta x_5 = -1$

Similarly $s_5(t) = \int_t^6 2(7 - t)\, dt = 14t - t^2 - 48$

Putting $t = 7,\ s_5 = 1$

On the basis of these obtained expressions w_x (r), x (;t) and s (t) plots can be easily plotted as shown in the figure of answersheet

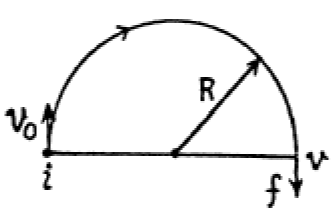
Q. 19. A point traversed half a circle of radius R = 160 cm during time interval τ = 10.0 s. Calculate the following quantities averaged over that time:
(a) the mean velocity (v);
(b) the modulus of the mean velocity vector |(v)|;
(c) the modulus of the mean vector of the total acceleration |(w)|

if the point moved with constant tangent acceleration.

Ans. (a) Mean velocity

$$\langle v \rangle = \frac{\text{Total distance covered}}{\text{Time elapsed}}$$

$$= \frac{s}{t} = \frac{\pi R}{\tau} = 50 \text{ cm/s} \qquad (1)$$



(b) Modulus of mean velocity vector

$$|\langle \vec{v} \rangle| = \frac{|\Delta \vec{r}|}{\Delta t} = \frac{2R}{\tau} = 32 \text{ cm/s} \qquad (2)$$

(c) Let the point moves from i to /along the half circle (Fig.) and v_0 and v be the spe at the points respectively.

We have $\frac{dv}{dt} = w_t$

or, $v = v_0 + w_t t$ (as wt is constant, according to the problem)

So, $$<v> = \frac{\int_0^t (v_0 + w_t t)\, dt}{\int_0^t dt} = \frac{v_0 + (v_0 + w_t t)}{2} = \frac{v_0 + v}{2}$$

So, from (1) and (3) $\frac{v_0 + v}{2} = \frac{\pi R}{\tau}$

Now the modulus of the mean vector of total acceleration

$$|<\vec{w}>| = \frac{|\Delta \vec{v}|}{\Delta t} = \frac{|\vec{v} - \vec{v}_0|}{\tau} = \frac{v_0 + v}{\tau} \quad \text{(see Fig.)}$$

Using (4) in (5), we get : $|<\vec{w}>| = \frac{2\pi R}{\tau^2}$

Q.20. A radius vector of a particle varies with time t as r = at (1 — αt), where a is a constant vector and α is a positive factor.
Find:
(a) the velocity v and the acceleration w of the particle as functions of time;
(b) the time interval Δt taken by the particle to return to the initial points, and the distance s covered during that time.

Ans. (a) we have $\vec{r} = \vec{a} t (1 - \alpha t)$

So $\vec{v} = \frac{d\vec{r}}{dt} = \vec{a}(1 - 2\alpha t)$

and $\vec{w} = \frac{d\vec{v}}{dt} = -2\alpha \vec{a}$

(b) From the equation

$\vec{r} = \vec{a} t (1 - \alpha t)$,

$\vec{r} = 0$, at $t = 0$ and also at $t = \Delta t = \frac{1}{\alpha}$

So, the sought time $\Delta t = \frac{1}{\alpha}$

As $\vec{v} = \vec{a}(1 - 2\alpha t)$

So $v = |\vec{v}| = \left.\begin{matrix} a(1-2\alpha t) \\ a(2\alpha t - 1) \end{matrix}\right\} \begin{matrix} \text{for } t \le \frac{1}{2\alpha} \\ \text{for } t > \frac{1}{2\alpha} \end{matrix}$

Hence, the sought distance

$$s = \int v\,dt = \int_0^{1/2\alpha} a(1-2\alpha t)\,dt + \int_{1/2\alpha}^{1/\alpha} a(2\alpha t - 1)\,dt$$

Simplifying, we get, $s = \frac{a}{2\alpha}$

Q.21. At the moment t = 0 a particle leaves the origin and moves in the positive direction of the x axis. Its velocity varies with time as $v = v_0$, $(1 — t/\tau)$, where v_0 is the initial velocity vector whose modulus equals v_0 = 10.0 cm/s; τ = 5.0 s.
Find:

(a) the x coordinate of the particle at the moments of time 6.0, 10, and 20 s;
(b) the moments of time when the particle is at the distance 10.0 cm from the origin;
(c) the distance s covered by the particle during the first 4.0 and 8.0 s; draw the approximate plot s (t).

Ans. (a) As the particle leaves the origin at t = 0

So, $\Delta x = x = \int v_x\,dt$ (1)

As $\vec{v} = \vec{v_0}\left(1 - \frac{t}{\tau}\right)$,

where $\vec{v_0}$ is s directed towards the +ve x-axis

So, $v_x = v_0\left(1 - \frac{t}{\tau}\right)$ (2)

From (1) and (2),

$$x = \int_0^t v_0\left(1 - \frac{t}{\tau}\right)dt = v_0 t\left(1 - \frac{t}{2\tau}\right) \quad (3)$$

Hence x coordinate of the particle at t = 6 s.

$$x = 10 \times 6\left(1 - \frac{6}{2 \times 5}\right) = 24 \text{ cm} = 0{\cdot}24 \text{ m}$$

Similarly at

$$x = 10 \times 10\left(1 - \frac{10}{2 \times 5}\right) = 0$$

and at

$$x = 10 \times 20\left(1 - \frac{20}{2 \times 5}\right) = -200 \text{ cm} = -2 \text{ m}$$

(b) At the moments the particle is at a distance of 10 cm from the origin, x = ± 10 cm. Putting x = +10 in Eq. (3)

$$10 = 10t\left(1 - \frac{t}{10}\right) \quad \text{or,} \quad t^2 - 10t + 10 = 0,$$

So, $t = t = \frac{10 \pm \sqrt{100 - 40}}{2} = 5 \pm \sqrt{15}$ s

Now putting x = -10in Eqn (3)

$$-10 = 10\left(1 - \frac{t}{10}\right),$$

On solving, $t = 5 \pm \sqrt{35}$ s

As t cannot be negative, so,

$$t = (5 + \sqrt{35}) \text{ s}$$

Hence the particle is at a distance of 10 cm from the origin at three moments of time :

$$t = 5 \pm \sqrt{15} \text{ s}, \ 5 + \sqrt{35} \text{ s}$$

(c) We have $\vec{v} = \vec{v_0}\left(1 - \frac{t}{\tau}\right)$

So, $v = |\vec{v}| = \left.\begin{matrix} v_0\left(1 - \frac{t}{\tau}\right) \\ v_0\left(\frac{t}{\tau} - 1\right) \end{matrix}\right\} \begin{matrix} \text{for } t \le \tau \\ \text{for } t > \tau \end{matrix}$

So $s = \int_0^t v_0\left(1 - \frac{t}{\tau}\right) dt$ for $t \le \tau = v_0 t (1 - t/2\tau)$

and

$$s = \int_0^\tau v_0\left(1 - \frac{t}{2}\right) dt + \int_\tau^t v_0\left(\frac{t}{\tau} - 1\right) dt \text{ for } t > \tau \qquad \text{(A)}$$

$= v_0 \tau [1 + (1 - t/\tau)^2] / 2$ for $t > \tau$

$$s = \int_0^4 v_0\left(1 - \frac{t}{\tau}\right) dt = \int_0^4 10\left(1 - \frac{t}{5}\right) dt = 24 \text{ cm.}$$

And for t = 8 s

$$s = \int_0^5 10\left(1 - \frac{t}{5}\right) dt + \int_5^8 10\left(\frac{t}{5} - 1\right) dt$$

On integrating and simplifying, we get s = 34 cm.

On the basis of Eqs. (3) and (4), x (t) and s (t) plots can be drawn as shown in the answer sheet.

Q.22. The velocity of a particle moving in the positive direction of the x axis varies a $v = \alpha\sqrt{x}$, here α is a positive constant. Assuming that at the moment t = 0 the particle was located at the point x = 0, find:
(a) the time dependence of the velocity and the acceleration of the particle;
(b) the mean velocity of the particle averaged over the time that the particle takes to cover the first s metres of the path.

Ans. As particle is in unidirectional motion it is directed along the jc-axis all the time. As at t = 0, x = 0

So, $\Delta x = x = s$, and $\frac{dv}{dt} = w$

Therefore, $v = \alpha\sqrt{x} = \alpha\sqrt{s}$

or, $w = \frac{dv}{dt} = \frac{\alpha}{2\sqrt{s}}\frac{ds}{dt} = \frac{\alpha}{2\sqrt{s}}$

$= \frac{\alpha v}{2\sqrt{s}} = \frac{\alpha\alpha\sqrt{s}}{2\sqrt{s}} = \frac{\alpha^2}{2}$ (1)

As, $w = \frac{dv}{dt} = \frac{\alpha^2}{2}$

On integrating, $\int_0^v dv = \int_0^t \frac{\alpha^2}{2} dt$ or, $v = \frac{\alpha^2}{2} t$ (2)

(b) Lets be the time to cover first s m of the path. From the Eq.

$$s = \int v\, dt$$

$$s = \int_0^t \frac{\alpha^2}{2}\, dt = \frac{\alpha^2}{2}\frac{t^2}{2} \qquad \text{(using 2)}$$

or $t = \frac{2}{\alpha}\sqrt{s}$ (3)

The mean velocity of particle

$$\langle v \rangle = \frac{\int v(t)\, dt}{\int dt} = \frac{\int_0^{2\sqrt{s}/\alpha} \frac{\alpha^2}{2} t\, dt}{2\sqrt{s}/\alpha} = \frac{\alpha\sqrt{s}}{2}$$

Q.23. A point moves rectilinearly with deceleration whose modulus depends on the velocity v of the particle as $w = a\sqrt{v}$, where a is a positive constant. At the initial moment the velocity of the point is equal to v_0. 'What distance will it traverse before it stops? What time will it take to cover that distance ?

Ans. According to the problem

$-\frac{v\,dv}{ds} = a\sqrt{v}$ (as v decreases with time)

or, $-\int_{v_0}^{0} \sqrt{v}\,dv = a\int_0^s ds$

On integrating w e get $s = \frac{2}{3a} v_0^{3/2}$

Again according to the problem

$-\frac{dv}{dt} = a\sqrt{v}$ or $-\frac{dv}{\sqrt{v}} = a\,dt$

or, $\int_{v_0}^{0} \frac{dv}{\sqrt{v}} = a\int_0^t dt$

Thus $t = \frac{2\sqrt{v_0}}{a}$

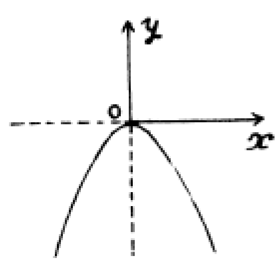
Q.24. A radius vector of a point A relative to the origin varies with time t as r = ati — bt^{2}j, where a and b are positive constants, and i and j are the unit vectors of the x and y axes.
Find:

(a) the equation of the point's trajectory y (x); plot this function;
(b) the time dependence of the velocity v and acceleration w vectors, as well as of the moduli of these quantities;
(c) the time dependence of the angle α between the vectors w and v;
(d) the mean velocity vector averaged over the first t seconds of motion, and the modulus of this vector

Ans. (a) As $\vec{r} = a t \vec{i} - b t^2 \vec{j}$

So, $x = a t$, $y = -b t^2$

and therefore $y = \frac{-b x^2}{a^2}$



which is Eq. of a parabola, whose graph is shown in the Fig.

(b) As $\vec{r} = at\vec{i} - bt^2\vec{j}$

$$\vec{v} = \frac{d\vec{r}}{dt} = a\vec{i} - 2bt\vec{j} \quad (1)$$

So, $v = \sqrt{a^2(-2bt)^2} = \sqrt{a^2 + 4b^2t^2}$

Diff. Eq. (1) w.r.t. time, we get

$$\vec{w} = \frac{d\vec{v}}{dt} = -2b\vec{j}$$

So, $|\vec{w}| = w = 2b$

(c) $\cos\alpha = \frac{\vec{v}\cdot\vec{w}}{vw} = \frac{(a\vec{i} - 2bt\vec{j})\cdot(-2b\vec{j})}{\left(\sqrt{a^2 + 4b^2t^2}\right)2b}$

or, $\cos\alpha = \frac{2bt}{\sqrt{a^2 + 4b^2t^2}}$,

so, $\tan\alpha = \frac{a}{2bt}$

or, $\alpha = \tan^{-1}\left(\frac{a}{2bt}\right)$

(d) The mean velocity vector

$$<\vec{v}> = \frac{\int \vec{v}dt}{\int dt} = \frac{\int_0^t (a\vec{i} - 2bt\vec{j})dt}{t} = a\vec{i} - bt\vec{j}$$

Hence, $|<\vec{v}>| = \sqrt{a^2 + (-bt)^2} = \sqrt{a^2 + b^2t^2}$

Q.25. A point moves in the plane xy according to the law x = at, y = at (1. — αt), where a and a are positive constants, and t is time.
Find:

(a) the equation of the point's trajectory y (x); plot this function;
(b) the velocity v and the acceleration w of the point as functions of time;
(c) the moment t_0 at which the velocity vector forms an angle π/4 with the acceleration vector.

Ans. (a) We have

x = at and y = αt (1 - αt)   (1)

Hence, y (x) becomes,

(b) Diiferentiating Eq. (1) we get

v_x = a and v_y = a (1 - 2αt)   (2)

So, $v = \sqrt{v_x^2 + v_y^2} = a\sqrt{1+(1-2\alpha t)^2}$

Diff. Eq. (2) with respect to time

$w_x = 0$ and $w_y = -2a\alpha$

So, $w = \sqrt{w_x^2 + w_y^2} = 2a\alpha$

(c) From Eqs. (2) and (3)

We have

$\vec{v} = a\vec{i} + a(1-2\alpha t)\vec{j}$ and $\vec{w} = 2a\alpha\vec{j}$

So, $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\vec{v}\cdot\vec{w}}{vw} = \frac{-a(1-2\alpha t_0)2a\alpha}{a\sqrt{1+(1-2\alpha t_0)^2}\,2a\alpha}$

On simplifying. $1-2\alpha t_0 = \pm 1$

As, $t_0 = 0,\ t_0 = \frac{1}{\alpha}$

Q.26. A point moves in the plane xy according to the law x = a sin cot, y = a (1 — cos ωt), where a and co are positive constants.
Find:

(a) the distance s traversed by the point during the time T;
(b) the angle between the point's velocity and acceleration vectors.

Ans. Differentiating motion law : x = a sin ωt , y = a (1 - cos ωt), with respect to time,
v_x = a ω cos ωt, v_y= a ω sin ωt

So, $\vec{v} = a\,\omega \cos \omega t\, \vec{i} + a\,\omega \sin \omega t\, \vec{j}$ (1)

and $v = a\,\omega = \text{Const.}$ (2)

Differentiating Eq. (1) with respect to time

$$\vec{w} = \frac{d\vec{v}}{dt} = -a\,\omega^2 \sin \omega t\, \vec{i} + a\,\omega^2 \cos \omega t\, \vec{j} \quad (3)$$

(a) The distance s traversed by the point during the time τ is given by

$$s = \int_0^{\tau} v\, dt = \int_0^{\tau} a\,\omega\, dt = a\,\omega\,\tau \quad (\text{using } 2)$$

(b) Taking inner product of $\vec{v}$ and $\vec{w}$

We get, $\vec{v} \cdot \vec{w} = (a\,\omega \cos \omega t\, \vec{i} + a\,\omega \sin \omega t\, \vec{j}) \cdot (a\,\omega^2 \sin \omega t\,(-i) + a\,\omega^2 \cos \omega t - \vec{j})$

So, $\vec{v} \cdot \vec{w} = -a^2\,\omega^2 \sin \omega t \cos \omega t + a^2\,\omega^3 \sin \omega t\, \cos \omega t = 0$

Thus, $\vec{v} \perp \vec{w}$, i.e., the angle between velocity vector and acceleration vector equals $\frac{\pi}{2}$.

Q.27. A particle moves in the plane xy with constant acceleration w directed along the negative y axis. The equation of motion of the particle has the form y = ax — bx^2, where a and b are positive constants. Find the velocity of the particle at the origin of coordinates.

Ans. Accordiing to the problem

$$\vec{w} = w(-\vec{j})$$

So, $w_x = \frac{dv_x}{dt} = 0$ and $w_y = \frac{dv_y}{dt} = -w$ (1)

Differentiating Eq. of trajectory, y - ax - bx^2, with respect to time

$$\frac{dy}{dt} = \frac{a\,dx}{dt} - 2bx\,\frac{dx}{dt} \quad (2)$$

So,

$$\left.\frac{dy}{dt}\right|_{x=0} = a\left.\frac{dx}{dt}\right|_{x=0}$$

Again differentiating with respect to time

$$\frac{d^2y}{dt^2} = \frac{a\,d^2x}{dt^2} - 2b\left(\frac{dx}{dt}\right)^2 - 2bx\,\frac{d^2x}{dt^2}$$

or, $-w = a(0) - 2b\left(\frac{dx}{dt}\right)^2 - 2bx(0)$ (using 1)

or, $\frac{dx}{dt} = \sqrt{\frac{w}{2b}}$ (using 1)

Using (3) in (2) $\left.\frac{dy}{dt}\right|_{x=0} = a\sqrt{\frac{w}{2b}}$

Hence, the velocity of the particle at the origin

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2_{x=0} + \left(\frac{dy}{dt}\right)^2_{x=0}} = \sqrt{\frac{w}{2b} + a^2\frac{w}{2b}} \text{ (using Eqns (3) and (4))}$$

Hence, $v = \sqrt{\frac{w}{2b}(1+a^2)}$

Q.28. A small body is thrown at an angle to the horizontal with the initial velocity v_0. Neglecting the air drag,
find:

(a) the displacement of the body as a function of time r (t);
(b) the mean velocity vector (v) averaged over the first t seconds and over the total time of motion.

Ans. As the body is under gravity of constant accelration $\vec{g}$, it's ve lo c ity vector and displacemen vectors are:

$\vec{v} = \vec{v_0} + \vec{g}t$ (1)

and $\Delta\vec{r} = \vec{r} = \vec{v_0}t + \frac{1}{2}gt^2$ ($\vec{r} = 0$ at $t = 0$) (2)

So, $<\vec{v}>$ over the first t seconds

$$<\vec{v}> = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}}{t} = \vec{v_0} + \frac{\vec{g}t}{2} \qquad (3)$$

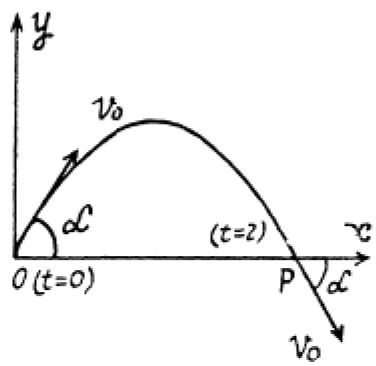
Hence from Eq. (3), $<\vec{v}>$ over the first t seconds

$$<\vec{v}> = \vec{v_0} + \frac{\vec{g}}{2}\tau \qquad (4)$$

For evaluating ty take

$$\vec{v}\cdot\vec{v} = (\vec{v_0}+\vec{g}t)\cdot(\vec{v_0}+\vec{g}t) = v_0^2 + 2(\vec{v_0}\cdot\vec{g})t + g^2t^2$$

or, $v^2 = v_0^2 + (\vec{v_0}\cdot\vec{g})t + g^2t^2$



But we have v = v_0 at t = 0 and

Also at t - x (Fig.) (also from energy conservation)

Hence using this propety in Eq. (5)

$$v_0^2 = v_0^2 + 2(\vec{v_0}\cdot\vec{g})\tau + g^2\tau^2$$

As $\tau \neq 0$, so, $\tau = -\frac{2(\vec{v}_o\cdot\vec{g})}{g^2}$

Putting this value of τ in Eq. (4), the average velocity over the time of flight

$$<\vec{v}> = \vec{v_0} - \vec{g}\frac{(\vec{v_0}\cdot\vec{g})}{g^2}$$

Q.29. A body is thrown from the surface of the Earth at an angle α to the horizontal with the initial velocity v_0. Assuming the air drag to be negligible, find:

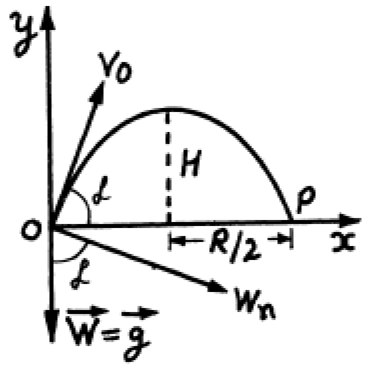
(a) the time of motion;
(b) the maximum height of ascent and the horizontal range; at what value of the angle α they will be equal to each other;
(c) the equation of trajectory y (x), where y and x are displacements of the body along the vertical and the horizontal respectively;
(d) the curvature radii of trajectory at its initial point and at its peak.

Ans. The body thrown in air with velocity v_0 at an angle a from the horizontal lands at point P on the Earth's surface at same horizontal level (Fig.). The point of projection is taken as origin, so, Δx = x and Δy = y

(a) From the Eq. $\Delta y = v_{0y} t + \frac{1}{2} w_y t^2$

$$0 = v_0 \sin\alpha\, \tau - \frac{1}{2} g\tau^2$$

As $\tau \neq 0$, so, time of motion $\tau = \frac{2 v_0 \sin\alpha}{g}$



(b) At the maximum height of ascent, $v_y = 0$

so, from the Eq. $v_y^2 = v_{0y}^2 + 2 w_y \Delta y$

$$0 = (v_0 \sin\alpha)^2 - 2 g H$$

Hence maximum height $H = \frac{v_0^2 \sin^2\alpha}{2g}$

During the time of motion the net horizontal displacement or horizontal range, will be obtained by the equation

$$\Delta x = v_{0x} t + \frac{1}{2} w_x \tau^2$$

or, $R = v_0 \cos\alpha\, \tau - \frac{1}{2}(0)\tau^2 = v_0 \cos\alpha\, \tau = \frac{v_0^2 \sin 2\alpha}{g}$

when R = H

$$\frac{v_0^2 \sin^2\alpha}{g} = \frac{v_0^2 \sin^2\alpha}{2g} \quad \text{or} \quad \tan\alpha = 4, \quad \text{so,} \quad \alpha = \tan^{-1} 4$$

(c) For the body, x (t) and y (t) are

$x = v_0 \cos\alpha\, t$ (1)

and $y = v_0 \sin\alpha\, t - \frac{1}{2} g t^2$ (2)

Hence putting the value of t from (1) into (2) we get,

$$y = v_0 \sin\alpha \left(\frac{x}{v_0 \cos\alpha}\right) - \frac{1}{2} g \left(\frac{x}{v_0 \cos\alpha}\right)^2 = x \tan\alpha - \frac{g x^2}{2 v_0^2 \cos^2\alpha},$$

Which is the sought equation of trajectory i.e. y (x)

(d) As the body thrown in air follows a curve, it has some normal acceleration at all the moments of time during it's motion in air.

At the initial point (x = 0, y = 0), from the equation :

$$w_n = \frac{v^2}{R}, \text{ (where } R \text{ is the radius of curvature)}$$

$$g \cos\alpha = \frac{v_0^2}{R_0} \text{ (see Fig.) or } R_0 = \frac{v_0^2}{g \cos\alpha}$$

At the peak point $v_y = 0,\ v = v_x = v_0 \cos\alpha$ and the angential acceleration is zero.

Now from the Eq. $w_n = \dfrac{v^2}{R}$

$$g = \frac{v_0^2 \cos^2\alpha}{R}, \quad \text{or} \quad R = \frac{v_0^2 \cos^2\alpha}{g}$$

Note : We may use the formula of curvature radius of a trajectory y (x), to solve part (d)y

$$R = \frac{\left[1 + (dy/dx)^2\right]^{\frac{3}{2}}}{\left| d^2 y / dx^2 \right|}$$

Q. 30. Using the conditions of the foregoing problem, draw the approximate time dependence of moduli of the normal w_n and tangent w_τ, acceleration vectors, as well as of the projection of the total acceleration vector w_v on the velocity vector direction.

Ans. We have, $v_x = v_0 \cos\alpha$, $v_y = v_0 \sin\alpha - gt$

As $\vec{v} \uparrow\uparrow \hat{u}_t$ all the moments of time.

Thus $v^2 = v_t^2 - 2 gt\, v_0 \sin\alpha + g^2 t^2$

Now, $w_t = \dfrac{dv_t}{dt} = \dfrac{1}{2 v_t} \dfrac{d}{dt}(v_t^2) = \dfrac{1}{v_t}(g^2 t - g\, v_0 \sin\alpha)$

$$= -\frac{g}{v_t}(v_0 \sin\alpha - g\,t) = -g\frac{v_y}{v_t}$$

Hence $|w_t| = g\frac{|v_y|}{v}$

Now $w_n = \sqrt{w^2 - w_t^2} = \sqrt{g^2 - g^2\frac{v_y^2}{v_t^2}}$

or $w_n = g\frac{v_x}{v_t}$ $\left(\text{where } v_x = \sqrt{v_t^2 - v_y^2}\right)$

As $\vec{v} \uparrow\uparrow \hat{v}_t$, during time o f motion

$$w_v = w_t = -g\frac{v_y}{v}$$

On the basis of obtained expressions or facts the sought plots can be drawn as shown in the figure of answer sheet

Kinematics (Part - 3)

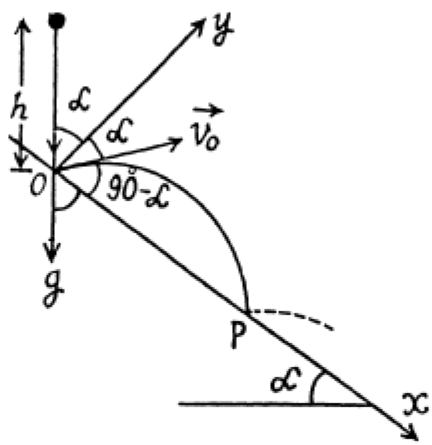
Q. 31. A ball starts falling with zero initial velocity on a smooth inclined plane forming an angle a with the horizontal. Having fallen the distance h, the ball rebounds elastically off the inclined plane. At what distance from the impact point will the -ball rebound for the second time?

Ans. The ball strikes the inclined plane (Ox) at point O (origin) with

velocity $v_0 = \sqrt{2gh}$ As the ball elastically rebounds, it recalls with same velocity v_0, at the same angle a from the normal or y axis (Fig.). Let the ball strikes the incline second time at P, which is at a distance l (say) from the point O, along the incline. From the equation $y = v_{0y}t + \frac{1}{2}w_y t^2$

where τ is the time of motion of ball in air while moving from O to P.

$$\text{As} \quad \tau \neq 0, \;\; \text{so,} \;\; \tau = \frac{2v_0}{g} \qquad (2)$$



Now from me equation.

$$x = v_{0x}t + \frac{1}{2}w_x t^2$$

$$l = v_0 \sin\alpha\,\tau + \frac{1}{2} g \sin\alpha\,\tau^2$$

so, $l = v_0 \sin\alpha \left(\frac{2 v_0}{g}\right) + \frac{1}{2} g \sin\alpha \left(\frac{2 v_0}{g}\right)$

$= \frac{4 v_0^2 \sin\alpha}{g}$ (using 2)

Hence the sought distance, $l = \frac{4 (2 gh) \sin\alpha}{g} = 8h \sin\alpha$ (Using Eq. 1)

Q. 32. A cannon and a target are 5.10 km apart and located at the same level. How soon will the shell launched with the initial velocity 240 m/s reach the target in the absence of air drag?

Ans. Total time of motion

$$\tau = \frac{2 v_0 \sin\alpha}{g} \text{ or } \sin\alpha = \frac{\tau g}{2 v_0} = \frac{9{\cdot}8\,\tau}{2 \times 240} \quad (1)$$

and horizontal range

$$R = v_0 \cos\alpha\,\tau \text{ or } \cos\alpha = \frac{R}{v_0 \tau} = \frac{5100}{240\,\tau} = \frac{85}{4\tau} \quad (2)$$

From Eqs. (1) and (2)

$$\frac{(9{\cdot}8)^2 \tau^2}{(480)^2} + \frac{(85)^2}{(4\tau^2)^2} = 1$$

On simplifying $\tau^4 - 2400\,\tau^2 + 1083750 = 0$

$$\tau^2 = \frac{2400 \pm \sqrt{1425000}}{2} = \frac{2400 \pm 1194}{2}$$

Solving for τ^2 wc get :

Thus

$\tau =$ 42.39 s = 0.71 min and

τ = 24-55 s = 0.41 min depending on the angle α.

Q. 33. A cannon fires successively two shells with velocity v_0 = 250 m/s; the first at the angle θ_1 = 60° and the second at the angle θ_2 = 45° to the horizontal, the azimuth being the same. Neglecting the air drag, find the time interval between firings leading to the collision of the shells.

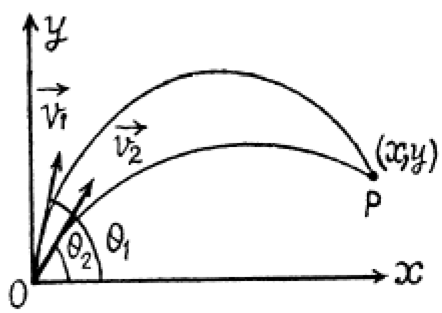
Ans. Let the shells collide at the point P (x, y). If the first shell takes t s to collide with second and Δt be the time interval between the firings, then

$$x = v_0 \cos\theta_1 \, t = v_0 \cos\theta_2 \,(t - \Delta t) \quad (1)$$

and $y = v_0 \sin\theta_1 \, t - \frac{1}{2} g t^2$

$$= v_0 \sin\theta_2 \,(t - \Delta t) - \frac{1}{2} g \,(t - \Delta t)^2$$

From Eq. (1) $t = \dfrac{\Delta t \cos\theta_2}{\cos\theta_2 - \cos\theta_1}$ (3)



From Eqs. (2) and (3)

$$\Delta t = \frac{2 v_0 \sin(\theta_1 - \theta_2)}{g(\cos\theta_2 + \cos\theta_1)} \quad \text{as } \Delta t \neq 0$$

Q. 34. A balloon starts rising from the surface of the Earth. The ascension rate is constant and equal to v_0. Due to the wind the balloon gathers the horizontal velocity component v_x = ay, where a is a constant and y is the height of ascent. Find how the following quantities depend on the height of ascent:

(a) the horizontal drift of the balloon x (y);
(b) the total, tangential, and normal accelerations of the balloon.

Ans. According to the problem

(a) $\frac{dy}{dt} = v_0$ or $dy = v_0 \, dt$

Integrating $\int_0^y dy = v_0 \int_0^t dt$ or $y = v_0 t$ (1)

And also we have $\frac{dx}{dt} = ay$ or $dx = a\,y\,dt = a\,v_0\,t\,dt$ (using 1)

So, $\int_0^x dx = a\,v_0 \int_0^t t\,dt$, or, $x = \frac{1}{2} a\,v_0\,t^2 = \frac{1}{2}\frac{a\,y^2}{v_0}$ (using 1)

(b) According to the problem

$v_y = v_0$ and $v_x = a\,y$ (2)

So, $v = \sqrt{v_x^2 + v_y^2} = \sqrt{v_0^2 + a^2 y^2}$

Therefore $w_t = \frac{dv}{dt} = \frac{a^2 y}{\sqrt{v_0^2 + a\,y^2}}\frac{dy}{dt} = \frac{a^2 y}{\sqrt{1 + (ay/v_0)^2}}$

Diff. Eq. (2) with respect to time.

$\frac{d\,v_y}{dt} = w_y = 0$ and $\frac{d\,v_x}{dt} = w_x = a\frac{dy}{dt} = a\,v_0$

So, $w = |w_x| = a\,v_0$

Hence $w_n = \sqrt{w^2 - w_t^2} = \sqrt{a^2 v_0^2 - \frac{a^4 y^2}{1 + (ay/v_0)^2}} = \frac{a\,v_0}{\sqrt{1 + (ay/v_0)^2}}$

Q. 35. A particle moves in the plane xy with velocity v = ai + bxj, where i and j are the unit vectors of the x and y axes, and a and b are constants. At the initial moment of time the particle was located at the point x = y = 0. Find:

(a) the equation of the particle's trajectory y (x);
(b) the curvature radius of trajectory as a function of x.

Ans. (a) The velocity vector of the particle

$\vec{v} = a\vec{i} + bx\vec{j}$

So, $\frac{dx}{dt} = a : \frac{dy}{dt} = bx$ (1)

From (1) $\int_0^x dx = a\int_0^t dt$ or, $x = at$ (2)

And $dy = bx\,dt = bat\,dt$

Integrating $\int_0^y dy = ab\int_0^t t\,dt$ or, $y = \frac{1}{2}ab\,t^2$ (3)

From Eqs. (2) and (3) , we get, $y = \frac{b}{2a}x^2$ (4)

$$R = \frac{\left[1 + (dy/dx)^2\right]^{\frac{3}{2}}}{\left|d^2y/dx^2\right|}$$

Let us differentiate the path Eq. $y = \frac{b}{2a}x^2$ w ith resp ect to x,

$\frac{dy}{dx} = \frac{b}{a}x$ and $\frac{d^2y}{dx^2} = \frac{b}{a}$

From Eqs. (5) and (6), the sought curvature radius :

$$R = \frac{a}{b}\left[1 + \left(\frac{b}{a}x\right)^2\right]^{\frac{3}{2}}$$

Q. 36. A particle A moves in one direction along a given trajectory with a tangential acceleration wτ = aτ, where a is a constant vector coinciding in direction with the x axis (Fig. 1.4), and τ is a unit vector coinciding in direction with the velocity vector at a given point. Find how the velocity of the particle depends on x provided that its velocity is negligible at the point x = 0.

Ans. In accordance with the problem $w_t = \vec{a} \cdot \vec{\tau}$

But $w_t = \frac{v\,dv}{ds}$ or $v\,dv = w_t\,ds$

So, $v\,dv = (\vec{a} \cdot \vec{\tau})\,ds = \vec{a} \cdot d\vec{r}$

or, $v\,dv = a\,\vec{i} \cdot d\vec{r} = a\,dx$ (because $\vec{a}$ is directed towards the jc-axis)

So, $\int_0^v v\,dv = a \int_0^x dx$

Hence $v^2 = 2\,ax$ or, $v = \sqrt{2\,ax}$

Q. 37. A point moves along a circle with a velocity v = at, where a = 0.50 m/s^2. Find the total acceleration of the point at the mo- merit when it covered the n-th (n = 0.10) fraction of the circle after the beginning of motion.

Ans. The velocity of the particle v = at

So, $\frac{dv}{dt} = w_t = a$ (1)

And $w_n = \frac{v^2}{R} = \frac{a^2 t^2}{R}$ (using $v = at$) (2)

From $s = \int v\,dt$

$\cdot 2\,\pi R\,\eta = a \int_0^t v\,dt = \frac{1}{2} at^2$

So, $\frac{4\,\pi\,\eta}{a} = \frac{t^2}{R}$ (3)

From Eqs. (2) and (3) $w_n = 4\pi a\eta$

Hence $w = \sqrt{w_t^2 + w_n^2}$

$= \sqrt{a^2 + (4\,\pi\,a\,\eta)^2} = a\sqrt{1 + 16\,\pi^2\,\eta^2} = 0{\cdot}8\text{ m/s}^2$

Q. 38. A point moves with deceleration along the circle of radius R so that at any moment of time its tangential and normal accelerations are equal in moduli. At the initial moment t = 0 the velocity of the point equals v_0. Find:

(a) the velocity of the point as a function of time and as a function of the distance covered s;
(b) the total acceleration of the point as a function of velocity and the distance covered.

Fig. 1.4.

Ans. (a) According to the problem

$$|w_t| = |w_n|$$

For v (t), $\frac{-dv}{dt} = \frac{v^2}{R}$

Integrating this equation from $v_0 \le v \le v$ and $0 \le t \le t$

$$-\int_{v_0}^{v} \frac{dv}{v^2} = \frac{1}{R}\int_0^t dt \quad \text{or,} \quad v = \frac{v_0}{\left(1 + \frac{v_0 t}{R}\right)}$$

Now for $v(s)$, $-\frac{v\,dv}{ds} = \frac{v^2}{R}$, Integrating this equation from $v_0 \le v \le v$ and $0 \le s \le s$

So, $\int_{v_0}^{v} \frac{dv}{v} = -\frac{1}{R}\int_0^s ds$ or, $\ln\frac{v}{v_0} = -\frac{s}{R}$

Hence $v = v_0 e^{-s/R}$

(b) The normal acceleration of the point

$$w_n = \frac{v^2}{R} = \frac{v_0^2 e^{-2s/R}}{R} \quad \text{(using 2)}$$

And as accordance with the problem

$|w_t| = |w_n|$ and $w_t \hat{u}_t \perp w_n \hat{u}_n$

so, $w = \sqrt{2}\, w_n = \sqrt{2}\,\frac{v_0^2}{R} e^{-2s/R} = \sqrt{2}\,\frac{v^2}{R}$

Q. 39. A point moves along an arc of a circle of radius R. Its velocity depends on the distance covered s as $v = a\sqrt{s}$, where a is a constant. Find the angle α between the vector of the total acceleration and the vector of velocity as a function of s.

Ans. From the equation $v = a\sqrt{s}$

$$w_t = \frac{dv}{dt} = \frac{a}{2\sqrt{s}}\frac{ds}{dt} = \frac{a}{2\sqrt{s}}\, a\sqrt{s} = \frac{a^2}{2}, \text{ and}$$

$$w_n = \frac{v^2}{R} = \frac{a^2 s}{R}$$

As w, is a positive constant, the speed of the particle increases with time, and the tangential acceleration vector and velocity vector coincides in direction.

Hence the angle between $\vec{v}$ and $\vec{w}$ is equal to between $w_t \hat{u}_t$ an $\vec{w}$, and α can be found by means of the formula : $\tan\alpha = \frac{|w_n|}{|w_t|} = \frac{a^2 s/R}{a^2/2} = \frac{2s}{R}$

Q. 40. A particle moves along an arc of a circle of radius R according to the law l = a sin ωt, where l is the displacement from the initial position measured along the arc, and a and ω are constants. Assuming R = 1.00 m, a = 0.80 m, and co = 2.00 rad/s, find:

(a) the magnitude of the total acceleration of the particle at the points l = 0 and l = ±a;
(b) the minimum value of the total acceleration w_{min} and the corresponding displacement l_m.

Ans. From the equation $l = a \sin \omega t$

$$\frac{dl}{dt} = v = a\omega \cos \omega t$$

So, $w_t = \frac{dv}{dt} = -a\omega^2 \sin \omega t$, and (1)

$$w_n = \frac{v^2}{R} = \frac{a^2\omega^2 \cos^2 \omega t}{R} \quad (2)$$

(a) At the point $l = 0, \sin \omega t = 0$ and $\cos \omega t = \pm 1$ so, $\omega t = 0, \pi$ etc.

Hence $w = w_n = \frac{a^2\omega^2}{R}$

Similarly at $l = \pm a$, $\sin \omega t = \pm 1$ and $\cos \omega t = 0$, so, $w_n = 0$

Hence $w = |w_t| = a\omega^2$

Q. 41. A point moves in the plane so that its tangential acceleration $w_\tau = a$, and its normal acceleration $w_n = bt^4$, where a and b are positive constants, and t is time. At the moment t = 0 the point was at rest. Find how the curvature radius R of the point's trajectory and the total acceleration w depend on the distance covered s.

Ans. As $w_t = a$ and at t = 0, the point is at rest

So, $v(t)$ and $s(t)$ are, $v = at$ and $s = \frac{1}{2}at^2$ (1)

Let R be the curvature radius, then

$$w_n = \frac{v^2}{R} = \frac{a^2t^2}{R} = \frac{2\,as}{R} \text{ (using 1)}$$

But according to the problem

$$w_n = bt^4$$

So, $bt^4 = \frac{a^2t^2}{R}$ or, $R = \frac{a^2}{bt^2} = \frac{a^2}{2\,bs}$ (using 1)

Therefore $w = \sqrt{w_t^2 + w_n^2} = \sqrt{a^2 + (2\,as/R)^2} = \sqrt{a^2 + \left(4\,bs^2/a^2\right)^2}$ (using 2)

Hence $w = a\sqrt{1 + \left(4\,bs^2/a^3\right)^2}$

Q. 42. A particle moves along the plane trajectory y (x) with velocity v whose modulus is constant. Find the acceleration of the particle at the point x = 0 and the curvature radius of the trajectory at that point if the trajectory has the form

(a) of a parabola $y = ax^2$;
(b) of an ellipse $(xla)^2$ $(y/b)^2$ = 1; a and b are constants here.

Ans. (a) Let us differentiate twice the path equation y (x) with respect to time.

$$\frac{dy}{dt} = 2\,ax\frac{dx}{dt};\ \frac{d^2y}{dt^2} = 2a\left[\left(\frac{dx}{dt}\right)^2 + x\frac{d^2x}{dt^2}\right]$$

Since the particle moves uniformly, its acceleration at all points of the path is normal and at the point x

= 0 it coincides with the direction of derivative d^2y/dt^2 Keeping in mind that at the point $x = 0,\ \left|\frac{dx}{dt}\right| = v,$

We get $w = \left|\frac{d^2y}{dt^2}\right|_{x=0} = 2\,a\,v^2 = w_n$

So, $w_n = 2\,a\,v^2 = \frac{v^2}{R}$, or $R = \frac{1}{2a}$

Note that we can also calculate it from the formula of problem (1.35 b)

(b) Differentiating the equation of the trajectory with respect to time we see that

$$b^2 x \frac{dx}{dt} + a^2 y \frac{dy}{dt} = 0 \quad (1)$$

which implies that the vector $(b^2 x \vec{i} + a^2 y \vec{j})$ is normal to the velocity vector $\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}$ which, of course, is along the tangent. Thus the former vactor is along the normal and the normal component of acceleration is clearly

$$w_n = \frac{b^2 x \frac{d^2x}{dt^2} + a^2 y \frac{d^2y}{dt^2}}{(b^4x^2 + a^4y^2)^{1/2}}$$

on using $w_n = \vec{w} \cdot \vec{n} / |\vec{n}|$. At $x = 0$, $y = \pm b$ and so at $x = 0$

$$w_n = \pm \left.\frac{d^2y}{dt^2}\right|_{x=0}$$

Differentiating (1)

$$b^2\left(\frac{dx}{dt}\right)^2 + b^2x\left(\frac{d^2x}{dt^2}\right) + a^2\left(\frac{dy}{dt}\right)^2 + a^2y\left(\frac{d^2y}{dt^2}\right) = 0$$

Also from (1) $\frac{dy}{dt} = 0$ at $x = 0$

So, $\left(\frac{dx}{dt}\right) = \pm v$ (since tangential velocity is constant = v)

Thus $\left(\frac{d^2y}{dt^2}\right) = \pm \frac{b}{a^2} v^2$

and $|w_n| = \frac{bv^2}{a^2} = \frac{v^2}{R}$

This gives $R = a^2/b$.

Q. 43. A particle A moves along a circle of radius R = 50 cm so that its radius vector r relative to the point O (Fig. 1.5) rotates with the constant angular velocity ω = 0.40 rad/s. Find the modulus of the velocity of the particle, and the modulus and direction of its total acceleration.

Ans. Let us fix the co-ordinate system at the point O as shown in the figure, such that the radius vector $\vec{r}$ of point A makes an angle θ with x axis at the moment shown.
Note that the radius vector of the particle A rotates clockwise and we here take line ox

as reference line, so in this case obviously the angular velocity $\omega = \left(-\frac{d\theta}{dt}\right)$ taking anticlockwise sense of angular displacement as positive.

Also from the geometry of the triangle OAC $\frac{R}{\sin\theta} = \frac{r}{\sin(\pi - 2\theta)}$ or, $r = 2R\cos\theta.$

Let us write,

$$\vec{r} = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} = 2R\cos^2\theta\,\vec{i} + R\sin 2\theta\,\vec{j}$$

Differentiating with respect to time.

$$\frac{d\vec{r}}{dt} \text{ or } \vec{v} = 2R\,2\cos\theta(-\sin\theta)\frac{d\theta}{dt}\vec{i} + 2R\cos 2\theta\frac{d\theta}{dt}\vec{j}$$

$$\text{or, } \vec{v} = 2R\left(\frac{-d\theta}{dt}\right)[\sin 2\theta\,\vec{i} - \cos 2\theta\,\vec{j}]$$

$$\text{or, } \vec{v} = 2R\omega(\sin 2\theta\,\vec{i} - \cos^2\theta\,\vec{j})$$

$$\text{So, } \qquad |\vec{v}| \text{ or } v = 2\omega R = 0{\cdot}4\text{ m/s.}$$

As ω is constant, v is also constant and $w_t = \frac{dv}{dt} = 0,$

So, $w = w_n = \frac{v^2}{R} = \frac{(2\omega R)^2}{R} = 4\omega^2 R = 0{\cdot}32 \text{ m/s}^2$

Thus we have the problem of finding the velocity and acceleration of a particle moving along a circle of radius R with constant angular velocity 2 ω

Hence

$\cdot\ v = 2\omega R$ and

$$w = w_n = \frac{v^2}{R} = \frac{(2\omega R)^2}{R} = 4\omega^2 R$$

Q. 44. A wheel rotates around a stationary axis so that the rotation angle φ varies with time as φ = at^2, where a = 0.20 rad/s2. Find the total acceleration w of the point A at the rim at the moment t = 2.5 s if the linear velocity of the point A at this moment v = 0.65 m/s.

Ans. Differentiating φ (t) with respect to time

$$\frac{d\varphi}{dt} = \omega_z = 2at \quad (1)$$

For fixed axis rotation, the speed of the point A:

$$v = \omega R = 2atR \text{ or } R = \frac{v}{2at} \quad (2)$$

Differentiating with respect to time

$$w_t = \frac{dv}{dt} = 2aR = \frac{v}{t}, \text{ (using 1)}$$

But $w_n = \frac{v^2}{R} = \frac{v^2}{v/2at} = 2atv$ (using 2)

So, $w = \sqrt{w_t^2 + w_n^2} = \sqrt{(v/t)^2 + (2atv)^2}$

$$= \frac{v}{t}\sqrt{1 + 4a^2t^4}$$

Q. 45. A shell acquires the initial velocity v = 320 m/s, having made n = 2.0 turns inside the barrel whose length is equal to l = 2.0 m. Assuming that the shell moves inside the barrel with a uniform acceleration, find the angular velocity of its axial rotation at the moment when the shell escapes the barrel.

Ans. The shell acquires a constant angular acceleration at the same time as it accelerates linearly. The two are related by (assuming both are constant)

$$\frac{w}{l} = \frac{\beta}{2\pi n}$$

Where w = linear acceleration and β = angular acceleration

Then, $\omega = \sqrt{2 \cdot \beta \, 2\pi n} = \sqrt{2 \cdot \frac{w}{l}(2\pi n)^2}$

But $v^2 = 2wl$, hence finally

$$\omega = \frac{2\pi n v}{l}$$

Kinematics (Part - 4)

Q. 46. A solid body rotates about a stationary axis according to the law φ = at - bt^3, where a = 6.0 rad/s and b = 2.0 rad/s^3. Find:

(a) the mean values of the angular velocity and angular acceleration averaged over the time interval between t = 0 and the complete stop;
(b) the angular acceleration at the moment when the body stops.

Ans. Let us take the rotation axis as z-axis whose positive direction is associated with the positive direction o f the cordinate <p, the rotation angle, in accordance w ith the right-hand screw rule (Fig.)

(a) Defferentiating φ (t) with respect to time.

$$\frac{d\varphi}{dt} = a - 3bt^2 = \omega_z \qquad \text{(1) and}$$

$$\frac{d^2\varphi}{dt^2} = \frac{d\omega_z}{dt} = \beta_z = -6bt \qquad (2)$$

From (1) the solid comes to stop at $\Delta t = t = \sqrt{\frac{a}{3b}}$

The angular velocity $\omega = a - 3bt^2$, for $0 \le t \le \sqrt{a/3b}$

So, $$< \omega > = \frac{\int \omega \, dt}{\int dt} = \frac{\int_0^{\sqrt{a/3b}} (a - 3bt^2)\, dt}{\int_0^{\sqrt{a/3b}} dt} = [at - bt^3]_0^{\sqrt{a/3b}} \Big/ \sqrt{a/3b} = 2a/3$$

Sim ilarly $\beta = |\beta_z| = 6bt$ for all values of t.

So, $$< \beta > = \frac{\int \beta\, dt}{\int dt} = \frac{\int_0^{\sqrt{a/3b}} 6bt\, dt}{\int_0^{\sqrt{a/3b}} dt} = \sqrt{3ab}$$

(b) From Eq. (2) $\beta_z = -6bt$

So, $$(\beta_z)_{t = \sqrt{a/3b}} = -6b\sqrt{\frac{a}{3b}} = -2\sqrt{ab}$$

Hence $$\beta = \left|(\beta_z)_{t = \sqrt{a/3b}}\right| = 2\sqrt{3ab}$$

Q. 47. A solid body starts rotating about a stationary axis with an angular acceleration β = at, where a = 2.0.10^{-2} rad/s^3. How soon after the beginning of rotation will the total acceleration vector of an arbitrary point of the body form an angle α = 60° with its velocity vector?

Ans. Angle a is related with |wt| and wn by means of the fomula :

$$\tan\alpha = \frac{w_n}{|w_t|}, \text{ where } w_n = \omega^2 R \text{ and } |w_t| = \beta R \quad (1)$$

where R is die radius of die circle which an arbitrary point of the body circumscribes.

From die given equation $\beta = \frac{d\omega}{dt} = at$ (here $\beta = \frac{d\omega}{dt}$, as β is positive for all values of t)

Integrating within the limit $\int_0^{\omega} d\omega = a\int_0^t t\,dt$ or, $\omega = \frac{1}{2}at^2$

So, $$w_n = \omega^2 R = \left(\frac{at^2}{2}\right)^2 R = \frac{a^2t^4}{4}R$$

and $|w_t| = \beta R = atR$

Putting the values of $|w_t|$ and w_n in Eq. (1), we get,

$$\tan\alpha = \frac{a^2t^4R/4}{atR} = \frac{at^3}{4} \text{ or, } t = \left[\left(\frac{4}{a}\right)\tan\alpha\right]^{1/3}$$

Q. 48. A solid body rotates with deceleration about a stationary axis with an angular deceleration $\beta \propto \sqrt{\omega}$, where ω is its angular velocity. Find the mean angular velocity of the body averaged over the whole time of rotation if at the initial moment of time its angular velocity was equal to ω_0.

Ans. In accordance with the problem, $\beta_z < 0$

Ans. In accordance with the problem, $\beta_z < 0$

Thus $-\frac{d\omega}{dt} = k\sqrt{\omega}$, where A: is proportionality constant

or, $-\int_{\omega_0}^{\omega} \frac{d\omega}{\sqrt{\omega}} = k\int_0^t dt$ or, $\sqrt{\omega} = \sqrt{\omega_0} - \frac{kt}{2}$ (1)

When $\omega = 0$, total time o f rotation $t = \tau = \frac{2\sqrt{\omega_0}}{k}$

Average angular velocity

$$\langle \omega \rangle = \frac{\int \omega\, dt}{\int dt} = \frac{\int_0^{2\sqrt{\omega_0}/k} \left(\omega_0 + \frac{k^2 t^2}{4} - kt\sqrt{\omega_0}\right) dt}{2\sqrt{\omega_0}/k}$$

Hence $\langle \omega \rangle = \left[\omega_0 t + \frac{k^2 t^3}{12} - \frac{k}{2}\sqrt{\omega_0}\, t^2\right]_0^{2\sqrt{\omega_0}/k} \Big/ 2\frac{\sqrt{\omega_0}}{k} = \omega_0/3$

Q.49. A solid body rotates about a stationary axis so that its angular velocity depends on the rotation angle φ as $\omega = \omega_0 - a\varphi$, where ω_0 and a are positive constants. At the moment t = 0 the angle = 0. Find the time dependence of

(a) the rotation angle;
(b) the angular velocity.

Ans. We have $\omega = \omega_0 - a\varphi = \frac{d\varphi}{dt}$

Integratin this Eq. within its limit for (φ) f

$$\int_0^{\varphi} \frac{d\varphi}{\omega_0 - k\varphi} = \int_0^t dt \text{ or, } \ln\frac{\omega_0 - k\varphi}{\omega_0} = -kt$$

Hence $\varphi = \frac{\omega_0}{k}(1 - e^{-kt})$ (1)

(b) From the Eq., $\omega = \omega_0 - k\varphi$ and Eq. (1) or by differentiating Eq. (1)

$\omega = \omega_0 e^{-kt}$

Q.50. A solid body starts rotating about a stationary axis with an angular acceleration $\beta = \beta_0 \cos\varphi$, where β_0 is a constant vector and φ is an angle of rotation from the initial position. Find the angular velocity of the body as a function of the angle φ. Draw the plot of this dependence.

Ans. Let us choose the positive direction of z-axis (stationary rotation axis) along the vector $\vec{\beta}_0$.

In accordance with the equation $\frac{d\omega_z}{dt} = \beta_z$ or $\omega_z \frac{d\omega_z}{d\varphi} = \beta_z$

or, $\omega_z \, d\omega_z = \beta_z \, d\varphi = \beta \cos\varphi \, d\varphi$,

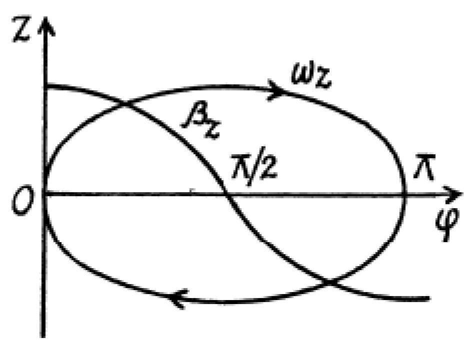
Integrating this Eq. within its limit for

$\omega_z(\varphi)$

or, $\int_0^{\omega_z} d\omega_z = \beta_0 \int_0^{\varphi} \cos\varphi \, d\varphi$

or, $\frac{\omega_z^2}{2} = \beta_0 \sin\varphi$

Hence $\omega_z = \pm\sqrt{2\beta_0 \sin\varphi}$



The plot $\omega_z(\varphi)$ is shown in the Fig. It can be seen that as the angle qp grows, the vector $\vec{\omega}$ first increases, coinciding with the direction of the vector $\vec{\beta}_0 (\omega_z > 0)$, reaches the maximum at $\varphi = \varphi/2$, then starts decreasing and finally turns into zero at $\varphi = \pi$. After that the body starts rotating in the opposite direction in a similar fashion ($\omega_z < 0$). As a result, the body will oscillate about the position $\varphi = \varphi/2$, with an amplitude equal to $\pi/2$.

Q. 51. A rotating disc (Fig. 1.6) moves in the positive direction of the x axis. Find the equation y (x) describing the position of the instantaneous axis of rotation, if at the initial moment the axis C of the disc was located at the point O after which it moved

(a) with a constant velocity v, while the disc started rotating counterclockwise with a constant angular acceleration β (the initial angular velocity is equal to zero);
(b) with a constant acceleration w (and the zero initial velocity), while the disc rotates counterclockwise with a constant angular velo- city ω.

Ans. Rotating disc moves along the x-axis, in plane motion in x - y plane. Plane motion of a solid can be imagined to be in pure rotation about a point (say 7) at a certain instant known as instantaneous centre of rotation. The instantaneous axis whose positive sense is directed along $\vec{\omega}$ of the solid and which passes through the point/, is known as instantaneous axis of rotation.

Therefore the velocity vector of an aibitrary point (P) of the solid can be represented as :

$$\vec{v}_P = \vec{\omega} \times \vec{r}_{PI} \quad (1)$$

On the basis of Eq. (1) for the C. M. (C) of the disc

$$\vec{v}_c = \vec{\omega} \times \vec{r}_{cI} \quad (2)$$

According to the problem $\vec{v}_c \uparrow\uparrow \vec{i}$ and $\vec{\omega} \uparrow\uparrow \vec{k}$ i.e. $\vec{\omega} \perp x - y$ plane, so to satisy the Eqn. (2) $\vec{r}_{CI}$ is directed along $(-\vec{j})$. Hence point l is at a distance $r_{cI} = y$, above the centre of the disc along y - axis. Using all these facts in Eq. (2), we get

$v_C = \omega y$ or $y = \frac{v_c}{\omega}$ (3)

(a) From the angular kinematical equation

$\omega_z = \omega_{0z} + \beta_z t$

$\omega = \beta t.$ (4)

On the other hand x = v t, (where x is the x coordinate of the C.M.)

or, t = x/y (5)

From Eqs. (4) and (5), $\omega = \frac{\beta x}{v}$

Using this value o f co in Eq. (3) w e get $y = \frac{v_c}{\omega} = \frac{v}{\beta x/v} = \frac{v^2}{\beta x}$ (hyperbola)

(b) As centre C moves with constant acceleration w, with zero initial velocity

So, $x = \frac{1}{2} w t^2$ and $v_c = w t$

Therefore, $v_c = w\sqrt{\frac{2x}{w}} = \sqrt{2xw}$

Hence $y = \frac{v_c}{\omega} = \frac{\sqrt{2wx}}{\omega}$ (parabola)

Q. 52. A point A is located on the rim of a wheel of radius R = 0.50 m which rolls without slipping along a horizontal surface with velocity v = 1.00 m/s. Find:

](a) the modulus and the direction of the acceleration vector of the point A;
(b) the total distance s traversed by the point A between the two successive moments at which it touches the surface.

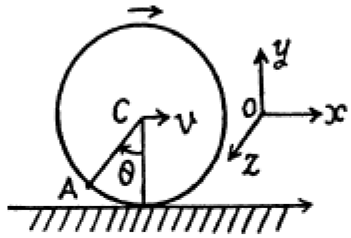
Ans. The plane motion of a solid can be imagined as the combination of translation of the C.M . and rotation about C.M.

So, we may write $\vec{v}_A = \vec{v}_C + \vec{v}_{AC}$

$$= \vec{v}_C + \vec{\omega} \times \vec{r}_{AC} \quad (1) \text{ and}$$

$$\vec{w}_A = \vec{w}_C + \vec{w}_{AC}$$

$$= \vec{w}_C + \omega^2(-\vec{r}_{AC}) + (\vec{\beta} \times \vec{r}_{AC}) \quad (2)$$



$\vec{r}_{AC}$ is the position o f vector o f A with respect to C.

In the problem $v_C = v =$ constant, and the rolling is without slipping $. v_C = v = \omega R$,

So, $w_C = 0$ and $\beta = 0$. Using these conditions in Eq. (2)

$$\vec{w}_A = \omega^2(-\vec{r}_{AC}) = \omega^2 R(-\hat{u}_{AC}) = \frac{v^2}{R}(-\hat{u}_{AC})$$

Here, $\hat{u}_{AC}$ is the unit vector directed along $\vec{r}_{AC}$.

Hence $w_A = \frac{v^2}{R}$ and $\vec{w}_A$ is directed along $(-\hat{u}_{AC})$ or directed toward the centre of the wheel.

(b) Let the centre o f the wheel m ove toward right (positive jc-axis) then for pure tolling on the rigid horizontal surface, wheel will have to rotate in clockwise sense. If co be the angular velocity o f the wheel then $\omega = \frac{v_C}{R} = \frac{v}{R}$.

Let the point A touches the horizontal surface at t = 0, further let us locate the point A at t = ty

When it makes θ = ω t at the centre of the wheel.

From Eqn. (1) $\vec{v}_A = \vec{v}_C + \vec{\omega} \times \vec{r}_{AC}$

$$= v\vec{i} + \omega(-\vec{k}) \times [R\cos\theta(-\vec{j}) + R\sin\theta(-\vec{i})]$$

or, $\vec{v}_A = v\vec{i} + \omega R[\cos\omega t(-\vec{i}) + \sin\omega t\vec{j}]$

$$= (v - \cos\omega t)\vec{i} + v\sin\omega t\vec{j} \quad (\text{as } v = \omega R)$$

or, $\vec{v}_A = v\vec{i} + \omega R[\cos\omega t(-\vec{i}) + \sin\omega t\vec{j}]$

$$= (v - \cos\omega t)\vec{i} + v\sin\omega t\vec{j} \quad (\text{as } v = \omega R)$$

So, $v_A = \sqrt{(v - v\cos\omega t)^2 + (v\sin\omega t)^2}$

$$= v\sqrt{2(1 - \cos\omega t)} = 2v\sin(\omega t/2)$$

Hence distance covered by the point A during $T = 2\pi/\omega$

$$s = \int v_A\, dt = \int_0^{2\pi/\omega} 2v \sin(\omega t/2)\, dt = \frac{8v}{\omega} = 8R.$$

Q. 53. A ball of radius R = 10.0 cm rolls without slipping down an inclined plane so that its centre moves with constant acceleration w = 2.50 cm/s^2; t = 2.00 s after the beginning of motion its position corresponds to that shown in Fig. 1.7. Find:

(a) the velocities of the points A, B, and 0;
(b) the accelerations of these points.

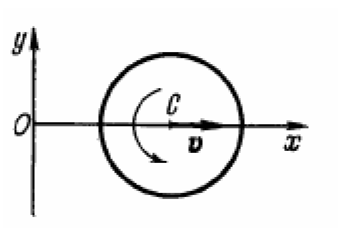


Fig. 1.6.

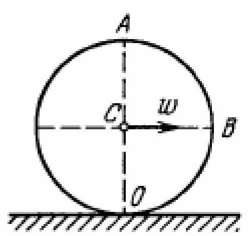


Fig. 1.7.

Ans. Let us fix the co-ordinate axis xyz as shown in the fig. As the ball rolls without slipping along the rigid surface so, on the basis of the solution of problem Q.52 :

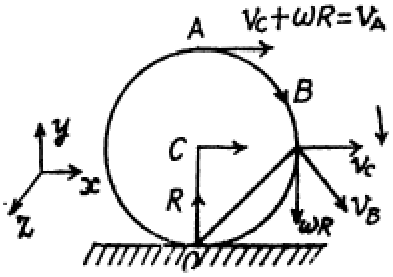
Thus
$$\left.\begin{array}{c} \vec{v}_0 = \vec{v}_c + \vec{\omega} \times \vec{r}_{oc} = 0 \\ v_c = \omega R \ \text{ and } \ \vec{\omega} \uparrow\uparrow (-\vec{k}) \ \text{ as } \ \vec{v}_c \uparrow\uparrow \vec{i} \end{array}\right\}$$

and
$$\left.\begin{array}{c} \vec{\omega}_c + \vec{\beta} \times \vec{r}_{oc} = 0 \\ w_c = \beta R \ \text{ and } \ \vec{\beta} \uparrow\uparrow (-\vec{k}) \ \text{ as } \ \vec{w}_c \uparrow\uparrow \vec{i} \end{array}\right\}$$

At the position corresponding to that of Fig., in accordance with the problem,

$w_c = w$, so $v_c = wt$

and $\omega = \frac{v_c}{R} = \frac{wt}{R}$ and $\beta = \frac{w}{R}$ (using 1)



(a) Let us fix the co-ordinate system with the frame attached with the rigid surface as shown in the Fig.

As point O is the instantaneous centre of rotation of the ball at the moment shown in Fig.

so, $\vec{v_0} = 0$,

Now, $\vec{v_A} = \vec{v_C} + \vec{\omega} \times \vec{r_{AC}}$

So, $= v_C \vec{i} + \omega(-\vec{k}) \times R(\vec{j}) = (v_C + \omega R)\vec{i}$

So, $\vec{v_A} = 2v_C\vec{i} = 2wt\vec{i}$ (using 1)

Similalry $\vec{v_B} = \vec{v_C} + \vec{\omega} \times \vec{r_{BC}} = v_C\vec{i} + \omega(-\vec{k}) \times R(\vec{i})$

$= v_C\vec{i} + \omega R(-\vec{j}) = v_C\vec{i} + v_C(-\vec{j})$

So, $v_B = \sqrt{2}\, v_c = \sqrt{2}\, wt$ and $\vec{v_B}$ is at an angle 45° from both $\vec{i}$ and $\vec{j}$ (Fig.)

(b) $\vec{w_0} = \vec{w_C} + \omega^2(-\vec{r_{oc}}) + \vec{\beta} \times \vec{r_{OC}}$

$= \omega^2(-\vec{r_{OC}}) = \frac{v_C^2}{R}(-\hat{u}_{OC})$ (using 1)

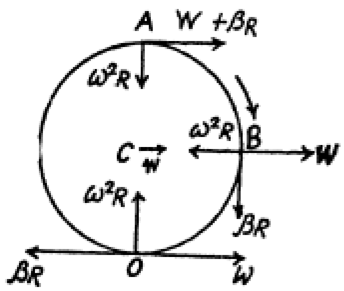
where $\hat{u}_{OC}$ is the unit vector along $\vec{r_{OC}}$

so, $w_0 = \frac{v_0^2}{R} = \frac{w^2t^2}{R}$ (using 2) and $\vec{w_0}$ is directed towards the centre of the ball

Now $\vec{w_A} = \vec{w_C} + \omega^2(-\vec{r_{AC}}) + \vec{\beta} \times \vec{r_{AC}}$

$= w\vec{i} + \omega^2 R(-\vec{j}) + \beta(-\vec{k}) \times R\vec{j}$

$= (w + \beta R)\vec{i} + \frac{v_c^2}{R}(-\vec{j})$ (using 1) $= 2w\vec{i} + \frac{w^2t^2}{R}(-\vec{j})$



So, $w_A = \sqrt{4w^2 + \frac{w^4t^4}{R^2}} = 2w\sqrt{1 + \left(\frac{wt^2}{2R}\right)^2}$

Similarly $\vec{w}_B = \vec{w}_c + \omega^2(-\vec{r}_{BC}) + \vec{\beta} \times \vec{r}_{BC}$

$= w\vec{i} + \omega^2 R(-\vec{i}) + \beta(-\vec{k}) \times R(\vec{i})$

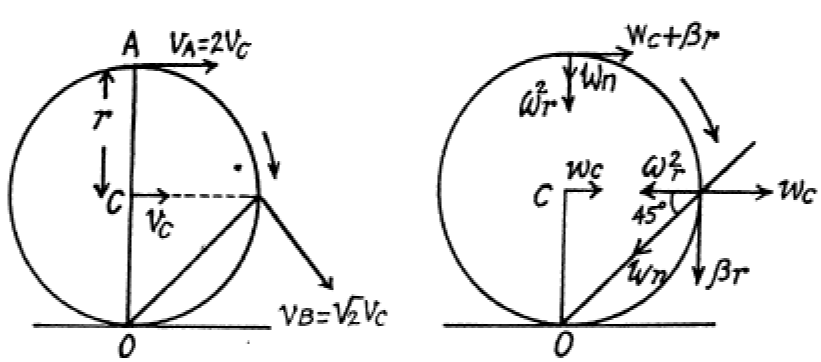
$= \left(v - \frac{v_C^2}{R}\right)\vec{i} + \beta R(-\vec{j})$ (using 1)

$= \left(w - \frac{w^2t^2}{R}\right)\vec{i} + w(-\vec{j})$ (using 2)

So, $w_B = \sqrt{\left(w - \frac{w^2t^2}{R}\right)^2 + w^2}$

Q. 54. A cylinder rolls without slipping over a horizontal plane. The radius of the cylinder is equal to r. Find the curvature radii of trajectories traced out by the points A and B (see Fig. 1.7).

Ans. Let us draw the kinematical diagram of the rolling cylinder on the basis of the solutioi of problem Q.53.



As, an arbitrary point of the cylinder follows a curve, its normal acceleration and radius of curvature are related by the well known equation

$$w_n = \frac{v^2}{R}$$

so, for point A, $w_{A(n)} = \frac{v_A^2}{R_A}$

or, $R_A = \frac{4 v_c^2}{\omega_r^2} = 4r$ (because $v_c = \omega r$, for pure rolling)

Similarly for point B,

$$w_{B(n)} = \frac{v_B^2}{R_B}$$

$$\omega^2 r \cos 45° = \frac{(\sqrt{2}\, v_c)^2}{R_B},$$

or, $R_B = 2\sqrt{2}\,\frac{v_C^2}{\omega^2 r} = 2\sqrt{2}\, r$

Q. 55. Two solid bodies rotate about stationary mutually perpendicular intersecting axes with constant angular velocities ω_1 = 3.0 rad/s and ω_2 = 4.0 rad/s. Find the angular velocity and angular acceleration of one body relative to the other.

Ans. The angular velocity is a vector as infinitesimal rotation commute. Then file relative angular velocity of the body 1 with respect to the body 2 is dearly.

$$\vec{\omega}_{12} = \vec{\omega}_1 - \vec{\omega}_2$$

as fo r relative lin ear velocity. The relative acceleration of 1 w.r.t 2 is

$$\left(\frac{d\vec{\omega}_1}{dt}\right)_{S'}$$

where S ' is a fram e corotating with the second body and S is a space fixed frame with origin coinciding with the point of intersection of the two axes,

but $\left(\frac{d\vec{\omega}_1}{dt}\right)_S = \left(\frac{d\vec{\omega}_1}{dt}\right)_{S'} + \vec{\omega}_2 \times \vec{\omega}_1$

Since S ' rotates with angular velocity $\vec{w}_2$. However $\left(\frac{d\vec{\omega}_1}{dt}\right)_S = 0$ as the first b ody rotates with constant angular velocity in space, thus

$$\vec{\beta}_{12} = \vec{\omega}_1 \times \vec{\omega}_2.$$

Note that for any vector $\vec{b}$, the relation in space forced frame (k) and a frame (Jd) rotating with angular velocity $\vec{\omega}$ is

$$\left.\frac{d\vec{b}}{dt}\right|_K = \left.\frac{d\vec{b}}{dt}\right|_{K'} + \vec{\omega} \times \vec{b}$$

Q. 56. A solid body rotates with angular velocity ω = ati + bt^{2}j, where a = 0.50 rad/s^2, b = 0.060 rad/s^3, and i and j are the unit vectors of the x and y axes. Find:

(a) the moduli of the angular velocity and the angular acceleration at the moment t = 10.0 s;
(b) the angle between the vectors of the angular velocity and the angular acceleration at that moment.

Ans. We have $\vec{\omega} = at\vec{i} + bt^2\vec{j}$ (1)

So, $\omega = \sqrt{(at)^2 + (bt^2)^2}$, thus, $\omega|_{t=10s} = 7.81$ rad/s

D ifferen tia tin g E q. (1) with respect to time

$$\vec{\beta} = \frac{d\vec{\omega}}{dt} = a\vec{i} + 2bt\vec{j} \quad (2)$$

So, $\beta = \sqrt{a^2 + (2bt)^2}$

and $\beta|_{t=10s} = 1{\cdot}3$ rad/s^2

(b) $\cos\alpha = \frac{\vec{\omega}\cdot\vec{\beta}}{\omega\beta} = \frac{(at\vec{i} + bt^2\vec{j})\cdot(a\vec{i} + 2bt\vec{j})}{\sqrt{(at)^2 + (bt^2)^2}\sqrt{a^2 + (2bt)^2}}$

Putting the values of (a) and (b) and ' taking t =10s, we get
α = 17°

Q. 57. A round cone with half-angle α = 30° and the radius of the base R = 5.0 cm rolls uniformly and without slipping over a horizontal plane as shown in Fig.

1.8. The cone apex is hinged at the point O which is on the same level with the point C, the cone base centre. The velocity of point C is v = 10.0 cm/s. Find the moduli of

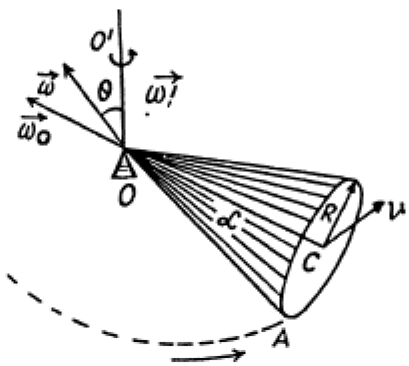
(a) the vector of the angular velocity of the cone and the angle it forms with the vertical;
(b) the vector of the angular acceleration of the cone.

Ans. Let the axis of the cone (OC) rotates in an ticlockw ise sense w ith constant angular velocity $\vec{\omega}'$ and the cone itself about it's own axis (OC) in clockwise sense with angular velocity $\vec{\omega}_0$ (Fig.). Then the resultant angular velocity of the cone.

$$\vec{\omega} = \vec{\omega}' + \vec{\omega}_0 \qquad (1)$$

As the rolling is pure the magnitudes of the vectors

$\vec{\omega}'$ and $\vec{\omega}_0$ can be easily found from Fig.



$$\omega' = \frac{v}{R\cot\alpha},\ \omega_0 = v/R \quad (2)$$

As $\vec{\omega}' \perp \vec{\omega}_0$, from Eq. (1) and (2)

$$\omega = \sqrt{\omega'^2 + \omega_0^2}$$

$$\sqrt{\left(\frac{v}{R\cot\alpha}\right)^2 + \left(\frac{v}{R}\right)^2} = \frac{v}{R\cos\alpha} = 2{\cdot}3 \text{ rad/s}$$

(b) Vector of angular acceleration

$$\vec{\beta} = \frac{d\vec{\omega}}{dt} = \frac{d(\vec{\omega}' + \vec{\omega}_0)}{dt} \quad (\text{as } \vec{\omega}' = \text{constant.})$$

The vector $\vec{\omega}_0$ which rotates about the OO' axis with the angular velocity $\vec{\omega}'$, retains i magnitude. This increment in the time interval dt is equal to

$|d\vec{\omega}_0| = \omega_0 \cdot \omega' \, dt$ or in vector form $d\vec{\omega}_0 = (\vec{\omega}' \times \vec{\omega}_0)\, dt.$

Thus $\vec{\beta} = \vec{\omega}' \times \vec{\omega}_0$

The magnitude of the vector $\vec{\beta}$ is equal to

$\beta = \omega' \omega_0$ (as $\vec{\omega}' \perp \vec{\omega}_0$)

So, $\beta = \frac{v}{R \cot \alpha} \frac{v}{R} = \frac{v^2}{R^2} \tan \alpha = 2{\cdot}3 \text{ rad/s}$

Q. 58. A solid body rotates with a constant angular velocity ω_0 = 0.50 rad/s about a horizontal axis AB. At the moment t = 0 the axis AB starts turning about the vertical with a constant angular acceleration β_0 = 0.10 rad/s^2. Find the angular velocity and angular acceleration of the body after t = 3.5 s.

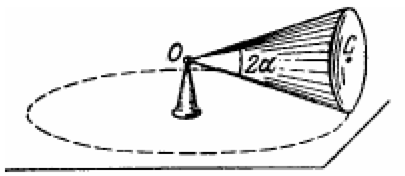


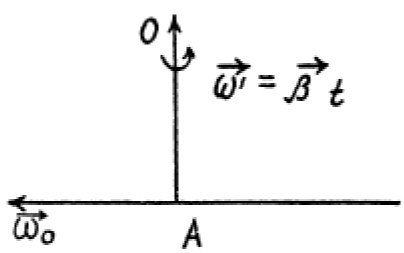
Fig. 1.8.

Ans. The axis AB acquired the angular velocity

$\vec{\omega}' = \vec{\beta}_o t$ (1)

Using the facts of the solution of 1.57, the angular velocity of the body

$$\omega = \sqrt{\omega_o^2 + \omega'^2}$$
$$= \sqrt{\omega_0^2 + \beta_0^2 t^2} = 0{\cdot}6 \text{ rad/s}$$



And the angular acceleration.

$$\vec{\beta} = \frac{d\vec{\omega}}{dt} = \frac{d(\vec{\omega}' + \vec{\omega}_0)}{dt} = \frac{d\vec{\omega}'}{dt} + \frac{d\vec{\omega}_0}{dt}$$

But $\dfrac{d\vec{\omega}_0}{dt} = \vec{\omega}' \times \vec{\omega}_0$, and $\dfrac{d\vec{\omega}'}{dt} = \vec{\beta}_0 t$

So, $\vec{\beta} = (\vec{\beta}_0 t \times \vec{\omega}_0) + \vec{\beta}_0$

As, $\vec{\beta}_0 \perp \vec{\omega}_0$ so, $\beta = \sqrt{(\omega_0 \beta_0 t)^2 + \beta_0^2} = \beta_0 \sqrt{1 + (\omega_0 t)^2} = 0{\cdot}2 \text{ rad/s}^2$