1. $\displaystyle\int_{\frac{1}{3}}^{1} \frac{\left(x - x^3\right)^{\frac{1}{3}}}{x^4}\, dx$ **નું મૂલ્ય .... છે.**

(A) 6 (B) 0 (C) 3 (D) 4

જવાબ (B) 0

➡ $$I = \int_{\frac{1}{3}}^{1} \frac{(x-x^3)^{\frac{1}{3}}}{x^4}\, dx$$

$$\therefore I = \int_{\frac{1}{3}}^{1} \frac{x\left(\frac{1}{x^2} - 1\right)^{\frac{1}{3}} dx}{x^4}$$

$$= \int_{\frac{1}{3}}^{1} \frac{\left(\frac{1}{x^2} - 1\right)^{\frac{1}{3}}}{x^3}\, dx$$

ધારો કે $\frac{1}{x^2} - 1 = t \Rightarrow \frac{-2}{x^3}\, dx = dt$

$$\Rightarrow \frac{1}{x^3}\, dx = -\frac{dt}{2}$$

$x = \frac{1}{3}$ હોય ત્યારે $t = 8$ તથા $x = 1$ હોય ત્યારે $t = 0$

જવાબ (B) 0

➡ $$I = \int_{\frac{1}{3}}^{1} \frac{(x-x^3)^{\frac{1}{3}}}{x^4}\, dx$$

$$\therefore I = \int_{\frac{1}{3}}^{1} \frac{x\left(\frac{1}{x^2} - 1\right)^{\frac{1}{3}} dx}{x^4}$$

$$= \int_{\frac{1}{3}}^{1} \frac{\left(\frac{1}{x^2} - 1\right)^{\frac{1}{3}}}{x^3}\, dx$$

ધારો કે $\frac{1}{x^2} - 1 = t \Rightarrow \frac{-2}{x^3}\,dx = dt$

$$\Rightarrow \frac{1}{x^3}\,dx = -\frac{dt}{2}$$

$x = \frac{1}{3}$ હોય ત્યારે $t = 8$ તથા $x = 1$ હોય ત્યારે $t = 0$

2. **જો $f(x) = \int_0^x t \sin t\, dt$, તો $f'(x) = \dots\dots\dots$.**

(A) $\cos x + x \sin x$ (B) $x \sin x$ (C) $x \cos x$ (D) $\sin x + x \cos x$

જવાબ (B) $x \sin x$

➡ $f(x) = \int_0^x t \sin t\, dt$

$$\int t \sin t\, dt = t \int \sin t\, dt - \int \left(\frac{d}{dt}(t) \int \sin t\, dt \right) dt$$
$$= -t \cos t - \int (-\cos t)\, dt$$
$$= -t \cos t + \sin t + c$$

$f(x) = \int_0^x t \sin t\, dt$

$= [-t \cos t + \sin t]_0^x$

$= -x \cos x + \sin x$

$\therefore f'(x) = -\cos x + x \sin x + \cos x$

$= x \sin x$

$\therefore$ વિકલ્પ (B) આવે.

3. **વિધેયનું સંકલન મેળવો : $\int_2^1 \frac{-\tan^{-1} x}{1 + x^2}\, dx$**

➡ $\frac{\pi^2}{32}$

4. **વિધેયનું સંકલન મેળવો : $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\operatorname{cosec} x \cdot \cot x}{1 + \operatorname{cosec}^2 x}\, dx$**

➡ $\tan^{-1} \frac{1}{3}$

5. **વિધેયનું સંકલન મેળવો : $\int_0^{\frac{\pi}{2}} \frac{dx}{4 \sin^2 x + 5\cos^2 x)}$**

➡ $\frac{\pi}{4\sqrt{5}}$

6. **વિધેયનું સંકલન મેળવો :** $\int_{0}^{\sqrt{2}} \sqrt{2 - x^2}\ dx$

➠ $\frac{\pi}{2}$

7. **વિધેયનું સંકલન મેળવો :** $\int_{1}^{2} \frac{1}{x(1 + x^2)}\ dx$

➠ $\frac{3}{2} \log 2 - \frac{1}{2} \log 5$

8. **વિધેયનું સંકલન મેળવો :** $\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{dx}{\sqrt{x - x^2}}$

➠ $\frac{\pi}{6}$

9. **વિધેયનું સંકલન મેળવો :** $\int_{0}^{\frac{\pi}{2}} \cos^4 x\ dx$

➠ $\frac{3\pi}{16}$

10. **વિધેયનું સંકલન મેળવો :** $\int_{1}^{3} \frac{\log x}{(x + 1)^2}\ dx$

➠ $\frac{3}{4} \log 3 - \log 2$

11. **વિધેયનું સંકલન મેળવો :** $\int_{0}^{\pi} \frac{1}{1 + \sin x}\ dx$

➠ 2

12. **વિધેયનું સંકલન મેળવો :** $\int_{0}^{1} \frac{1 - x}{1 + x}\ dx$

➠ $2 \log 2 - 1$

13. મૂલ્ય આદેશની રીતનો ઉપયોગ કરીને મેળવો : $\int_{0}^{1} \frac{x}{x^2 + 1}\ dx$

➠ $$\int_{0}^{1} \frac{x}{x^2 + 1}\ dx = \frac{1}{2} \int_{0}^{1} \frac{2x}{x^2 + 1}\ dx$$

$$= \frac{1}{2} \left[\log (x^2 + 1)\right]_{0}^{1}$$

$$= \frac{1}{2} \left[\log 2 - \log 0\right]$$

$$= \frac{1}{2} \log_e 2$$

14. મૂલ્ય આદેશની રીતનો ઉપયોગ કરીને મેળવો : $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x} dx$

➡ ધારો કે $\cos x = t \Rightarrow -\sin x \, dx = dt$

જ્યારે $x = 0$ ત્યારે $t = 1$ તથા $x = \frac{\pi}{2}$ ત્યારે $t = 0$

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x} dx = \int_1^0 \frac{-dt}{1 + t^2}$$

$$= -[\tan^{-1} t]_1^0$$

$$= -[\tan^{-1}0 - \tan^{-1}1]$$

$$= -\left[0 - \frac{\pi}{4}\right]$$

$$= \frac{\pi}{4}$$

15. મૂલ્ય આદેશની રીતનો ઉપયોગ કરીને મેળવો : $\int_{-1}^{1} \frac{dx}{x^2 + 2x + 5}$

➡ $$\int_{-1}^{1} \frac{dx}{x^2 + 2x + 5} = \int_{-1}^{1} \frac{dx}{x^2 + 2x + 1 + 4}$$

$$= \int_{-1}^{1} \frac{dx}{(x + 1)^2 + (2)^2}$$

$$= \frac{1}{2}\left[\tan^{-1} \frac{(x + 1)}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{2}{2}\right) - \tan^{-1}0\right]$$

$$= \frac{1}{2} \tan^{-1} 1$$

$$= \frac{1}{2} \cdot \frac{\pi}{4} = \frac{\pi}{8}$$

16. મૂલ્ય આદેશની રીતનો ઉપયોગ કરીને મેળવો : $\int_0^{\frac{\pi}{2}} \sqrt{\sin \phi} \; \cos^5\phi \; d\phi$

➡ ધારો કે $\sin \phi = t^2$

$\therefore \cos \phi \; d\phi = 2t \; dt$

જ્યારે $\phi = 0$ ત્યારે $t = 0$ તથા $\phi = \frac{\pi}{2}$ ત્યારે $t = 1$, $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi} \; \cos^5 \phi \; d\phi$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\ \cos^4\ \phi \cdot \cos\phi \cdot d\phi$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\ (1-\sin^2\phi)^2 \cdot \cos\phi \cdot d\phi$$

$$= \int_0^1 t \cdot (1 - t^4)^2 \cdot 2t\ dt$$

$$= \int_0^1 2t^2\ (1 - 2t^4 + t^8)\ dt$$

$$= \int_0^1 (2t^2 - 4t^6 + 2t^{10})\ dt$$

$$= \left[\frac{2t^3}{3} - \frac{4t^7}{7} + \frac{2t^{11}}{11}\right]_0^1$$

$$= \frac{2}{3} - \frac{4}{7} + \frac{2}{11}$$

$$= \frac{64}{231}$$

17. **મૂલ્ય આદેશની રીતનો ઉપયોગ કરીને મેળવો :** $\int_0^2 x\sqrt{x+2}\ dx$ ($x + 2 = t^2$ લો.)

➥ $x + 2 = t^2 \Rightarrow dx = 2t\ dt$

$\therefore\ x = t^2 - 2$

$x = 0$ હોય ત્યારે $t = \sqrt{2}$ તથા $x = 2$ હોય ત્યારે $t = 2$

$$\int_0^2 x\sqrt{x+2}\ dx = \int_{\sqrt{2}}^2 (t^2 - 2)\cdot t\cdot 2t\ dt$$

$$= \int_{\sqrt{2}}^2 (2t^4 - 4t^2)\ dt$$

$$= \left[\frac{2t^5}{5} - \frac{4t^3}{3}\right]_{\sqrt{2}}^2$$

$$= \left(\frac{64}{5} - \frac{32}{3}\right) - \left(\frac{8\sqrt{2}}{5} - \frac{8\sqrt{2}}{3}\right)$$

$$= \frac{32}{15} + \frac{16\sqrt{2}}{15}$$

$$= \frac{16\sqrt{2}}{15}\left(\sqrt{2} + 1\right)$$

18. **મૂલ્ય આદેશની રીતનો ઉપયોગ કરીને મેળવો :** $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x}\ dx$

➥ ધારો કે $2x = t \Rightarrow 2dx = dt \Rightarrow dx = \frac{dt}{2}$ જ્યારે $x = 1$ ત્યારે $t = 2$ તથા $x = 2$ ત્યારે $t = 4$

$$\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x}\ dx$$

$$= \int_2^4 \left(\frac{2}{t} - \frac{4}{2t^2}\right) e^t\ \frac{dt}{2}$$

$$= \frac{1}{2}\int_2^4 e^t \cdot 2\left(\frac{1}{t} - \frac{1}{t^2}\right) dt$$

$$= \int_2^4 e^t\left(\frac{1}{t} - \frac{1}{t^2}\right) dt$$

$$= \left[e^t \cdot \frac{1}{t}\right]_2^4$$ $\left(\because \int e^x\ (f(x) + f(x))\ dx\right.$

$= f(x)e^x + c$ નો ઉપયોગ કરતાં$\left.\right)$

$$= \frac{e^4}{4} - \frac{e^2}{2} = \frac{e^2}{2}\left(\frac{e^2}{2} - 1\right)$$

$$= e^2\ \frac{(e^2 - 2)}{4}$$

19. **મૂલ્ય આદેશની રીતનો ઉપયોગ કરીને મેળવો :** $\int_0^1 \sin^{-1}\left(\frac{2x}{1 + x^2}\right) dx$

➥ ધારો કે $x = \tan\theta \Rightarrow dx = \sec^2\theta\ d\theta$

$x = 0$ હોય ત્યારે $\theta = 0$ તથા $x = 1$ હોય ત્યારે $\theta = \frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1 + \tan^2\theta}\right) \cdot \sec^2\theta . d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sin^{-1}\ (\sin 2\theta) \cdot \sec^2\theta . d\theta$$

$\frac{\pi}{}$

$$= \int_0^4 2\theta \cdot \sec^2\theta \, d\theta \qquad ....(i)$$

હવે $\int 2\theta \sec^2\theta \, d\theta$ માટે ખંડશઃ સંકલનના નિયમનો ઉપયોગ કરતાં $u = 2\theta$, $v = \sec^2\theta$

$$\int 2\theta \sec^2\theta \, d\theta = 2\theta \int \sec^2\theta \, d\theta - \int \left(\frac{d}{d\theta} (2\theta) \int \sec^2\theta \, d\theta \right) d\theta$$

$= 2\theta \tan\theta - \int 2 \tan\theta \, d\theta$

$= 2\theta \tan\theta - 2 \log |\sec\theta| + c$

આ કિંમત (1) માં મૂકતાં,

$$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx = \left[2\theta \tan\theta - 2\log|\sec\theta|\right]_0^{\frac{\pi}{4}}$$

$$= \left(2 \cdot \frac{\pi}{4} \tan\frac{\pi}{4} - 2\log\left|\sec\frac{\pi}{4}\right|\right) - (0 - 2\log|\sec 0|)$$

$$= \frac{\pi}{2} - 2\log\sqrt{2} \quad (\because \sec 0 = 1 \ \log 1 = 0)$$

$$= \frac{\pi}{2} - \log 2$$

20. **મૂલ્ય આદેશની રીતનો ઉપયોગ કરીને મેળવો :** $\displaystyle\int_0^2 \frac{dx}{x + 4 - x^2}$

➥ $$\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{\frac{1}{4} + 4 - \left(x^2 - x + \frac{1}{4}\right)} \quad (\because \text{ પૂર્ણ વર્ગ બનાવતાં})$$

$$= \int_0^2 \frac{dx}{\frac{17}{4} - \left(x - \frac{1}{2}\right)^2}$$

$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}$$

$$= \frac{1}{2 \cdot \frac{\sqrt{17}}{2}} \left[\log\left|\frac{\frac{\sqrt{17}}{2} - \left(x - \frac{1}{2}\right)}{\frac{\sqrt{17}}{2} + \left(x - \frac{1}{2}\right)}\right|\right]_0^2$$

$$= \frac{1}{\sqrt{17}} \left[\log\left(\frac{\sqrt{17} + 2x - 1}{\sqrt{17} - 2x + 1}\right)\right]_0^2$$

$$= \frac{1}{\sqrt{17}} \left[\log\left(\frac{\sqrt{17} + 4 - 1}{\sqrt{17} - 4 + 1}\right) - \log\left(\frac{\sqrt{17} - 1}{\sqrt{17} + 1}\right)\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log\left(\frac{\sqrt{17}+3}{\sqrt{17}-3}\times\frac{\sqrt{17}+1}{\sqrt{17}-1}\right)\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log\left(\frac{17+\sqrt{17}+3\sqrt{17}+3}{17-\sqrt{17}-3\sqrt{17}+3}\right)\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log\left(\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right)\right]$$

➥ $\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{\frac{1}{4}+4-\left(x^2-x+\frac{1}{4}\right)}$ (∵ પૂર્ણ વર્ગ બનાવતાં)

$$= \int_0^2 \frac{dx}{\frac{17}{4}-\left(x-\frac{1}{2}\right)^2}$$

$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$$

$$= \frac{1}{2\cdot\frac{\sqrt{17}}{2}}\left[\log\left|\frac{\frac{\sqrt{17}}{2}-\left(x-\frac{1}{2}\right)}{\frac{\sqrt{17}}{2}+\left(x-\frac{1}{2}\right)}\right|\right]_0^2$$

$$= \frac{1}{\sqrt{17}}\left[\log\left(\frac{\sqrt{17}+2x-1}{\sqrt{17}-2x+1}\right)\right]_0^2$$

$$= \frac{1}{\sqrt{17}}\left[\log\left(\frac{\sqrt{17}+4-1}{\sqrt{17}-4+1}\right)-\log\left(\frac{\sqrt{17}-1}{\sqrt{17}+1}\right)\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log\left(\frac{\sqrt{17}+3}{\sqrt{17}-3}\times\frac{\sqrt{17}+1}{\sqrt{17}-1}\right)\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log\left(\frac{17+\sqrt{17}+3\sqrt{17}+3}{17-\sqrt{17}-3\sqrt{17}+3}\right)\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log\left(\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right)\right]$$