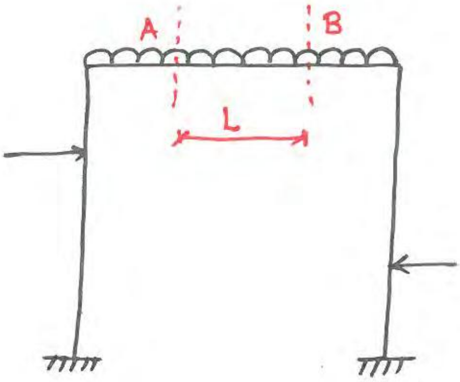
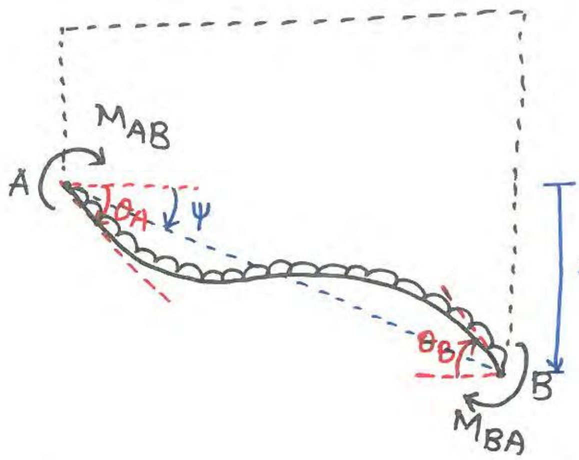
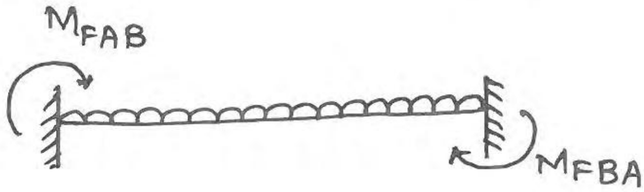
6. Slope - Deflection Method

6.1 Derivation:





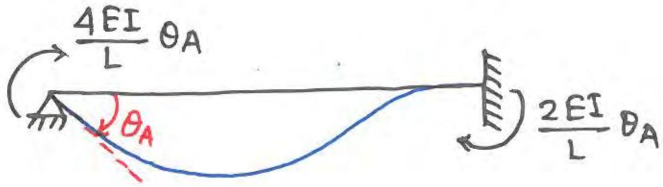


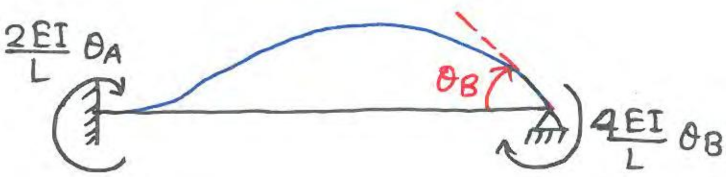
M_{AB} = End Moment

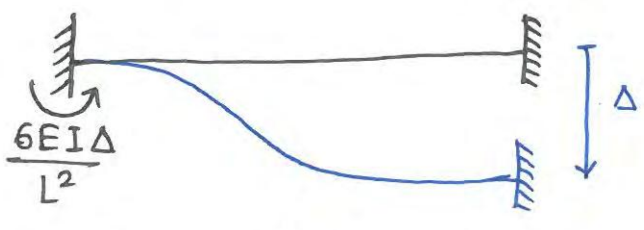
M_{FAB} = Fixed End Moment

θ_A & θ_B = Rotation of Ends of Member

ψ = Rotation of Axis of member.







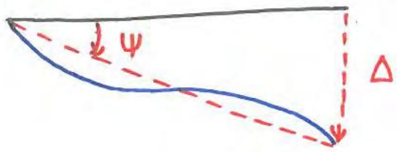
$$M_{AB} = M_{FAB} + \frac{4EI}{L}\theta_A + \frac{2EI}{L}\theta_B - \frac{6EI\Delta}{L^2}$$

$$M_{AB} = M_{FAB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

$$M_{AB} = M_{FAB} + \frac{2EI}{L}\left(2\theta_{near\,end} + \theta_{far\,end} - 3\psi\right)$$

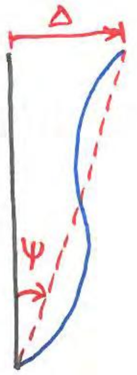
6.2 Sign Convention:

- All clockwise moments are +ve
- All clockwise rotations are positive
- Δ is +ve if it produces clockwise ψ.

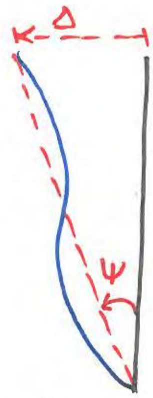


Δ is +ve becoz ψ is clockwise

Δ is -ve becoze ψ is anticlockwise



Δ is +ve becoze ψ is clockwise



Δ is -ve becoze ψ is anticlockwise

6.3 Procedure:

Step I: Calculate KI and Identify them.

Step II: Calculate fixed end moment (FEM) of each span

Step III: Draw FBD of each span and joint (without shear force and axial force unless required)

Step IV: Write equilibrium equation corresponding to each KI.

Step V: Express all end moments in terms of FEM and joint displacements using slope deflection equation.

Step VI: Use equations of step IV to formulate simultaneous equations in terms of unknown displacements.

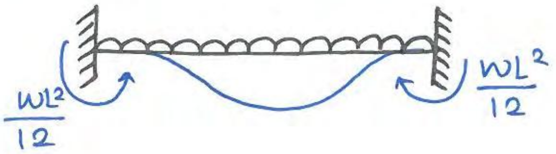
Step VII: Solve simultaneous equations of step IV to calculate unknown displacements.

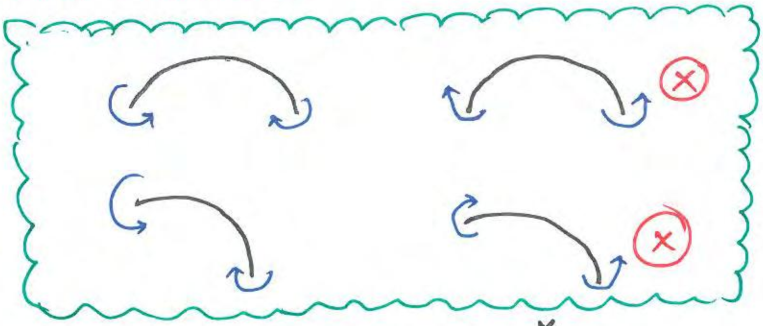
Step VIII: Substitute joint displacements of step VII in slope deflection equation of step V to calculate end moments

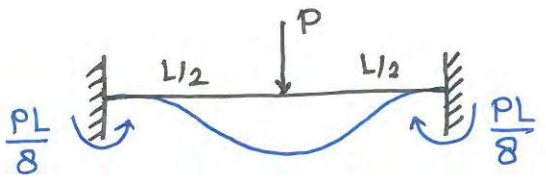
Step IX Draw BMD using method of super position.

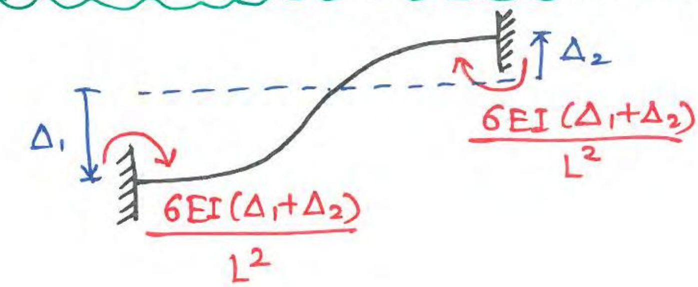
Step X: Calculate other reactions and draw SFD

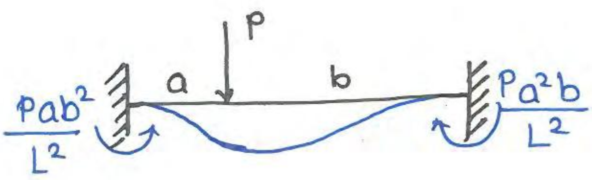
6.4 Standard Results of Fixed End Moment:

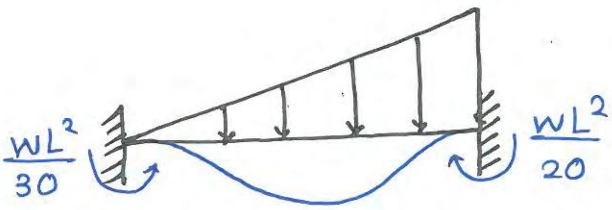


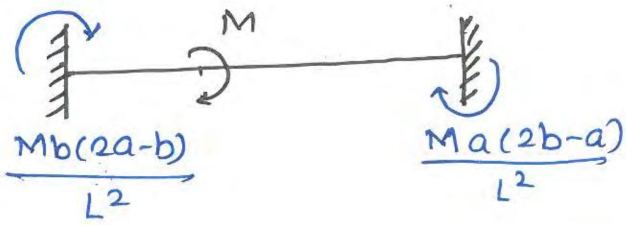


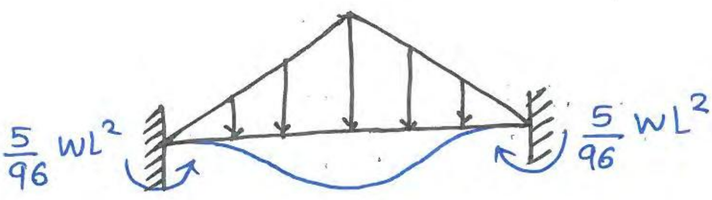


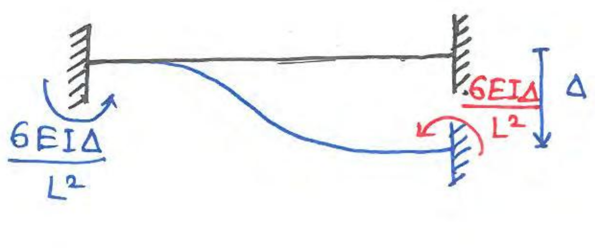




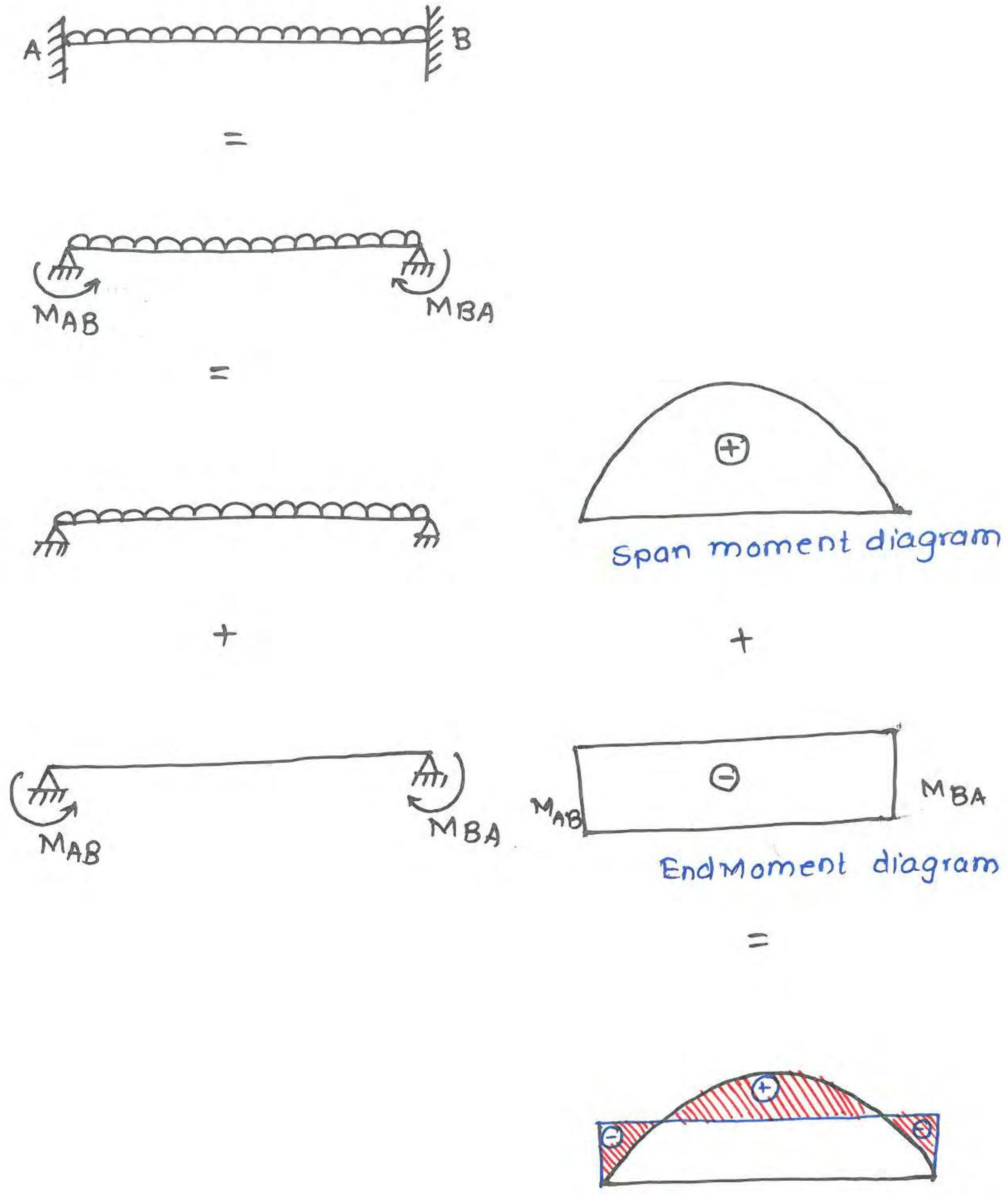








6.5 BMD by Method of Superposition:



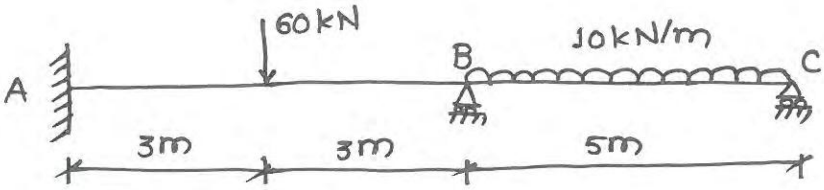
StepI: Draw BMD of span for given Loading by considering end condition as pin support and use usual sign convention. (+ve on reference side)

StepII: Draw BMD of span for end moments by considering end condition as pin support and use opposite to usual sign convention (-ve on reference side)

StepIII: Common area of above two BMD is cancelled out and remaining area is Final BMD of that span.

*Note: BMD of overhang portion is plotted by using sign convention as of end moment diag. (-ve on ref. side)

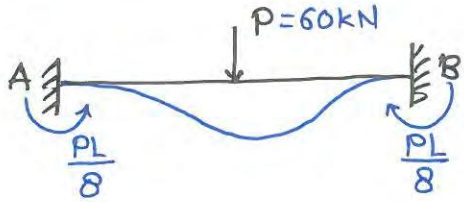
Ex. Analyze the continuous beam given below. Rotational settlement of support A is $\frac{20}{EI}$ clockwise and vertical settlement of support B is $\frac{10}{EI}$ downward.



Step I: KI

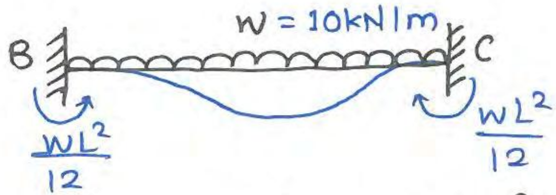
$KI = 2 \quad (\theta_B, \theta_C)$

Step II: Fixed End Moments for each span.



$M_{FAB} = -\frac{PL}{8} = -\frac{60\times 6}{8} = -45$ kNm (-ve becoz anticlockwise

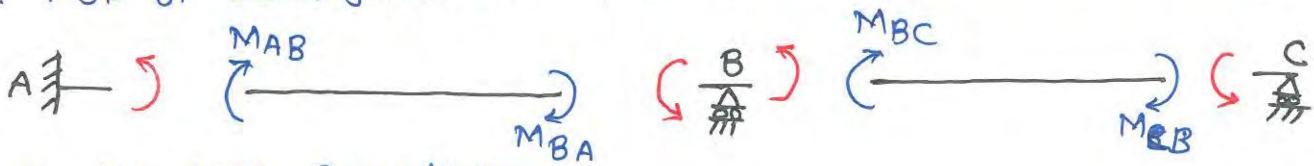
$M_{FBA} = 45$ kNm (+ve becoz clockwise)



$M_{FBC} = -\frac{WL^2}{12} = -\frac{10\times 5^2}{12} = -20.83$ (-ve becoz anticlockwise)

$M_{FCB} = \frac{WL^2}{12} = 20.83$ (+ve becoz clockwise)

Step III: FBD of each joint and span (without axial force & SF)



Step IV: Equilibrium Equations:-

$\Sigma M_B = 0$

$-M_{BA} - M_{BC} = 0$

$M_{BA} + M_{BC} = 0 \quad \cdots\cdots (i)$

$\Sigma M_C = 0$

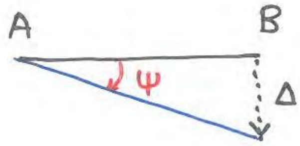
$-M_{CB} = 0$

$M_{CB} = 0 \quad \cdots\cdots (ii)$

Step V:

$$M_{AB} = M_{FAB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

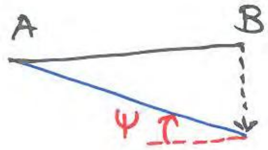
$$= -45 + \frac{2EI}{6}\left(2 \times \frac{20}{EI} + \theta_B - \frac{3(10/EI)}{6}\right)$$



$$M_{AB} = -33.33 + 0.33\,EI\,\theta_B$$

$$M_{BA} = M_{FBA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right)$$

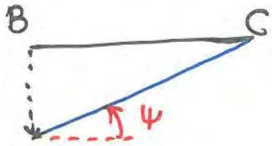
$$= 45 + \frac{2EI}{6}\left(2\theta_B + \frac{20}{EI} - \frac{3(10/EI)}{6}\right)$$



$$M_{BA} = 50 + 0.67\,EI\,\theta_B$$

$$M_{BC} = M_{FBC} + \frac{2EI}{L}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right)$$

$$= -20.83 + \frac{2EI}{5}\left(2\theta_B + \theta_C - \frac{3(-10/EI)}{5}\right)$$

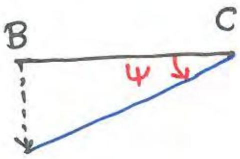


$\cancel{M_{BC} = 50 + 0.67\,EI\,\theta_B}$

$M_{BC}\ \ -18.43 + 0.8\,EI\,\theta_B + 0.4\,EI\,\theta_C$

$$M_{CB} = M_{FCB} + \frac{2EI}{L}\left(2\theta_C + \theta_B - \frac{3\Delta}{L}\right)$$

$$= 20.83 + \frac{2EI}{5}\left(2\theta_C + \theta_B - \frac{3(-10/EI)}{5}\right)$$



$$M_{CB} = 23.23 + 0.4\,EI\,\theta_B + 0.8\,EI\,\theta_C$$

Step VI:

from equation (i) :-

$$31.57 + 1.4\,EI\,\theta_B + 0.4\,EI\,\theta_C = 0 \quad \cdots \text{(iii)}$$

from equation (ii)

$$28.23 + 0.4\,EI\,\theta_B + 0.8\,EI\,\theta_C = 0 \quad \cdots \text{(iv)}$$

Step VII:

$$\theta_B = -15.71/EI$$

$$\theta_C = -21.18/EI$$

Step VIII:

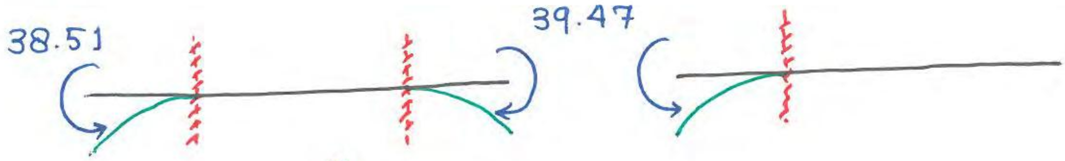
$M_{AB} = -38.51\,kNm$

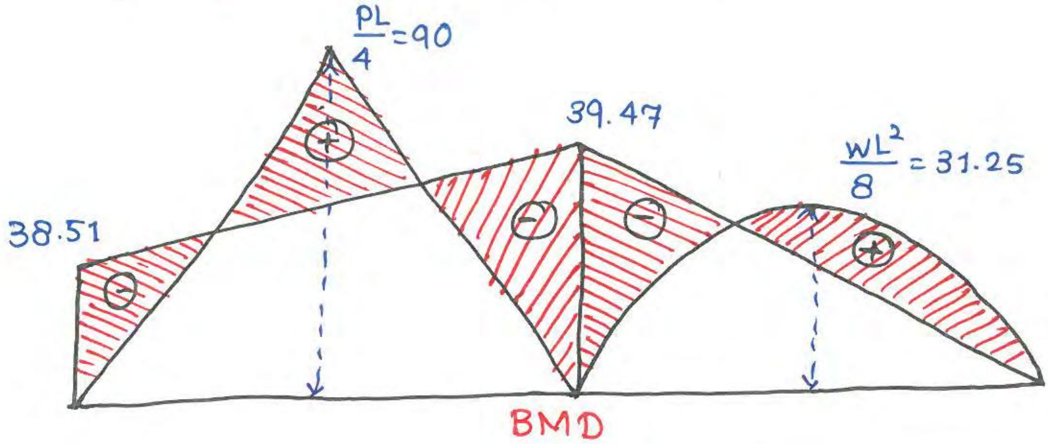
$M_{BA} = 39.47\,kNm$

$M_{BC} = -39.47\,kNm.$

$M_{CB} = 0$

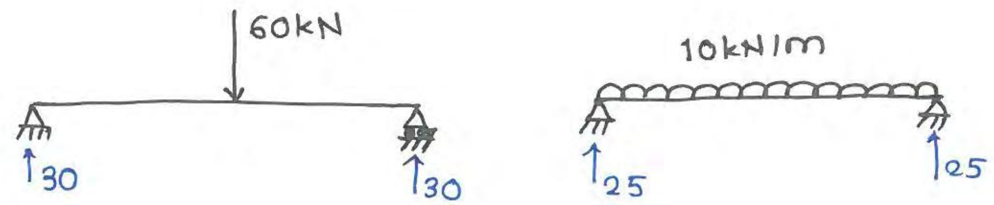
Step IX: BMD.



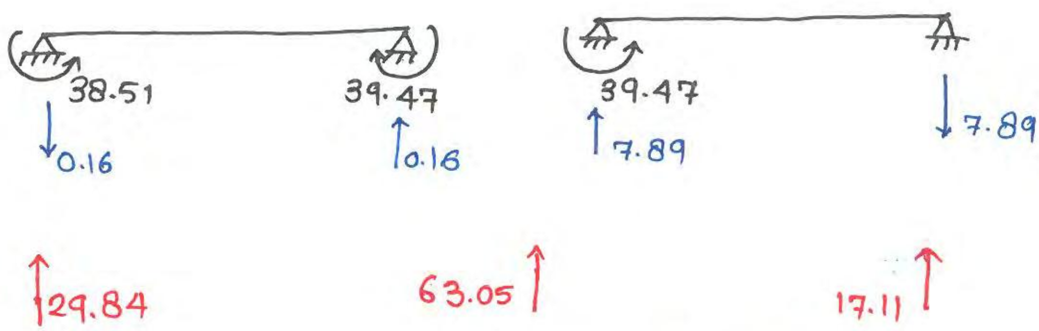


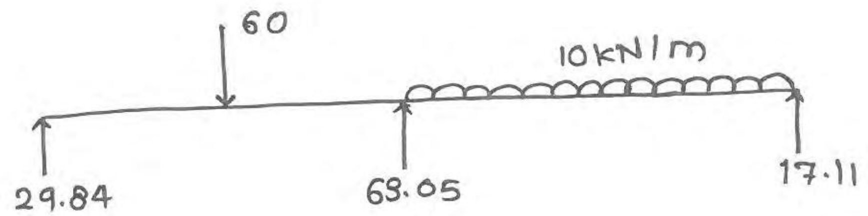
Step X: SFD.

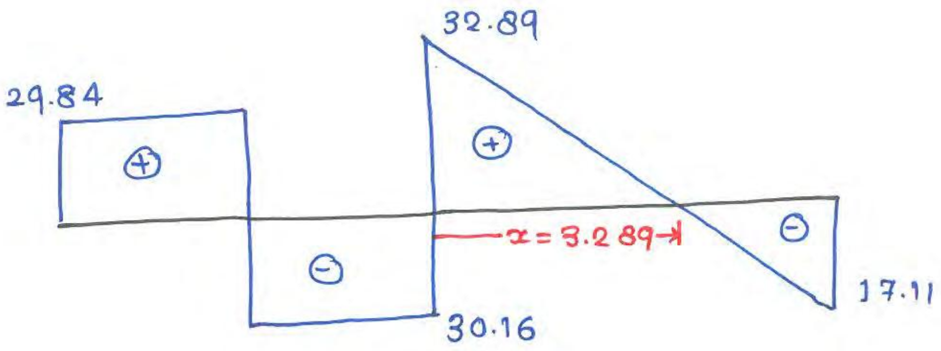
Due to Loading



Due to End Moments



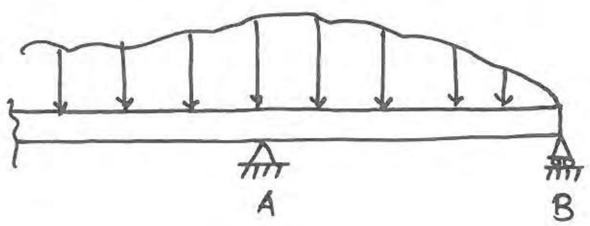




SFD

6.6 Modified Slope Deflection Equation:-

If far end of member has zero final end moment then modified slope deflection eqn can be used to reduce number of unknowns in the simultaneous equation.



$$M_{AB} = M_{FAB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right) \quad \text{----(i)}$$

$$M_{BA} = M_{FBA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right)$$

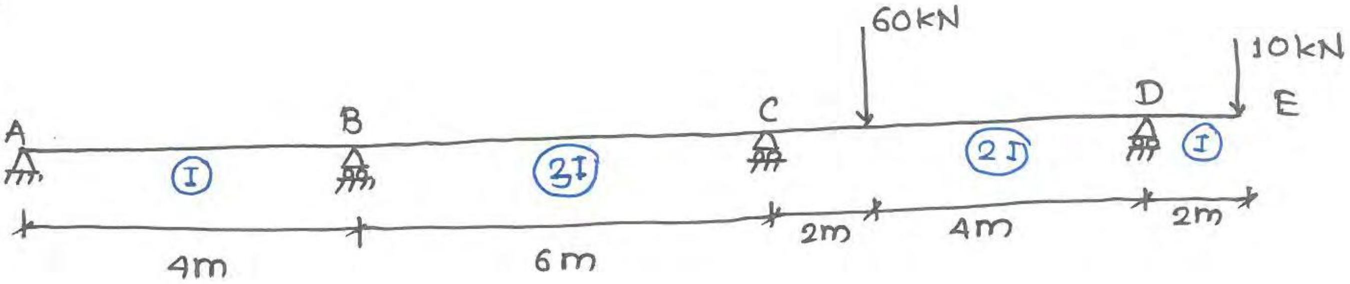
$$\Rightarrow \quad 0 = M_{FBA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right) \quad \text{---(ii)}$$

From equation (i) and (ii):-

$$(i) - \frac{1}{2}(ii)$$

$$M_{AB} = M_{FAB} - \frac{M_{FBA}}{2} + \frac{3EI}{L}\left(\theta_A - \frac{\Delta}{L}\right)$$

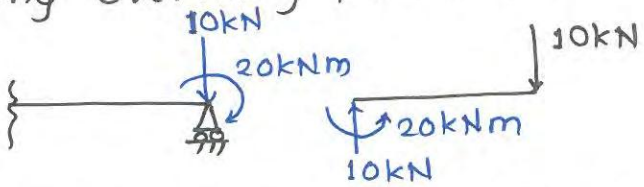
Ex.



Step I: KI

KI = 6 $(\theta_A, \theta_B, \theta_C, \theta_D, \theta_E, \Delta_{YE})$.

By removing over hang portion, KI can be reduced to 4. $(\theta_A, \theta_B, \theta_C, \theta_D)$



By using modified slope deflection equation for M_{BA}, unknowns can be reduced to 3 $(\theta_B, \theta_C, \theta_D)$

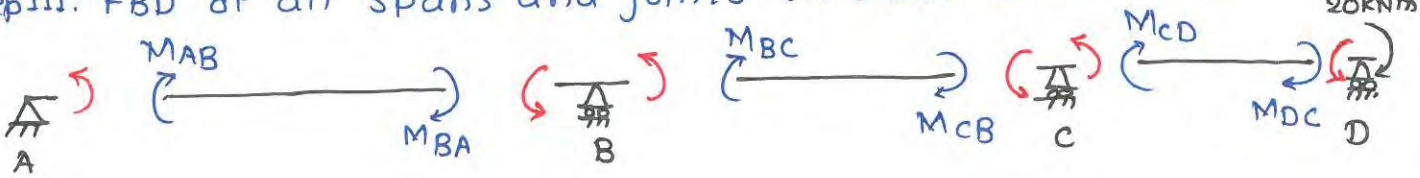
Step II: Fixed End Moments.

$$M_{FAB} = M_{FBA} = M_{FBC} = M_{FCB} = 0$$

$$M_{FCD} = -\frac{Pab^2}{L^2} = \frac{-60 \times 2 \times 4^2}{6^2} = -53.33 \text{ kNm}$$

$$M_{FDC} = \frac{Pa^2b}{L^2} = \frac{60 \times 2^2 \times 4}{6^2} = 26.67 \text{ kNm}$$

Step III: FBD of all spans and joints (without SF and AF)



Step IV: Formulate Equilibrium Equations:

$\Sigma M_B = 0$

$\Rightarrow -M_{BA} - M_{BC} = 0$

$M_{BA} + M_{BC} = 0$ ... (i)

$\Sigma M_C = 0$

$-M_{CB} - M_{CD} = 0$

$M_{CB} + M_{CD} = 0$ ... (ii)

$\Sigma M_D = 0$

$-M_{DC} + 20 = 0$

$M_{DC} = 20$ ... (iii)

Step V:

$M_{AB} = 0$

M_{BA} = Modified slope deflection equation.

$$M_{BA} = M_{FBA} - \frac{M_{FAB}}{2} + \frac{3EI}{L}\left(\theta_B - \frac{\Delta}{L}\right)$$

$$= 0 - 0 + \frac{3EI}{4}(\theta_B - 0)$$

$$M_{BA} = 0.75EI\,\theta_B$$

$$M_{BC} = M_{FBC} + \frac{2EI}{L}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2E(3I)}{6}(2\theta_B + \theta_C - 0)$$

$$M_{BC} = 2EI\,\theta_B + EI\,\theta_C$$

$$M_{CB} = M_{FCB} + \frac{2EI}{L}\left(2\theta_C + \theta_B - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{6}(2\theta_C + \theta_B - 0)$$

$$M_{CB} = 2EI\,\theta_C + EI\,\theta_B$$

$$M_{CD} = M_{FCD} + \frac{2EI}{L}\left(2\theta_C + \theta_D - \frac{3\Delta}{L}\right)$$

$$= -53.33 + \frac{2E(2I)}{6}(2\theta_C + \theta_D - 0)$$

$$M_{CD} = -53.33 + 1.33\,EI\,\theta_C + 0.67\,EI\theta_D$$

$$M_{DC} = M_{FDC} + \frac{2EI}{L}\left(2\theta_D + \theta_C - \frac{3\Delta}{L}\right)$$

$$= 26.67 + \frac{2E(2I)}{6}(2\theta_D + \theta_C - 0)$$

$$M_{DC} = 26.67 + 1.33EI\theta_D + 0.67EI\theta_C$$

Step VI:

from equation (i) :-

$$M_{BA} + M_{BC} = 0$$

$$0.75EI\theta_B + 2EI\theta_B + EI\theta_C = 0$$

$$2.75\,\theta_B + \theta_C = 0 \quad \text{.....} \quad \text{(iv)}$$

from equation (ii) :-

$$M_{CB} + M_{CD} = 0$$

$$2EI\theta_C + EI\theta_B - 53.33 + 1.33\,EI\theta_C + 0.67\,EI\theta_D = 0.$$

$$\theta_B + 3.33\,\theta_C + 0.67\,\theta_D = \frac{53.33}{EI} \quad \text{....} \quad \text{(v)}$$

from equation (iii) :-

$$M_{DC} = 20$$

$$26.67 + 1.33\,EI\theta_D + 0.67\,EI\theta_C = 20.$$

$$0.67\,\theta_C + 1.33\,\theta_D = \frac{-6.67}{EI} \quad \text{.....} \quad \text{(vi)}$$

Step VII:

From equation (iv), (v) and (vi)

$$\theta_B = -7.84/EI$$

$$\theta_C = 21.56/EI$$

$$\theta_D = -15.87/EI$$

Step VIII:

$$M_{AB} = 0$$

$$M_{BA} = -5.88$$

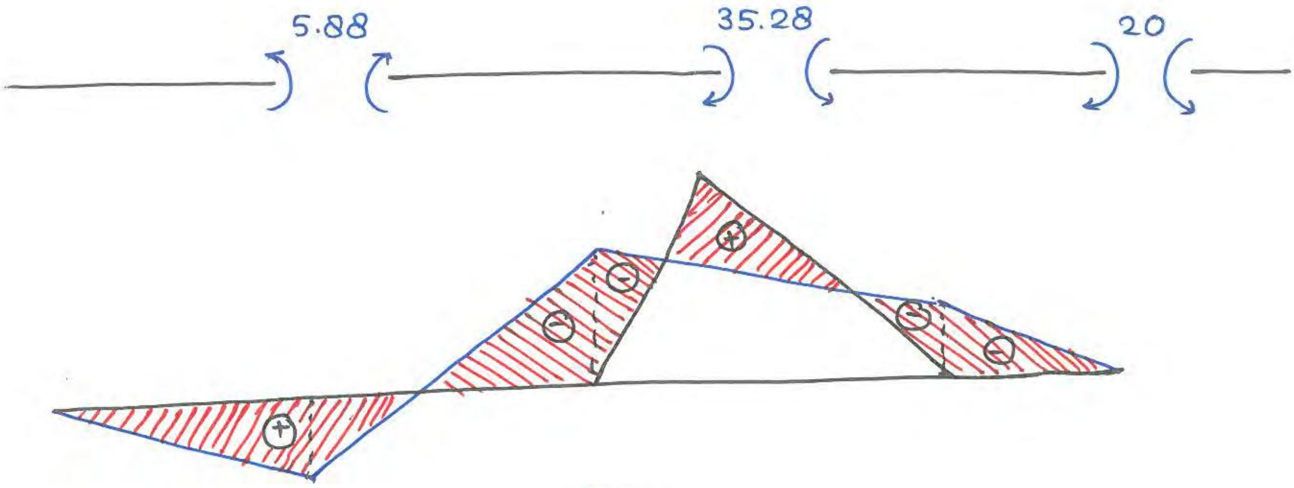
$$M_{BC} = 5.88$$

$$M_{CB} = 35.28$$

$$M_{CD} = -35.28$$

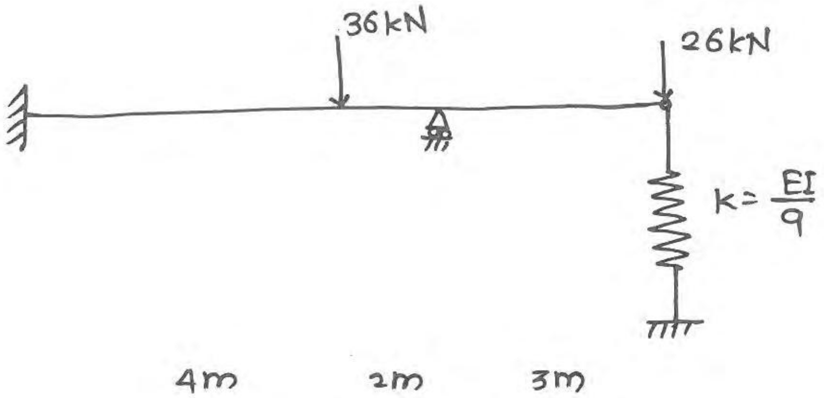
$$M_{DC} = 20$$

Step IX:



BMD

Ex



step I: KI

KI= 3

By using modified slope-deflection equation for M_{BC} number of unknowns can be reduced to 2 (θ_B, $\Delta_{YC} = \Delta(\downarrow)$)

Step II: Fixed End Moments:

$$M_{FAB} = -\frac{Pab^2}{L^2} = \frac{-36\times 4\times 2^2}{6^2} = -16\ kNm$$

$$M_{FBA} = +\frac{Pa^2b}{L^2} = \frac{36\times 4^2\times 2}{6^2} = 32\ kNm$$

$$M_{FBC} = M_{FCB} = 0$$

Step III: FBD of each spans and joint

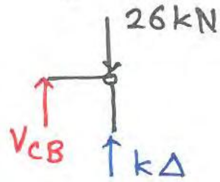
Step IV:

$\Sigma M_B = 0$

$\Rightarrow -M_{BA} - M_{BC} = 0$

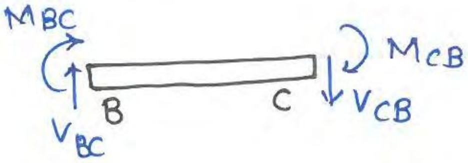
$M_{BA} + M_{BC} = 0$ ......(i)

$\Sigma F_{YC} = 0$



$V_{CB} + k\Delta - 26 = 0$ ..(ii)

For V_{CB} :-



$\Sigma M_B = 0$

$\Rightarrow V_{CB} = -\left(\dfrac{M_{BC} + \cancel{M_{CB}}^{\,0}}{3}\right)$

$= -\dfrac{M_{BC}}{3}$

from eq^n (ii)

$-M_{BC} + \dfrac{EI\Delta}{3} = 78$ .....(iii)

Step V:

$$M_{AB} = M_{FAB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

$$= -16 + \frac{2EI}{6}(0 + \theta_B - 0)$$

$$M_{AB} = -16 + 0.33\,EI\theta_B$$

$$M_{BA} = M_{FBA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right)$$

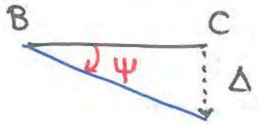
$$= 32 + \frac{2EI}{6}(2\theta_B + 0 - 0)$$

$$M_{BA} = 32 + 0.67\,EI\theta_B$$

M_{BC} = Mod. slope deflection equation.

$$= M_{FBC} - \frac{M_{FCB}}{2} + \frac{3EI}{L}\left(\theta_B - \frac{\Delta}{L}\right)$$

$$= 0 - 0 + \frac{3EI}{3}\left(\theta_B - \frac{\Delta}{L}\right)$$



Step VI:

from equation (i)

$$1.67\,\theta_B - 0.33\Delta = \frac{-32}{EI} \quad \text{....(iv)}$$

from equation (iii)

$$\theta_B - 0.66\Delta = \frac{-78}{EI} \quad \text{.....(v)}$$

Step VII:

from equation (iv) and (v)

$$\theta_B = \frac{5.98}{EI}$$

$$\Delta = \frac{127.24}{EI}$$

Step VIII:

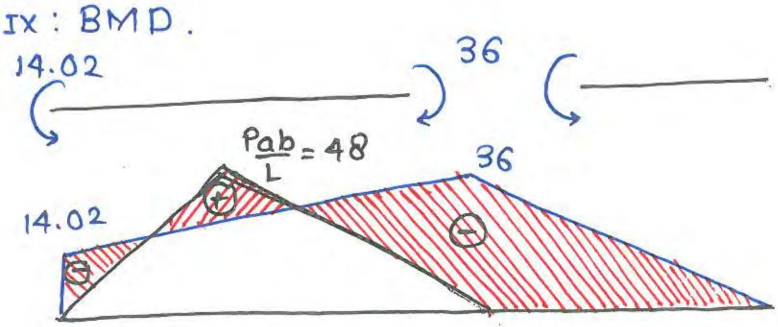
$M_{AB} = -14.02$

$M_{BA} = 36$

$M_{BC} = -36$

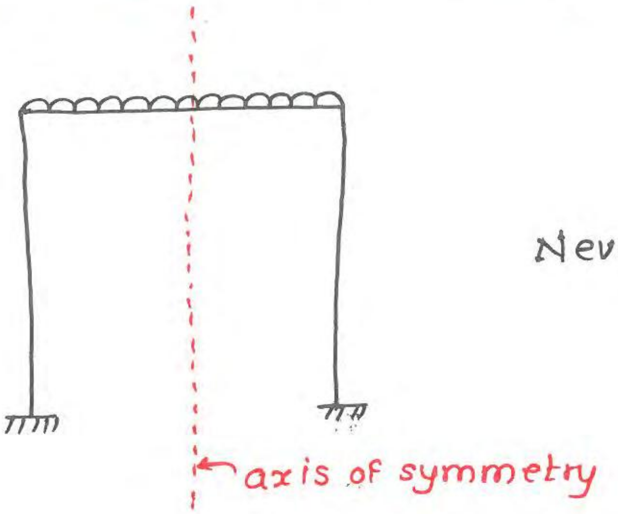
$M_{CB} = 0$

Step IX : BMD.



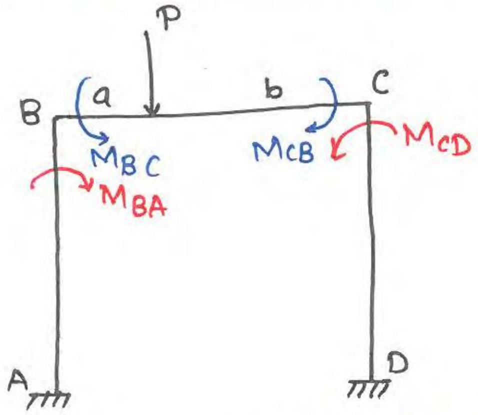
6.7 Conditions of Sway:

Case I: Symmetrical Structure with symmetrical loading.



Never Sway.

Case II: Symmetrical Structure with unsymmetrical loading.



$a < b$

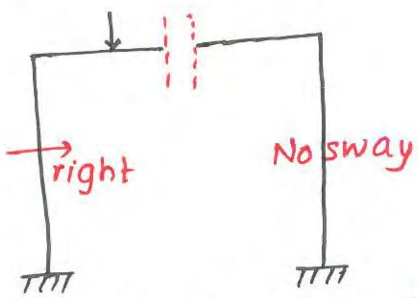
$\Rightarrow \quad M_{BC} > M_{CB}$

$\Rightarrow \quad M_{BA} > M_{CD}$

Since M_{BA} has tendency of right sway and M_{CD} has left so resultant sway will be towards right.

Trick:

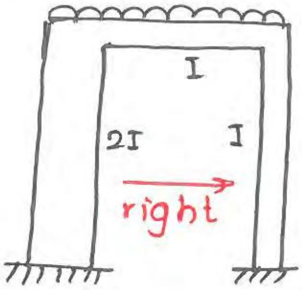
Cut the structure from axis of symmetry and based on direction and magnitude of s~~e~~way of both portion, resultant sway is decided.

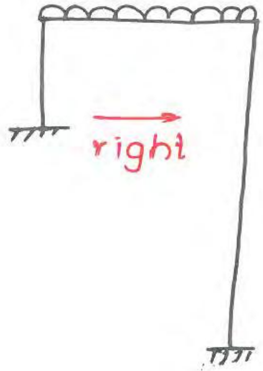


So resultant sway toward right.

Case III : Unsymmetrical structure with symmetrical loading :

Sway is always from higher stiffness to lower stiffness.

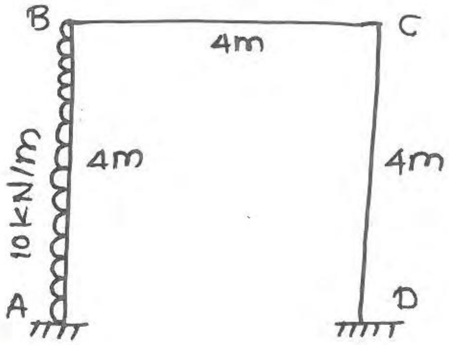




Case IV: Unsymmetrical Structure with Unsymmetrical loading:

Can't say anything about direction of sway.

Ex.



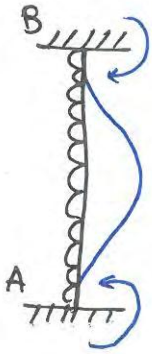
Step I: KI:

$KI = 3 \quad (\theta_B, \theta_C, \Delta_{xB} = \Delta_{xC})$

$$M_{FAB} = -\frac{WL^2}{12}$$

$$= -\frac{10 \times 4^2}{12}$$

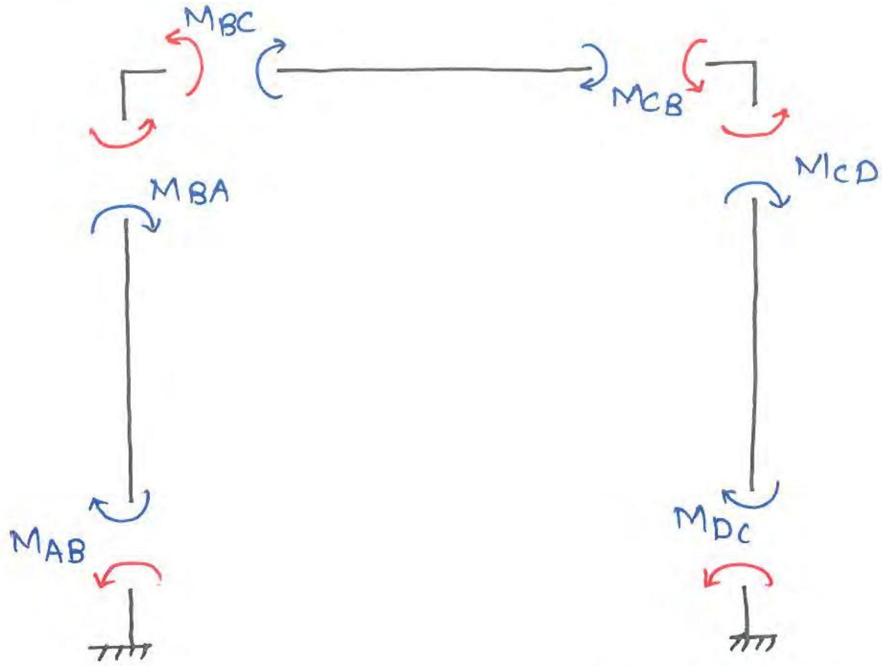
$$M_{FAB} = -13.33$$



$$M_{FBA} = 13.33$$

$$M_{FBC} = M_{FCB} = M_{FCD} = M_{FDC} = 0$$

Step III: FBD:



Step IV: Formulating Equilibrium Equation.

$$\Sigma M_B = 0$$

$$\Rightarrow -M_{BA} - M_{BC} = 0$$

$$\Rightarrow M_{BA} + M_{BC} = 0 \quad \text{.... (i)}$$

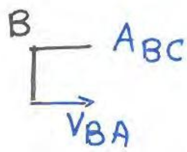
$$\Sigma M_c = 0$$

$$\Rightarrow -M_{CB} - M_{CD} = 0$$

$$\Rightarrow M_{CB} + M_{CD} = 0 \quad \text{.... (ii)}$$

$$\Sigma F_{xB} = 0$$

$$\Rightarrow V_{BA} + A_{BC} = 0$$

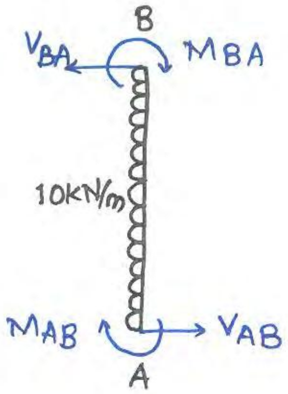


Since it is difficult to convert axial force of member (A_{BC}) in terms of end moments so writing shear equation as follows:

$$\Sigma F_x = 0$$

$$\Rightarrow V_{BC} + V_{CD} = 0 \quad \text{.... (iii)}$$

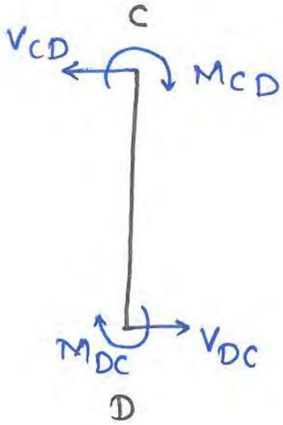
For V_{BA}:-



$$\Sigma M_A = 0$$

$$\Rightarrow V_{BA} = \frac{M_{AB} + M_{BA} + 10 \times 4 \times 2}{4}$$

For V_{CD}:-



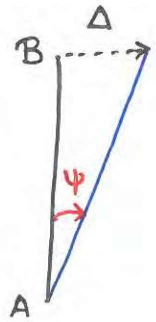
$$\Sigma M_D = 0$$

$$V_{CD} = \frac{M_{CD} + M_{DC}}{4}$$

from equation (iii)

$$M_{AB} + M_{BA} + M_{CD} + M_{DC} = -80 \quad \text{-- (iv)}$$

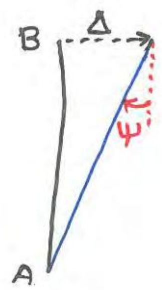
Step V: Slope-Deflection Equation:-



$$M_{AB} = M_{FAB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

$$= -13.33 + \frac{2EI}{4}\left(0 + \theta_B - \frac{3\Delta}{4}\right)$$

$$M_{AB} = -13.33 + 0.5\,EI\theta_B - 0.375\,EI\Delta$$



$$M_{BA} = M_{FBA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right)$$

$$= 13.33 + \frac{2EI}{4}\left(2\theta_B + 0 - \frac{3\Delta}{L}\right)$$

$$M_{BA} = 13.33 + EI\theta_B - 0.375\,EI\Delta$$

$$M_{BC} = M_{FBC} + \frac{2EI}{L}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(2\theta_B + \theta_C - 0\right)$$

$$M_{BC} = EI\,\theta_B + 0.5EI\,\theta_C$$

$$M_{CB} = M_{FCB} + \frac{2EI}{L}\left(2\theta_c + \theta_B - \frac{3\Delta}{L}\right)$$

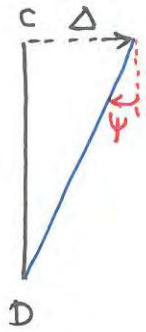
$$= 0 + \frac{2EI}{4}\left(2\theta_c + \theta_B - 0\right)$$

$$M_{CB} = EI\theta_c + 0.5EI\theta_B$$

$$M_{CD} = M_{FCD} + \frac{2EI}{L}\left(2\theta_c + \theta_D - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(2\theta_c + 0 - \frac{3\Delta}{L}\right)$$

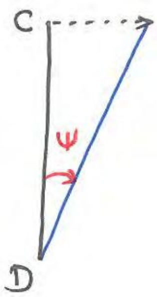
$$M_{CD} = EI\theta_c - 0.375EI\Delta$$



$$M_{DC} = M_{FDC} + \frac{2EI}{L}\left(2\theta_D + \theta_c - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \theta_c - \frac{3\Delta}{L}\right)$$

$$M_{DC} = 0.5EI\theta_c - 0.375EI\Delta$$



Step VI:

From equation (i)

$$2\theta_B + 0.5\theta_c - 0.375\Delta = \frac{-13.33}{EI} \quad \text{------(v)}$$

from equation (ii)

$$0.5\theta_B + 2\theta_c - 0.375\Delta = 0 \quad \text{----(vi)}$$

from equation (iv)

$$1.5\theta_B + 1.5\theta_c - 1.5\Delta = \frac{-80}{EI} \quad \text{.....(vii)}$$

Step VII:

From equation (v), (vi) and (vii)

$$\theta_B = 3.17/EI$$

$$\theta_c = 12.06/EI$$

$$\Delta_{xB} = 68.57/EI$$

Step VIII :

$M_{AB} = -37.45$

$M_{BA} = -9.21$

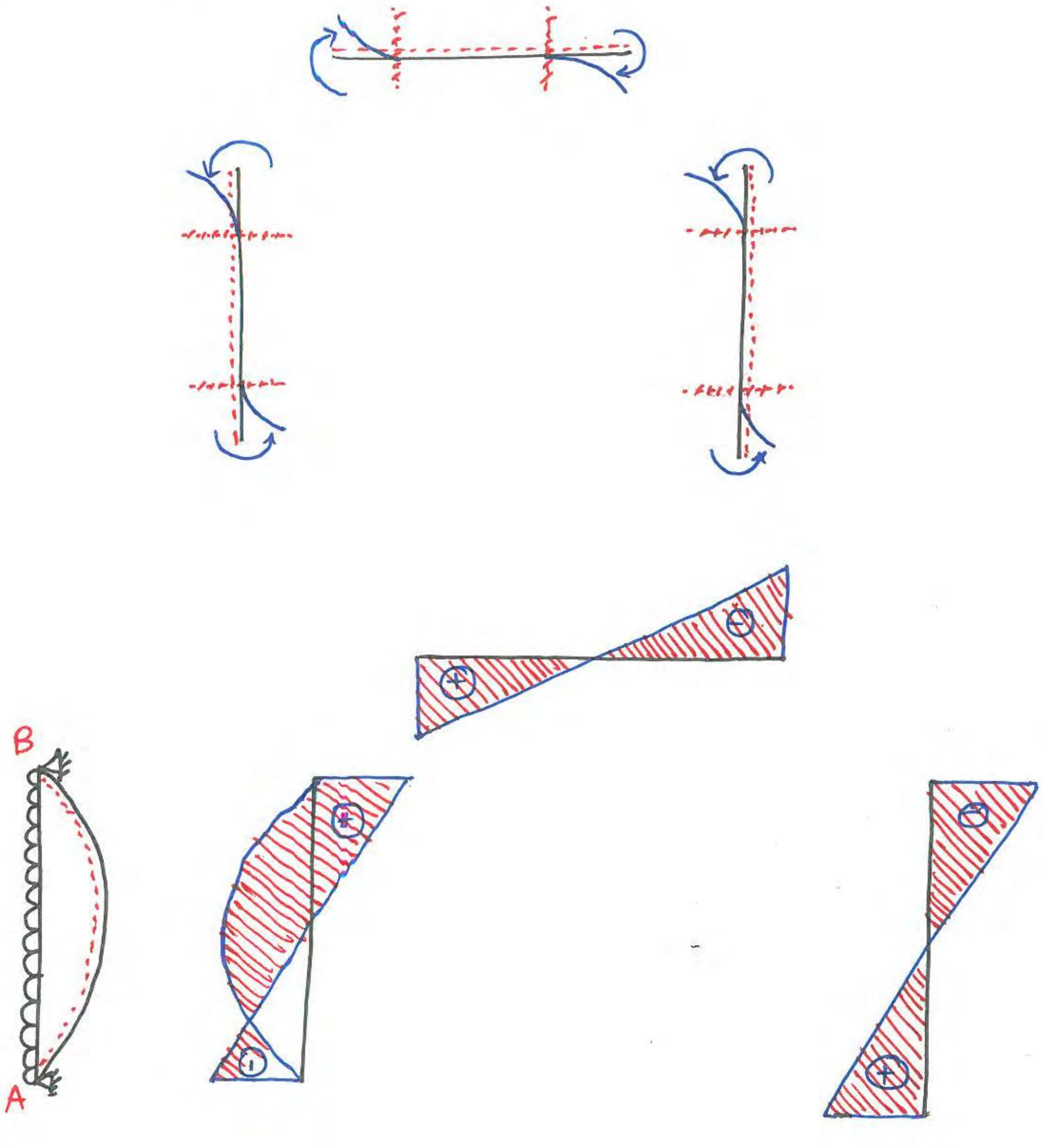
$M_{BC} = 9.21$

$M_{CB} = 13.65$

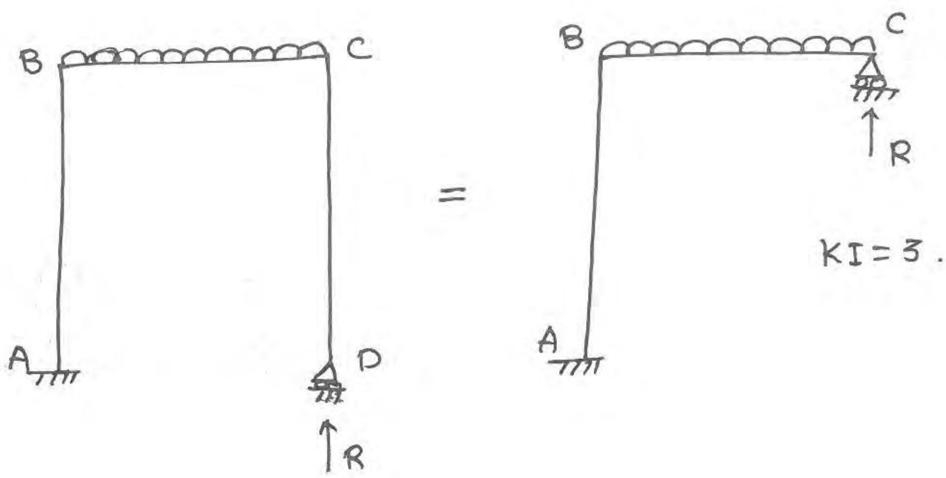
$M_{CD} = -13.65$

$M_{DC} = -19.68$

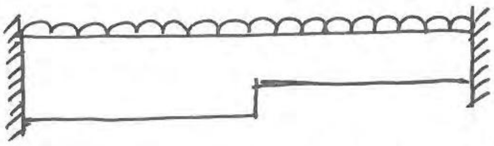
Step IX : BMD



Ex.

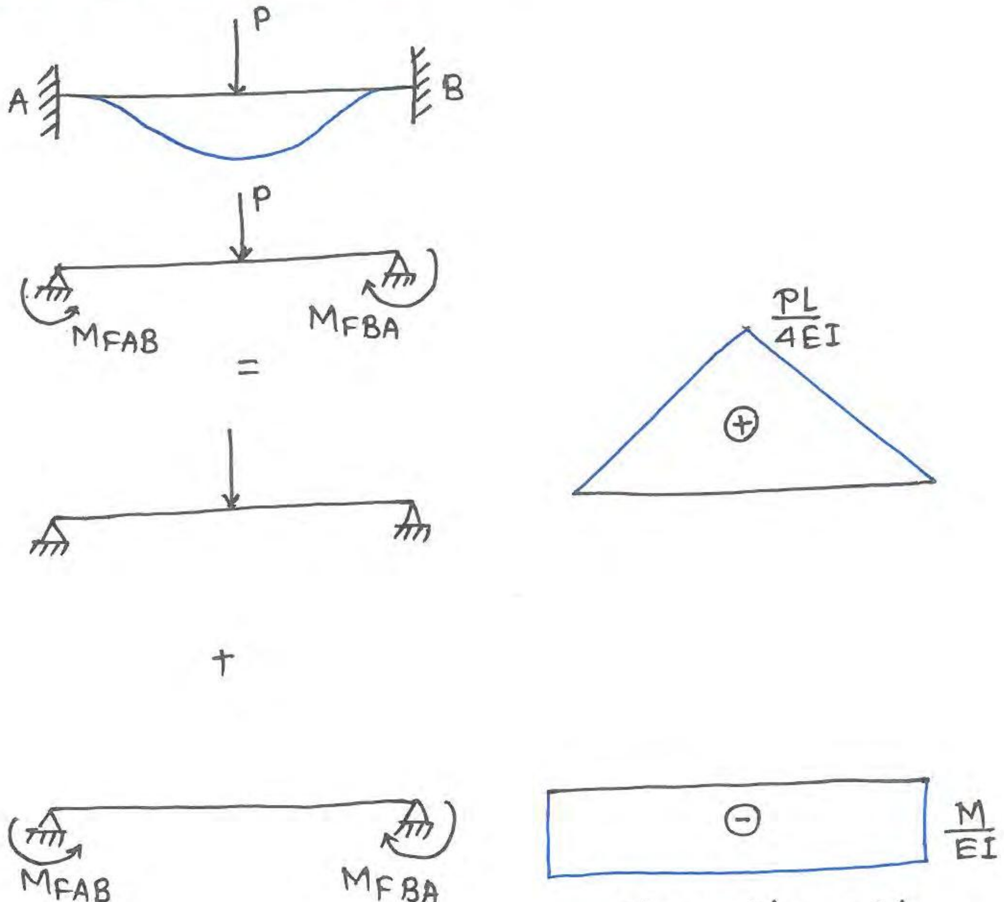


Ex



If cross-section area of member varies then a joint is considered at junction of variation of cross-section.

6.8 Calculation of Fixed End Moment :-



$M_{FAB} = M_{FBA} = M$

from moment area theorem:-

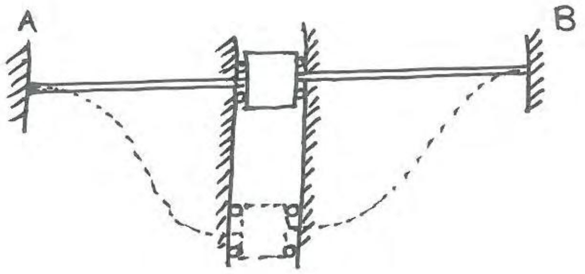
$$\theta_{AB}=0$$

$\Rightarrow$ Area of $\frac{M}{EI}$ diagram between A & B = 0

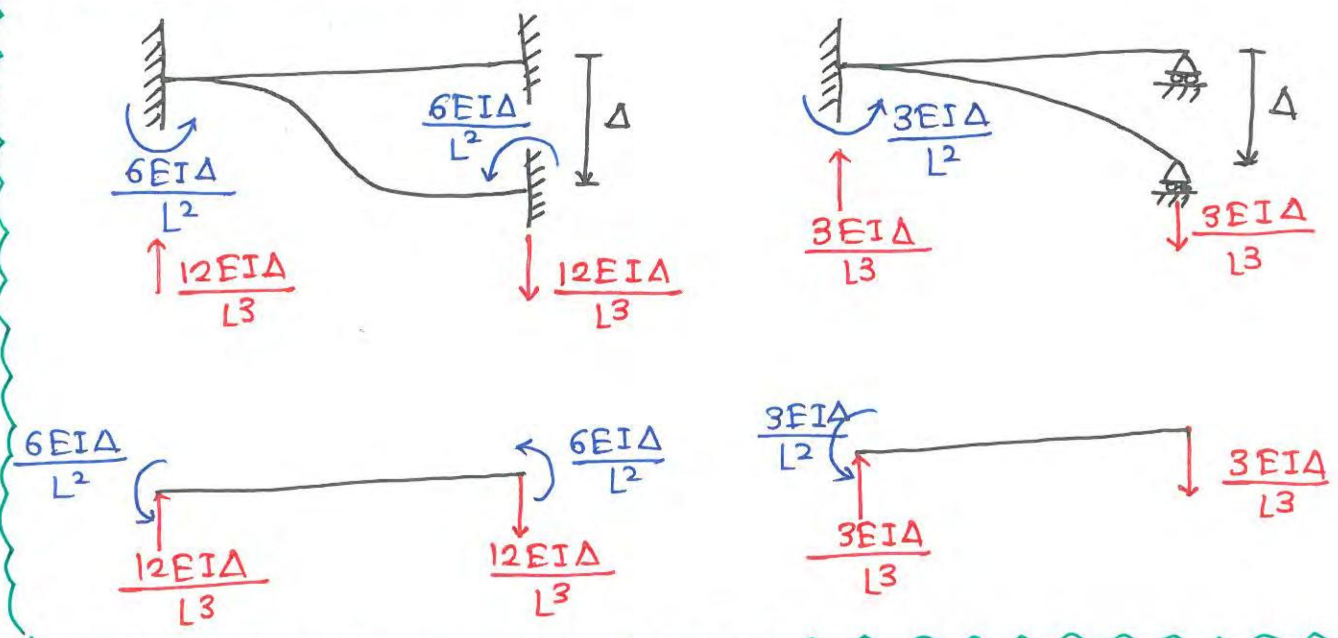
$$\Rightarrow \left(\frac{1}{2}\times L\times\frac{PL}{4EI}\right)+\left(-\frac{M}{EI}\times L\right)=0$$

$$\Rightarrow \boxed{M=\frac{PL}{8}}$$

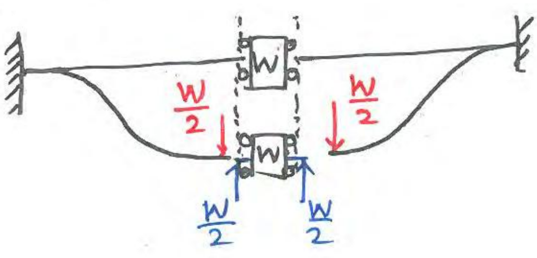
Q. 4 / Pg. 58



Standard Formulae

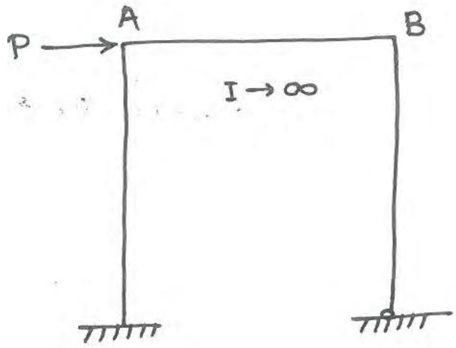


Solution:-

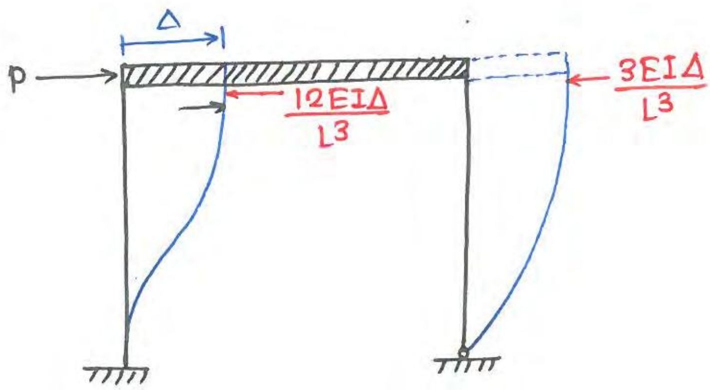


$$\frac{W}{2}=\frac{12EI\Delta}{L^3}\Rightarrow\boxed{\Delta=\frac{WL^3}{24EI}}$$

Q.44.
Pg.64



⇒



$$P = \frac{12EI\Delta}{L^3} + \frac{3EI\Delta}{L^3}$$

$$\Rightarrow \quad P = \frac{15EI\Delta}{L^3}$$