MISCELLANEOUS EXERCISE ON CHAPTER 8

Q.No 1: Find the area under the given curves and given lines.

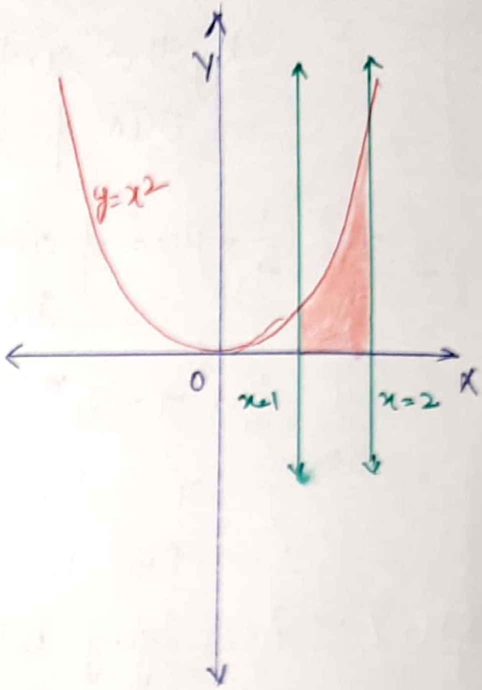
(i) $y = x^2 \quad x = 1 \quad x = 2$ and x-axis

(ii) $y = x^4 \quad x = 1 \quad x = 5$ and x-axis

Sol: (i) Required Area (shaded)

$$= \int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2$$

$$= \left[\frac{2^3}{3} - \frac{1^3}{3}\right] = \frac{8}{3} - \frac{1}{3} = \frac{7}{3} \text{ sq. units.}$$



(ii) Required Area (shaded)

$$= \int_1^5 y\,dx = \int_1^5 x^4\,dx$$

$$= \left[\frac{x^5}{5}\right]_1^5 = \frac{1}{5}\left[x^5\right]_1^5$$

$$= \frac{1}{5}\left[5^5 - 1\right] = \frac{1}{5}(3125 - 1) = \frac{3124}{5} \text{ sq. units.}$$

Q.No. 2: Find the area between the curves $y = x$ and $y = x^2$.

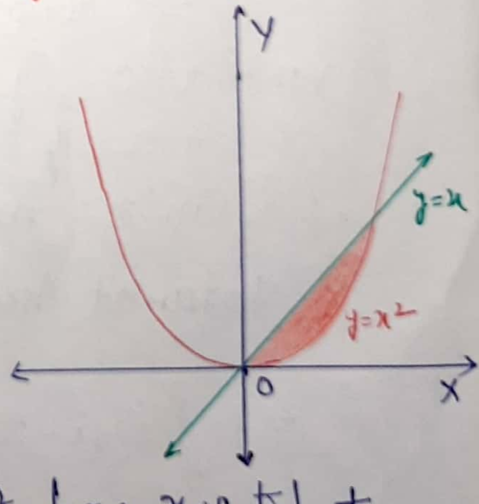
Soln: The parabola $y = x^2$ and line $y = x$ meet wher $x = x^2$

ie $x(x-1) = 0$

ie $x = 0$ or $x = 1$

when $x = 0$ $y = 0$; $x = 1$, $y = 1$

$\therefore$ The two curves meet in point $(0,0)$ and $(1,1)$



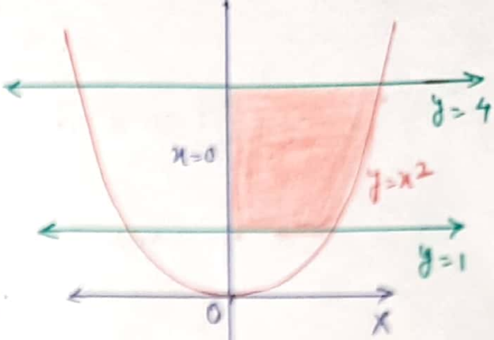
Required Area (shaded) = Area under $y = x$ from $x = 0$ to 1 − Area under line $y = x^2$ from $x = 0$ to $x = 1$

$$= \int_0^1 x\,dx - \int_0^1 x^2\,dx = \int_0^1 (x - x^2)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \left(\frac{1}{2} - \frac{1}{3}\right) - 0 = \frac{1}{6} \text{ sq. units.}$$

Q.No.3: Find the area of the region lying in the first quadrant and bounded by $y = 4x^2$, $x = 0$, $y = 1$ and $y = 4$.

Sol.: The given curve is $y = 4x^2$ which is an upward parabola.



Considering area with Y-axis

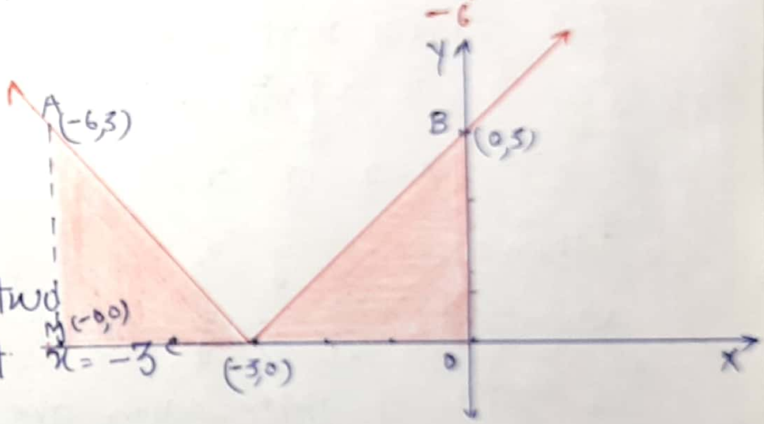
Required Area (shaded) =

$$\int_1^4 x\,dy = \int_1^4 \sqrt{\frac{y}{4}}\,dy = \frac{1}{2}\int_1^4 \sqrt{y}\,dy$$

$$= \frac{1}{2}\left[\frac{y^{3/2}}{3/2}\right]_1^4 = \frac{1}{2}\times\frac{2}{3}\left[y^{3/2}\right]_1^4 = \frac{1}{3}\left(4^{3/2} - 1^{3/2}\right) = \frac{1}{3}(8-1) = \frac{7}{3} \text{ sq.u.}$$

Q.No 4. Sketch the graph of $y = |x+3|$ and evaluate $\int_{-6}^{0} |x+3|\,dx$.

Sol.: The given fn. is $y = |x+3|$



$$y = \begin{cases} x+3 \;; & x+3 \geq 0 \\ -(x+3) \;; & x+3 < 0 \end{cases}$$

∴ graph of this fn. has two half lines meeting at $x = -3$ $M(-3,0)$

∴ Required Area $= \int_{-6}^{0} |x+3|\,dx$ = Area bounded by $y = |x+3|$, x-axis and lines $x = -6$ and $x = 0$

= Area under line $y = -(x+3)$ from $x = -6$ to $x = -3$

\+ Area under line $y = x+3$ from $x = -3$ to $x = 0$

$$= \int_{-6}^{-3} -(x+3)\,dx + \int_{-3}^{0} (x+3)\,dx = -\left[\frac{x^2}{2} + 3x\right]_{-6}^{-3} + \left[\frac{x^2}{2} + 3x\right]_{-3}^{0}$$

$$= -\left[\left(\frac{9}{2} - 9\right) - (18 - 18)\right] + \left[(0+0) - \left(\frac{9}{2} - 9\right)\right] = \frac{9}{2} - 0 + 0 + \frac{9}{2} = 9 \text{ sq. units}$$

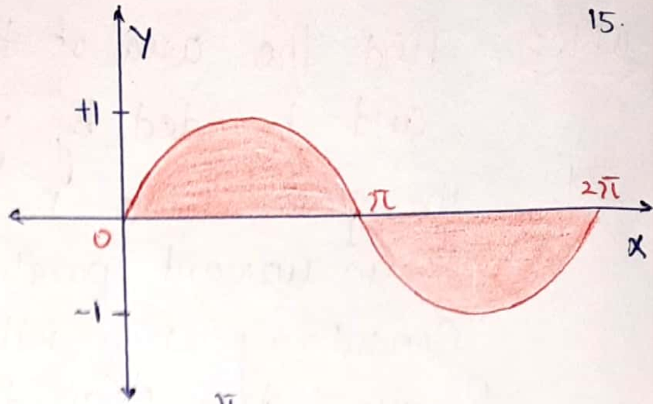
Q.No 5: Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.

Soln: The equation of given curve is

$$y = \sin x.$$

Now For $x \in [0, \pi]$; $\sin x \geq 0$

and for $x \in [\pi, 2\pi]$, $\sin x \leq 0$



$\therefore$ Required Area $= \int_0^{2\pi} |\sin x|\, dx$

$= \int_0^{\pi} |\sin x|\, dx + \int_{\pi}^{2\pi} |\sin x|\, dx$

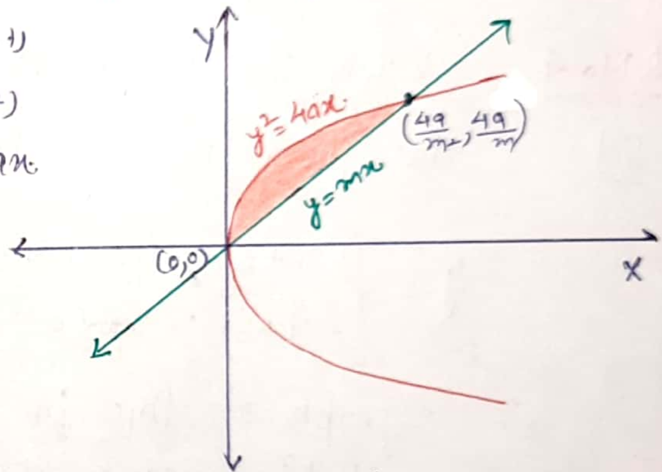
$= \int_0^{\pi} \sin x\, dx + \int_{\pi}^{2\pi} -\sin x \cdot dx = [\cos x]_0^{\pi} + [+\cos x]_{\pi}^{2\pi}$

$= [-\cos\pi + \cos 0] + (\cos 2\pi - \cos\pi) = -(-1)+1+1-(-1) = 4$ square units.

Q No 6: Find the area enclosed between the parabola $y^2 = 4ax$ and line $y = mx$.

Soln: Given parabola is $y^2 = 4ax$ —(1)

and line is $y = mx$ —(2)



Both curves meet where $(mx)^2 = 4ax$

ie $x[m^2x - 4a] = 0$

$\Rightarrow x = 0$ or $x = \frac{4a}{m^2}$

When $x = 0$, $y = 0$

when $x = \frac{4a}{m^2}$; $y = \frac{4a}{m}$

$\therefore$ Pts of intersection are $(0,0)$ and $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$

$\therefore$ Required Area (shaded) = Area Under parabola $y^2 = 4ax$ from $x = 0$ to $\frac{4a}{m^2}$ — Area under line $y = mx$ from $x = 0$ to $\frac{4a}{m^2}$

$= \int_0^{4a/m^2} (\sqrt{4ax} - mx)\, dx = 2\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^{4a/m^2} - \left[m\frac{x^2}{2}\right]_0^{4a/m^2}$

$= \frac{4}{3}\sqrt{a}\left(\frac{4a}{m^2}\right)^{3/2} - \frac{m}{2}\left(\frac{4a}{m^2}\right)^2 = \frac{4}{3}\sqrt{a}\left(\frac{4a}{m^2}\right)\sqrt{\frac{4a}{m^2}} - \frac{m(16a^2)}{2m^4}$

$= \frac{32}{3}\frac{a^2}{m^3} - \frac{8a^2}{m^3} = \frac{8a^2}{3m^3}$.

Q No 7: Find the area enclosed by the parabola $4y = 3x^2$ and $2y = 3x + 12$.

Sol: Given parabola is $4y = 3x^2$ —(1)

and line is $2y = 3x + 12$ —(2)

Solving (1) and (2) for pts of intersection.

$2(3x + 12) = 3x^2$

or $3x^2-6x-24=0$ or $x^2-2x-8=0$

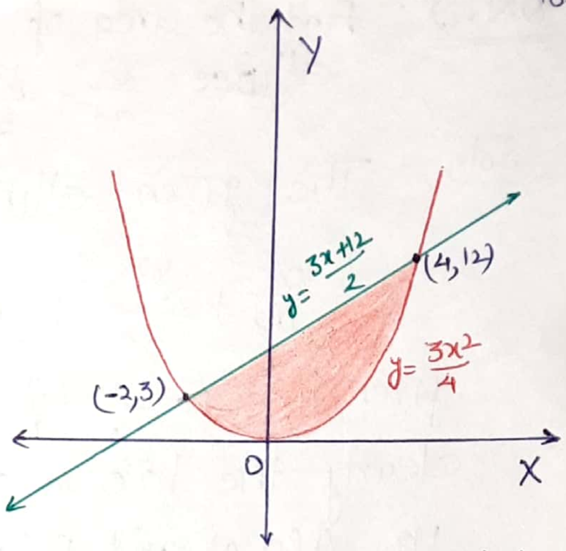
or $(x-4)(x+2)=0$

$\Rightarrow x = 4, -2.$

When $x=4$, $y=\frac{3\times16}{4}=12$

$x=-2$ $y=\frac{3\times4}{4}=3.$

$\therefore$ The curves meet at pts. $(4,12)$, $(-2,3)$



$\therefore$ Required Area (shaded) =

Area under the line between $x=-2$ and $x=4$ − Area under the parabola between lines $x=-2$ and $x=4$.

$$= \int_{-2}^{4}\left(\frac{3x+12}{2}-\frac{3x^2}{4}\right)dx = \left[\frac{1}{2}\left(\frac{3x^2}{2}+12x\right)-\frac{3}{4}\times\frac{x^3}{3}\right]_{-2}^{4}$$

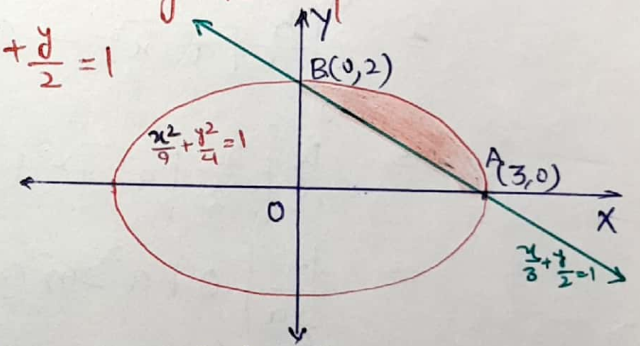
$$= \left[\frac{3x^2}{4}+6x-\frac{x^3}{4}\right]_{-2}^{4} = \left[\frac{3\times16}{4}+6\times4-\frac{64}{4}-\frac{3\times4}{4}-6\times(-2)+\frac{-8}{4}\right]$$

$$= 12+24-16-3+12-2 = 27 \text{ square units.}$$

Q No 8: Find the area of smaller region bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the line $\frac{x}{3}+\frac{y}{2}=1$

Sol: Given ellipse is $\frac{x^2}{9}+\frac{y^2}{4}=1$ and.

given line is $\frac{x}{3}+\frac{y}{2}=1$



As shown in fig. Required Area =

Area under parabola $\frac{x^2}{9}+\frac{y^2}{4}=1$ from $x=0$ and $x=3$ —
Area under line $\frac{x}{3}+\frac{y}{2}=1$ from $x=0$ and $x=3$.

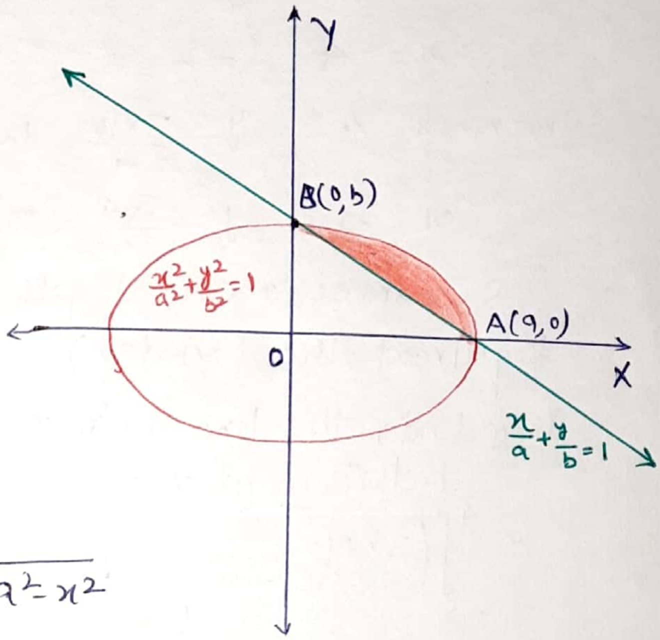
$$= \int_0^3 2\sqrt{1-\frac{x^2}{9}}\,dx - \int_0^3 2\left(1-\frac{x}{3}\right)dx = \frac{2}{3}\int_0^3\sqrt{3^2-x^2}\,dx - \frac{2}{3}\int_0^3(3-x)\,dx$$

$$= \frac{2}{3}\left[\frac{x}{2}\sqrt{3^2-x^2}+\frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3 - \frac{2}{3}\left[3x-\frac{x^2}{2}\right]_0^3$$

$$= \frac{2}{3}\left[0+\frac{9}{2}\sin^{-1}(1)-0\right]-\frac{2}{3}\left[9-\frac{9}{2}-0\right] = 3\times\frac{\pi}{2}-3$$

$$= 3\left(\frac{\pi}{2}-1\right) \text{ square units.}$$

Q No. 9: Find the area of smaller region enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and line $\frac{x}{a}+\frac{y}{b}=1$.



Soln The given ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad -(1)$$

and the given line is $\frac{x}{a}+\frac{y}{b}=1. \quad -(2)$

clearly the line (2) meets (1) in $A(a,0)$ and $B(0,b)$.

Also in first quadrant

$$\frac{y^2}{a^2}=1-\frac{x^2}{a} \Rightarrow |y|=\frac{b}{a}\sqrt{a^2-x^2}$$

ie $y=\frac{b}{a}\sqrt{a^2-x^2}$.

Now Required Area (Shaded) = Area Under ellipse from $x=0$ to $x=a$ − Area Under line from $x=0$ to $x=a$.

$$= \int_0^a \frac{b}{a}\sqrt{a^2-x^2} - ar(\Delta \text{ rt } \angle d \ \Delta AOB)$$

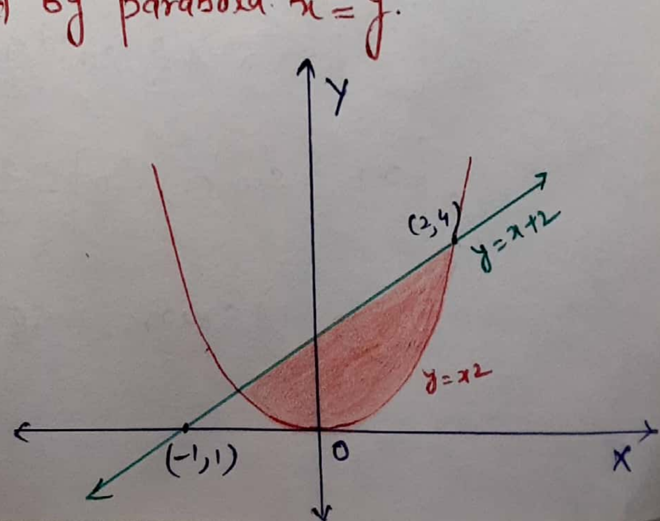
$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a - \frac{1}{2}\times|OA|\times|OB|$$

$$= \frac{b}{2a}\left[0+a^2\sin^{-1}(1)-0-a^2\sin^{-1}(0)\right] - \frac{1}{2}\times a\times b.$$

$$= \frac{b}{2a}\left[a^2\times\pi/2 - a^2\times 0\right] - \frac{1}{2}ab$$

$$= \frac{\pi ab}{4} - \frac{1}{2}ab = \frac{(\pi-2)ab}{4} \text{ Square units.}$$

Q No. 10. Find the area of region enclosed by parabola $x^2=y$, the line $y=x+2$ and x-axis.



Soln: Given parabola is $y=x^2 \quad -(1)$

and given line is $y=x+2 \quad -(2)$

Solving (1) and (2) for pts. of intersection:

$x^2=x+2$. or $x^2-x-2=0$

or $(x-2)(x+1)=0$

$\Rightarrow x=2,-1$

When $x = 2$, $y = 2+2 = 4$

when $x = -1$, $y = 2+(-1) = 1$

$\therefore$ The line meets the parabola at the pts $(-1,1)$ and $(2,4)$

$\therefore$ Required Area (shaded) $= \int_{-1}^{2}(x+2)\,dx - \int_{-1}^{2} x^2\,dx$

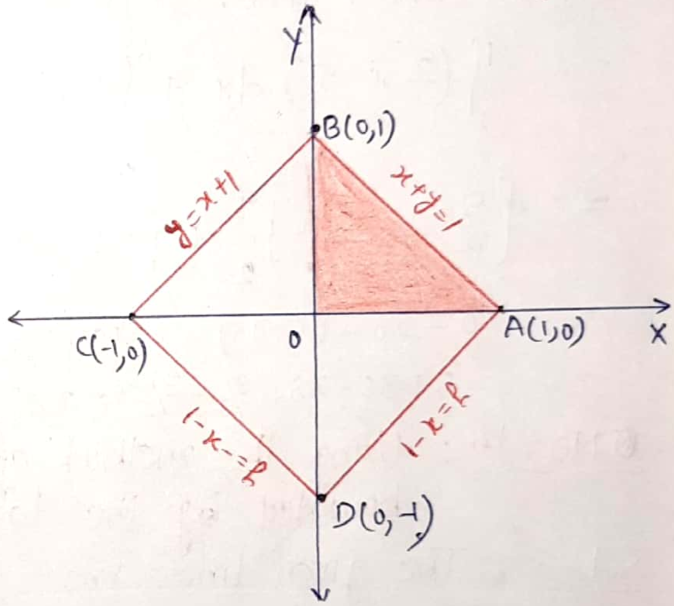
$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \left(\frac{4}{2} + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

$$= \frac{10}{3} + \frac{7}{6} = \frac{20+7}{6} = \frac{27}{6} = \frac{9}{2} \text{ square units.}$$

Q No. 11: Using method of integration, find the area bounded by the curve $|x| + |y| = 1$.

Soln: Given eqn. $|x| + |y| = 1$

$\Rightarrow \pm x \pm y = 1$.



The equation represents four lines

$x + y = 1 \quad x - y = 1$

$-x + y = 1 \quad -x - y = 1$.

The graph of these lines show that they enclose a square having diagonal 2 units

$\therefore$ Required area is symmetrical in all quadrants.

$=$ 4 × Area shown shaded in first quadrant (By Symmetry)

$$= 4 \times \int_{0}^{1}(1-x)\,dx = 4\left[x - \frac{x^2}{2}\right]_0^1 = 4\left[\left(1-\frac{1}{2}\right) - 0\right] = 2 \text{ square units.}$$

Q No 12: Find the area bounded by the curves $\{(x,y) ; y \geq x^2\}$ and $y = |x|$.

Same as Q No. 9 Exercise 8.1.

Q No 13: Using the method of Integration, find the area of the triangle ABC, coordinates of whose vertices are $A(2,0)$, $B(4,5)$, $C(6,3)$

Soln The given pts are $A(2,0)$, $B(4,5)$, $C(6,3)$

Eqn of AB is

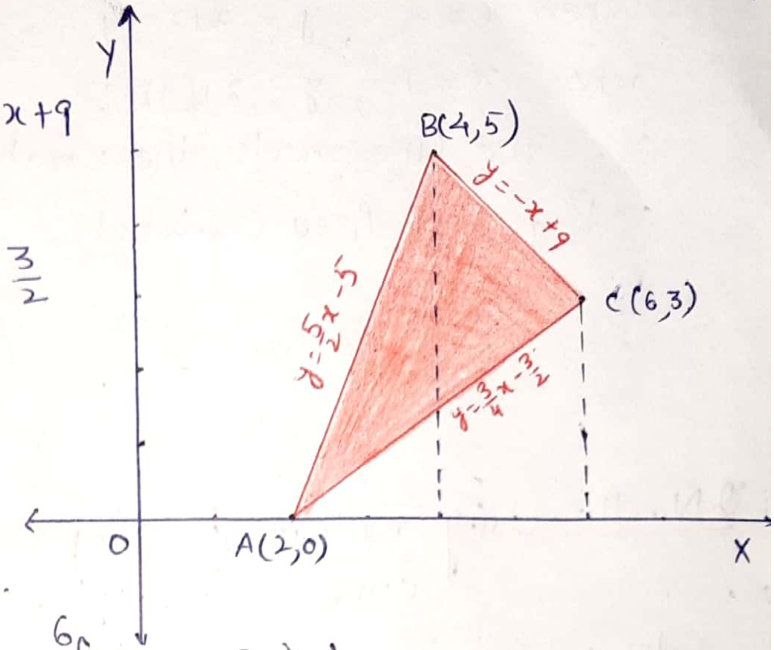
$$y - 0 = \frac{5-0}{4-2}(x-2) \quad \text{or} \quad y = \frac{5}{2}x - 5$$

Eqn of BC is

$$y-5=\frac{3-5}{6-4}(x-4) \text{ or } y=-x+9$$

Eqn. of AC is

$$y-0=\frac{3-0}{6-2}(x-2) \text{ or } y=\frac{3}{4}x-\frac{3}{2}$$



∴ Required Area of Δ ABC =

Area Under line segment AB + Area under line segment BC − Area under line segment AC.

$$= \int_2^4 \left(\frac{5}{2}x-5\right)dx + \int_4^6 (-x+9)\,dx - \int_2^6 \left(\frac{3}{4}x-\frac{3}{2}\right)dx$$

$$= \left[\frac{5}{2}\cdot\frac{x^2}{2}-5x\right]_2^4 + \left[-\frac{x^2}{2}+9x\right]_4^6 - \left[\frac{3}{4}\frac{x^2}{2}-\frac{3}{2}x\right]_2^6$$

$$= 20-20-(5-10)+(-18+54)-(-8+36)-\left(\frac{27}{2}+9\right)+\left(\frac{3}{2}-3\right)$$

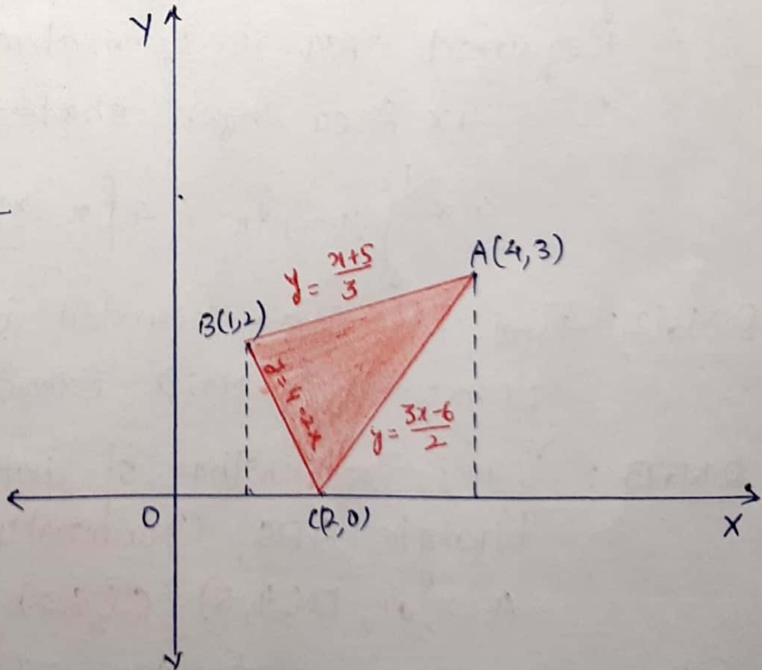
$$= 5+36-28-9/2-3/2 = 7 \text{ square unit.}$$

Q.No. 14: Using the method of integration find Area of Δ region bounded by the lines $2x+y=4$, $3x-2y=6$ and $x-3y+5=0$

Soln: The given lines are

$$2x+y=4 \quad —(1)$$
$$3x-2y=6 \quad —(2)$$
$$x-3y+5=0 \quad —(3)$$

Solving (1), (2), (3) for pts of intersection.



Lines (1) and (2) meet where

$$3x-2(4-2x)=6$$

ie $3x-8+4x=6$

ie $7x=14$

ie $x=2$

∴ $y=4-2x=4-2\times2=0$

∴ (1) and (2) meet in $(2,0)$

Lines (2) and (3) meet where $3(3y-5)-2y=6$

or $9y-15-2y=6$ or $7y=21$ or $y=3$

∴ $x=3y-5=3\times3-5=4$

∴ Lines (2) and (3) meet in (4,3)

Lines (1) and (3) meet where $x-3(4-2x)+5=0$

or $x-12+6x+5=0$ or $7x-7=0$ or $x=1$.

∴ $y = 4-2x = 4-2\times1 = 4-2 = 2$

∴ (1) and (3) meets in (1,2)

∴ Required Area of ΔABC = Area Under Segment BA − Area Under Segment BC − Area under segment AC

$$= \int_1^4 \left(\frac{x+5}{3}\right)dx - \int_1^2 (4-2x)\,dx - \int_2^4 \frac{3x-6}{2}\,dx$$

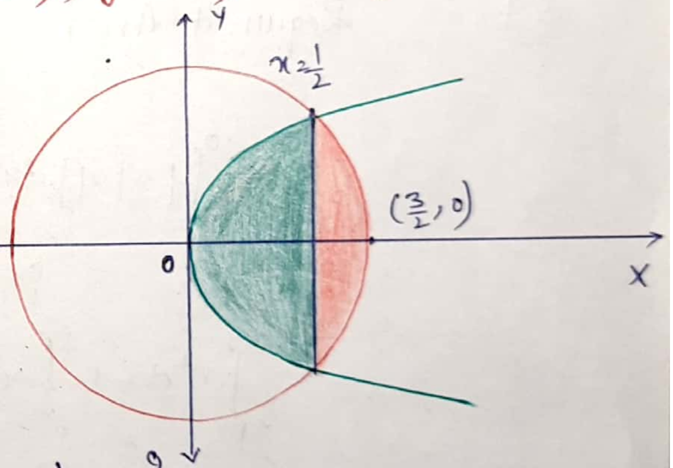
$$= \frac13\left[\frac{x^2}{2}+5x\right]_1^4 - \left[4x-\frac{2x^2}{2}\right]_1^2 - \frac12\left[\frac{3x^2}{2}-6x\right]_2^4$$

$$= \frac13\left[\frac{16}{2}+20-\frac12-5\right] - \left[(8-4)-(4-1)\right] - \frac12\left[\left(\frac{3\times4^2}{2}-6\times4\right)-\left(\frac{3\times2^2}{2}-6\times2\right)\right]$$

$$= \frac{45}{6}-1-\frac12(0-6+12) - \frac{15}{2}-1-3 = \frac72 \text{ square units.}$$

Q No. 15: Find the area of the region $\{(x,y);\ y^2\le 4x,\ 4x^2+4y^2\le 9\}$

Soln: Required Area is bounded by the circle $4x^2+4y^2=9$ and the parabola $y^2=4x$.



Given Circle is $4x^2+4y^2=9$ —(1)

and the given parabola is $y^2=4x$ —(2)

(1) and (2) meet where

$4x^2+16x=9$ i.e. where

$$x = \frac{-16\pm\sqrt{256+144}}{8} = \frac{-16\pm20}{8} = \frac12,\ -\frac92$$

But as $y^2 \not< 0$ ∴ $x\not<0$ ∴ $x=\frac12$

Since required area is symmetrical about x-axis

∴ Required Area $= 2\left[\int_0^{1/2}\sqrt{4x}\,dx + \int_{1/2}^{3/2}\sqrt{\frac{9-4x^2}{4}}\,dx\right]$.

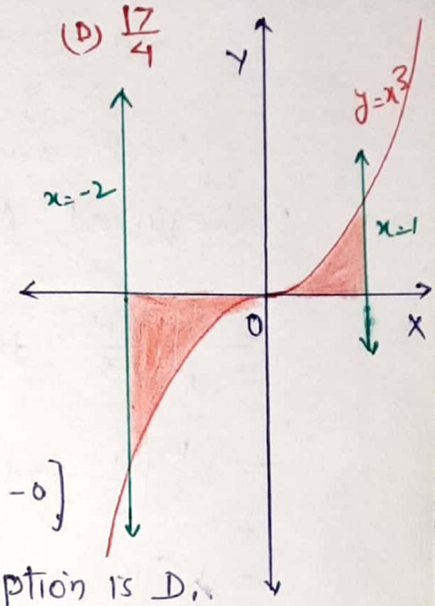
$$= 2\times2\left[\frac{x^{3/2}}{3/2}\right]_0^{1/2} + 2\int_{1/2}^{3/2}\sqrt{\left(\frac32\right)^2-x^2}\,dx = \frac83\left[\left(\frac12\right)^{3/2}-0\right] + 2\left[\frac x2\sqrt{\frac94-x^2}+\frac{9/4}{2}\sin^{-1}\frac{x}{3/2}\right]_{1/2}^{3/2}$$

$$= \frac83\cdot\frac1{2\sqrt2} + 2\left[0+\frac98\sin^{-1}(1) - \frac14\sqrt{\frac94-\frac14} - \frac98\sin^{-1}\left(\frac13\right)\right]$$

$$= \frac{2\sqrt2}{3}+\frac94\times\frac\pi2 - \frac{\sqrt2}{2} - \frac94\sin^{-1}\left(\frac13\right) = \frac{9\pi}{8}+\frac{\sqrt2}{6}-\frac94\sin^{-1}(1/3)$$

Choose the correct answer in the following questions 16 to 20

Q No 16: Area bounded by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$ is (A) -9 (B) $\frac{-15}{4}$ (C) $\frac{15}{4}$ (D) $\frac{17}{4}$



Sol: Required Area $= \int_{-2}^{1} |x^3|\, dx$

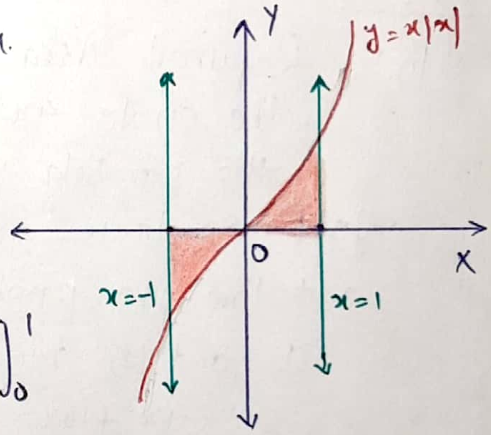
$$= \int_{-2}^{0} |x^3|\, dx + \int_{0}^{1} |x^3|\, dx$$

$$= \int_{-2}^{0} -x^3\, dx + \int_{0}^{1} x^3\, dx =$$

$$= -\left[\frac{x^4}{4}\right]_{-2}^{0} + \left[\frac{x^4}{4}\right]_{0}^{1} = -\left[0 - \frac{(-2)^4}{4}\right] + \left[\frac{1}{4} - 0\right]$$

$$= -(-4) + \frac{1}{4} = \frac{17}{4} \text{ units.}$$ So correct option is D.

Q No 17: The area bounded by the curve $y = x|x|$, x-axis and the ordinates $x = -1$ and $x = 1$ is given by (A) 0 (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{4}{3}$



Sol: Required Area $= \int_{-1}^{1} y\, dx = \int_{-1}^{1} x|x|\, dx$

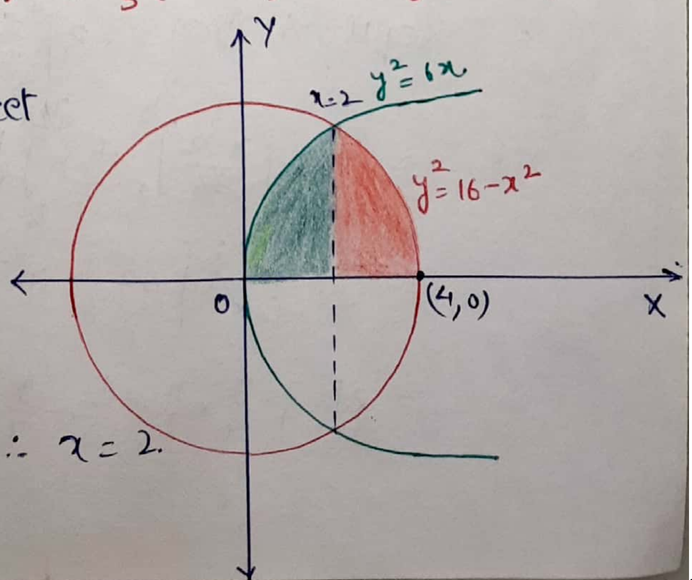
$$= \int_{-1}^{0} |x|x||\, dx + \int_{0}^{1} x|x|\, dx$$

$$\{\because |x|x|| = |x||x| = x^2\}$$

$$= \int_{-1}^{0} x^2\, dx + \int_{0}^{1} x^2\, dx = \left[\frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^3}{3}\right]_{0}^{1}$$

$$= \left[0 - \frac{(-1)^3}{3}\right] + \left(\frac{1}{3} - 0\right) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$ So correct option is C.

Q No 18: The area of circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$ is (A) $\frac{4}{3}(4\pi - \sqrt{3})$ (B) $\frac{4}{3}(4\pi + \sqrt{3})$ (C) $\frac{4}{3}(8\pi - \sqrt{3})$ (D) $\frac{4}{3}(8\pi + \sqrt{3})$



Sol: The circle $x^2 + y^2 = 16$ and the parabola $y^2 = 6x$ meet where $x^2 + 6x = 16$

$$x^2 + 6x - 16 = 0$$

$$\Rightarrow (x+8)(x-2) = 0$$

$$\Rightarrow x = -8, 2.$$

But x can not be $-$ve $\therefore x = 2$.

Now Required Area = Area of circle − 2 × Shaded Area (By Symmetry)

$= \pi(4)^2 - 2\left\{ \int_0^2 \sqrt{6x}\,dx + \int_2^4 \sqrt{16-x^2}\,dx \right\}$

$= 16\pi - 2\sqrt{6}\left[\frac{x^{3/2}}{3/2}\right]_0^2 - 2\left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_2^4$

$= 16\pi - \frac{4\sqrt{6}}{3}\left(2^{3/2} - 0\right) - 2\left[0 + 8\sin^{-1}(1) - \sqrt{16-4} - 8\sin^{-1}\left(\tfrac{1}{2}\right)\right]$

$= 16\pi - \frac{4\sqrt{6}\times 2\sqrt{2}}{3} - 2\left[8\times\frac{\pi}{2} - \sqrt{12} - 8\times\frac{\pi}{6}\right] = 16\pi - \frac{16\sqrt{3}}{3} - \frac{16\pi}{3} + 4\sqrt{3}$

$= \frac{32\pi}{3} - \frac{4\sqrt{3}}{3} = \frac{4}{3}\left(8\pi - \sqrt{3}\right)$ square units. ∴ Correct option is (C)

Q No 19: The area bounded by the y-axis, $y = \cos x$ and $y = \sin x$, $0 \le x \le \frac{\pi}{2}$ is

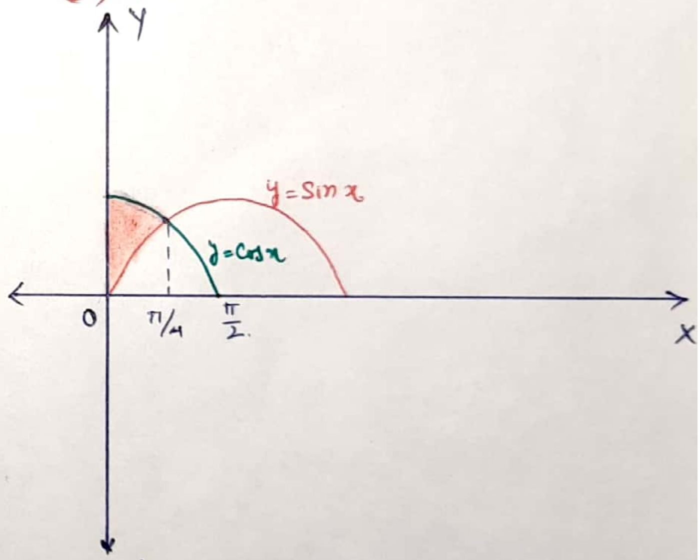
(A) $2(\sqrt{2}-1)$ (B) $\sqrt{2}-1$ (C) $\sqrt{2}+1$ (D) $\sqrt{2}$.

Sol:

The two curves are

$y = \sin x$

and $y = \cos x$



Now Two curves meet where

$\sin x = \cos x$

ie $\frac{\sin x}{\cos x} = 1$

ie $\tan x = 1$

$\Rightarrow x = \pi/4$.

∴ Required Area (shown shaded) $= \int_0^{\pi/4} (\cos x - \sin x)\,dx$.

$= \left[\sin x + \cos x\right]_0^{\pi/4}$

$= \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - \left(\sin 0 + \cos 0\right)$

$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - 1$

$= \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$

∴ Correct option is (B)

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