CBSE Board
Class XII Mathematics
Board Paper 2011
Delhi Set – 1

Time: 3 hrs **Total Marks: 100**

General Instructions:

1. All questions are compulsory.
2. The question paper consists of 29 questions divided into three Section A, B and C. Section A comprises of 10 questions of one mark each, Section B comprises of 12 questions of four marks each and Section C comprises of 7 questions of six marks each.
3. All questions in section A are to be answered in one word, one sentence or as per the exact requirement of the question.
4. There is no overall choice. However, internal choice has been provided in 4 questions of four marks each and 2 questions of six mark each. You have to attempt only one of the alternatives in all such questions.
5. Use of calculators is not permitted. You may ask for logarithmic tables, if required.

SECTION – A

1. State the reason for the relation R in the set {1, 2, 3} given by R = {(1, 2), (2, 1)} not to be transitive.

2. Write the value of $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$

3. For a 2 × 2 matrix, A = $[a_{ij}]$ whose elements are given by $a_{ij} = \frac{i}{j}$, write the value of a_{12}.

4. For what value of x, the matrix $\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$ is singular?

5. Write A^{-1} for A = $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$.

6. Write the value of $\int \sec x\,(\sec x + \tan x)\,dx$

7. Write the value of $\int \frac{dx}{x^2+16}$

8. For what value of 'a' the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $a\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear?

9. Write the direction cosines of the vector $-2\hat{i} + \hat{j} - 5\hat{k}$.

10. Write the intercept cut off by the plane 2x + y – z = 5 on x-axis.

SECTION – B

11. Consider the binary operation * on the set {1, 2, 3, 4, 5} defined by a * b = min {a, b}. Write the operation table of the operation *.

12. Prove the following:

$$\cot^{-1}\left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right] = \frac{x}{2}, x \in \left(0, \frac{\pi}{4}\right)$$

OR

Find the value of $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$

13. Using properties of determinants, prove that

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

14. Find the value of 'a' for which the function f defined as

$$f(x) = \begin{cases} a\sin\frac{\pi}{2}(x+1), & x \le 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$$

is continuous at x = 0.

15. Differentiate $x^{x\cos x} + \frac{x^2+1}{x^2-1}$ w.r.t. x

OR

If $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$, find $\frac{d^2y}{dx^2}$

16. Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the sand cone increasing when the height is 4 cm?

OR

Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to x-axis.

17. Evaluate: $\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx$

OR

Evaluate: $\int \frac{2x}{\sqrt{(x^2+1)(x^2+3)}}dx$

18. Solve the following differential equation:

$e^x \tan y\, dx + (1-e^x)\sec^2 y\, dy = 0$

19. Solve the following differential equation:

$\cos^2 x \frac{dy}{dx} + y = \tan x$

20. Find a unit vector perpendicular to each of the vector $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.

21. Find the angle between the following pair of lines:

$\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{2y-8}{4} = \frac{z-5}{4}$

And check whether the lines are parallel or perpendicular.

22. Probabilities of solving problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that (i) the problem is solved (ii) exactly one of them solves the problem.

SECTION – C

23. Using matrix method, solve the following system of equations:

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4, \frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1, \frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2; x, y, z \neq 0$$

OR

Using elementary transformations, find the inverse of the matrix $\begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$

24. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

25. Using integration find the area of the triangular region whose sides have equations $y=2x+1$, $y=3x+1$ and $x=4$.

26. Evaluate: $\int_0^{\frac{\pi}{2}} 2\sin x \cos x \tan^{-1}(\sin x)dx$

OR

Evaluate: $\int_0^{\frac{\pi}{2}} \frac{x\sin x \cos x}{\sin^4 x+\cos^4}dx$

27. Find the equation of the plane which contains the line of intersection of the planes $\vec{r}.\left(\hat{i}+2\hat{j}+3k\right)-4=0$, $\vec{r}.\left(2\hat{i}+\hat{j}-k\right)+5=0$ and which is perpendicular to the plane $\vec{r}.\left(5\hat{i}+3\hat{j}-6k\right)+8=0$.

28. A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftsman's time in its making while a cricket bat takes 3 hours of machine time and 1 hour of craftsman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftsman's time. If the profit on a racket and on a bat is Rs20 and Rs10 respectively, find the number of tennis rackets and crickets bats that the factory must manufacture to earn the maximum profit. Make it as an L.P.P and solve graphically.

29. Suppose 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability of this person being male? Assume that there are equal number of males and females.

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SECTION – A

1. A relation R in a set A is transitive if $(a_1, a_2) \in R$ and $(a_2, a_3) \in$ implies $(a_1, a_3) \in R$, where $a_1, a_2, a_3 \in A$

Now, $(1, 2), (2, 1) \in R$, but $(1, 1) \notin R$

Thus, the given relation R is not transitive.

2. $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right]$

Let $\sin^{-1}\left(\frac{-1}{2}\right) = x$

$\Rightarrow \left(\frac{-1}{2}\right) = \sin x$

$\Rightarrow \sin x = -\sin\frac{\pi}{6} = \sin\left(-\frac{\pi}{6}\right) = \sin\left(2\pi - \frac{\pi}{6}\right)$

$\Rightarrow x = 2\pi - \frac{\pi}{6}$

$$\begin{aligned}
\therefore \sin\left[\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right] &= \sin\left[\frac{\pi}{3} - \left(2\pi - \frac{\pi}{6}\right)\right] \\
&= \sin\left[-\frac{9\pi}{6}\right] \\
&= -\sin\left[\frac{3\pi}{2}\right] \\
&= -\sin\left[\pi + \frac{\pi}{2}\right] \\
&= -\left[-\sin\frac{\pi}{2}\right] \\
&= +\sin\frac{\pi}{2} \\
&= 1
\end{aligned}$$

Thus, $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right] = 1$

3. It is given that the elements of the matrix A = [a_{ij}] are given by $a_{ij} = \frac{i}{j}$

For a_{12}, the value of i = 1 and j = 2.

$\therefore\ a_{12} = \frac{1}{2}$

4.

Let A = $\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$

It is given that the matrix A is singular, therefore |A| = 0

$\Rightarrow \begin{vmatrix} 5-x & x+1 \\ 2 & 4 \end{vmatrix} = 0$

$\Rightarrow 4(5 - x) - 2(x + 1) = 0$

$\Rightarrow 20 - 4x - 2x - 2 = 0$

$\Rightarrow -6x + 18 = 0$

$\Rightarrow x = \frac{-18}{-6} = 3$

Thus, when x = 3, the given matrix A is singular.

5. A = $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

$= \frac{1}{[(2\times3)-(1\times5)]}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

$= \frac{1}{(6-5)}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

6.

$\int \sec x\,(\sec x + \tan x)dx$

$= \int\left(\sec^2 x + \sec x \tan x\right) dx$

$= \int \sec^2 x\ dx + \int \sec x \tan x\ dx$

= tan x + sec x + c, where c is a constant

7. $\int \frac{dx}{x^2+16}$

$= \int \frac{dx}{(x)^2 + (4)^2}$

$= \frac{1}{4} \tan^{-1}\left(\frac{x}{4}\right)$ + c, where c is a constant

8. Two vectors $\vec{x}$ and $\vec{y}$ are collinear if $\vec{x} = \lambda \vec{y}$, where λ is a constant.

Now, the vectors $2\hat{i} - 3\hat{j} + 4k$ and $a\hat{i} + 6\hat{j} - 8k$ are collinear,

$\therefore 2\hat{i} - 3\hat{j} + 4k = \lambda.\left(a\hat{i} + 6\hat{j} - 8k\right)$, where λ is a constant.

$\Rightarrow 2 = \lambda a, -3 = 6\lambda, 4 = -8\lambda$

Now, $-3 = 6\lambda$ or $4 = -8\lambda \Rightarrow \lambda = -\frac{1}{2}$

$2 = \lambda a$

$\Rightarrow 2 = -\frac{1}{2} \times a$

$\Rightarrow a = -4$

9. The direction cosines of the given vector $-2\hat{i} + \hat{j} - 5k$ is given by

$$\left(\frac{-2}{\sqrt{(-2)^2 + (1)^2 + (-5)^2}}, \frac{1}{\sqrt{(-2)^2 + (1)^2 + (-5)^2}}, \frac{-5}{\sqrt{(-2)^2 + (1)^2 + (-5)^2}}\right)$$

$$= \left(\frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{-5}{\sqrt{30}}\right)$$

10. 2x + y – z = 5

Dividing both sides by 5,

$\frac{2x}{5} + \frac{y}{5} - \frac{z}{5} = 1$

$\Rightarrow \frac{x}{\frac{5}{2}} + \frac{y}{5} + \frac{z}{-5} = 1$

It is known that the equation of a plane in intercept form is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c}$ = 1, where a, b and c are the intercepts cut off by the plane at x, y, and z-axes respectively.

Thus, the intercept cut off by the given plane on the x-axis is $\frac{5}{2}$.

SECTION – B

11. The binary operation * on the set {1, 2, 3, 4, 5} is defined by a * b = min {a, b}

The operation table for the given operation * on the given set is as follows:

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

12. $\cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$

$$= \cot^{-1}\left[\frac{\sqrt{\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)+\sin 2\left(\frac{x}{2}\right)}+\sqrt{\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)-\sin 2\left(\frac{x}{2}\right)}}{\sqrt{\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)+\sin 2\left(\frac{x}{2}\right)}-\sqrt{\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)-\sin 2\left(\frac{x}{2}\right)}}\right]$$

[Since, $\sin^2 A + \cos^2 A = 1$]

$$=\cot^{-1}\left[\frac{\sqrt{\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)+2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}+\sqrt{\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)-2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}}{\sqrt{\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)+2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}-\sqrt{\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)-2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}}\right]$$

[Since, $\sin 2A = 2 \sin A \cos A$]

$$= \cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right]$$

$$= \cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right) = \cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$= \frac{x}{2}$$

Hence proved.

OR

$$\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$$

$$= \tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{\frac{x}{y}-1}{\frac{x}{y}+1}\right) = \tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{\frac{x}{y}-1}{1+\frac{x}{y}}\right)$$

$$= \tan^{-1}\left(\frac{x}{y}\right) - \left[\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}(1)\right] \left[\because \tan^{-1} a - \tan^{-1} b = \tan^{-1}\left(\frac{a-b}{1+ab}\right)\right]$$

$$= \tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1}(1)$$

$$= \tan^{-1}(1) = \frac{\pi}{4}$$

Thus, $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right) = \frac{\pi}{4}$

13.

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$$

$$= abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

[Taking out a, b, and c common from R_1, R_2, and R_3 respectively]

$$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

[Taking out a, b, and c common from C_1, C_2, and C_3 respectively]

$$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix}$$ [Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$]

$= a^2b^2c^2\,[(-1)\,(0 \times 0 - 2 \times 2)]$

$= a^2b^2c^2\,[-\,(0 - 4)] = 4a^2b^2c^2$

Hence proved.

14. $f(x) = \begin{cases} a\sin\frac{\pi}{2}(x+1), & x \le 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$

The given function f is defined for all $x \in \mathbf{R}$.

It is known that a function f is continuous at x = 0, if $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^+} f(x) = f(0)$

$$\lim_{x\to0^-} f(x) = \lim_{x\to0}\left[a\sin\frac{\pi}{2}(x+1)\right] = a\sin\frac{\pi}{2} = a(1) = a$$

$$\lim_{x\to0^+} f(x) = \lim_{x\to0}\frac{\tan x-\sin x}{x^3} = \lim_{x\to0}\frac{\frac{\sin x}{\cos x}-\sin x}{x^3}$$

$$= \lim_{x\to0}\frac{\sin x(1-\cos x)}{x^3\cos x} = \lim_{x\to0}\frac{\sin x.2\sin^2\frac{x}{2}}{x^3\cos x}$$

$$= 2\lim_{x\to0}\frac{1}{\cos x}\times\lim_{x\to0}\frac{\sin x}{x}\times\lim_{x\to0}\left[\frac{\sin\frac{x}{2}}{x}\right]^2$$

$$= 2\times1\times1\times\frac{1}{4}\times\lim_{\frac{x}{2}\to0}\left[\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right]^2$$

$$= 2\times1\times1\times\frac{1}{4}\times1 = \frac{1}{2}$$

Now, $f(0) = a\sin\frac{\pi}{2}(0+1) = a\sin\frac{\pi}{2} = a\times1 = a$

Since f is continuous at x = 0, $a = \frac{1}{2}$

15. Let $y = x^{x\cos x}$ and $z = \dfrac{x^2+1}{x^2-1}$

Consider $y = x^{x\cos x}$

Taking log on both sides,

$\log y = \log(x^{x\cos x})$

$\log y = x\cos x\log x$

Differentiating with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = (x\cos x)\frac{1}{x} + \log x\frac{d}{dx}(x\cos x)$$

$$\frac{1}{y}\frac{dy}{dx} = \cos x + \log x(\cos x - x\sin x)$$

$$\frac{dy}{dx} = y[\cos x + \log x(\cos x - x\sin x)]$$

$$\frac{dy}{dx} = x^{x\cos x}[\cos x + \log x(\cos x - x\sin x)] \quad \ldots (1)$$

Consider $z = \dfrac{x^2+1}{x^2-1}$

Differentiating with respect to x,

$$\frac{dz}{dx} = \frac{(x^2-1).\frac{d}{dx}(x^2+1)-(x^2+1).\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2}$$

$$= \frac{2x^3-2x-2x^3-2x}{(x^2-1)^2}$$

$$= \frac{-4x}{(x^2-1)^2} \text{....(2)}$$

Adding (1) and (2):

$$\frac{d}{dx}\left\{x^{x\cos x} + \frac{x^2+1}{x^2-1}\right\} = \frac{dy}{dx} + \frac{dz}{dx}$$

$$= x^{x\cos x}[\cos x + \log x\,(\cos x - x\sin x)] - \frac{4x}{\left(x^2-1\right)^2}$$

OR

$x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$

Differentiating x and y w.r.t. θ,

$$\frac{dx}{d\theta} = a(1 - \cos\theta) \quad \text{...(1)}$$

$$\frac{dy}{d\theta} = -a\sin\theta \quad \text{...(2)}$$

Dividing (2) by (1),

$$\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-a\sin\theta}{a\left(1-\cos\theta\right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin\theta}{1-\cos\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$\Rightarrow \frac{dy}{dx} = -\cot\frac{\theta}{2} \text{....(3)}$$

Differentiating w.r.t. x,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \times \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \times \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(-\cot\frac{\theta}{2}\right) \times \frac{d\theta}{dx} \quad \left[\text{from equation (3)}\right]$$

$$\frac{d^2y}{dx^2} = -\left(-\operatorname{cosec}^2\frac{\theta}{2} \times \frac{1}{2}\right) \times \frac{d\theta}{dx}$$

$$= \frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2} \times \frac{1}{\left(\frac{dx}{d\theta}\right)}$$

$$= \frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2} \times \frac{1}{a(1-\cos\theta)} \text{ ....}\left[\text{from equation (1)}\right]$$

$$= \frac{\operatorname{cosec}^2\frac{\theta}{2}}{2a(1-\cos\theta)}$$

$$= \frac{\operatorname{cosec}^2\frac{\theta}{2}}{2a\left(2\sin^2\frac{\theta}{2}\right)}$$

$$= \frac{1}{4a} \times \operatorname{cosec}^4\frac{\theta}{2}$$

16. The volume of a cone with radius r and height h is given by the formula,

$V = \frac{1}{3}\pi r^2 h$

According to the question,

$h = \frac{1}{6}\ r \Rightarrow r = 6h$

Substituting in the formula,

$\therefore\ V = \frac{1}{3}\pi(6h)^2 h = 12\pi h^3$

The rate of change of the volume with respect to time is

$$\frac{dV}{dt} = 12\pi\frac{d}{dh}\left(h^3\right) \times \frac{dh}{dt} \quad \left[\text{By chain rule}\right]$$

$$= 12\pi\left(3h^2\right) \times \frac{dh}{dt}$$

$= 36\pi h^2 \times \frac{dh}{dt}$ ....(1)

Given that $\frac{dV}{dt} = 12\ cm^3/s$

Substituting the values $\frac{dV}{dt}$=12 and h=4 in equation (1), we have,

$12 = 36\pi(4)^2 \times \frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{12}{36\pi \times (16)}$

$\Rightarrow \frac{dh}{dt} = \frac{1}{48\pi}$

Hence, the height of the sand cone is increasing at the rate of $\frac{1}{48\pi}$ cm/s.

OR

Let P(x, y) be any point on the given curve $x^2 + y^2 - 2x - 3 = 0$.

Tangent to the curve at the point (x, y) is given by $\frac{dy}{dx}$.

Differentiating the equation of the curve w .r. t. x we get

$2x + 2y\frac{dy}{dx} - 2 = 0$

$\Rightarrow \frac{dy}{dx} = \frac{2-2x}{2y} = \frac{1-x}{y}$

Let $P(x_1, y_1)$ be the point on the given curve at which the tangents are parallel to the x-axis.

$\therefore \left.\frac{dy}{dx}\right]_{(x_1, y_1)} = 0$

$\Rightarrow \frac{1-x_1}{y_1} = 0$

$\Rightarrow 1 - x_1 = 0$

$\Rightarrow x_1 = 1$

To get the value of y_1 just substitute $x_1 = 1$ in the equation $x^2 + y^2 - 2x - 3 = 0$, we get

$(1)^2 + {y_1}^2 - 2 \times 1 - 3 = 0$

$\Rightarrow {y_1}^2 - 4 = 0$

$\Rightarrow {y_1}^2 = 4$

$\Rightarrow y_1 = \pm 2$

So, the points on the given curve at which the tangents are parallel to the x-axis are (1, 2) and (1, -2).

17. $\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx$

$$\text{Now}, 5x + 3 = A\frac{d}{dx}\left(x^2+4x+10\right) + B$$

$$\Rightarrow 5x + 3 = A\left(2x+4\right) + B$$

$$\Rightarrow 5x + 3 = 2Ax + 4A + B$$

$$\Rightarrow 2A = 5 \text{ and } 4A + B = 3$$

$$\Rightarrow A = \frac{5}{2}$$

$$\text{Thus, } 4\left(\frac{5}{2}\right) + B = 3$$

$$\Rightarrow 10 + B = 3$$

$$\Rightarrow B = 3 - 10 = -7$$

On substituting the values of A and B, we get

$$\int \frac{(5x+3)}{\sqrt{x^2+4x+10}}dx = \int \frac{\left[\frac{5}{2}\frac{d}{dx}\left(x^2+4x+10\right)-7\right]}{\sqrt{x^2+4x+10}}dx$$

$$= \int \left[\frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}\right]dx$$

$$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}dx - 7\int \frac{dx}{\sqrt{x^2+4x+10}}$$

$$= \frac{5}{2}I_1 - 7I_2 \qquad \ldots(1)$$

$$I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$$

$$\text{Put } x^2+4x+10 = z^2$$

$$(2x+4)dx = 2z\,dz$$

$$\text{Thus}, I_1 = \int \frac{2z}{z}dz = 2z = 2\sqrt{x^2+4x+10} + C_1$$

$$I_2 = \int \frac{dx}{\sqrt{x^2+4x+10}}$$

$$= \int \frac{dx}{\sqrt{x^2+4x+4+6}}$$

$$= \int \frac{dx}{\sqrt{(x+2)^2+\left(\sqrt{6}\right)^2}}$$

$$= \log\left|(x+2) + \sqrt{x^2+4x+10}\right| + C_2$$

Substituting I_1 and I_2 in (1), we get

$$\therefore \int \frac{5x+3}{\sqrt{x^2+4x+10}} = \frac{5}{2}(2\sqrt{x^2+4x+10}+C_1) - 7\left[\log\left|(x+2)+\sqrt{x^2+4x+10}\right|+C_2\right]$$

$$= 5\sqrt{x^2+4x+10} - 7\left[\log\left|(x+2)+\sqrt{x^2+4x+10}\right|\right] + \frac{5}{2}C_1 - 7C_2$$

$$= 5\sqrt{x^2+4x+10} - 7\left[\log\left|(x+2)+\sqrt{x^2+4x+10}\right|\right] + C, \quad \text{where } C = \frac{5}{2}C_1 - 7C_2$$

OR

$$I = \int \frac{2x}{(x^2+1)(x^2+3)}dx$$

Let $x^2 = z$

$\therefore 2xdx = dz$

$$\therefore I = \int \frac{dz}{(z+1)(z+3)}$$

By partial fraction,

$$\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$$

$\Rightarrow 1 = A(z+3) + B(z+1)$

Putting $z = -3$, we obtain:

$1 = -2B$

$B = -\frac{1}{2}$

$\therefore A = \frac{1}{2}$

$$\therefore \frac{1}{(z+1)(z+3)} = \frac{\frac{1}{2}}{z+1} + \frac{\left(-\frac{1}{2}\right)}{z+3}$$

$$\Rightarrow \int \frac{dz}{(z+1)(z+3)} = \frac{1}{2}\int \frac{dz}{z+1} - \frac{1}{2}\int \frac{dz}{z+3}$$

$$= \frac{1}{2}\log|z+1| - \frac{1}{2}\log|z+3| + C$$

$$\therefore \int \frac{2xdx}{(x^2+1)(x^2+3)} = \frac{1}{2}\log\left|x^2+1\right| - \frac{1}{2}\log\left|x^2+3\right| + C$$

18. The given differential equation is:

$$e^x \tan y\, dx + (1-e^x)\sec^2 y\, dy = 0$$

$\Rightarrow e^x \tan y \, dx = -\left(1-e^x\right)\sec^2 y \, dy$

$\Rightarrow e^x \tan y \, dx = \left(e^x - 1\right)\sec^2 y \, dy$

$\Rightarrow \dfrac{e^x}{e^x - 1} dx = \dfrac{\sec^2 y}{\tan y} dy$

On integrating on both sides, we get

$\int \dfrac{e^x}{e^x - 1} dx = \int \dfrac{\sec^2 y}{\tan y} dy \quad \ldots(i)$

Let $I_1 = \int \dfrac{\sec^2 y}{\tan y} dy$

Put $\tan y = t$

$\Rightarrow \sec^2 y \, dy = dt$

$\therefore \int \dfrac{\sec^2 y}{\tan y} dy = \int \dfrac{dt}{t} = \log|t| = \log \tan y \ldots.(ii)$

Let $I_2 = \int \dfrac{e^x}{e^x - 1} dx$

Put $e^x - 1 = u$

$\therefore e^x \, dx = du$

$\int \dfrac{e^x}{e^x - 1} dx = \int \dfrac{du}{u}$

$= \log u$

$= \log\left(e^x - 1\right) \quad \ldots..(iii)$

From (i), (ii) and (iii), we get

$\log \tan y = \log\left(e^x - 1\right) + \log C$

$\Rightarrow \log \tan y = \log C\left(e^x - 1\right)$

$\Rightarrow \tan y = C\left(e^x - 1\right)$

The solution of the given differential equation is tan y = C (e^x - 1).

19. $\cos^2 x \dfrac{dy}{dx} + y = \tan x$

$\Rightarrow \dfrac{dy}{dx} + \sec^2 x.y = \sec^2 x \tan x$

This equation is in the form of $\dfrac{dy}{dx} + py = Q$

here $p = \sec^2 x$ and $Q = \sec^2 x \tan x$

Integrating Factor, I.F $= e^{\int pdx} = e^{\int \sec^2 xdx} = e^{\tan x}$

The general solution can be given by

$$y(\text{I.F}) = \int (Q \times \text{I.F})dx + C \qquad ...(1)$$

Let $\tan x = t$

$$\Rightarrow \frac{d}{dx}(\tan x) = \frac{dt}{dx}$$

$$\Rightarrow \sec^2 x = \frac{dt}{dx}$$

$$\Rightarrow \sec^2 xdx = dt$$

Therefore, equation (1) becomes:

$$y.e^{\tan x} = \int (e^t.t)dt$$

$$\Rightarrow y.e^{\tan x} = \int (e^t.t)dt + C$$

$$\Rightarrow y.e^{\tan x} = t.\int e^t dt - \int \left(\frac{d}{dt}(t).\int e^t dt\right)dt + C$$

$$\Rightarrow y.e^{\tan x} = t.e^t - \int e^t dt + C$$

$$\Rightarrow y.e^{\tan x} = t.e^t - e^t + C$$

$$\Rightarrow y.e^{\tan x} = (t-1)e^t + C$$

$$\Rightarrow y.e^{\tan x} = (\tan x - 1)e^{\tan x} + C$$

$\Rightarrow y = (\tan x - 1) + Ce^{-\tan x}$, where C is an arbitary constant.

20.

$\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k} \qquad \vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}.$

$\therefore \vec{a} + \vec{b} = 4\hat{i} + 4\hat{j}$ and $\vec{a} - \vec{b} = 2\hat{i} + 4\hat{k}$

$$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix} = \hat{i}(16) - \hat{j}(16) + \hat{k}(-8) = 16\hat{i} - 16\hat{j} - 8\hat{k}$$

$$\therefore \left|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})\right| = \sqrt{16^2 + (-16)^2 + (-8)^2}$$

$$= \sqrt{256 + 256 + 64}$$

$$= \sqrt{576} = 24$$

So the unit vector, perpendicular to each of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ is given by

$$\pm \frac{(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})}{\left|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})\right|} = \pm \frac{16\hat{i} - 16\hat{j} - 8\hat{k}}{24} = \pm \frac{2\hat{i} - 2\hat{j} - \hat{k}}{3} = \pm \frac{2}{3}\hat{i} \mp \frac{2}{3}\hat{j} \mp \frac{1}{3}\hat{k}$$

21.

Let $\vec{b}_1$ and $\vec{b}_2$ be the vector parallel to the pair to lines,

$\frac{-x+2}{-2}=\frac{y-1}{7}=\frac{z+3}{-3}$ and $\frac{x+2}{-1}=\frac{2y-8}{4}=\frac{z-5}{4}$, respectively.

Now, $\frac{-x+2}{-2}=\frac{y-1}{7}=\frac{z+3}{-3}$

$\Rightarrow \frac{x-2}{2}=\frac{y-1}{7}=\frac{z+3}{-3}$

$\frac{x+2}{-1}=\frac{2y-8}{4}=\frac{z-5}{4}$

$\Rightarrow \frac{x+2}{-1}=\frac{y-4}{2}=\frac{z-5}{4}$

$\therefore \vec{b}_1 = 2\hat{i}+7\hat{j}-3\hat{k}$ and $\vec{b}_2 = -\hat{i}+2\hat{j}+4\hat{k}$

$\left|\vec{b}_1\right| = \sqrt{(2)^2+(7)^2+(-3)^2} = \sqrt{62}$

$\left|\vec{b}_2\right| = \sqrt{(-1)^2+(2)^2+(4)^2} = \sqrt{21}$

$$\begin{aligned}\vec{b}_1.\vec{b}_2 &= \left(2\hat{i}+7\hat{j}-3\hat{k}\right).\left(-\hat{i}+2\hat{j}+4\hat{k}\right)\\ &= 2(-1)+7\times 2+(-3).4\\ &= -2+14-12\\ &= 0\end{aligned}$$

The angle θ between the given pair of lines is given by the relation,

$$\cos\theta = \left|\frac{\vec{b}_1.\vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$$

$$\Rightarrow \cos\theta = \frac{0}{\sqrt{62}\times\sqrt{21}} = 0$$

$$\Rightarrow \theta = \cos^{-1}(0) = \frac{\pi}{2}$$

Thus, the given lines are perpendicular to each other and the angle between them is 90°.

22. The probability of solving the problem independently by A and B are given as $\frac{1}{2}$ and $\frac{1}{3}$ respectively.

i.e. $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$.

$\therefore P(A \cap B) = P(A).P(B)$

[Since the events corresponding to A and B are independent]

$$= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

(i) Probability that the problem is solved

$= P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= \frac{1}{2} + \frac{1}{3} - \frac{1}{6}$

$= \frac{3+2-1}{6}$

$= \frac{4}{6}$

$= \frac{2}{3}$

Thus, the probability that the problem is solved is $\frac{2}{3}$.

(ii) Probability that exactly one of them solves the problem

$= P(A - B) + P(B - A)$

$= \left[P(A) - P(A \cap B) + \left[P(B) - P(A \cap B)\right]\right]$

$= \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{3} - \frac{1}{6}\right)$

$= \frac{3-1+2-1}{6}$

$= \frac{3}{6}$

$= \frac{1}{2}$

Thus, the probability that exactly one of them solves the problem is $\frac{1}{2}$.

SECTION – C

23. The given system of equation is $\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4, \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1, \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$

The given system of equation can be written as

$$\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix} \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\text{or } AX = B, \text{Where } A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, X = \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\text{Now, } |A| = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$$

$$= 2(120-45) - 3(-80-30) + 10(36+36)$$

$$= 1200 \neq 0$$

Hence, the unique solution of the system of equation is given by $X=A^{-1}B$

Now, the cofactors of A are computed as:

$$C_{11} = (-1)^2(120-45) = 75, C_{12} = (-1)^3(-80-30) = 110, C_{13} = (-1)^4(36+36) = 72$$

$$C_{21} = (-1)^3(-60-90) = 150, C_{22} = (-1)^4(-40-60) = -100, C_{23} = (-1)^5(18-18) = 0$$

$$C_{31} = (-1)^4(15+60) = 75, C_{32} = (-1)^5(10-40) = 30, C_{33} = (-1)^6(-12-12) = -24$$

$$\therefore \text{Adj}A = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}^T = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{\text{Adj}A}{|A|} = \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{1200} \begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix} = \frac{1}{1200} \begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$$

$$X=\begin{bmatrix}\frac{600}{1200}\\\frac{400}{1200}\\\frac{240}{1200}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\\frac{1}{3}\\\frac{1}{5}\end{bmatrix}\Rightarrow\begin{bmatrix}\frac{1}{x}\\\frac{1}{y}\\\frac{1}{z}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\\frac{1}{3}\\\frac{1}{5}\end{bmatrix}$$

$$\Rightarrow\frac{1}{x}=\frac{1}{2},\frac{1}{y}=\frac{1}{3}\text{ and }\frac{1}{z}=\frac{1}{5}$$

$$\Rightarrow x=2,\ y=3\text{ and }z=5$$

Thus, solution of given system of equation is given by $x=2$, $y=3$ and $z=5$.

OR

The given matrix is $A=\begin{bmatrix}1&3&-2\\-3&0&-1\\2&1&0\end{bmatrix}$.

We have $AA^{-1} = I$

Thus, $A = IA$

Or, $\begin{bmatrix}1&3&-2\\-3&0&-1\\2&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A$

Applying $R_2\to R_2+3R_1$ and $R_3\to R_3-2R_1$

$$\begin{bmatrix}1&3&-2\\0&9&-7\\0&-5&4\end{bmatrix}=\begin{bmatrix}1&0&0\\3&1&0\\-2&0&1\end{bmatrix}A$$

Now,applying $R_2\to\frac{1}{9}R_2$

$$\begin{bmatrix}1&3&-2\\0&1&-\frac{7}{9}\\0&-5&4\end{bmatrix}=\begin{bmatrix}1&0&0\\\frac{1}{3}&\frac{1}{9}&0\\-2&0&1\end{bmatrix}A$$

Applying $R_1\to R_1-3R_2$ and $R_3\to R_3+5R_2$

$$\begin{bmatrix}1&0&\frac{1}{3}\\0&1&-\frac{7}{9}\\0&0&\frac{1}{9}\end{bmatrix}=\begin{bmatrix}0&-\frac{1}{3}&0\\\frac{1}{3}&\frac{1}{9}&0\\-\frac{1}{3}&\frac{5}{9}&1\end{bmatrix}A$$

Applying $R_3 \to 9R_3$

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{7}{9} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -3 & 5 & 9 \end{bmatrix} A$$

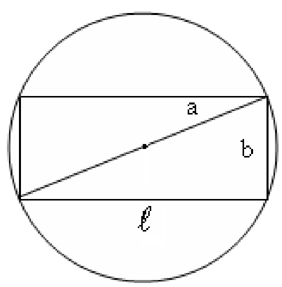
Applying $R_1 \to R_1 - \frac{1}{3}R_3$ and $R_2 \to R_2 + \frac{7}{9}R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix} A \Rightarrow I = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix} A$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix}$$

Hence, inverse of the matrix A is $\begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix}$.

24. Let the rectangle of length ℓ and breadth b be inscribed in circle of radius a.



Then, the diagonal of the rectangle passes through the centre and is of length 2a cm.
Now, by applying the Pythagoras Theorem, we have:

$$(2a)^2 = \ell^2 + b^2$$

$$\Rightarrow b^2 = 4a^2 - \ell^2$$

$$\Rightarrow b = \sqrt{4a^2 - \ell^2}$$

$$\therefore \text{Area of rectangle, } A = \ell b = \ell\sqrt{4a^2 - \ell^2}$$

$$\therefore \frac{dA}{d\ell} = \sqrt{4a^2 - \ell^2} + \ell \frac{1}{2\sqrt{4a^2 - \ell^2}}(-2\ell) = \sqrt{4a^2 - \ell^2} - \frac{\ell^2}{\sqrt{4a^2 - \ell^2}}$$

$$= \frac{4a^2 - 2\ell^2}{\sqrt{4a^2 - \ell^2}}$$

$$\frac{d^2A}{d\ell^2}=\frac{\sqrt{4a^2-\ell^2}\,(-4\ell)-\left(4a^2-2\ell^2\right)\frac{(-2\ell)}{2\sqrt{4a^2-\ell^2}}}{\left(4a^2-\ell^2\right)}$$

$$=\frac{\left(4a^2-\ell^2\right)(-4\ell)+\ell\left(4a^2-2\ell^2\right)}{\left(4a^2-\ell^2\right)^{\frac{3}{2}}}$$

$$=\frac{-12a^2\ell+2\ell^3}{\left(4a^2-\ell^2\right)^{\frac{3}{2}}}=\frac{-2\ell\left(6a^2-\ell^2\right)}{\left(4a^2-\ell^2\right)^{\frac{3}{2}}}$$

Now, $\frac{dA}{d\ell}=0$ gives $4a^2=2\ell^2\Rightarrow\ell=\sqrt{2}a$

$$\Rightarrow b=\sqrt{4a^2-2a^2}=\sqrt{2a^2}=\sqrt{2}a$$

when $\ell=\sqrt{2}a$,

$$\frac{d^2A}{d\ell^2}=\frac{-2\left(\sqrt{2}a\right)\left(6a^2-2a^2\right)}{2\sqrt{2}a^3}=\frac{-8\sqrt{2}a^3}{2\sqrt{2}a^3}=-4<0$$

$\therefore$ Thus, from the second derivative test, when $\ell=\sqrt{2}a$, the area of the rectangle is maximum.

Since ℓ = b = $\sqrt{2}a$, the rectangle is a square.

25. Hence, of all the rectangles inscribed in the given circle, the square has the maximum area.

Equations of the lines are y = 2x + 1, y = 3x + 1 and x + 4

Let y_1 = 2x + 1, y_2 = 3x + 1

Now area of the triangle bounded by the given lines,

$$=\int_0^4\left(y_2-y_1\right)dx$$

$$=\int_0^4\left[(3x+1)-(2x+1)\right]dx$$

$$=\int_0^4 x\,dx$$

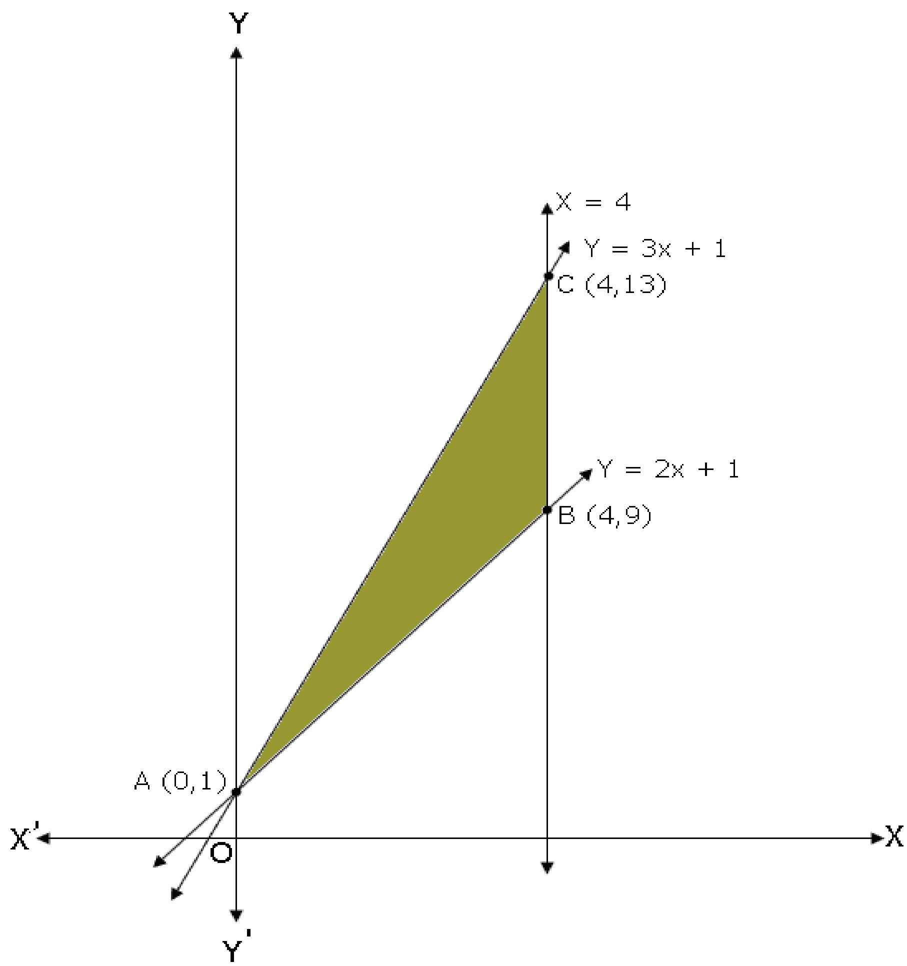
$$=\frac{1}{2}\left[x^2\right]_0^4$$

$$=\frac{1}{2}\left(4^2-0^2\right)$$

$$=\frac{1}{2}\times16$$

$$=8\text{ sq.units}$$

Thus, the area of the required triangular region is 8 square units.



26. Consider the given integral

$$I=\int_0^{\frac{\pi}{2}} 2\sin x\cos x\tan^{-1}(\sin x)dx$$

$$\text{Let } t=\sin x$$
$$\Rightarrow dt=\cos x\,dx$$
$$\text{When } x=\frac{\pi}{2},\ t=1$$
$$\text{When } x=0,\ t=0$$
$$\text{Now,} \int 2\sin x\cos x\tan^{-1}(\sin x)dx$$
$$=\int 2t\tan^{-1}t\,dt$$
$$=\left[\tan^{-1}t\right]\int 2t\,dt-\int\left[\frac{d}{dt}.\left(\tan^{-1}t\right)\int 2tdt\right]dt$$
$$=\left[\tan^{-1}t\right]\left[2.\frac{t^2}{2}\right]-\int\left(\frac{1}{1+t^2}x2.\frac{t^2}{2}\right)dt$$
$$=t^2\tan^{-1}t-\int\frac{t^2}{1+t^2}dt$$

$$= t^2 \tan^{-1} t - \int \left[1 - \frac{1}{1+t^2} \right] dt$$
$$= t^2 \tan^{-1} t - t + \tan^{-1} t$$

$$\therefore I = \int_0^{\frac{\pi}{2}} 2 \sin x \cos x \tan^{-1}(\sin x) dx$$
$$= \left[t^2 \tan^{-1} t - t + \tan^{-1} t \right]_0^1$$
$$= \left[1^2 \tan^{-1} 1 - 1 + \tan^{-1} 1 \right] - \left[0^2 \tan^{-1} 0 - 0 + \tan^{-1} 0 \right]$$
$$= \left[1 \times \frac{\pi}{4} - 1 + \frac{\pi}{4} \right] - 0$$
$$= \frac{\pi}{4} - 1 + \frac{\pi}{4}$$
$$= \frac{\pi}{2} - 1$$

OR

$$I = \int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx \qquad ...(1)$$

Using the proporty $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

$$I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right) \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right) \cos x \sin x}{\cos^4 x + \sin^4 x} dx \qquad ...(2)$$

Adding(1)and(2),

$$2I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} . \sin x \cos x\right)}{\sin^4 x + \cos^4 x} dx$$

$$\Rightarrow I = \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \left[\frac{\sin x \cos x}{\sin^4 x + \cos^4 x} \right] dx$$

$$= \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \left[\frac{\frac{\sin x \cos x}{\cos^4 x}}{\frac{\sin^4 x}{\cos^4 x} + 1} \right] dx$$

$$=\frac{\pi}{4}\int_0^{\frac{\pi}{2}}\frac{\tan x\sec^2 x}{\tan^4 x+1}dx$$

Put $\tan^2 x=z$

$\therefore$ $2\tan x\sec^2 x\, dx = dz$

$\Rightarrow \tan x\sec^2 x\, dx = \frac{dz}{2}$

When x = 0, z = 0 and when $x = \frac{\pi}{2}$, $z = \infty$

$$\therefore I=\frac{\pi}{4}\int_0^{\infty}\frac{\frac{dz}{2}}{z^2+1}$$

$$\Rightarrow I=\frac{\pi}{8}\int_0^{\infty}\frac{dz}{1+z^2}$$

$$=\frac{\pi}{8}\left|\tan^{-1}(z)\right|_0^{\infty}$$

$$=\frac{\pi}{8}\ \tan^{-1}\ \infty\ -\tan^{-1}\ 0$$

$$=\frac{\pi}{8}\left(\frac{\pi}{2}-0\right)$$

$$=\frac{\pi^2}{16}$$

27. The equations of the given planes are

$\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)$ - 4 = 0 ... (1)

$\vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)$ + 5 = 0 ... (2)

The equation of the plane passing through the line of intersection of the given planes is

$\left[\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4\right]+\lambda\left[\vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)+5\right]$ = 0

$\vec{r}.\left[(1+2\lambda)\hat{i}+(2+\lambda)\hat{j}+(3-\lambda)\hat{k}\right]+(-4+5\lambda)$ = 0 ... (3)

The plane in equation (3) is perpendicular to the plane, $\vec{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)$ + 8 = 0.

$\therefore$ $5(1 + 2\lambda) + 3(2 + \lambda) - 6(3 - \lambda) = 0$

$\Rightarrow 5 + 10\lambda + 6 + 3\lambda - 18 + 6\lambda = 0$

$\Rightarrow 19\lambda - 7 = 0$

$\Rightarrow \lambda = \frac{7}{19}$

Substituting $\lambda=\frac{7}{19}$ in equation (3),

$$\vec{r}.\left[\frac{33}{19}\hat{i}+\frac{45}{19}\hat{j}+\frac{50}{19}k\right]-\frac{41}{19}\;0$$

$$\Rightarrow \vec{r}.\left(33\hat{i}+45\hat{j}+50k\right)\text{ - }41=0$$

This is the vector equation of the required plane.

28. Let the number of rackets and the number of bats to be made be x and y respectively.
The given information can be tabulated as below:

	Tennis Racket	Cricket Bat
Machine Time (h)	1.5	3
Craftsman's Time (h)	3	1

In a day, the machine time is not available for more than 42 hours.

$\therefore\; 1.5\,x + 3y \le 42$

In a day, the craftsman's time cannot be more than 24 hours.

$\therefore\; 3x + y \le 24$

Let the total profit be Rs. Z.

The profit on a racket is Rs. 20 and on a bat is Rs. 10.

$\therefore Z = 20x + 10y$

Thus, the given linear programming problem can be stated as follows:

Maximise $Z = 20x + 10y$ ... (1)

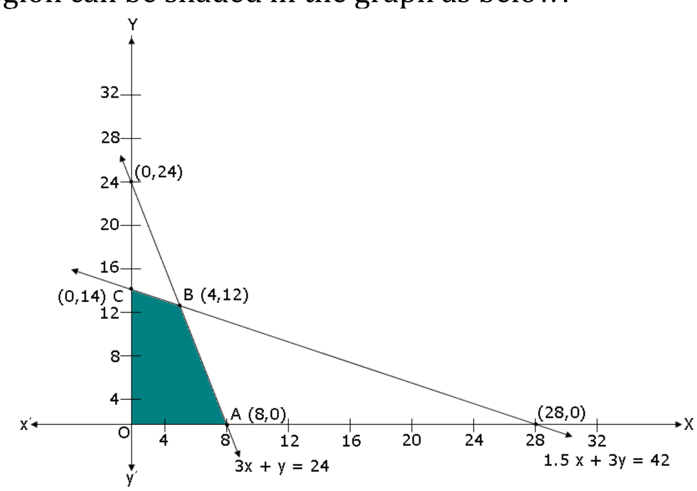
Subject to

$1.5x + 3y \le 42$ ... (2)

$3x + y \le 24$ ... (3)

$x, y \ge 0$ ... (4)

The feasible region can be shaded in the graph as below:



The corner points are A(8,0), B(4,12), C(0,14) and 0(0,0).

The values of Z at these corner points are tabulated as follows:

Corner point	$Z = 20x + 10y$
A(8,0)	160
B(4, 12)	200
C(0,14)	140
O(0,0)	0

$\longrightarrow$ Maximum

The maximum value of Z is 200, which occurs at x = 4 and y = 12.
Thus, the factory must produce 4 tennis rackets and 12 cricket bats to earn the maximum profit of Rs. 200.

29. Let the events M, F and G be defined as follows:

M: A male is selected

F: A female is selected

G: A person has grey hair

It is given that the number of males = the number of females

$\therefore$ P (M) = P(F) = $\frac{1}{2}$

Now, P (G/M) = Probability of selecting a grey haired person given that the person is a:

Male = 5% = $\frac{5}{100}$

Similarly, P (G/F) = 0.25% = $\frac{0.25}{100}$

A grey haired person is selected at random, the probability that this person is a male

= P(M|G)

$$= \frac{P(M)\times P(G|M)}{P(M)\times P(G|M)+P(F)\times P(G|F)} \quad \text{[Using Baye's Theorem]}$$

$$= \frac{\frac{1}{2}\times\frac{5}{100}}{\frac{1}{2}\times\frac{5}{100}+\frac{1}{2}\times\frac{0.25}{100}}$$

$$= \frac{\frac{5}{100}}{\frac{5}{100}+\frac{0.25}{100}}$$

$$= \frac{5}{5.25}$$

$$= \frac{20}{21}$$