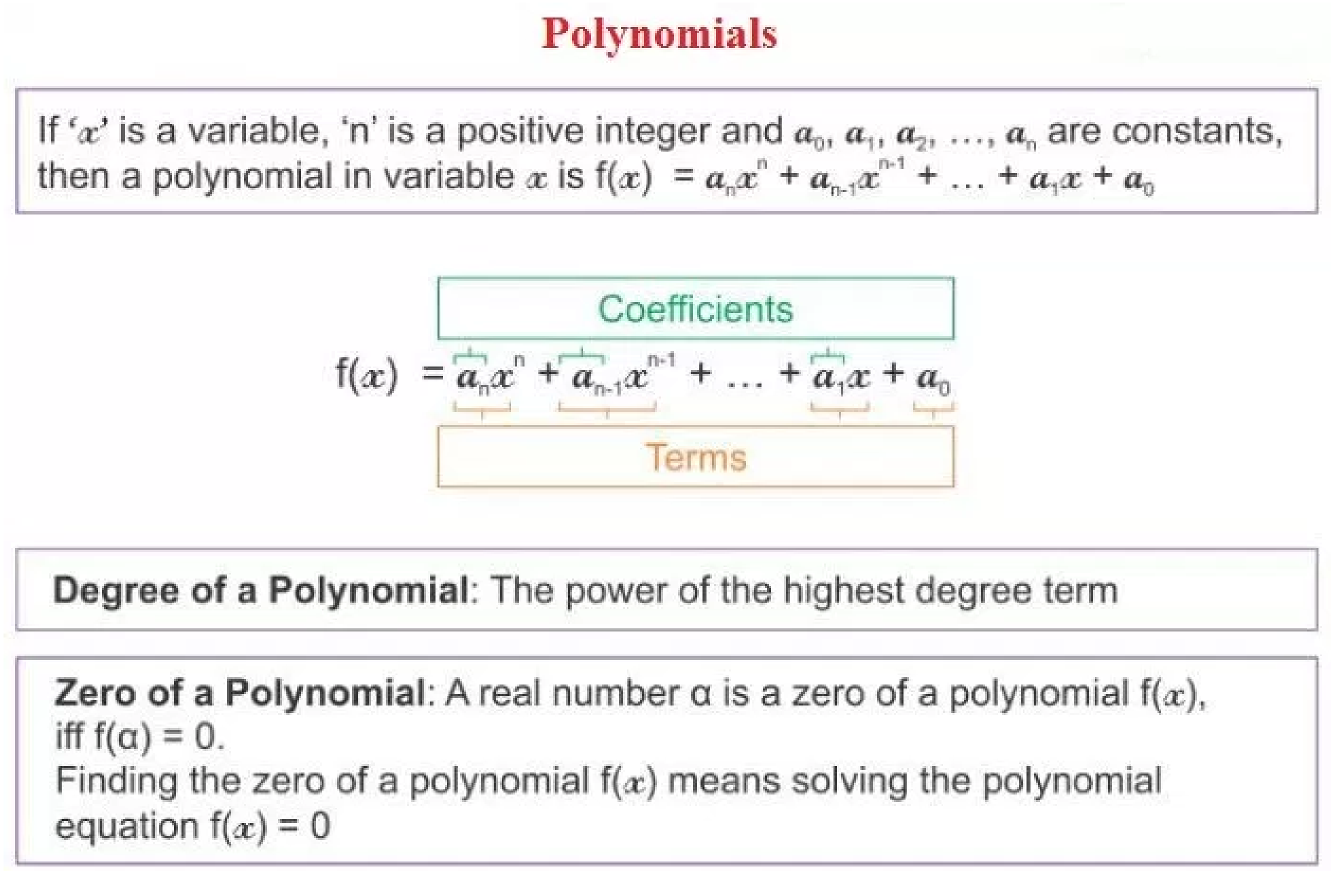
5. Polynomials

Polynomials

Algebraic expression containing many terms of the form ax^n, n being a non-negative integer is called a polynomial. i.e., $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots. + a_{n-1}x^{n-1} + a_nx_n$, where x is a variable, a_0, a_1, a_2, …… a_n are constants and $a_n \neq 0$.

Example: $4x^4 + 3x^3 - 7x^2 + 5x + 3$, $3x^3 + x^2 - 3x + 5$.



(1) Real polynomial:
$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots. + a_nx_n$ is called real polynomial of real variable x with real coefficients.

Example: $3x^3 - 4x^2 + 5x - 4$, $x^2 - 2x + 1$ etc. are real polynomials.

(2) Complex polynomial:
$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots. + a_nx_n$ is called complex polynomial of complex variable x with complex coefficients.

Example: $3x^2 - (2+4i)x + (5i-4)$, $x^3 - 5ix^2 + (1+2i)x + 4$ etc. are complex polynomials.

(3) Degree of polynomial:
Highest power of variable x in a polynomial is called degree of polynomial.

Example: $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots. + a_nx_n$ is a n degree polynomial.
$f(x) = 4x^3 + 3x^2 - 7x + 5$ is a 3 degree polynomial.
A polynomial of second degree is generally called a quadratic polynomial. Polynomials of degree 3 and 4 are known as cubic and biquadratic polynomials respectively.

(4) Polynomial equation:

If f(x) is a polynomial, real or complex, then f(x) = 0 is called a polynomial equation.

Types Of Polynomials

(i) Based on degree :

If degree of polynomial is

			Examples
1.	One	Linear	$x + 3$, $y - x + 2$, $\sqrt{3}x - 3$
2.	Two	Quadratic	$2x^2 - 7$, $\frac{1}{3}x^2 + y^2 - 2xy$, $x^2 + 1 + 3y$
3.	Three	Cubic	$x^3 + 3x^2 - 7x + 8$, $2x^2 + 5x^3 + 7$,
4.	Four	bi-quadratic	$x^4 + y^4 + 2x^2y^2$, $x^4 + 3$,…

(ii) Based on Terms :
If number of terms in polynomial is

			Examples
1.	One	Monomial	$7x$, $5x^9$, $3x^{16}$, xy, ……
2.	Two	Binomial	$2 + 7y^6$, $y^3 + x^{14}$, $7 + 5x^9$, …
3.	Three	Trinomial	$x^3 - 2x + y$, $x^{31} + y^{32} + z^{33}$, …..

Note:
(1) Degree of constant polynomials (Ex.5, 7, −3, 8/5, …) is zero.
(2) Degree of zero polynomial (zero = 0 = zero polynomial) is not defined.

Monomials, Binomials, and Polynomials

1. A **monomial** is the product of non-negative integer powers of variables. Consequently, a monomial has NO variable in its denominator. It has one term. (mono implies one)
   $13,\quad 3x,\quad -57,\quad x^2,\quad 4y^2,\quad -2xy,\ \text{or}\ 520x^2y^2$
   (notice: no negative exponents, no fractional exponents)
2. A **binomial** is the sum of two monomials. It has two unlike terms.
   (bi implies two)
   $3x + 1,\quad x^2 - 4x,\quad 2x + y,\quad \text{or}\quad y - y^2$
3. A **trinomial** is the sum of three monomials. It has three unlike terms. (tri implies three)
   $x^2 + 2x + 1,\quad 3x^2 - 4x + 10,\quad 2x + 3y + 2$
4. A **polynomial** is the sum of one or more terms. (poly implies many)
   $x^2 + 2x,\quad 3x^3 + x^2 + 5x + 6,\quad 4x - 6y + 8$

Polynomials are in simplest form when they contain no like terms.
$x^2 + 2x + 1 + 3x^2 - 4x$ when simplified becomes $4x^2 - 2x + 1$

Polynomials are generally written in descending order.

Adding Polynomials

Adding Polynomials

Add: $(x^2 + 3x + 1) + (4x^2 + 5)$

Step 1: Underline like terms:

$$(\underline{x^2} + 3x + \underline{\underline{1}}) + (\underline{4x^2} + \underline{\underline{5}})$$

Notice: '3x' doesn't have a like term.

Step 2: Add the coefficients of *like terms*, do not change the powers of the variables:

$$(x^2 + 4x^2) + 3x + (1 + 5)$$

$$\boxed{5x^2 + 3x + 6}$$

Add like terms by adding the numerical portion of the terms, following the rules for adding signed numbers.
(The numerical portion of an expression is called the coefficient.)

Example: Add: $(2x^2 - 4) + (x^2 + 3x - 3)$

Below are several different ways to attack this example:

- Using a Horizontal Method to add like terms:

1. Using a horizontal method to add like terms:

Remove parentheses. Identify like terms. Group the like terms together. Add the like terms.

$$(2x^2 - 4) + (x^2 + 3x - 3)$$

$= 2x^2 - 4 + x^2 + 3x - 3$ .. identify like terms

$= 2x^2 + x^2 + 3x - 4 - 3$ .. group the like terms together

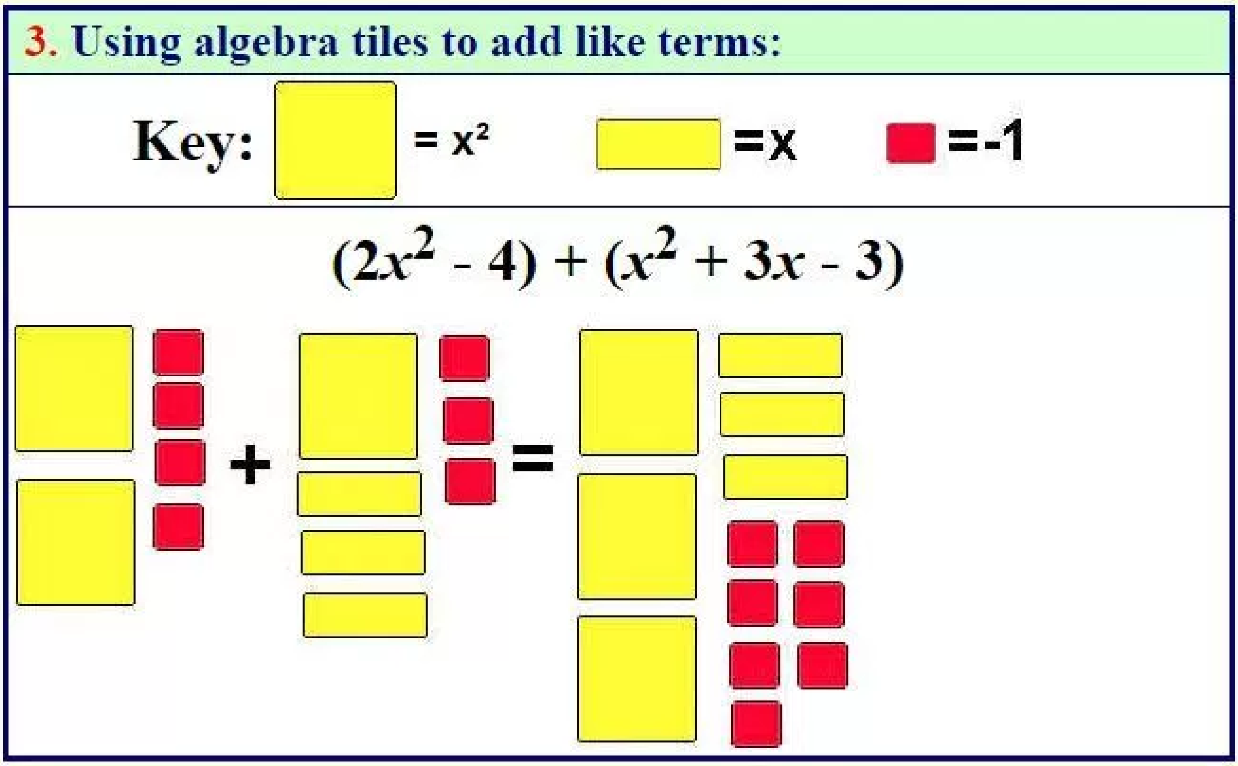
$= 3x^2 + 3x - 7$ ... add the like terms

- Using a Vertical Method to add like terms:

2. Using a vertical method to add like terms:

Arrange the like terms so that they are lined up under one another in vertical columns, adding 0 place holders if necessary. Add the like terms in each column following the rules for adding signed numbers.

$$\begin{array}{r} 2x^2 + 0x - 4 \\ + x^2 + 3x - 3 \\ \hline 3x^2 + 3x - 7 \end{array}$$



Subtracting Polynomials

Subtract like terms by changing the signs of the terms being subtracted, and following the rules for adding polynomials.

Example: Simplify: $(2x^2 - 4) - (x^2 + 3x - 3)$

Below are several different ways to attack this example:

1. Using a horizontal method to subtract like terms:

Change the signs of ALL of the terms being subtracted. Change the subtraction sign to addition. Follow the rules for adding signed numbers.

$$(2x^2 - 4) - (x^2 + 3x - 3)$$

$$= (2x^2 - 4) + (-x^2 - 3x + 3)$$ ... change signs of terms being subtracted and change subtraction to addition.

$$= 2x^2 - 4 + -x^2 - 3x + 3$$ ... identify like terms

$$= 2x^2 - x^2 - 3x - 4 + 3$$ ... group the like terms

$$= x^2 - 3x - 1$$ ... add the like terms

2. Using the vertical method to subtract like terms:

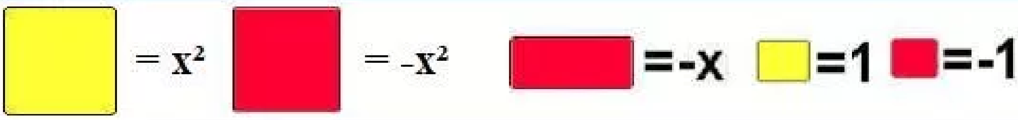
$$\begin{array}{r} 2x^2 + 0x - 4 \\ -(x^2 + 3x - 3) \\ \hline \end{array}$$

Now, change signs of all terms being subtracted and follow rules for add.

$$\begin{array}{r} 2x^2 + 0x - 4 \\ -x^2 - 3x + 3 \\ \hline x^2 - 3x - 1 \end{array}$$ (signs changed)

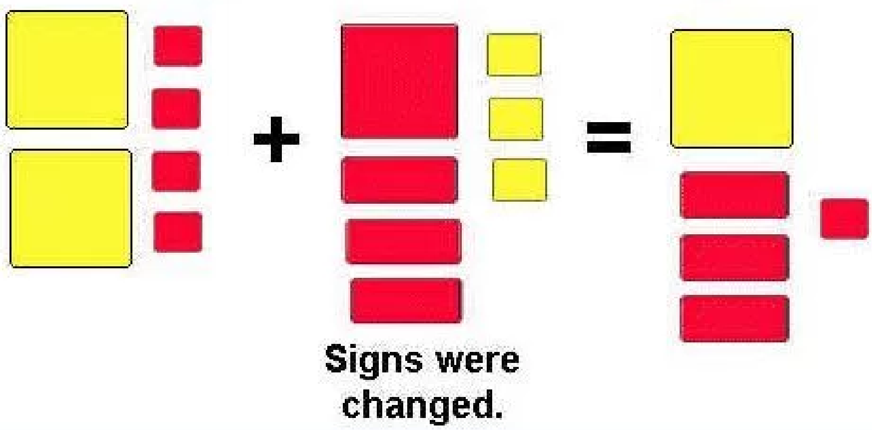
3. Using algebra tiles to subtract like terms:

Key:



$$(2x^2 - 4) - (x^2 + 3x - 3)$$

The diagram below shows the signs changed to $-x^2 - 3x + 3$ and addition occurring.



4. Using the Distributive Property to subtract like terms:

When you are subtracting the coefficients (the numbers in front of the variables) of like terms, you are actually using the distributive property in reverse.

$$4x^2 - 5x^2 = (4 - 5)\,x^2 = -x^2$$

Dividing Polynomials

We will be examining polynomials divided by monomials and by binomials.

Steps for Dividing a Polynomial by a Monomial:

1. Divide each term of the polynomial by the monomial.
   a) Divide numbers (coefficients)
   b) Subtract exponents
   * The number of terms in the polynomial equals the number of terms in the answer when dividing by a monomial.
2. Remember that numbers do not cancel and disappear! A number divided by itself is **1.** It reduces to the number 1.
3. Remember to write the appropriate sign in between the terms.

Example:

$$\frac{16x^6 - 12x^4 + 4x^2}{4x^2}$$

$$\frac{\overset{4}{\cancel{16}}x^6}{\underset{1}{\cancel{4}}\,x^2} - \frac{\overset{3}{\cancel{12}}x^4}{\underset{1}{\cancel{4}}x^2} + \frac{\overset{1}{\cancel{4x^2}}}{\underset{1}{\cancel{4x^2}}}$$

Notice how the numbers (the coefficients) were divided.

Notice how the exponents were subtracted.

Answer:

$$4x^4 - 3x^2 + 1$$

Notice how the last term reduced to one.

The polynomial on the top has 3 terms and the answer has 3 terms.

Think about it:

$$\frac{16x^6 - 12x^4 + 4x^2}{4x^2}$$

Dividing by a number is the same process as multiplying by the reciprocal of that number.

$$\frac{1}{4x^2}\left(16x^6 - 12x^4 + 4x^2\right)$$

Notice how we used the reciprocal of $4x^2$

$$\frac{16x^6}{4x^2} - \frac{12x^4}{4x^2} + \frac{4x^2}{4x^2}$$

Now the reciprocal was distributed across the parentheses and the problem proceeds as in the example above.

Answer:

$$4x^4 - 3x^2 + 1$$

Steps for Dividing a Polynomial by a Binomial:

1. Remember that the terms in a binomial cannot be separated from one another when reducing. For example, in the binomial 2x + 3, the 2x can never be reduced unless the entire expression 2x + 3 is reduced.
2. Factor completely both the numerator and denominator before reducing.
3. Divide both the numerator and denominator by their greatest common factor.

Example 1:

$$\frac{2x+2}{x+1}=\frac{2(\cancel{x+1})}{\cancel{x+1}}=2$$

Notice that the x+1 was reduced as a "set".

Example 2:

$$\frac{x^2-16}{x-4}=\frac{(x+4)(\cancel{x-4})}{\cancel{x-4}}=x+4$$

Example 3:

$$\frac{x^2+7x+12}{x^2-9}=\frac{(\cancel{x+3})(x+4)}{(\cancel{x+3})(x-3)}=\frac{x+4}{x-3}$$

Example 4:

$$\frac{2-x}{4x-8}=\frac{2-x}{4(x-2)}=\frac{-1(\cancel{x-2})}{4(\cancel{x-2})}=\frac{-1}{4}$$

Tricky strategy: Notice that the -1 was factored out of the numerator to create a binomial compatible with the one in the denominator.
2 – x = -1(x – 2)

Multiplying Binomials

There are numerous ways to set up the multiplication of two binomials. The first three methods shown here work for multiplying **ALL** polynomials, not just binomials. All methods, of course, give the same answer.

Multiply $(x + 3)(x + 2)$ **1. "Distributive" Method:**

The most universal method. Applies to all polynomial multiplications, not just to binomials.
Start with the first term in the first binomial – the circled blue X. Multiply (distribute) this term times EACH of the terms in the second binomial.

$$(x + 3)\ (x + 2) \quad = x\bullet x + x\bullet 2$$

Now, take the second term in the first binomial – the circled red +3 (notice we take the sign also). Multiply this term times EACH of the terms in the second binomial.

$$(x + 3)(x + 2) \quad = 3 \cdot x + 3 \cdot 2$$

Add the results: $x \cdot x + x \cdot 2 + 3 \cdot x + 3 \cdot 2$

$$x^2 + 2x + 3x + 6$$

$$x^2 + 5x + 6$$ **Answer**

Do you see the "distributive property" at work?
$(x + 3)(x + 2) = x(x + 2) + 3(x + 2)$

Before we move on to the next set up method, let’s look at an example of the “distributive” method involving negative values.

"Distributive" Method: Dealing with **negative values.**

Notice how the negative sign is treated as part of the term following the sign.

$$(x - 4)(x - 5) \quad = x \cdot x + x \cdot (-5)$$

$$(x - 4)(x - 5) \quad = (-4) \cdot x + (-4) \cdot (-5)$$

Add the results:

$$x \cdot x + x \cdot (-5) + (-4) \cdot x + (-4) \cdot (-5)$$

$$x^2 - 5x - 4x + 20$$

$$x^2 - 9x + 20$$ Answer

2. “Vertical” Method:

This is a vertical “picture” of the distributive method.
This style applies to all polynomial multiplications.

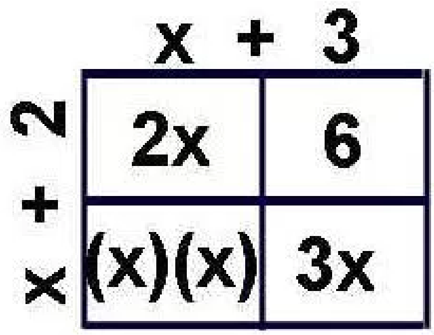
$$\begin{array}{rrl} & x + 2 & \\ & x + 3 & \\ \hline x^2 + 2x & & \text{multiply "x" from bottom term times "x+2"} \\ & 3x + 6 & \text{multiply "3" from bottom term times "x+2"} \\ \hline x^2 + 5x + 6 & & \text{add the like terms} \end{array}$$

Be sure to line up the like terms.

3. “Grid” Method

This is a “table” version of the distributive method.
This style applies to all polynomial multiplications.

To multiply by the grid method, place one binomial at the top of a 2×2 grid (for binomials) and the second binomial on the side of the grid. Place the terms such that each term with its sign lines up with a row or column of the grid. Multiply the rows and columns of the grid to complete the interior of the grid. Finish by adding together the entries inside the grid.



$$2x + 6 + (x)(x) + 3x$$

$$2x + 6 + x^2 + 3x$$

Answer: $x^2 + 5x + 6$

C A U T I O N !!!

There are set up methods that work **ONLY** for binomials. While these set ups may be helpful to understanding binomial multiplication, you must remember that they do not extend to other types of multiplications, such as a binomial times a trinomial. You will have to go back to the "distributive method" for these other polynomial multiplications.

4. "FOIL" Method: multiply First Outer Inner Last

For Binomial Multiplication ONLY!
The words/letters used to describe the FOIL process pertain to the distributive method for multiplying two binomials. These words/letters do not apply to other multiplications such as a binomial times a trinomial.

F: $(\mathbf{x} + 3)(\mathbf{x} + 2)$

O: $(\mathbf{x} + 3)(x + \mathbf{2})$

I: $(x + \mathbf{3})(\mathbf{x} + 2)$

L: $(x + \mathbf{3})(x + \mathbf{2})$

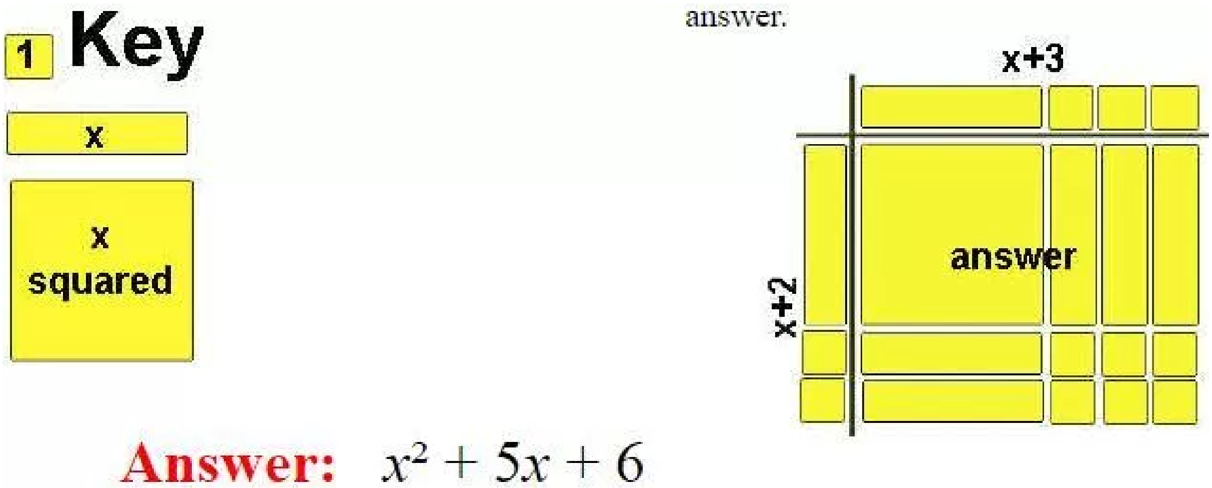
$$\begin{aligned}(x+3)(x+2) &= x^2 + 2x + 3x + 6 \\ &= x^2 + 5x + 6\end{aligned}$$

The drawback to using the FOIL lettering is that it ONLY WORKS on binomial multiplication.

5. "Algebra Tile" Method

While this method is helpful for understanding how binomials are multiplied, it is not easily applied to ALL multiplications and may not be practical for overall use.

The example shown here is for binomial multiplication only!
To multiply binomials using algebra tiles, place one expression at the top of the grid and the second expression on the side of the grid. You MUST maintain straight lines when you are filling in the center of the grid. The tiles needed to complete the inner grid will be your answer.



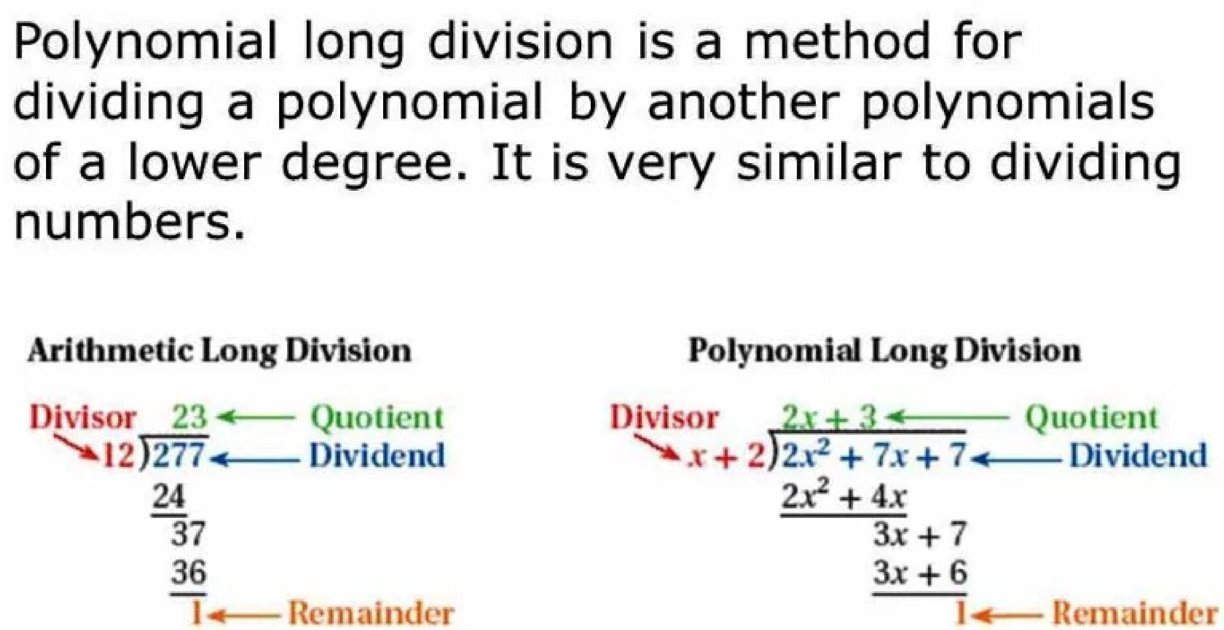
Division Algorithm For Polynomials

If p(x) and g(x) are any two polynomials with
g(x) ≠ 0, then we can find polynomials q(x) and r(x) such that
p(x) = q(x) × g(x) + r(x)
where r(x) = 0 or degree of r(x) < degree of g(x).
The result is called Division Algorithm for polynomials.
Dividend = Quotient × Divisor + Remainder

Polynomials – Long Division

Working rule to Divide a Polynomial by Another Polynomial:

Step 1: First arrange the term of dividend and the divisor in the decreasing order of their degrees.
Step 2: To obtain the first term of quotient divide the highest degree term of the dividend by the highest degree term of the divisor.
Step 3: To obtain the second term of the quotient, divide the highest degree term of the new dividend obtained as remainder by the highest degree term of the divisor.
Step 4: Continue this process till the degree of remainder is less than the degree of divisor.



Division Algorithm For Polynomials With Examples

Example 1: Divide $3x^3 + 16x^2 + 21x + 20$ by $x + 4$.
Sol.

$$
\begin{array}{r|l}
 & 3x^2 + 4x + 5 \\ \hline
x+4 & 3x^3 + 16x^2 + 21x + 20 \\
 & 3x^3 + 12x^2 \\
 & - \quad - \\ \hline
 & 4x^2 + 21x + 20 \\
 & 4x^2 + 16x \\
 & - \quad - \\ \hline
 & 5x + 20 \\
 & 5x + 20 \\
 & - \quad - \\ \hline
 & 0
\end{array}
\qquad
\begin{array}{l}
\text{First term of } q(x) = \dfrac{3x^3}{x} = 3x^2 \\[2ex]
\text{Second term of } q(x) = \dfrac{4x^2}{x} = 4x \\[2ex]
\text{Third term of } q(x) = \dfrac{5x}{x} = 5
\end{array}
$$

Quotient = $3x^2 + 4x + 5$
Remainder = 0

Example 2: Apply the division algorithm to find the quotient and remainder on dividing p(x) by g(x) as given below :
$p(x) = x^3 - 3x^2 + 5x - 3$ and $g(x) = x^2 - 2$
Sol. We have,
$p(x) = x^3 - 3x^2 + 5x - 3$ and $g(x) = x^2 - 2$

$$
\begin{array}{r|l}
 & x - 3 \\ \hline
x^2-2 & x^3 - 3x^2 + 5x - 3 \\
 & x^3 \qquad\quad - 2x \\
 & - \qquad\quad + \\ \hline
 & -3x^2 + 7x - 3 \\
 & -3x^2 \qquad + 6 \\
 & + \qquad\quad - \\ \hline
 & 7x - 9
\end{array}
\qquad
\begin{array}{l}
\text{First term of quotient is } \dfrac{x^3}{x^2} = x \\[2ex]
\text{Second term of quotient is } \dfrac{-3x^2}{x^2} = -3
\end{array}
$$

We stop here since
degree of $(7x - 9)$ < degree of $(x^2 - 2)$
So, quotient = $x - 3$, remainder = $7x - 9$
Therefore,
Quotient × Divisor + Remainder
= $(x - 3)(x^2 - 2) + 7x - 9$
= $x3 - 2x - 3x^2 + 6 + 7x - 9$
= $x^3 - 3x^2 + 5x - 3$ = Dividend
Therefore, the division algorithm is verified.

Example 3: Apply the division algorithm to find the quotient and remainder on dividing p(x) by g(x) as given below
$p(x) = x^4 - 3x^2 + 4x + 5$, $g(x) = x^2 + 1 - x$
Sol. We have,
$p(x) = x^4 - 3x^2 + 4x + 5$, $g(x) = x^2 + 1 - x$

$$
\begin{array}{r|l}
 & x^2 + x - 3 \\ \hline
x^2 - x + 1 & x^4 - 3x^2 + 4x + 5 \\
 & x^4 - x^3 + x^2 \\
 & - \quad + \quad - \\ \hline
 & x^3 - 4x^2 + 4x + 5 \\
 & x^3 - x^2 + x \\
 & - \quad + \quad - \\ \hline
 & -3x^2 + 3x + 5 \\
 & -3x^2 + 3x - 3 \\
 & + \quad - \quad + \\ \hline
 & 8
\end{array}
$$

We stop here since
degree of (8) < degree of $(x^2 - x + 1)$.

So, quotient = $x^2 + x - 3$, remainder = 8
Therefore,
Quotient × Divisor + Remainder
= $(x^2 + x - 3)(x^2 - x + 1) + 8$
= $x^4 - x^3 + x^2 + x^3 - x^2 + x - 3x^2 + 3x - 3 + 8$
= $x^4 - 3x^2 + 4x + 5$ = Dividend
Therefore the Division Algorithm is verified.

Example 4: Check whether the first polynomial is a factor of the second polynomial by applying the division algorithm. $t^2 - 3$; $2t^4 + 3t^3 - 2t^2 - 9t - 12$.
Sol. We divide $2t^4 + 3t^3 - 2t^2 - 9t - 12$ by $t^2 - 3$

$$
\begin{array}{r|l}
 & 2t^2 + 3t + 4 \\
\hline
t^2 - 3 & 2t^4 + 3t^3 - 2t^2 - 9t - 12 \\
 & 2t^4 \qquad\quad - 6t^2 \\
 & - \qquad\qquad + \\
\hline
 & \qquad 3t^3 + 4t^2 + 9t - 12 \\
 & \qquad 3t^3 \qquad\quad - 9t \\
 & \qquad - \qquad\quad + \\
\hline
 & \qquad\quad 4t^2 \qquad\quad - 12 \\
 & \qquad\quad 4t^2 \qquad\quad - 12 \\
 & \qquad\quad - \qquad\qquad + \\
\hline
 & \qquad\qquad\quad 0
\end{array}
$$

Here, remainder is 0, so $t^2 - 3$ is a factor of $2t^4 + 3t^3 - 2t^2 - 9t - 12$.
$2t^4 + 3t^3 - 2t^2 - 9t - 12 = (2t^2 + 3t + 4)(t^2 - 3)$

Example 5: Obtain all the zeroes of $3x^4 + 6x^3 - 2x^2 - 10x - 5$, if two of its zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$.

Sol. Since two zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$
$x = \sqrt{\frac{5}{3}}$, $x = -\sqrt{\frac{5}{3}}$
$\Rightarrow \left(x - \sqrt{\frac{5}{3}}\right)\left(x + \sqrt{\frac{5}{3}}\right) = x^2 - \frac{5}{3}$ Or $3x^2 - 5$ is a factor of the given polynomial.
Now, we apply the division algorithm to the given polynomial and $3x^2 - 5$.

$$
\begin{array}{r|l}
 & x^2 + 2x + 1 \\
\hline
3x^2 - 5 & 3x^4 + 6x^3 - 2x^2 - 10x - 5 \\
 & 3x^4 \qquad\quad - 5x^2 \\
 & - \qquad\qquad + \\
\hline
 & \qquad 6x^3 + 3x^2 - 10x - 5 \\
 & \qquad 6x^3 \qquad\quad - 10x \\
 & \qquad - \qquad\quad + \\
\hline
 & \qquad\quad 3x^2 \qquad\quad - 5 \\
 & \qquad\quad 3x^2 \qquad\quad - 5 \\
 & \qquad\quad - \qquad\qquad + \\
\hline
 & \qquad\qquad\quad 0
\end{array}
$$

So, $3x^4 + 6x^3 - 2x^2 - 10x - 5$
= $(3x^2 - 5)(x^2 + 2x + 1) + 0$
Quotient = $x^2 + 2x + 1 = (x + 1)^2$
Zeroes of $(x + 1)^2$ are −1, −1.
Hence, all its zeroes are $\sqrt{\frac{5}{3}}$, $-\sqrt{\frac{5}{3}}$, −1, −1.

Example 6: On dividing $x^3 - 3x^2 + x + 2$ by a polynomial g(x), the quotient and remainder were $x - 2$ and $-2x + 4$, respectively. Find g(x).

Sol. $p(x) = x^3 - 3x^2 + x + 2$ $q(x) = x - 2$ and $r(x) = -2x + 4$
By Division Algorithm, we know that
$p(x) = q(x) \times g(x) + r(x)$
Therefore,
$x^3 - 3x^2 + x + 2 = (x - 2) \times g(x) + (-2x + 4)$
$\Rightarrow x^3 - 3x^2 + x + 2 + 2x - 4 = (x - 2) \times g(x)$
$\Rightarrow g(x) = \frac{x^3-3x^2+3x-2}{x-2}$
On dividing $x^3 - 3x^2 + x + 2$ by $x - 2$,
we get g(x)

$$
\begin{array}{r|l}
 & x^2 - x + 1 \\
\hline
x-2 & x^3 - 3x^2 + 3x - 2 \\
 & x^3 - 2x^2 \\
 & - \quad + \\
\hline
 & -x^2 + 3x - 2 \\
 & -x^2 + 2x \\
 & + \quad - \\
\hline
 & x - 2 \\
 & x - 2 \\
 & - \quad + \\
\hline
 & 0
\end{array}
$$

First term of quotient is $\frac{x^3}{x} = x$

Second term of quotient is $\frac{-x^2}{x} = -x$

Third term of quotient is $\frac{x}{x} = 1$

Hence, $g(x) = x^2 - x + 1$.

Example 7: Give examples of polynomials p(x), q(x) and r(x), which satisfy the division algorithm and
(i) deg p(x) = deg q(x)
(ii) deg q(x) = deg r(x)
(iii) deg q(x) = 0
Sol.
(i) Let $q(x) = 3x^2 + 2x + 6$, degree of q(x) = 2
$p(x) = 12x^2 + 8x + 24$, degree of p(x) = 2
Here, deg p(x) = deg q(x)
(ii) $p(x) = x^5 + 2x^4 + 3x^3 + 5x^2 + 2$
$q(x) = x^2 + x + 1$, degree of q(x) = 2
$g(x) = x^3 + x^2 + x + 1$
$r(x) = 2x^2 - 2x + 1$, degree of r(x) = 2
Here, deg q(x) = deg r(x)
(iii) Let $p(x) = 2x^4 + x^3 + 6x^2 + 4x + 12$
q(x) = 2, degree of q(x) = 0
$g(x) = x^4 + 4x^3 + 3x^2 + 2x + 6$
r(x) = 0
Here, deg q(x) = 0

Example 8: If the zeroes of polynomial $x^3 - 3x^2 + x + 1$ are a − b, a , a + b. Find a and b.
Sol. $\because$ a − b, a, a + b are zeros
$\therefore$ product $(a - b)\,a(a + b) = -1$
$\Rightarrow (a^2 - b^2)\,a = -1$ …(1)
and sum of zeroes is $(a - b) + a + (a + b) = 3$
$\Rightarrow 3a = 3 \Rightarrow a = 1$ …(2)
by (1) and (2)
$(1 - b^2)1 = -1$
$\Rightarrow 2 = b^2 \Rightarrow b = \pm\sqrt{2}$
$\therefore$ $a = -1$ & $b = \pm\sqrt{2}$

Example 9: If two zeroes of the polynomial $x^4 - 6x^3 - 26x^2 + 138x - 35$ are $2 \pm \sqrt{3}$, find other zeroes.
Sol. $\because$ $2 \pm \sqrt{3}$ are zeroes.
$\therefore$ $x = 2 \pm \sqrt{3}$

⇒ $x - 2 = \pm$(squaring both sides)
⇒ $(x - 2)^2 = 3$ ⇒ $x^2 + 4 - 4x - 3 = 0$
⇒ $x^2 - 4x + 1 = 0$, is a factor of given polynomial
∴ other factors $= \frac{x^4 - 6x^3 - 26x^2 + 138x - 35}{x^2 - 4x + 1}$

$$
\begin{array}{r l}
 & x^2 - 2x - 35 \\
x^2 - 4x + 1 \,\big) & x^4 - 6x^3 - 26x^2 + 138x - 35 \\
 & x^4 - 4x^3 + x^2 \\
 & \underline{- \quad + \quad -} \\
 & -2x^3 - 27x^2 + 138x - 35 \\
 & -2x^3 + 8x^2 - 2x \\
 & \underline{+ \quad - \quad +} \\
 & -35x^2 + 140x - 35 \\
 & -35x^2 + 140x - 35 \\
 & \underline{+ \quad - \quad +} \\
 & \times
\end{array}
$$

∴ other factors $= x^2 - 2x - 35$
$= x^2 - 7x + 5x - 35 = x(x - 7) + 5(x - 7)$
$= (x - 7)(x + 5)$
∴ other zeroes are $(x - 7) = 0 \Rightarrow x = 7$
$x + 5 = 0 \Rightarrow x = -5$

Example 10: If the polynomial $x^4 - 6x^3 + 16x^2 - 25x + 10$ is divided by another polynomial $x^2 - 2x + k$, the remainder comes out to be $x + a$, find k & a.
Sol.

$$
\begin{array}{r l}
 & x^2 - 4x + (8 - k) \\
x^2 - 2x + k \,\big) & x^4 - 6x^3 + 16x^2 - 25x + 10 \\
 & x^4 - 2x^3 + x^2k \\
 & \underline{- \quad + \quad -} \\
 & -4x^3 + x^2(16 - k) - 25x + 10 \\
 & -4x^3 + x^2(8) - 4xk \\
 & \underline{+ \quad - \quad +} \\
 & x^2[8 - k] + x[4k - 25] + 10 \\
 & x^2[8 - k] - 2x[8 - k] + k(8 - k) \\
 & \underline{- \quad + \quad -} \\
 & x[4k - 25 + 16 - 2k] + 10 - 8k + k^2
\end{array}
$$

According to questions, remainder is $x + a$
∴ coefficient of $x = 1$
⇒ $2k - 9 = 1$
⇒ $k = (10/2) = 5$
Also constant term = a
⇒ $k^2 - 8k + 10 = a$ ⇒ $(5)^2 - 8(5) + 10 = a$
⇒ $a = 25 - 40 + 10$
⇒ $a = -5$
∴ $k = 5, a = -5$

How Do You Determine The Degree Of A Polynomial

Degree Of A Polynomial

The greatest power (exponent) of the terms of a polynomial is called degree of the polynomial.
For example :
In polynomial $5x^2 - 8x^7 + 3x$:
(i) The power of term $5x^2 = 2$
(ii) The power of term $-8x^7 = 7$
(iii) The power of $3x = 1$
Since, the greatest power is 7, therefore degree of the polynomial $5x^2 - 8x^7 + 3x$ is 7
The degree of polynomial :
(i) $4y^3 - 3y + 8$ is 3

(ii) $7p + 2$ is $1(p = p^1)$
(iii) $2m - 7m^8 + m^{13}$ is 13 and so on.

Degree Of A Polynomial With Example Problems With Solutions

Example 1: Find which of the following algebraic expression is a polynomial.
(i) $3x^2 - 5x$ (ii) $x + \frac{1}{x}$ (iii) $\sqrt{y} - 8$ (iv) $z^5 - \sqrt[3]{z} + 8$

Sol.
(i) $3x^2 - 5x = 3x^2 - 5x^1$
It is a polynomial.
(ii) $x + \frac{1}{x} = x^1 + x^{-1}$
It is not a polynomial.
(iii) $\sqrt{y} - 8 = y^{1/2} - 8$
Since, the power of the first term ($\sqrt{y}$) is $\frac{1}{2}$, which is not a whole number.
(iv) $z^5 - \sqrt[3]{z} + 8 = z^5 - z^{1/3} + 8$
Since, the exponent of the second term is 1/3, which in not a whole number. Therefore, the given expression is not a polynomial.

Example 2: Find the degree of the polynomial :
(i) $5x - 6x^3 + 8x^7 + 6x^2$ (ii) $2y^{12} + 3y^{10} - y^{15} + y + 3$ (iii) x (iv) 8

Sol.
(i) Since the term with highest exponent (power) is $8x^7$ and its power is 7.
∴ The degree of given polynomial is 7.
(ii) The highest power of the variable is 15
∴ degree = 15
(iii) $x = x^1 \Rightarrow$ degree is 1.
(iv) $8 = 8x^0 \Rightarrow$ degree = 0

Special Pattern Binomials

The following are special multiplications involving binomials that you will want to try to remember. Be sure to notice the patterns in each situation. You will be seeing these patterns in numerous problems. Don't panic! If you cannot remember these patterns, you can arrive at your answer by simply multiplying with the distributive method. These patterns are, however, very popular. If you can remember the patterns, you can save yourself some work.

Let's examine these patterns:

Squaring a Binomial – multiplying times itself

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$

Notice the middle terms in both of these problems. In each problem, the middle term is twice the multiplication of the terms used to create the binomial expression.

Example 1:

$$(x + 3)^2 = (x + 3)(x + 3)$$
$$= x^2 + 3x + 3x + 9 \quad \text{Distributive method}$$
$$= x^2 + 6x + 9$$

* Notice the middle term.

Example 2:

$$(x-4)^2 = (x-4)(x-4)$$
$$= x^2 - 4x - 4x + 16 \quad \text{Distributive method}$$
$$= x^2 - 8x + 16$$

* Again, notice the middle term.

Product of Sum and Difference
(notice that the binomials differ only by the sign between the terms)

$(a+b)(a-b) = a^2 - b^2$ Notice that there appears to be no "middle" term to forma a trinomial answer, as was seen in the problems above. When multiplication occurs, the values that would form the middle term of a trinomial actually add to zero.

Example 3:

$$(x+3)(x-3) = x^2 - 3x + 3x - 9 \quad \text{Distributive method}$$
$$= x^2 - 9$$

*Notice how the middle term is zero.

Example 4:

$$(2x+3y)(2x-3y) = 4x^2 - 6xy + 6xy - 9y^2 \quad \text{Distributive method}$$
$$= 4x^2 - 9y^2$$

* Again, notice how the middle term is zero.

Multiplying Binomials

There are numerous ways to set up the multiplication of two binomials. The first three methods shown here work for multiplying **ALL** polynomials, not just binomials. All methods, of course, give the same answer.

Multiply $(x+3)(x+2)$

1. "Distributive" Method:

The most universal method. Applies to all polynomial multiplications, not just to binomials.
Start with the first term in the first binomial – the circled blue X. Multiply (distribute) this term times EACH of the terms in the second binomial.

$$(x + 3)\ (x + 2) \quad = x \cdot x + x \cdot 2$$

Now, take the second term in the first binomial – the circled red +3 (notice we take the sign also). Multiply this term times EACH of the terms in the second binomial.

$$(x + 3)(x + 2) \quad = 3 \cdot x + 3 \cdot 2$$

Add the results: $x \cdot x + x \cdot 2 + 3 \cdot x + 3 \cdot 2$

$$x^2 + 2x + 3x + 6$$

$$x^2 + 5x + 6$$ **Answer**

Do you see the "distributive property" at work?
$(x + 3)(x + 2) = x(x + 2) + 3(x + 2)$

Before we move on to the next set up method, let's look at an example of the "distributive" method involving negative values.

"Distributive" Method: Dealing with **negative values.**

Notice how the negative sign is treated as part of the term following the sign.

$$(x - 4)(x - 5) \quad = x \cdot x + x \cdot (-5)$$

$$(x - 4)(x - 5) \quad = (-4) \cdot x + (-4) \cdot (-5)$$

Add the results:

$$x \cdot x + x \cdot (-5) + (-4) \cdot x + (-4) \cdot (-5)$$

$$x^2 - 5x - 4x + 20$$

$$x^2 - 9x + 20$$ Answer

2. "Vertical" Method:

This is a vertical "picture" of the distributive method.
This style applies to all polynomial multiplications.

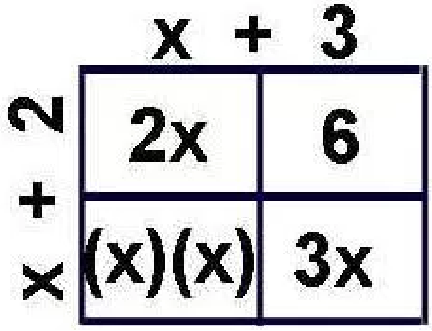
$$\begin{array}{rl} x + 2 & \\ x + 3 & \\ \hline x^2 + 2x & \text{multiply "}x\text{" from bottom term times "}x+2\text{"} \\ 3x + 6 & \text{multiply "3" from bottom term times "}x+2\text{"} \\ \hline x^2 + 5x + 6 & \text{add the like terms} \end{array}$$

Be sure to line up the like terms.

3. "Grid" Method

This is a "table" version of the distributive method.
This style applies to all polynomial multiplications.

To multiply by the grid method, place one binomial at the top of a 2×2 grid (for binomials) and the second binomial on the side of the grid. Place the terms such that each term with its sign lines up with a row or column of the grid. Multiply the rows and columns of the grid to complete the interior of the grid. Finish by adding together the entries inside the grid.



$$2x + 6 + (x)(x) + 3x$$

$$2x + 6 + x^2 + 3x$$

Answer: $x^2 + 5x + 6$

C A U T I O N !!!
There are set up methods that work **ONLY** for binomials. While these set ups may be helpful to understanding binomial multiplication, you must remember that they do not extend to other types of multiplications, such as a binomial times a trinomial. You will have to go back to the "distributive method" for these other polynomial multiplications.

4. "FOIL" Method: multiply First Outer Inner Last

For Binomial Multiplication ONLY!
The words/letters used to describe the FOIL process pertain to the distributive method for multiplying two binomials. These words/letters do not apply to other multiplications such as a binomial times a trinomial.

F: $(\mathbf{x} + 3)(\mathbf{x} + 2)$

O: $(\mathbf{x} + 3)(x + \mathbf{2})$

I: $(x + \mathbf{3})(\mathbf{x} + 2)$

L: $(x + \mathbf{3})(x + \mathbf{2})$

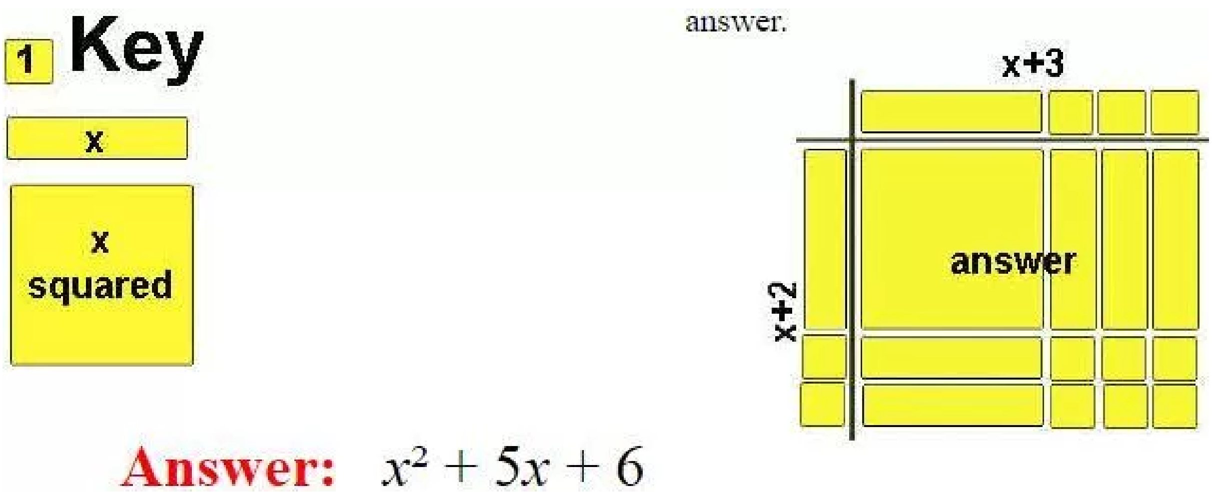
$$(x+3)(x+2) = x^2 + 2x + 3x + 6$$
$$= x^2 + 5x + 6$$

The drawback to using the FOIL lettering is that it ONLY WORKS on binomial multiplication.

5. "Algebra Tile" Method

While this method is helpful for understanding how binomials are multiplied, it is not easily applied to ALL multiplications and may not be practical for overall use.

The example shown here is for binomial multiplication only!
To multiply binomials using algebra tiles, place one expression at the top of the grid and the second expression on the side of the grid. You MUST maintain straight lines when you are filling in the center of the grid. The tiles needed to complete the inner grid will be your answer.



Algebraic Identities Of Polynomials

1. $(a + b)^2 = a^2 + 2ab + b^2 = (-a - b)^2$
2. $(a - b)^2 = a^2 - 2ab + b^2$
3. $(a - b)(a + b) = a^2 - b^2$
4. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
5. $(a + b - c)^2 = a^2 + b^2 + c^2 + 2ab - 2bc - 2ca$
6. $(a - b + c)^2 = a^2 + b^2 + c^2 - 2ab - 2bc + 2ca$
7. $(-a + b + c)^2 = a^2 + b^2 + c^2 - 2ab + 2bc - 2ca$
8. $(a - b - c)^2 = a^2 + b^2 + c^2 - 2ab + 2bc - 2ca$
9. $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$
10. $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$
11. $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$
    $= (a + b)(a^2 - ab + b^2)$
12. $a^3 - b^3 = (a - b)^3 + 3ab(a - b)$
    $= (a - b)(a^2 + ab + b^2)$
13. $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

if $a + b + c = 0$ then $a^3 + b^3 + c^3 = 3abc$

Algebraic Identities

If an equality holds true for all values of the variable, then it is called an Identity.

Identity 1: $(x + y)^2 = x^2 + 2xy + y^2$

Identity 2: $(x - y)^2 = x^2 - 2xy + y^2$

Identity 3: $(x + y)(x - y) = x^2 - y^2$

Identity 4: $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

Identity 5: $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$

Identity 6: $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$

Identity 7: $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

Identity 8: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Identity 9: $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$
$= \frac{1}{2}(x + y + z)\{(x - y)^2 + (y - z)^2 + (z - x)^2\}$

Algebraic Identities Of Polynomials Example Problems With Solutions

Example 1: Expand each of the following

(i) $(3x - 4y)^2$ (ii) $\left(\frac{x}{2} + \frac{y}{3}\right)^2$

Solution: (i) We have,

$$(3x - 4y)^2 = (3x)^2 - 2 \times 3x \times 4y + (4y)^2$$
$$= 9x^2 - 24xy + 16y^2$$

(ii) We have,

$$\left(\frac{x}{2} + \frac{y}{3}\right)^2 = \left(\frac{x}{2}\right)^2 + 2 \times \frac{x}{2} \times \frac{y}{3} + \left(\frac{y}{3}\right)^2$$
$$= \frac{x^2}{4} + \frac{1}{3}xy + \frac{y^2}{9}$$

Example 2: Find the products

(i) (2x + 3y) (2x – 3y)

(ii) $\left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right)\left(x^4 + \frac{1}{x^4}\right)$

Solution: (i) We have,

$(2x + 3y)(2x - 3y)$

$= (2x)^2 - (3y)^2$ [Using: $(a+b)(a-b) = a^2 - b^2$]

$= (2x)^2 - (3y)^2 = 4x^2 - 9y^2$

(ii) We have,

$$\left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right)\left(x^4 + \frac{1}{x^4}\right)$$

$$= \left(x^2 - \frac{1}{x^2}\right)\left(x^2 + \frac{1}{x^2}\right)\left(x^4 + \frac{1}{x^4}\right)$$

$$= \left\{(x^2)^2 - \left(\frac{1}{x^2}\right)^2\right\}\left(x^4 + \frac{1}{x^4}\right)$$

$$= \left(x^4 - \frac{1}{x^4}\right)\left(x^4 + \frac{1}{x^4}\right) = (x^4)^2 - \left(\frac{1}{x^4}\right)^2$$

$$= x^8 - \frac{1}{x^8}$$

Example 3: Evaluate each of the following by using identities

(i) 103×97 (ii) 103×103

(iii) $(97)^2$ (iv) $185 \times 185 - 115 \times 115$

Solution: (i) We have,

$103 \times 97 = (100 + 3)(100 - 3)$

$= (100)^2 - (3)^2 = 10000 - 9 = 9991$

(ii) We have,

$103 \times 103 = (103)^2$

$= (100 + 3)^2 = (100)^2 + 2 \times 100 \times 3 + (3)^2$

$= 10000 + 600 + 9 = 10609$

(iii) We have,

$(97)^2 = (100 - 3)^2$

$= (100)^2 - 2 \times 100 \times 3 + (3)^2$

$= 10000 - 600 + 9 = 9409$

(iv) We have,

$185 \times 185 - 115 \times 115$

$= (185)^2 - (115)^2 = (185 + 115)(185 - 115)$

$= 300 \times 70 = 21000$

Example 4: If $x + \frac{1}{x} = 6$, find $x^4 + \frac{1}{x^4}$

Solution: We have,

$$x^2 + \frac{1}{x^2} = 34 \quad \Rightarrow \quad \left(x^2 + \frac{1}{x^2}\right)^2 = (34)^2$$

$$\Rightarrow \quad (x^2)^2 + \left(\frac{1}{x^2}\right)^2 + 2 \times x^2 \times \frac{1}{x^2} = 1156$$

$$\Rightarrow x^4 + \frac{1}{x^4} + 2 = 1156 \quad \Rightarrow x^4 + \frac{1}{x^4} = 1156 - 2$$

$$\Rightarrow \quad x^4 + \frac{1}{x^4} = 1154$$

Example 5: If $x^2 + \frac{1}{x^2} = 27$, find the value of the $x - \frac{1}{x}$

Solution: We have,

$$\left(x - \frac{1}{x}\right)^2 = x^2 - 2 \times x \times \frac{1}{x} + \frac{1}{x^2}$$

$$\Rightarrow \quad \left(x - \frac{1}{x}\right)^2 = x^2 - 2 + \frac{1}{x^2}$$

$$\Rightarrow \quad \left(x - \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} - 2$$

$$\Rightarrow \quad \left(x - \frac{1}{x}\right)^2 = 27 - 2$$

$$\left[\therefore x^2 + \frac{1}{x^2} = 27 \text{ (given)}\right]$$

$$\Rightarrow \quad \left(x - \frac{1}{x}\right)^2 = 25$$

$$\Rightarrow \quad \left(x - \frac{1}{x}\right)^2 = (\pm 5)^2 \quad \Rightarrow \quad x - \frac{1}{x} = \pm 5$$

Example 6: If $x + y = 12$ and $xy = 32$, find the value of $x^2 + y^2$

Solution: We have,

$$(x + y)^2 = x^2 + y^2 + 2xy$$

$$\Rightarrow \quad 144 = x^2 + y^2 + 2 \times 32$$

$$[\text{Putting } x + y = 12 \text{ and } xy = 32]$$

$$\Rightarrow \quad 144 = x^2 + y^2 + 64$$

$$\Rightarrow \quad 144 - 64 = x^2 + y^2$$

$$\Rightarrow \quad x^2 + y^2 = 80$$

Example 7: Prove that:

$2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = [(a - b)^2 + (b - c)^2 + (c - a)^2]$

Solution: We have,

$$\text{L.H.S.} = 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca$$

$$= (a^2 - 2ab + b^2) + (b^2 - 2bc + c^2)$$

$$+ (c^2 - 2ca + a^2) \text{ [Re-arranging the terms]}$$

$$= (a - b)^2 + (b - c)^2 + (c - a)^2 = \text{R.H.S.}$$

Hence, $2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca$

$$= [(a - b)^2 + (b - c)^2 + (c - a)^2]$$

Example 8: If $a^2 + b^2 + c^2 - ab - bc - ca = 0$, prove that $a = b = c$.

Solution: We have,

If $a^2 + b^2 + c^2 - ab - bc - ca = 0$

$$\Rightarrow 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 2 \times 0$$

[Multiplying both sides by 2]

$$\Rightarrow (a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ac + a^2) = 0$$

$$\Rightarrow (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$$

$$\Rightarrow a - b = 0,\ b - c = 0,\ c - a = 0$$

[$\therefore$ Sum of positive quantities is zero if and only if each quantity is zero]

$$\Rightarrow a = b,\ b = c \text{ and } c = a$$

$$\Rightarrow a = b = c$$

Example 9: Write the following in expanded form :

(i) $(9x + 2y + z)^2$ (ii) $(3x + 2y - z)^2$

(iii) $(x - 2y - 3z)^2$ (iv) $(-x + 2y + z)^2$

Solution: Using the identity

(i) We have,

$$(9x + 2y + z)^2$$

$$= (9x)^2 + (2y)^2 + z^2 + 2 \times 9x \times 2y$$

$$+ 2 \times 2y \times z + 2 \times 9x \times z$$

$$= 81x^2 + 4y^2 + z^2 + 36xy + 4yz + 18xz$$

(ii) We have,

$$(3x + 2y - z)^2$$

$$= [3x + 2y + (-z)]^2$$

$$= (3x)^2 + (2y)^2 + (-z)^2 + 2 \times 3x \times 2y$$

$$+ 2 \times 2y \times (-z) + 2 \times 3x \times (-z)$$

$$= 9x^2 + 4y^2 + z^2 + 12xy - 4yz - 6xz$$

(iii) We have,

$(x - 2y - 3z)^2$

$= [x + (-2y) + (-3z)]^2$

$= x^2 + (-2y)^2 + (-3z)^2 + 2 \times x \times (-2y) + 2 \times (-2y) \times (-3z) + 2 \times (-3z) \times x$

$= x^2 + 4y^2 + 9z^2 - 4xy + 12yz - 6zx$

(iv) We have,

$(-x + 2y + z)^2$

$= [(-x) + 2y + z]^2$

$= (-x)^2 + (2y)^2 + z^2 + 2 \times (-x) \times (2y) + 2 \times 2y \times z + 2 \times (-x) \times z$

$= x^2 + 4y^2 + z^2 - 4xy + 4yz - 2zx$

Example 10: If $a^2 + b^2 + c^2 = 20$ and $a + b + c = 0$, find $ab + bc + ca$.

Solution: We have,

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

$$\Rightarrow (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$

$$\Rightarrow 0^2 = 20 + 2(ab + bc + ca)$$

$$\Rightarrow -20 = 2(ab + bc + ca)$$

$$\Rightarrow -\frac{20}{2} = \left\{\frac{2(ab+bc+ca)}{2}\right\}$$

$$\Rightarrow ab + bc + ca = -10$$

Example 11: If $a + b + c = 9$ and $ab + bc + ca = 40$, find $a^2 + b^2 + c^2$.
Solution: We know that

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$

$$\Rightarrow 9^2 = a^2 + b^2 + c^2 + 2 \times 40$$

$$\Rightarrow 81 = a^2 + b^2 + c^2 + 80$$

$$\Rightarrow a^2 + b^2 + c^2 = 1$$

Example 12: If $a^2 + b^2 + c^2 = 250$ and $ab + bc + ca = 3$, find $a + b + c$.
Solution: We know that

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$

$$\Rightarrow (a + b + c)^2 = 250 + 2 \times 3$$

$$\Rightarrow (a + b + c)^2 = 256$$

$$\Rightarrow (a + b + c)^2 = (\pm 16)^2$$

[Taking square root of both sides]

$$\Rightarrow a + b + c = \pm 16$$

Example 13: Write each of the following in expanded form:
(i) $(2x + 3y)^3$ (ii) $(3x - 2y)^3$

Solution: (i) Replacing a by 2x and b by 3y in the identity

$(a + b)^3 = a^3 + b^3 + 3ab(a + b)$, we have

$$(2x + 3y)^3 = (2x)^3 + (3y)^3 + 3 \times 2x \times 3y \times (2x + 3y)$$
$$= 8x^3 + 27^3 + 18xy \times 2x + 18xy \times 3y$$
$$= 8x^3 + 27y^3 + 36x^2y + 54xy^2$$

(ii) Replacing a by 3x and b by 2y in the identity

$(a - b)^3 = a^3 - b^3 - 3ab(a - b)$, we have

$$(3x - 2y)^3 = (3x)^3 - (2y)^3 - 3 \times 3x \times 2y \times (3x - 2y)$$
$$= 27x^3 - 8y^3 - 18xy \times (3x - 2y)$$
$$= 27x^3 - 8y^3 - 54x^2y + 36xy^2$$

Example 14: If x + y = 12 and xy = 27, find the value of $x^3 + y^3$.

Solution: We know that

$$(x + y)^3 = x^3 + y^3 + 3xy(x + y)$$

Putting x + y = 12 and xy = 27 in the above identity, we get

$$12^3 = x^3 + y^3 + 3 \times 27 \times 12$$
$$\Rightarrow \quad 1728 = x^3 + y^3 + 972$$
$$\Rightarrow \quad x^3 + y^3 = 1728 - 972$$
$$\Rightarrow \quad x^3 + y^3 = 756$$

Example 15: If x – y = 4 and xy = 21, find the value of $x^3 - y^3$.

Solution: We know that

$$(x - y)^3 = x^3 - y^3 - 3xy(x - y)$$

Putting x – y = 4 and xy = 21, we get

$$4^3 = x^3 - y^3 - 3 \times 21 \times 4$$
$$\Rightarrow 64 = x^3 - y^3 - 252 \quad \Rightarrow \quad 64 + 252 = x^3 - y^3$$
$$\Rightarrow \quad x^3 - y^3 = 316$$

Example 16: If $x + \frac{1}{x} = 7$, find the value of $x^3 + \frac{1}{x^3}$

Solution: We have,

$$\left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3 \times x \times \frac{1}{x}\left(x + \frac{1}{x}\right)$$
$$\Rightarrow \left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right)$$

Putting $x + \frac{1}{x} = 7$, we get

$$7^3 = x^3 + \frac{1}{x^3} + 3 \times 7$$
$$\Rightarrow \quad 343 = x^3 + \frac{1}{x^3} + 21$$
$$\Rightarrow \quad x^3 + \frac{1}{x^3} = 343 - 21 \quad \Rightarrow x^3 + \frac{1}{x^3} = 322$$

Example 17: If a + b = 10 and a2 + b2 = 58, find the value of $a^3 + b^3$.

Solution: We know that

$(a + b)^2 = a^2 + b^2 + 2ab$

Putting $a + b = 10$ and $a^2 + b^2 = 58$, we get

$10^2 = 58 + 2\,ab \Rightarrow 100 = 58 + 2ab$

$\Rightarrow 100 - 58 = 2ab \Rightarrow 42 = 2ab$

$\Rightarrow ab = 21$ Thus, we have

$a + b = 10$ and $ab = 21$ Now,

$(a + b)^3 = a^3 + b^3 + 3ab\,(a + b)$

$\Rightarrow 10^3 = a^3 + b^3 + 3 \times 21 \times 10$

[Putting $a + b = 10$ and $ab = 21$]

$\Rightarrow 1000 = a^3 + b^3 + 630$

$\Rightarrow 1000 - 630 = a^3 + b^3$

$\Rightarrow a^3 + b^3 = 370$

Example 18: If $x^2 + \frac{1}{x^2} = 7$, find the value of $x^3 + \frac{1}{x^3}$

Solution: We have,

$$\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 \times x \times \frac{1}{x}$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = 7 + 2 \left[\text{Putting } x^2 + \frac{1}{x^2} = 7\right]$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = 9 \qquad \Rightarrow \left(x + \frac{1}{x}\right)^2 = 3^2$$

$$\Rightarrow x + \frac{1}{x} = 3 \text{ [Taking square root of both sides]}$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^3 = 3^3 \qquad \text{[Cubing both sides]}$$

$$\Rightarrow x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = 27$$

$$\Rightarrow \left(x^3 + \frac{1}{x^3}\right) + 3 \times 3 = 27$$

$$\Rightarrow x^3 + \frac{1}{x^3} = 27 - 9 \qquad \Rightarrow x^3 + \frac{1}{x^3} = 18$$

Example 19: If $x^4 + \frac{1}{x^4} = 47$, find the value of $x^3 + \frac{1}{x^3}$

Solution: We know that

$$\left(x^2 + \frac{1}{x^2}\right)^2 = x^4 + \frac{1}{x^4} + 2 \times x^2 + \frac{1}{x^2}$$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 = \left(x^4 + \frac{1}{x^4}\right) + 2$$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 = 47 + 2 \left[\text{Putting } x^4 + \frac{1}{x^4} = 47\right]$$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 = 7^2 \Rightarrow x^2 + \frac{1}{x^2} = 7$$

[Taking square root of both sides]

$$\text{Now, } \left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = 7 + 2 \quad \left[\text{using}: x^2 + \frac{1}{x^2} = 7\right]$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = 3^2 \quad \Rightarrow \quad x + \frac{1}{x} = 3$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^3 = 3^3 \quad \text{[Cubing both sides]}$$

$$\Rightarrow x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = 27$$

$$\Rightarrow x^3 + \frac{1}{x^3} + 3 \times 3 = 27 \left[\text{Putting } x + \frac{1}{x} = 3\right]$$

$$\Rightarrow x^3 + \frac{1}{x^3} = 27 - 9 \Rightarrow x^3 + \frac{1}{x^3} = 18$$

Example 20: If a + b = 10 and ab = 21, find the value of $a^3 + b^3$.

Solution: We know that

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$= (a + b)(a^2 + 2ab + b^2 - 2ab - ab)$

[Adding and subtracting 2ab in the second bracket]

$= (a + b)[(a + b)^2 - 3ab]$

$= 10 \times (10^2 - 3 \times 21)$

$= 10 \times (100 - 63) = 10 \times 37 = 370.$

Example 21: If a – b = 4 and ab = 45, find the value of $a^3 - b^3$.

Solution: We have,

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$= (a - b)(a^2 - 2ab + b^2 + 2ab + ab)$

$= (a - b)\{(a - b)^2 + 3ab\}$

$= 4 \times (4^2 + 3 \times 45) = 4 \times (16 + 135)$

$= 4 \times 151 = 604.$

Example 22: If a + b + c = 0, then prove that a3 + b3 + c3 = 3abc
Solution: We know that

$a^3 + b^3 + c^3 - 3\ abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

putting a + b + c = 0 on R.H.S., we get

$a^3 + b^3 + c^3 - 3abc = 0$

$\Rightarrow \quad a^3 + b^3 + c^3 = 3abc$

Example 23: Find the following product:
$(x + y + 2z)(x^2 + y^2 + 4z^2 - xy - 2yz - 2zx)$
Solution: We have,

$(x + y + 2z)(x^2 + y^2 + 4z^2 - xy - 2yz - 2zx)$

$= (x + y + 2z)(x^2 + y^2 + (2z)^2 - x \times y - y \times 2z - 2z \times x)$

$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca),$

where a = x, b = y, c = 2z

$= a^3 + b^3 + c^3 - 3abc$

$= x^3 + y^3 + (2z)^3 - 3 \times x \times y \times 2z$

$= x^3 + y^3 + 8z^3 - 6xyz$

Example 24: If a + b + c = 6 and ab + bc + ca = 11, find the value of $a^3 + b^3 + c^3 - 3abc$.
Solution: We know that

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$\Rightarrow \; a^3 + b^3 + c^3 - 3abc =$

$(a + b + c)\{(a^2 + b^2 + c^2) - (ab + bc + ca)\}$...(i)

Clearly, we require the values of $a + b + c$,
$a^2 + b^2 + c^2$ and $ab + bc + ca$ to obtain the value of $a^3 + b^3 + c^3 - 3abc$.

We are given the values of $a + b + c$ and $ab + bc + ca$.

So, let us first obtain the value of $a^2 + b^2 + c^2$.

We know that

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$\Rightarrow (a + b + c)^2 = (a^2 + b^2 + c^2) + 2(ab + bc + ca)$

$\Rightarrow \; 6^2 = a^2 + b^2 + c^2 + 2 \times 11$

[Putting the values of $a + b + c$ and $ab + bc + ca$]

$\Rightarrow \; 36 = a^2 + b^2 + c^2 + 22$

$\Rightarrow \; a^2 + b^2 + c^2 = 36 - 22$

$\Rightarrow \; a^2 + b^2 + c^2 = 14$

Now, putting $a + b + c = 6$, $ab + bc + ca = 1$ and $a^2 + b^2 + c^2 = 14$ in (i), we get

$a^3 + b^3 + c^3 - 3abc = 6 \times (14 - 11)$

$= 6 \times 3 = 18.$

Example 25: If $x + y + z = 1$, $xy + yz + zx = -1$ and $xyz = -1$, find the value of $x^3 + y^3 + z^3$.
Solution: We know that

$x^3 + y^3 + z^3 - 3xyz$

$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

$\Rightarrow \; x^3 + y^3 + z^3 - 3xyz$

$= (x + y + z)(x^2 + y^2 + z^2 + 2xy + 2yz + 2zx - 3xy - 3yz - 3zx)$

[Adding and subtracting $2xy + 2yz + 2zx$]

$\Rightarrow \; x^3 + y^3 + z^3 - 3xyz$

$= (x + y + z)\{(x + y + z)^2 - 3(xy + yz + zx)\}$

$\Rightarrow x^3 + y^3 + z^3 - 3 \times -1 = 1 \times \{(1)^2 - 3 \times -1\}$

[Putting the values of $x + y + z$, $xy + yz + zx$ and xyz]

$\Rightarrow \; x^3 + y^3 + z^3 + 3 = 4$

$\Rightarrow \; x^3 + y^3 + z^3 = 4 - 3$

$\Rightarrow \; x^3 + y^3 + z^3 = 1$

Polynomial Remainder Theorem

Remainder Theorem

Theorem: Let p(x) be any polynomial of degree greater than or equal to one and let a be any real number. If p(x) is divided by the linear polynomial x – a, then the remainder is p(a).

Proof: Let p(x) be any polynomial with degree greater than or equal to 1. Suppose that when p(x) is divided by x – a, the quotient is q(x) and the remainder is r(x), i.e., p(x) = (x – a) q(x) + r(x)
Since the degree of x – a is 1 and the degree of r(x) is less than the degree of x – a, the degree of r(x) = 0. This means that r(x) is a constant, say r.
So, for every value of x, r(x) = r.
Therefore, p(x) = (x – a) q(x) + r
In particular, if x = a, this equation gives us
p(a) = (a – a) q(a) + r
= r,
which proves the theorem.

1. Remainder obtained on dividing polynomial p(x) by x – a is equal to p(a) .
2. If a polynomial p(x) is divided by (x + a) the remainder is the value of p(x) at x = –a.
3. (x – a) is a factor of polynomial p(x) if p(a) = 0
4. (x + a) is a factor of polynomial p(x) if p(–a) = 0
5. (x – a) (x – b) is a factor of polynomial p(x), if p(a) = 0 and p(b) = 0.
6. If a polynomial p(x) is divided by (ax – b), the remainder is the value of p(x) at x = b/a
7. If a polynomial p(x) is divided by (b – ax), the remainder is equal to the value of p(x) at x = b/a.
8. (ax – b) is a factor of polynomial p(x) if p(b/a) = 0.

Remainder Theorem Example Problems With Solutions

Example 1: Find the remainder when $4x^3 – 3x^2 + 2x – 4$ is divided by x – 1
Solution: Let $p(x) = 4x^3 – 3x^2 + 2x – 4$
When p(x) is divided by (x – 1), then by remainder theorem, the required remainder will be p(1)
$p(1) = 4 (1)^3 – 3(1)^2 + 2(1) – 4$
= 4 × 1 – 3 × 1 + 2 × 1 – 4
= 4 – 3 + 2 – 4 = – 1

Example 2: Find the remainder when $4x^3 – 3x^2 + 2x – 4$ is divided by x + 1
Solution: Let $p(x) = 4x^3 – 3x^2 + 2x – 4$
When p(x) is divided by (x + 2), then by remainder theorem, the required remainder will be p (–2).
$p(–2) = 4 (–2)^3 – 3 (–2)^2 + 2(–2) – 4$
= 4 × (–8) – 3 × 4 – 4 – 4
= – 32 – 12 – 8 = – 52

Example 3: Find the remainder when $4x^3 – 3x^2 + 2x – 4$ is divided by $x+\frac{1}{2}$
Solution: Let $p(x) = 4x^3 – 3x^2 + 2x – 4$
When p(x) is divided by $x+\frac{1}{2}$, then by remainder theorem, the required remainder will be
$$p\left(-\frac{1}{2}\right)=\left(-\frac{1}{2}\right)^3-3\left(-\frac{1}{2}\right)^2+2\left(-\frac{1}{2}\right)-4$$
$$=4\times\left(-\frac{1}{8}\right)-3\times\frac{1}{4}-2\times\frac{1}{2}-4$$
$$=-\frac{1}{2}-\frac{3}{4}-1-4=\frac{-2-3-20}{4}$$
$$=-\frac{25}{4}$$

Example 4: Determine the remainder when the polynomial $p(x) = x^4 – 3x^2 + 2x + 1$ is divided by x – 1.
Solution: By remainder theorem, the required remainder is equal to p(1).
Now, $p(x) = x^4 – 3x^2 + 2x + 1$
$p(1) = (1)^4 – 3\times1^2 + 2 \times 1 + 1$
= 1 – 3 + 2 + 1 = 1
Hence required remainder = p(1) = 1

Example 5: Find the remainder when the polynomial $f(x) = 2x^4 - 6x^3 + 2x^2 - x + 2$ is divided by $x + 2$.

Solution: We have, $x + 2 = x - (-2)$. So, by remainder theorem, when f(x) is divided by $(x-(-2))$ the remainder is equal to $f(-2)$.

Now, $f(x) = 2x^4 - 6x^3 + 2x^2 - x + 2$

$\Rightarrow f(-2) = 2(-2)^4 - 6(-2)^3 + 2(-2)^2 - (-2) + 2$

$\Rightarrow f(-2) = 2\times16 - 6 \times -8 + 2 \times 4 + 2 + 2$

$\Rightarrow f(-2) = 32 + 48 + 8 + 2 + 2 = 92$

Hence, required remainder = 92

Example 6: Find the remainder when $p(x) = 4x^3 - 12x^2 + 14x - 3$ is divided by $g(x) = x - 1/2$

Solution:

By remainder theorem, we know that

p(x) when divided by $g(x) = \left(x - \frac{1}{2}\right)$

gives a remainder equal to $p\left(\frac{1}{2}\right)$

Now, $p(x) = 4x^3 - 12x^2 + 14x - 3$

$$\Rightarrow p\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 12\left(\frac{1}{2}\right)^3 + 14\left(\frac{1}{2}\right) - 3$$

$$\Rightarrow p\left(\frac{1}{2}\right) = \frac{4}{8} - \frac{12}{4} + \frac{14}{2} - 3$$

$$\Rightarrow p\left(\frac{1}{2}\right) = \frac{1}{2} - 3 + 7 - 3$$

$$\Rightarrow p\left(\frac{1}{2}\right) = \frac{3}{2}$$

Hence, required remainder = $p\left(\frac{1}{2}\right) = \frac{3}{2}$

Example 7: If the polynomials $ax^3 + 4x^2 + 3x - 4$ and $x^3 - 4x + a$ leave the same remainder when divided by $(x-3)$, find the value of a.

Solution: Let $p(x) = ax^3 + 4x^2 + 3x - 4$ and

$q(x) = x^3 - 4x + a$ be the given polynomials. The remainders when p(x) and q(x) are divided by $(x-3)$ are p(3) and q(3) respectively.

By the given condition, we have

$p(3) = q(3)$

$$\Rightarrow a \times 3^3 + 4\times3^2 + 3\times 3 - 4 = 3^3 - 4\times3 + a$$

$$\Rightarrow 27a + 36 + 9 - 4 = 27 - 12 + a$$

$$\Rightarrow 26a + 26 = 0$$

$$\Rightarrow 26a = -26$$

$$\Rightarrow a = -1$$

Example 8: Let R_1 and R_2 are the remainders when the polynomials $x^3 + 2x^2 - 5ax - 7$ and $x^3 + ax^2 - 12x + 6$ are divided by $x + 1$ and $x - 2$ respectively. If $2R_1 + R_2 = 6$, find the value of a.

Solution: Let $p(x) = x^3 + 2x^2 - 5ax - 7$ and

$q(x) = x^3 + ax^2 - 12x + 6$ be the given polynomials.

Now, R_1 = Remainder when p(x) is divided by $x + 1$

$\Rightarrow R_1 = p(-1)$

$\Rightarrow R_1 = (-1)^3 + 2(-1)^2 - 5a \times -1 - 7$

$[\because p(x) = x^3 + 2x^2 - 5ax - 7]$

$\Rightarrow R_1 = -1 + 2 + 5a - 7$

$\Rightarrow R_1 = 5a - 6$

And, R_2 = Remainder when q(x) is divided by $x - 2$

$\Rightarrow R_1 = q(2)$

$\Rightarrow R_2 = (2)^3 + a \times 2^2 - 12 \times 2 + 6$

$[\because q(x) = x^3 + ax^2 - 12x - 6]$

$\Rightarrow R_2 = 8 + 4a - 24 + 6$

$\Rightarrow R_2 = 4a - 10$

Substituting the values of R_1 and R_2 in $2R_1 + R_2 = 6$, we get

$2(5a - 6) + (4a - 10) = 6$

$\Rightarrow 10a - 12 + 4a - 10 = 6$

$\Rightarrow 14a - 22 = 6$

$\Rightarrow 14a = 28$

$\Rightarrow a = 2$

What Are The Types Of Factorization

Types Of Factorization Example Problems With Solutions

Type I: Factorization by taking out the common factors.

Example 1: Factorize the following expression
$2x^2y + 6xy^2 + 10x^2y^2$

Solution: $2x^2y + 6xy^2 + 10x^2y^2$
$= 2xy(x + 3y + 5xy)$

Type II: Factorization by grouping the terms.

Example 2: Factorize the following expression

$a^2 - b + ab - a$

Solution: $a^2 - b + ab - a$

$= a^2 + ab - b - a = (a^2 + ab) - (b + a)$

$= a\,(a + b) - (a + b) = (a + b)\,(a - 1)$

Type III: Factorization by making a perfect square.

Example 3: Factorize of the following expression

$9x^2 + 12xy + 4y^2$

Solution: $9x^2 + 12xy + 4y^2$

$= (3x)^2 + 2 \times (3x) \times (2y) + (2y)^2$

$= (3x + 2y)^2$

Example 4: Factorize of the following expression

$\frac{x^2}{y^2} + 2 + \frac{y^2}{x^{2\prime}}, x \neq 0, y \neq 0$

Solution:

$$\frac{x^2}{y^2} + 2 + \frac{y^2}{x^{2\prime}}$$

$$= \left(\frac{x}{y}\right)^2 + 2\left(\frac{x}{y}\right).\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2$$

$$= \left(\frac{x}{y} + \frac{y}{x}\right)^2$$

Example 5: Factorize of the following expression

$\left(5x - \frac{1}{x}\right)^2 + 4\left(5x - \frac{1}{x}\right) + 4, x \neq 0$

Solution:

$$= \left(5x - \frac{1}{x}\right)^2 + 4\left(5x - \frac{1}{x}\right) + 4$$

$$= \left(5x - \frac{1}{x}\right)^2 + 2 \times \left(5x - \frac{1}{x}\right) \times 2 + 2^2$$

$$= \left(5x - \frac{1}{x} + 2\right)^2$$

Type IV: Factorizing by difference of two squares.

Example 6: Factorize the following expressions

(a) $2x^2y + 6\,xy^2 + 10\,x^2y^2$

(b) $2x^4 + 2x^3y + 3xy^2 + 3y^3$

Solution:

(a) $2x^2y + 6\,xy^2 + 10\,x^2y^2$

$= (2xy)\,(x + 3y + 5xy)$

(b) $2x^4 + 2x^3y + 3xy^2 + 3y^3$

$= (2x^4 + 2x^3y) + (3xy^2 + 3y^3)$

$= (2x^3 + 3y^2)\,(x + y)$

Example 7: Factorize $4x^2 + 12xy + 9y^2$

Solution:

Note that $4x^2 = (2x)^2 = a^2$ say, and
$9y^2 = (3y)^2 = b^2$ say, where $a = 2x$ and
$b = 3y$. This suggests the use of identity (i) may be used

and the given expression is equal to
$(a + b)^2$. Hence

$4x^2 + 12xy + 9y^2$

$= (2x)^2 + 2(x)(3y) + (3y)^2$

$= (2x + 3y)^2$

$= (2x + 3y)(2x + 3y)$

If the expression A can be reduced to an expression, three of whose terms are the squares of some expression, then the identity (vii) may be useful.

Example 8: Factorize each of the following expressions
(i) $9x^2 - 4y^2$
(ii) $x^3 - x$
Solution:

(i) $9x^2 - 4y^2 = (3x)^2 - (2y)^2$

$= (3x + 2y)(3x - 2y)$

(ii) $x^3 - x = x(x^2 - 1)$

$= x(x-1)(x+1)$

Example 9: Factorize each of the following expressions
(i) $36x^2 - 12x + 1 - 25y^2$
(ii) $a^2 - \frac{9}{a^2}, a \neq 0$

Solution:

(i) $36x^2 - 12x + 1 - 25y^2$

$= (6x)^2 - 2 \times 6x \times 1 + 1^2 - (5y)^2$

$= (6x-1)^2 - (5y)^2$

$= \{(6x - 1) - 5y\}\{(6x - 1) + 5y\}$

$= (6x - 1 - 5y)(6x - 1 + 5y)$

$= (6x - 5y - 1)(6x + 5y - 1)$

(ii) $a^2 - \frac{9}{a^2} = (a)^2 - \left(\frac{3}{a}\right)^2$

$= \left(a - \frac{3}{a}\right)\left(a + \frac{3}{a}\right)$

Example 10: Factorize the following algebraic expression
$x^4 - 81y^4$
Solution:

$$x^4 - 81y^4 = [(x)^2]^2 - (9y^2)^2$$
$$= (x^2 - 9y^2)(x^2 + 9y^2)$$
$$= \{x^2 - (3y)^2\}(x^2 + 9y^2)$$
$$= (x - 3y)(x + 3y)(x^2 + 9y^2)$$

Example 11: Factorize the following expression
$x(x+z) - y(y+z)$
Solution: $x(x+z) - y(y+z) = (x^2 - y^2) + (xz-yz)$
$= (x-y)(x+y) + z(x-y)$
$= (x-y)\{(x+y) + z\}$
$= (x-y)(x+ y + z)$

Example 12: Factorize the following expression
$x^4 + x^2 + 1$
Solution: $x^4 + x^2 + 1 = (x^4 + 2x^2 + 1) - x^2$
$= (x^2 + 1)^2 - x^2 = (x^2 + 1 - x)(x^2 + 1 + x)$
$= (x^2 - x + 1)(x^2 + x + 1)$

Type V: Factorizing the sum and difference of cubes of two quantities.
(i) $(a^3 + b^3) = (a + b)(a^2 - ab + b^2)$
(ii) $(a^3 - b^3) = (a - b)(a^2 + ab + b^2)$

Example 13: Factorize the following expression
$a^3 + 27$
Solution: $a^3 + 27 = a^3 + 3^3 = (a + 3)(a^2 - 3a + 9)$

Example 14: Simplify : $(x+ y)^3 - (x - y)^3 - 6y(x^2 - y^2)$
Solution:

Let $x + y = a$ and $x - y = b$.

Then, $ab = (x+y)(x-y) = x^2 - y^2$ and
$a - b = (x+y) - (x-y) = 2y$

$\therefore \quad (x+y)^3 - (x-y)^3 - 6y(x^2 - y^2)$
$= a^3 - b^3 - 3ab(a - b) = (a - b)^3$
$= \{(x+y) - (x-y)\}^3 = (2y)^3 = 8y^3$

Factorization Of Polynomials Using Factor Theorem

1. Obtain the polynomial p(x).
2. Obtain the constant term in p(x) and find its all possible factors. For example, in the polynomial $x^4 + x^3 - 7x^2 - x + 6$ the constant term is 6 and its factors are ± 1, ± 2, ± 3, ± 6.
3. Take one of the factors, say a and replace x by it in the given polynomial. If the polynomial reduces to zero, then (x – a) is a factor of polynomial.
4. Obtain the factors equal in no. to the degree of polynomial. Let these are (x–a), (x–b), (x–c.)…..
5. Write p(x) = k (x–a) (x–b) (x–c) ….. where k is constant.
6. Substitute any value of x other than a,b,c …… and find the value of k.

Factorization Of Polynomials Using Factor Theorem Example Problems With Solutions

Example 1: Factorize $x^2 + 4 + 9z^2 + 4x - 6xz - 12z$

Solution:

The presence of the three squares x^2, $(2)^2$, and $(3z)^2$ So we write.

$A = x^2 + (2)^2 + (3z)^2 + 4x - 6xz - 12z$

We note that the last two of the product terms are negative

and that both of these contain z. Hence we write A as

$A = x^2 + (2)^2 + (-3z)^2 + 2.2x - 2.x.(-3z) + 2.2(-3z) = (x+2-3z)^2$

$= (x + 2 - 3z)(x + 2 - 3z)$

Example 2: Using factor theorem, factorize the polynomial $x^3 - 6x^2 + 11x - 6$.

Solution:

Let $f(x) = x^3 - 6x^2 + 11x - 6$

The constant term in f(x) is equal to -6 and factors of -6 are $\pm 1, \pm 2, \pm 3, \pm 6$.

Putting $x = 1$ in f(x), we have

$f(1) = 1^3 - 6 \times 1^2 + 11 \times 1 - 6$

$= 1 - 6 + 11 - 6 = 0$

$\therefore$ $(x - 1)$ is a factor of f(x)

Similarly, $x - 2$ and $x - 3$ are factors of f(x).

Since f(x) is a polynomial of degree 3.

So, it can not have more than three linear factors.

Let $f(x) = k(x-1)(x-2)(x-3)$. Then,

$x^3 - 6x^2 + 11x - 6 = k(x-1)(x-2)(x-3)$

Putting $x = 0$ on both sides, we get

$-6 = k(0-1)(0-2)(0-3)$

$\Rightarrow -6 = -6k \Rightarrow k = 1$

Putting $k = 1$ in $f(x) = k(x-1)(x-2)(x-3)$, we get

$f(x) = (x-1)(x-2)(x-3)$

Hence, $x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$

Example 3: Using factor theorem, factorize the polynomial $x^4 + x^3 - 7x^2 - x + 6$.
Solution:

Let $f(x) = x^4 + x^3 - 7x^2 - x + 6$

the factors of constant term in f(x) are ± 1, ± 2, ± 3 and ± 6

Now,

$f(1) = 1 + 1 - 7 - 1 + 6 = 8 - 8 = 0$

$\Rightarrow$ $(x - 1)$ is a factor of f(x)

$f(-1) = 1 - 1 - 7 + 1 + 6 = 8 - 8 = 0$

$\Rightarrow$ $x + 1$ is a factor of f(x)

$f(2) = 2^4 + 2^3 - 7 \times 2^2 - 2 + 6$

$= 16 + 8 - 28 - 2 + 6 = 0$

$\Rightarrow$ $x - 2$ is a factor of f(x)

$f(-2) = (-2)^4 + (-2)^3 - 7(-2)^2 - (-2) + 6$

$= 16 - 8 - 28 + 2 + 6 = -12 \neq 0$

$\Rightarrow$ $x + 2$ is not a factor of f(x)

$f(-3) = (-3)^4 + (-3)^3 - 7(-3)^2 - (-3) + 6$

$= 81 - 27 - 63 + 3 + 6 = 90 - 90 = 0$

$\Rightarrow$ $x + 3$ is a factor of f (x)

Since f(x) is a polynomial of degree 4.

So, it cannot have more than 4 linear factors

Thus, the factors of f (x) are $(x-1)$, $(x+1)$, $(x-2)$ and $(x+3)$.

Let $f(x) = k(x-1)(x+1)(x-2)(x+3)$

$\Rightarrow$ $x^4 + x^3 - 7x^2 - x + 6$

$= k(x-1)(x+1)(x-2)(x+3)$

Putting $x = 0$ on both sides, we get

$6 = k(-1)(1)(-2)(3) \quad \Rightarrow 6 = 6k \Rightarrow k = 1$

Substituting $k = 1$ in (i), we get

$x^4 + x^3 - 7x^2 - x + 6 = (x-1)(x+1)(x-2)(x+3)$

Example 4: Factorize, $2x^4 + x^3 - 14x^2 - 19x - 6$

Solution:

Let $f(x) = 2x^4 + x^3 - 14x^2 - 19x - 6$ be the given polynomial.

The factors of the constant term – 6 are ±1, ±2, ±3 and ±6, we have,

$f(-1) = 2(-1)^4 + (-1)^3 - 14(-1)^2 - 19(-1) - 6$

$\quad = 2 - 1 - 14 + 19 - 6 = 21 - 21 = 0$

and,

$f(-2) = 2(-2)^4 + (-2)^3 - 14(-2)^2 - 19(-2) - 6$

$\quad = 32 - 8 - 56 + 38 - 6 = 0$

So, $x + 1$ and $x + 2$ are factors of $f(x)$.

$\Rightarrow$ $(x + 1)(x + 2)$ is also a factor of $f(x)$

$\Rightarrow$ $x^2 + 3x + 2$ is a factor of $f(x)$

Now, we divide

$f(x) = 2x^4 + x^3 - 14x^2 - 19x - 6$ by

$x^2 + 3x + 2$ to get the other factors.

$$
\begin{array}{r|l}
 & 2x^2 - 5x - 3 \\
\hline
x^2 + 3x + 2 & 2x^4 + x^3 - 14x^2 - 19x - 6 \\
 & 2x^4 + 6x^3 + 4x^2 \\
 & - \quad - \quad - \\
\hline
 & -5x^3 - 18x^2 - 19x - 6 \\
 & -5x^3 - 15x^2 - 10x \\
\hline
 & + \quad + \quad + \\
 & -3x^2 - 9x - 6 \\
 & -3x^2 - 9x - 6 \\
\hline
 & + \quad + \quad + \\
 & 0
\end{array}
$$

$\therefore \quad 2x^4 + x^3 - 14x^2 - 19x - 6$

$\quad = (x^2 + 3x + 2)(2x^2 - 5x - 3)$

$\quad = (x + 1)(x + 2)(2x^2 - 5x - 3)$

Now $2x^2 - 5x - 3 \quad = \; 2x^2 - 6x + x - 3$

$\quad = 2x(x - 3) + 1(x - 3)$

$\quad = (x - 3)(2x + 1)$

Hence, $2x^4 + x^3 - 14x^2 - 19x - 6$

$\quad = (x + 1)(x + 2)(x - 3)(2x + 1)$

Example 5: Factorize, $9z^3 - 27z^2 - 100z + 300$, if it is given that $(3z+10)$ is a factor of it.

Solution:

Let us divide $9z^3 - 27z^2 - 100z + 300$ by $3z + 10$ to get the other factors

$$
\begin{array}{r|l}
 & 3z^2 - 19z + 30 \\
\hline
3z + 10 & 9z^3 - 27z^2 - 100z + 300 \\
 & 9z^3 + 30z^2 \\
 & - \quad - \\
\hline
 & \quad - 57z^2 - 100z + 300 \\
 & \quad - 57z^2 - 190z \\
 & \quad + \qquad + \\
\hline
 & \qquad 90z + 300 \\
 & \qquad 90z + 300 \\
 & \qquad - \quad - \\
\hline
 & \qquad 0
\end{array}
$$

$$\therefore \quad 9z^3 - 27z^2 - 100z + 300$$

$$= (3z + 10)(3z^2 - 19z + 30)$$

$$= (3z + 10)(3z^2 - 10z - 9z + 30)$$

$$= (3z + 10)\{(3z^2 - 10z) - (9z - 30)\}$$

$$= (3z + 10)\{z(3z - 10) - 3(3z - 10)\}$$

$$= (3z + 10)(3z - 10)(z - 3)$$

Hence, $9z^3 - 27z^2 - 100z + 300$

$$= (3z + 10)(3z - 10)(z - 3)$$

Example 6: Simplify

$$\frac{4x-2}{x^2-x-2} + \frac{3}{2x^2-7x+6} - \frac{8x+3}{2x^2-x-3}$$

Solution:

$$\frac{2(2x-1)}{(x-2)(x+1)} + \frac{3}{(2x-3)(x-2)} - \frac{8x+3}{(2x-3)(x+1)}$$

The L.C.M. of the factors in the denominator is $(x - 2)(x + 1)(2x - 3)$

The given expression can be reduced to

$$\frac{2(2x-1)(2x-3) + 3(x+1) - (8x+3)(x-2)}{(x-2)(x+1)(2x-3)}$$

$$= \frac{2(4x^2 - 8x + 3) + 3(x+1) - (8x^2 - 13x - 6)}{(x-2)(x+1)(2x-3)}$$

$$= \frac{15}{(x-2)(x+1)(2x-3)}$$

Example 7: Establish the identity
$$\frac{6x^2+11x-8}{3x-2} = (2x+5) + \frac{2}{3x-2}$$
Solution:

$$
\begin{array}{l}
3x - 2 \;)\; 6x^2 + 11x - 8 \;(\; 2x + 5 \\
\quad\quad 6x^2 - 4x \\
\hline
\quad\quad\quad 15x - 8 \\
\quad\quad\quad 15x - 10 \\
\hline
\quad\quad\quad\quad 2
\end{array}
$$

$$\therefore \quad \frac{6x^2+11x-8}{3x-2} = (2x+5) + \frac{2}{3x-2}$$

How Do You Use The Factor Theorem

Factor Theorem

Theorem: If p(x) is a polynomial of degree $n \geq 1$ and a is any real number, then
(i) x − a is a factor of p(x), if p(a) = 0, and (ii) p(a) = 0, if x − a is a factor of p(x).

Proof: By the Remainder Theorem,
p(x) = (x − a) q(x) + p(a).
(i) If p(a) = 0, then p(x) = (x − a) q(x),
which shows that x − a is a factor of p(x).
(ii) Since x − a is a factor of p(x),
p(x) = (x − a) g(x) for same polynomial g(x).
In this case, p(a) = (a − a) g(a) = 0.

To use factor theorem

- **Step 1:** (x + a) is factor of a polynomial p(x) if p(−a) = 0.
- **Step 2:** (ax − b) is a factor of a polynomial p(x) if p(b/a) = 0
- **Step 3:** ax + b is a factor of a polynomial p(x) if p(−b/a) = 0.
- **Step 4:** (x − a) (x − b) is a factor of a polynomial p(x) if p(a) = 0 and p(b) = 0.

Factor Theorem Example Problems With Solutions

Example 1: Examine whether x + 2 is a factor of $x^3 + 3x^2 + 5x + 6$ and of 2x + 4.
Solution: The zero of x + 2 is −2.
Let $p(x) = x^3 + 3x^2 + 5x + 6$ and s(x) = 2x + 4
Then, $p(-2) = (-2)^3 + 3(-2)^2 + 5(-2) + 6$
= −8 + 12 − 10 + 6
= 0
So, by the Factor Theorem, x + 2 is a factor of $x^3 + 3x^2 + 5x + 6$.
Again, s(−2) = 2(−2) + 4 = 0
So, x + 2 is a factor of 2x + 4.

Example 2: Use the factor theorem to determine whether x − 1 is a factor of
(a) $x^3 + 8x^2 - 7x - 2$
(b) $2x^3 + 5x^2 - 7$
(c) $8x^4 + 12x^3 - 18x + 14$
Solution:

(a) Let $p(x) = x^3 + 8x^2 - 7x - 2$

By using factor theorem, $(x-1)$ is a factor of $p(x)$ only when $p(1) = 0$

$p(1) = (1)^3 + 8(1)^2 - 7(1) - 2$

$= 1 + 8 - 7 - 2$

$= 9 - 9 = 0$

Hence $(x - 1)$ is a factor of $p(x)$.

(b) Let $p(x) = 2\sqrt{2}\,x^3 + 5\sqrt{2}\,x^2 - 7\sqrt{2}$

By using factor theorem, $(x-1)$ is a factor of $p(x)$ only when $p(1) = 0$.

$p(1) = 2\sqrt{2}\,(1)^3 + 5\sqrt{2}\,(1)^2 - 7\sqrt{2}$

$= 2\sqrt{2} + 5\sqrt{2} - 7\sqrt{2}$

$= 7\sqrt{2} - 7\sqrt{2} = 0$

Hence $(x-1)$ is a factor of $p(x)$

(c) Let $p(x) = 8x^4 + 12x^3 - 18x + 14$

By using factor theorem, $(x-1)$ is a factor of $p(x)$ only when $p(1) = 0$

$p(1) = 8(1)^4 + 12(1)^3 - 18(1) + 14$

$= 8 + 12 - 18 + 14$

$= 34 - 18 = 16 \neq 0.$

Hence $(x-1)$ is not a factor of $p(x)$.

Example 3: Factorize each of the following expression, given that $x^3 + 13x^2 + 32x + 20$. $(x+2)$ is a factor.

Solution:

Let $p(x) = x^3 + 13x^2 + 32x + 20$

$= (x+2)$ is a factor of $p(x)$

$p(x) = (x+2)(x^2 + 11x + 10)$

$= (x + 2)(x^2 + 10x + x + 10)$

$= (x+2)(x + 10)(x + 1)$

Example 4: Factorize x3 – 23 x^2 + 142 x – 120

Solution:

Let $p(x) = x^3 - 23x^2 + 142x - 120$

Constant term, p(x) is -120

$\pm1, \pm2, \pm3, \pm4, \pm5, \pm10, \pm12$ ....±120

$P(1) = 1 - 23 + 142 - 120 = 0$

$\Rightarrow$ $x - 1$ is a factor of p(x). We find the other factor by dividing p(x) by $(x - 1)$

$p(x) = (x - 1)(x^2 - 22x + 120)$

$= (x - 1)(x^2 - 10x - 12x + 120)$

$= (x - 1)[x(x - 10) - 12(x - 10))$

$= (x - 1)(x - 10)(x - 12)$

Example 5: Show that $(x - 3)$ is a factor of the polynomial $x^3 - 3x^2 + 4x - 12$

Solution:

Let $p(x) = x^3 - 3x^2 + 4x - 12$ be the given polynomial.

By factor theorem, $(x - a)$ is a factor of a polynomial p(x) if $p(a) = 0$.

Therefore, in order to prove that $x - 3$ is a factor of p(x), it is sufficient to show that $p(3) = 0$. Now,

$p(x) = x^3 - 3x^2 + 4x - 12$

$\Rightarrow$ $p(3) = 3^3 - 3 \times 3^2 + 4 \times 3 - 12$

$= 27 - 27 + 12 - 12 = 0$

Hence, $(x - 3)$ is a factor of

$p(x) = x^3 - 3x^2 + 4x - 12$.

Example 6: Show that $(x - 1)$ is a factor of $x^{10} - 1$ and also of $x^{11} - 1$.

Solution:

Let $f(x) = x^{10} - 1$ and $g(x) = x^{11} - 1$.

In order to prove that $(x - 1)$ is a factor of both f(x) and g(x), it is sufficient to show that $f(1) = 0$ and $g(1) = 0$.

Now, $f(x) = x^{10} - 1$ and $g(x) = x^{11} - 1$

$\Rightarrow$ $f(1) = 1^{10} - 1 = 0$ and $g(1) = 1^{11} - 1 = 0$

$\Rightarrow$ $(x - 1)$ is a factor of both f(x) and g(x)

Example 7: Show that $x + 1$ and $2x - 3$ are factors of $2x^3 - 9x^2 + x + 12$.

Solution:

Let $p(x) = 2x^3 - 9x^2 + x + 12$ be the given polynomial.

In order to prove that $x + 1$ and $2x - 3$ are factors of $p(x)$,

it is sufficient to show that $p(-1)$ and $p(3/2)$ both are equal to zero.

Now, $p(x) = 2x^3 - 9x^2 + x + 12$

$\Rightarrow \; p(-1) = 2 \times (-1)^3 - 9 \times (-1)^2 + (-1) + 12$

and, $p\left(\frac{3}{2}\right) = 2 \times \left(\frac{3}{2}\right)^3 - 9 \times \left(\frac{3}{2}\right)^3 + \frac{3}{2} + 12$

$\Rightarrow \; p(-1) = -2 - 9 - 1 + 12$ and

$$p\left(\frac{3}{2}\right) = \frac{54}{8} - \frac{81}{4} + \frac{3}{2} + 12$$

$\Rightarrow \; p(-1) = -12 + 12$ and

$$p\left(\frac{3}{2}\right) = \frac{54 - 162 + 12 + 96}{8}$$

$\Rightarrow \; p(-1) = 0$ and $p\left(\frac{3}{2}\right) = 0$

Hence, $(x + 1)$ and $(3x - 2)$ are factors of the given polynomial.

Example 8: Find the value of k, if $x + 3$ is a factor of $3x^2 + kx + 6$.

Solution:

Let $p(x) = 3x^2 + kx + 6$ be the given polynomial. Then, $(x + 3)$ is a factor of $p(x)$

$\Rightarrow \; p(-3) = 0$

$\Rightarrow \; 3(-3)^2 + k \times (-3) + 6 = 0$

$\Rightarrow \; 27 - 3k + 6 = 0$

$\Rightarrow \; 33 - 3k = 0 \;\Rightarrow\; k = 11$

Hence, $x + 3$ is a factor of $3x^2 + kx + 6$ if $k = 11$.

Example 9: If $ax^3 + bx^2 + x - 6$ has $x + 2$ as a factor and leaves a remainder 4 when divided by $(x - 2)$, find the values of a and b.

Solution:

Let $p(x) = ax^3 + bx^2 + x - 6$ be the given polynomial. Then, $(x + 2)$ is a factor of $p(x)$

$\Rightarrow\ p(-2) = 0 \qquad [x + 2 = 0 \Rightarrow x = -2]$

$\Rightarrow\ a(-2)^3 + b(-2)^2 + (-2) - 6 = 0$

$\Rightarrow\ -8a + 4b - 2 - 6 = 0 \ \Rightarrow\ -8a + 4b = 8$

$\Rightarrow\ -2a + b = 2 \qquad ....(i)$

It is given that $p(x)$ leaves the remainder 4 when it is divided by $(x - 2)$. Therefore,

$p(2) = 4 \qquad [x - 2 = 0 \ \Rightarrow\ x = 2]$

$\Rightarrow\ a(2)^3 + b(2)^2 + 2 - 6 = 4$

$\Rightarrow\ 8a + 4b - 4 = 4 \qquad \Rightarrow\ 8a + 4b = 8$

$\Rightarrow\ 2a + b = 2 \qquad ....(ii)$

Adding (i) and (ii), we get

$2b = 4 \qquad \Rightarrow\ b = 2$

Putting $b = 2$ in (i), we get

$-2a + 2 = 2 \ \Rightarrow\ -2a = 0 \Rightarrow\ a = 0.$

Hence, $a = 0$ and $b = 2$.

Example 10: If both $x - 2$ and $x - 1/2$ are factors of $px^2 + 5x + r$, show that $p = r$.

Solution:

Let $f(x) = px^2 + 5x + r$ be the given polynomial.

Since $x - 2$ and $x - \frac{1}{2}$ are factors of $f(x)$.

Therefore, $\qquad f(2) = 0$ and $f\left(\frac{1}{2}\right) = 0$

$\left[\because\ x - 2 = 0 \Rightarrow x = 2 \text{ and } x - \frac{1}{2} = 0 \Rightarrow x = \frac{1}{2}\right]$

$\Rightarrow\ p \times 2^2 + 5 \times 2 + r = 0$ and $p\left(\frac{1}{2}\right)^2 + 5 \times \frac{1}{2} + r = 0$

$\Rightarrow\ 4p + 10 + r = 0$ and $\frac{p}{4} + \frac{5}{2} + r = 0$

$\Rightarrow\ 4p + r = -10$ and $\frac{p + 4r + 10}{4} = 0$

$\Rightarrow\ 4p + r = -10$ and $p + 4r + 10 = 0$

$\Rightarrow\ 4p + r = -10$ and $p + 4r = -10$

$\Rightarrow\ 4p + r = p + 4r \qquad [\because$ RHS of the two equations are equal$]$

$\Rightarrow\ 3p = 3r \qquad \Rightarrow\ p = r$

Example 11: If $x^2 - 1$ is a factor of $ax^4 + bx^3 + cx^2 + dx + e$, show that $a + c + e = b + d = 0$.
Solution:

Let $p(x) = ax^4 + bx^3 + cx^2 + dx + e$ be the given polynomial.

Then, $(x^2 - 1)$ is a factor of $p(x)$

$\Rightarrow$ $(x - 1)(x + 1)$ is a factor of $p(x)$

$\Rightarrow$ $(x - 1)$ and $(x + 1)$ are factors of $p(x)$

$\Rightarrow$ $p(1) = 0$ and $p(-1) = 0$

$[x - 1 = 0 \Rightarrow x = 1$ and $x + 1 = 0 \Rightarrow x = -1]$

$\Rightarrow a + b + c + d + e = 0$ and $a - b + c - d + e = 0$

Adding and subtracting these two equations, we get $2(a + c + e) = 0$ and $2(b + d) = 0$

$\Rightarrow$ $a + c + e = 0$ and $b + d = 0$

$\Rightarrow$ $a + c + e = b + d = 0$

Example 12: Using factor theorem, show that $a - b$, $b - c$ and $c - a$ are the factors of $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$.
Solution:

By factor theorem, $a - b$ will be a factor of the given expression if it vanishes

by substituting $a = b$ in it.

Substituting $a = b$ in the given expression, we have

$a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$

$= b(b^2 - c^2) + b(c^2 - b^2) + c(b^2 - b^2)$

$= b^3 - bc^2 + bc^2 - b^3 + c(b^2 - b^2) = 0$

$\therefore$ $(a - b)$ is a factor of

$a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$.

Similarly,

we can show that $(b - c)$ and $(c - a)$ are also factors of the given expression.

Hence, $(a - b)$, $(b - c)$ and $(c - a)$ are also factors of the given expression.

How To Factorise A Polynomial By Splitting The Middle Term

Factorise A Polynomial By Splitting The Middle Term Example Problems With Solutions

Type I: Factorization of Quadratic polynomials of the form $x^2 + bx + c$.
(i) In order to factorize $x^2 + bx + c$ we have to find numbers p and q such that $p + q = b$ and $pq = c$.
(ii) After finding p and q, we split the middle term in the quadratic as $px + qx$ and get desired factors by grouping the terms.

Example 1: Factorize each of the following expressions:
(i) $x^2 + 6x + 8$ (ii) $x^2 + 4x - 21$
Solution:
(i) In order to factorize $x^2 + 6x + 8$, we find two numbers p and q such that $p + q = 6$ and $pq = 8$.
Clearly, $2 + 4 = 6$ and $2 \times 4 = 8$.

We know split the middle term 6x in the given quadratic as 2x + 4x, so that
$x^2 + 6x + 8 = x^2 + 2x + 4x + 8$
$= (x^2 + 2x) + (4x + 8)$
$= x (x + 2) + 4 (x+ 2)$
$= (x + 2) (x + 4)$
(ii) In order to factorize $x^2 + 4x - 21$, we have to find two numbers p and q such that
$p + q = 4$ and $pq = - 21$
Clearly, $7 + (- 3) = 4$ and $7 \times - 3 = - 21$
We now split the middle term 4x of
$x^2 + 4x - 21$ as $7x - 3x$, so that
$x^2 + 4x - 21 = x^2 + 7x - 3 x - 21$
$= (x^2 + 7x) - (3x + 21)$
$= x (x + 7) - 3 (x + 7) = (x + 7) (x - 3)$

Example 2: Factorize each of the following quadratic polynomials: $x^2 - 21x + 108$
Solution: In order to factorize $x^2 - 21x + 108$,
we have to find two numbers such that their sum is – 21 and the product 108.
Clearly, $- 21 = - 12 - 9$ and $- 12 \times - 9 = 108$
$x^2 - 21 x + 108 = x^2 - 12 x - 9x + 108$
$= (x^2 - 12 x) - (9x - 108)$
$= x(x - 12) - 9 (x - 12) = (x-12) (x - 9)$

Example 3: Factorize the following by splitting the middle term : $x^2 + 3 \sqrt{3} x + 6$
Solution: In order to factorize $x^2 + 3 \sqrt{3} x + 6$, we have to find two numbers p and q such that

$p + q = 3\sqrt{3}$ and $pq = 6$

Clearly, $2\sqrt{3} + \sqrt{3} = 3\sqrt{3}$ and $2\sqrt{3} \times \sqrt{3} = 6$

So, we write the middle term $3\sqrt{3}\,x$ as
$2\sqrt{3}\,x + \sqrt{3}\,x$, so that

$$x^2 + 3\sqrt{3}\,x + 6$$

$$= x^2 + 2\sqrt{3}\,x + \sqrt{3}\,x + 6$$

$$= (x^2 + 2\sqrt{3}\,x) + (\sqrt{3}\,x + 6)$$

$$= (x^2 + 2\sqrt{3}\,x) + (\sqrt{3}\,x + 2\sqrt{3} \times \sqrt{3})$$

$$= x (x + 2\sqrt{3}) + \sqrt{3} (x + 2\sqrt{3})$$

$$= (x + 2\sqrt{3}) (x + \sqrt{3})$$

Type II: Factorization of polynomials reducible to the form $x^2 + bx + c$.

Example 4: Factorize $(a^2 - 2a)^2 - 23(a^2 - 2a) + 120$.

Solution:

Let $a^2 - 2a = x$. Then,

$(a^2 - 2a)^2 - 23(a^2 - 2a) + 120$

$= x^2 - 23x + 120$

Now, $x^2 - 23x + 120 = x^2 - 15x - 8x + 120$

$= (x^2 - 15x) - (8x - 120)$

$= x(x - 15) - 8(x - 15)$

$= (x - 15)(x - 8)$

Replacing x by $a^2 - 2a$ on both sides, we get

$(a^2 - 2a)^2 - 23(a^2 - 2a) + 120$

$= (a^2 - 2a - 15)(a^2 - 2a - 8)$

$= (a^2 - 5a + 3a - 15)(a^2 - 4a + 2a - 8)$

$= \{(a(a - 5) + 3(a - 5)\}\{a(a - 4) + 2(a - 4)\}$

$= \{(a - 5)(a + 3)\}\{(a - 4)(a + 2)\}$

$= (a - 5)(a + 3)(a - 4)(a + 2)$

Example 5: Factorize the following by splitting the middle term $x^4 - 5x^2 + 4$

Solution:

Let $x^2 = y$. Then, $x^4 - 5x^2 + 4$

$= y^2 - 5y + 4$

Now, $y^2 - 5y + 4$

$= y^2 - 4y - y + 4$

$= (y^2 - 4y) - (y - 4)$

$= y(y - 4) - (y - 4)$

$= (y - 4)(y - 1)$

Replacing y by x^2 on both sides, we get

$x^4 - 5x^2 + 4 = (x^2 - 4)(x^2 - 1)$

$= (x^2 - 2^2)(x^2 - 1^2) = (x - 2)(x + 2)(x - 1)(x + 1)$

Example 6: Factorize $(x^2 - 4x)(x^2 - 4x - 1) - 20$

Solution: The given expression is

$(x^2 - 4x)(x^2 - 4x - 1) - 20 = (x^2 - 4x)^2 - (x^2 - 4x) - 20$

Let $x^2 - 4x = y$. Then,

$(x^2 - 4x)^2 - (x^2 - 4x) - 20 = y^2 - y - 20$

Now, $y^2 - y - 20$

$= y^2 - 5y + 4y - 20$

$= (y^2 - 5y) + (4y - 20)$

$= y(y - 5) + 4(y - 5)$

$= (y - 5)(y + 4)$

Thus, $y^2 - y - 20 = (y - 5)(y + 4)$

Replacing y by $x^2 - 4x$ on both sides, we get

$(x^2 - 4x)^2 - (x^2 - 4x) - 20$

$= (x^2 - 4x - 5)(x^2 - 4x + 4)$

$= (x^2 - 5x + x - 5)(x^2 - 2 \times x \times 2 + 2^2)$

$= \{x(x - 5) + (x - 5)\}(x - 2)^2$

$= (x - 5)(x + 1)(x - 2)^2$

Type III: Factorization of Expressions which are not quadratic but can factorized by splitting the middle term.

Example 7: If $x^2 + px + q = (x + a)(x + b)$, then factorize $x^2 + pxy + qy^2$.
Solution: We have,

$x^2 + px + q = (x + a)(x + b)$

$\Rightarrow \quad x^2 + px + q = x^2 + x(a + b) + ab$

On equating the coefficients of like powers of x, we get

$p = a + b$ and $q = ab$

$\therefore \quad x^2 + pxy + qy^2 = x^2 + (a + b)xy + aby^2$

$= (x^2 + axy) + (bxy + aby^2)$

$= x(x + ay) + by(x + ay)$

$= (x + ay)(x + by)$

Example 8: Factorize the following expression $x^2y^2 - xy - 72$
Solution:

In order to factorize $x^2y^2 - xy - 72$,

we have to find two numbers p and q such that

$p + q = -1$ and $pq = -72$

clearly, $-9 + 8 = -1$ and $-9 \times 8 = -72$.

So, we write the middle term $-xy$ of

$x^2y^2 - xy - 72$ as $-9xy + 8xy$, so that

$x^2y^2 - xy - 72 = x^2y^2 - 9xy + 8xy - 72$

$= (x^2y^2 - 9xy) + (8xy - 72)$

$= xy(xy - 9) + 8(xy - 9)$

$= (xy - 9)(xy + 8)$

Factorization Of Polynomials Of The Form $ax^2 + bx + c$, $a \neq 0, 1$

Type I: Factorization of quadratic polynomials of the form $ax^2 + bx + c$, a 0, 1

(i) In order to factorize $ax^2 + bx + c$. We find numbers l and m such that $l + m = b$ and $lm = ac$

(ii) After finding l and m, we split the middle term bx as lx + mx and get the desired factors by grouping the terms.

Example 9: Factorize the following expression

$6x^2 - 5x - 6$

Solution: The given expression is of the form $ax^2 + bx + c$, where, $a = 6$, $b = -5$ and $c = -6$.
In order to factorize the given expression, we have to find two numbers l and m such that
$l + m = b$ = i.e., $l + m = -5$
and $lm = ac$ i.e. $lm = 6 \times -6 = -36$
i.e., we have to find two factors of -36
such that their sum is -5. Clearly,
$-9 + 4 = -5$ and $-9 \times 4 = -36$
$l = -9$ and $m = 4$
Now, we split the middle term $-5x$ of
$x^2 - 5x - 6$ as $-9x + 4x$, so that
$6x^2 - 5x - 6 = 6x^2 - 9x + 4x - 6$
$= (6x^2 - 9x) + (4x - 6)$
$= 3x(2x - 3) + 2(2x - 3) = (2x - 3)(3x + 2)$

Example 10: Factorize each of the following expressions:
(i) $\sqrt{3}x^2 + 11x + 6\sqrt{3}$
(ii) $4\sqrt{3}x^2 + 5x - 2\sqrt{3}$
(iii) $7\sqrt{2}x^2 - 10x - 4\sqrt{2}$

Solution: (i) The given quadratic expression is of the form $ax^2 + bx + c$,
where $a = \sqrt{3}$, $b = 11$ and $c = 6\sqrt{3}$.
In order to factorize it, we have to find two numbers l and m such that

$l + m = b = 11$ and $lm = ac = \sqrt{3} \times 6\sqrt{3} = 18$

Clearly, $9 + 2 = 11$ and $9 \times 2 = 18$

$\therefore$ $l = 9$ and $m = 2$

Now, $\sqrt{3}\,x^2 + 11\,x + 6\sqrt{3}$

$= \sqrt{3}\,x^2 + 9x + 2x + 6\sqrt{3}$

$= (\sqrt{3}\,x^2 + 9x) + (2x + 6\sqrt{3})$

$= (\sqrt{3}\,x^2 + 3\sqrt{3} \times \sqrt{3}\,x) + (2x + 6\sqrt{3})$

$= \sqrt{3}\,x\,(x + 3\sqrt{3}) + 2(x + 3\sqrt{3})$

$= (\sqrt{3}\,x + 2)\,(x + 3\sqrt{3})$.

Hence, $\sqrt{3}\,x^2 + 11\,x + 6\sqrt{3}$

$= (\sqrt{3}\,x + 2)\,(x + 3\sqrt{3})$

(ii) Here, $a = 4\sqrt{3}$, $b = 5$ and $c = -2\sqrt{3}$

In order to factorize $4\sqrt{3}\,x^2 + 5x - 2\sqrt{3}$,

we have to find two numbers l and m such that

$l + m = b = 5$ and $lm = ac$

$= 4\sqrt{3} \times -2\sqrt{3} = -24$

Clearly, $8 + (-3) = 5$ and $8 \times -3 = -24$

$\therefore$ $l = 8$ and $m = -3$

Now, $4\sqrt{3}\,x^2 + 5x - 2\sqrt{3}$

$= 4\sqrt{3}\,x^2 + 8x - 3x - 2\sqrt{3}$

$= (4\sqrt{3}\,x^2 + 8x) - (3x + 2\sqrt{3})$

$= 4x\,(\sqrt{3}\,x + 2) - \sqrt{3}\,(\sqrt{3}\,x + 2)$

$= (\sqrt{3}\,x + 2)\,(4x - \sqrt{3})$

(iii) The given quadratic polynomial is
$7\sqrt{2}\,x^2 - 10x - 4\sqrt{2}$.

Clearly, it is of the form $ax^2 + bx + c$, where
$a = 7\sqrt{2}$, $b = -10$ and $c = -4\sqrt{2}$.

In order to factorize $7\sqrt{2}\,x^2 - 10x - 4\sqrt{2}$,

we have to find two numbers l and m such that

$l + m = b = -10$ and

$l\,m = ac = 7\sqrt{2} \times -4\sqrt{2} = -56$

Clearly, $-14+4 = -10$ and $-14 \times 4 = -56$

$\therefore \quad l = -14$ and $m = 4$

Now, we split the middle term $-10x$ of
$7\sqrt{2}\,x^2 - 10x - 4\sqrt{2}$ as $-14x + 4x$ so that

$7\sqrt{2}\,x^2 - 10x - 4\sqrt{2}$

$$7\sqrt{2}\,x^2 - 10x - 4\sqrt{2}$$

$$= 7\sqrt{2}\,x^2 - 14x + 4x - 4\sqrt{2}$$

$$= (7\sqrt{2}\,x^2 - 14x) + (4x - 4\sqrt{2})$$

$$= (7\sqrt{2}\,x^2 - 7\sqrt{2} \times \sqrt{2}\,x) + (4x - 4\sqrt{2})$$

$$= 7\sqrt{2}\,x\,(x - \sqrt{2}) + 4(x - \sqrt{2})$$

$$= (x - \sqrt{2})(7\sqrt{2}\,x + 4)$$

Example 11: Factorize the following by splitting the middle term
$1/3\,x^2 - 2x - 9$

Solution:

In order to factorize $\frac{1}{3}x^2 - 2x - 9$,

we have to find to number l and m such that

$l + m = -2$ and $\quad lm = \frac{1}{3} \times -9 = -3$

Clearly, $-3 + 1 = -2$ and $-3 \times 1 = -3$

So, we write the middle term $-2x$ as

$-3x + x$, so that

$$\frac{1}{3}x^2 - 2x - 9 \quad = \frac{1}{3}x^2 - 3x + x - 9$$

$$= (\frac{1}{3}x^2 - 3x) + (x - 9) = (\frac{1}{3}x^2 - \frac{9}{3}x) + (x - 9)$$

$$= (x-9)\left(\frac{1}{3}x + 1\right)$$

Type II: Factorization of trinomial expressions which are not quadratic but can be factorized by splitting the middle term.

Example 12: Factorize the following trinomial by splitting the middle term
$8a^3 - 2a^2b - 15\,ab^2$

Solution: Here $a^3 \times ab^2 = (a^2b)^2$ i.e., the product of the variables in first and last term is same as the square of the variables in the middle term. So, in order to factorize the given trinomial, we split the middle term
$-2a^2b$ as $-12a^2b + 10\,a^2b$, so that
$8a^3 - 2a^2b - 15\,ab^2$
$= 8a^3 - 12a^2b + 10\,a^2b - 15\,ab^2$
$= 4a^2(2a - 3b) + 5\,ab\,(2a - 3b)$
$= (2a - 3b)\,(4a^2 + 5ab)$
$= (2a - 3b)\,a\,(4a + 5b)$
$= a\,(2a - 3\,b)\,(4a + 5b)$

Type III : Factorization of trinomial expressions reducible to quadratic expressions.

Example 13: Factorize each of the following expressions by splitting the middle term :
(i) $9(x - 2y)^2 - 4(x - 2y) - 13$
(ii) $2(x + y)^2 - 9(x + y) - 5$
(iii) $8(a + 1)^2 + 2(a + 1)\,(b + 2) - 15(b + 2)^2$

Solution: (i) The given expression is $9(x - 2y)^2 - 4(x - 2y) - 13$.
Putting $x - 2y = a$, we get
$9(x - 2y)^2 - 4(x - 2y) - 13 = 9a^2 - 4a - 13$
Now, $9a^2 - 4a - 13 = 9a^2 - 13a + 9a - 13$
$= (9a^2 - 13a) + (9a - 13)$
$= a(9a - 13) + (9a - 13)$
$= (a + 1)\,(9a - 13)$
Replacing a by $x - 2y$ on both sides, we get
$9(x - 2y)^2 - 4(x - 2y) - 13$
$= (x - 2y + 1)\,\{9(x - 2y) - 13\}$
$= (x - 2y + 1)\,(9x - 18y - 13)$

(ii) The given expression is
$2(x + y)^2 - 9(x + y) - 5$
Replacing x + y by a in the given expression, we have
$2(x + y)^2 - 9(x + y) - 5 = 2a^2 - 9a - 5$
Now, $2a^2 - 9a - 5 = 2a^2 - 10a + a - 5$
$= (2a^2 - 10a) + (a - 5)$
$= 2a(a - 5) + (a - 5) = (a - 5)(2a + 1)$
Replacing a by x + y on both sides, we get
$2(x + y)^2 - 9(x + y) - 5$
$= (x + y - 5)\{2(x + y) + 1\}$
$= (x + y - 5)(2x + 2y + 1)$.
(iii) The given trinomial is
$8(a + 1)^2 + 2(a + 1)(b + 2) - 15(b + 2)^2$
Putting a + 1 = x and b + 2 = y, we have
$8(a + 1)^2 + 2(a + 1)(b + 2) - 15(b + 2)^2$
$= 8x^2 + 2xy - 15y^2$
$= 8x^2 + 12xy - 10xy - 15y^2$
$= 4x(2x + 3y) - 5y(2x + 3y)$
$= (2x + 3y)(4x - 5y)$
Replacing x by a + 1 and y by b + 2, we get
$8(a + 1)^2 + 2(a + 1)(b + 2) - 15(b + 2)^2$
$= \{2(a + 1) + 3(b + 2)\}\{4(a + 1) - 5(b + 2)\}$
$= (2a + 3b + 8)(4a - 5b - 6)$

Review Factoring Polynomials

This lesson will review the process of factoring, which is used when solving equations and simplifying rational expressions.

To factor polynomial expressions, there are several approaches that can be used to simplify the process. While all of these approaches are not used for each problem, it is best to examine your expression for the possible existence of these situations. Ask yourself the following questions:

Are there Common Factors?

Factor out the Greatest Common Factor (GCF) of the expression, if one exists. This will make it simpler to factor the remaining expression.

Take care NOT to drop this GCF, as it is still part of the expression's answer.

Example 1:	Example 2:
$2x^2y - 6xy^2$	$9x^2 - 27x$
$2xy(x - 3y)$	$9x(x - 3)$

Does the expression have only 2 terms?

If it does, is the expression a DIFFERENCE of PERFECT SQUARES?
If so, you should be able to write the expression as a product of the sum and difference of the square roots of the terms.

Sometimes, as in Example 2 below, it is best to write the terms in square notation so
you can see what the terms will be in factored form. Be sure to use parentheses!

This process is also called Factoring with DOTS (Difference of Two Squares).

Example 1:

$$x^2 - 25$$
$$(x+5)(x-5)$$

Example 2:

$$16x^2 - 25y^2$$
$$(4x)^2 - (5y)^2$$
$$(4x+5y)(4x-5y)$$

Does the expression have exactly 3 terms?

If yes, then the expression may factor into the product of two binomials. One way to solve this type of problem is to use trial and error, keeping certain "hints" in mind.

Hints:
With the trinomial arranged in proper order (highest to lowest powers):
• if the leading coefficient is 1, you are looking for two numbers that multiply to the last term and add to the coefficient of the middle term.

• if the leading coefficient is not 1, you will have to look more carefully to find the answer. See Factoring Trinomials () – Set Up, Guess and Check Method and Factoring by Grouping Method.

For the examples below, use the hint above for factoring when the leading coefficient is 1, and the trial and error (guess and check) method when the leading coefficient is 2.

Always check your work by multiplying the binomials to see if your center term matches the original problem.

Example 1:

$$x^2 - 6x + 8$$

Factors:
x, x

Factors:
+8, +1   +2, +4
-8, -1   -2, -4

ANSWER
$$(x-2)(x-4)$$

Example 2:

$$2x^2 - x - 6$$

Factors:
2x, x

Factors:
+6, -1   +2, -3
-6, +1   -2, +3

ANSWER
$$(2x+3)(x-2)$$

Special trinomial: Perfect Square

Consider what happens when a binomial is squared:

where the center term is twice the product of a and b.

If you can recognize this pattern, it is very easy to factor a trinomial that is the perfect square of a binomial.

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+(-b))^2 = (a-b)^2 = a^2 - 2ab + b^2$$

where the center term is twice the product of a and b.

Does the first term have a Negative Coefficient?

If yes, then factor out the negative sign first, using the common factor method at the top of this page. Remember, if the leading term has a coefficient of (- 1), and there are
NO other common terms, then the GCF is = -1.

Example 1:	Example 2:
$-2x^2+18$	$-x^2+8x+20$
$-2(x^2-9)$	$-(x^2-8x-20)$
$-2(x+3)(x-3)$	$-(x-10)(x+2)$

Zeros Of A Polynomial Function

If for x = a, the value of the polynomial p(x) is 0 i.e., p(a) = 0; then x = a is a zero of the polynomial p(x).

For Example:
(i) For polynomial p(x) = x – 2; p(2) = 2 – 2 = 0
∴ x = 2 or simply 2 is a zero of the polynomial
p(x) = x – 2.
(ii) For the polynomial g(u) = u^2 – 5u + 6;
g(3) = $(3)^2$ – 5 × 3 + 6 = 9 – 15 + 6 = 0
∴ 3 is a zero of the polynomial g(u)
= u2 – 5u + 6.
Also, g(2) = $(2)^2$ – 5 × 2 + 6 = 4 – 10 + 6 = 0
∴ 2 is also a zero of the polynomial
g(u) = u^2 – 5u + 6
(a) Every linear polynomial has one and only one zero.
(b) A given polynomial may have more than one zeroes.
(c) If the degree of a polynomial is n; the largest number of zeroes it can have is also n.
For Example:
If the degree of a polynomial is 5, the polynomial can have at the most 5 zeroes; if the degree of a polynomial is 8; largest number of zeroes it can have is 8.
(d) A zero of a polynomial need not be 0.
For Example: If f(x) = x^2 – 4,
then f(2) = (2)2 – 4 = 4 – 4 = 0
Here, zero of the polynomial f(x) = x^2 – 4 is 2 which itself is not 0.
(e) 0 may be a zero of a polynomial.
For Example: If f(x) = x^2 – x,
then f(0) = 0^2 – 0 = 0
Here 0 is the zero of polynomial
f(x) = x^2 – x.

Zeros Of A Polynomial Function With Examples

Example 1: Verify whether the indicated numbers are zeroes of the polynomial corresponding to them in the following cases :
(i) p(x) = 3x + 1, x = $-\frac{1}{3}$
(ii) p(x) = (x + 1) (x – 2), x = – 1, 2
(iii) p(x) = x^2, x = 0
(iv) p(x) = lx + m, x = $-\frac{m}{l}$
(v) p(x) = 2x + 1, x = $\frac{1}{2}$
Sol.
(i) p(x) = 3x + 1
$\Rightarrow p\left(-\frac{1}{3}\right)=3\times-\frac{1}{3}+1=-1+1=0$

∴ x = $-\frac{1}{3}$ is a zero of p(x) = 3x + 1.
(ii) p(x) = (x + 1) (x − 2)
⇒ p(−1) = (−1 + 1) (−1 − 2) = 0 × −3 = 0
and, p(2) = (2 + 1) (2 − 2) = 3 × 0 = 0
∴ x = −1 and x = 2 are zeroes of the given polynomial.
(iii) p(x) = x^2
⇒ p(0) = 0^2 = 0
∴ x = 0 is a zero of the given polynomial
(iv) p(x) = lx + m
$\Rightarrow p\left(-\frac{m}{l}\right) = l\left(-\frac{m}{l}\right) + m$
= − m + m = 0
∴ x = $-\frac{m}{l}$ is a zero of the given polynomial.
(v) p(x) = 2x + 1
$\Rightarrow p\left(\frac{1}{2}\right) = 2 \times \frac{1}{2} + 1$
= 1 + 1 = 2 ≠ 0
∴ x = $\frac{1}{2}$ is not a zero of the given polynomial.

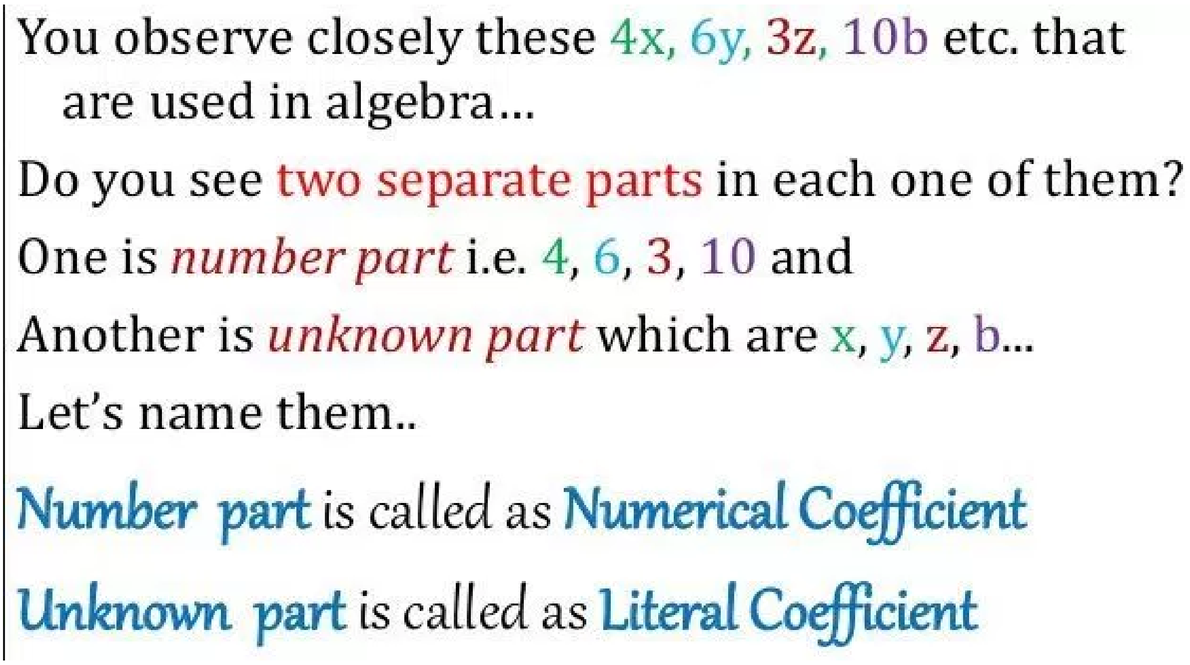
Example 2: Find the zero of the polynomial in each of the following cases :
(i) p(x) = x + 5
(ii) p(x) = 2x + 5
(iii) p(x) = 3x − 2
Sol.
To find the zero of a polynomial p(x) means to solve the polynomial equation p(x) = 0.
(i) For the zero of polynomial p(x) = x + 5
p(x) = 0 ⇒ x + 5 = 0 ⇒ x = −5
∴ x = −5 is a zero of the polynomial.
p(x) = x + 5.
(ii) p(x) = 0 ⇒ 2x + 5 = 0
⇒ 2x = −5 and x = $-\frac{5}{2}$
∴ $-\frac{5}{2}$ is a zero of p(x) = 2x + 5.
(iii) p(x) = 0 ⇒ 3x − 2 = 0
⇒ 3x = 2 and x = $\frac{2}{3}$.
∴ x = $\frac{2}{3}$ is zero of p(x) = 3x − 2.

Factors And Coefficients Of A Polynomial

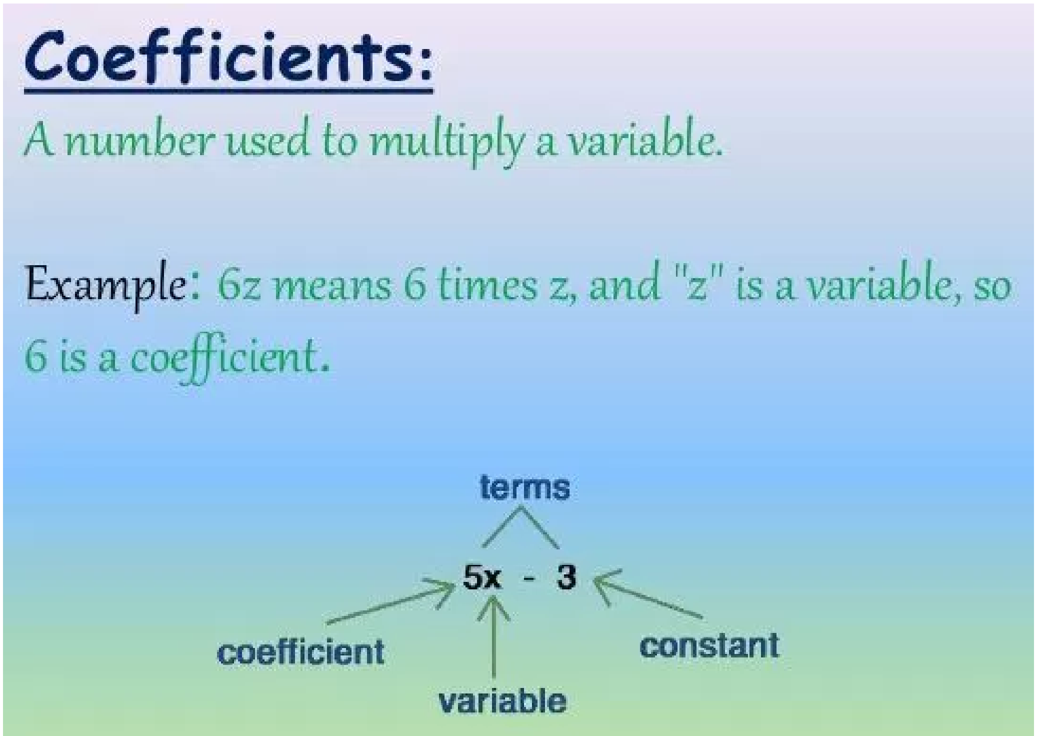
Factor:
When numbers (constants) and variables are multiplied to form a term, then each quantity multiplied is called a **factor** of the term. A constant factor is called a numerical factor while a variable factor is called a literal factor.

For Example:
(i) 7, x and 7x are factors of 7x, in which
7 is constant (numerical) factor and x is variable (literal) factor.
(ii) In $5x^2y$, the numerical factor is −5 and literal factors are : x, y, xy, x^2 and x^2y.

Coefficient:
Any factor of a term is called the **coefficient** of the product of the remaining factors.

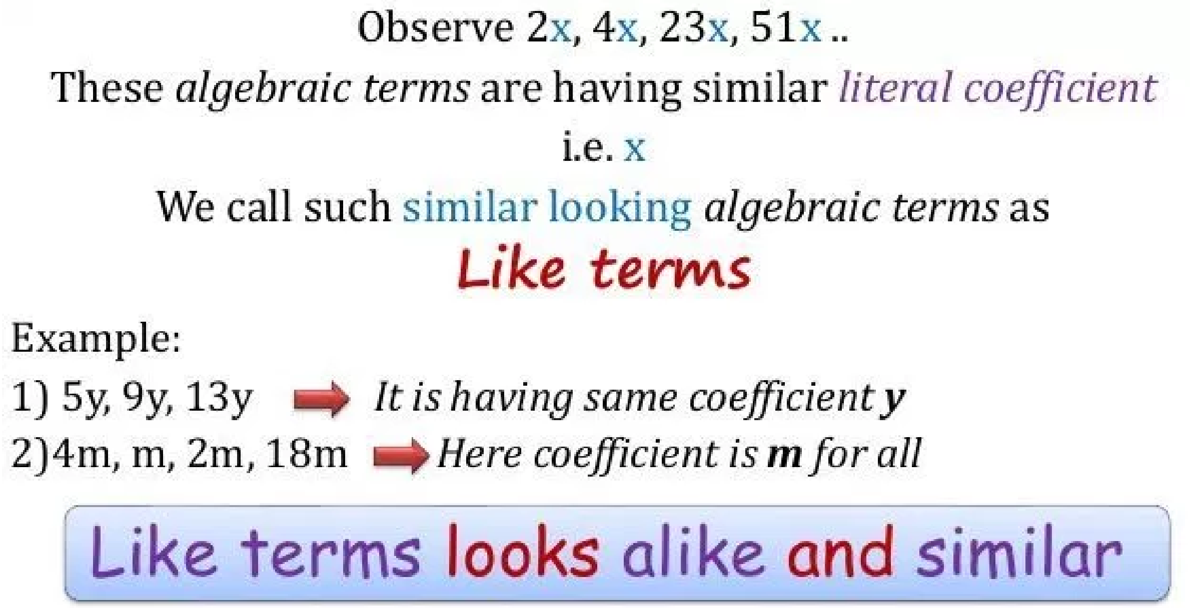


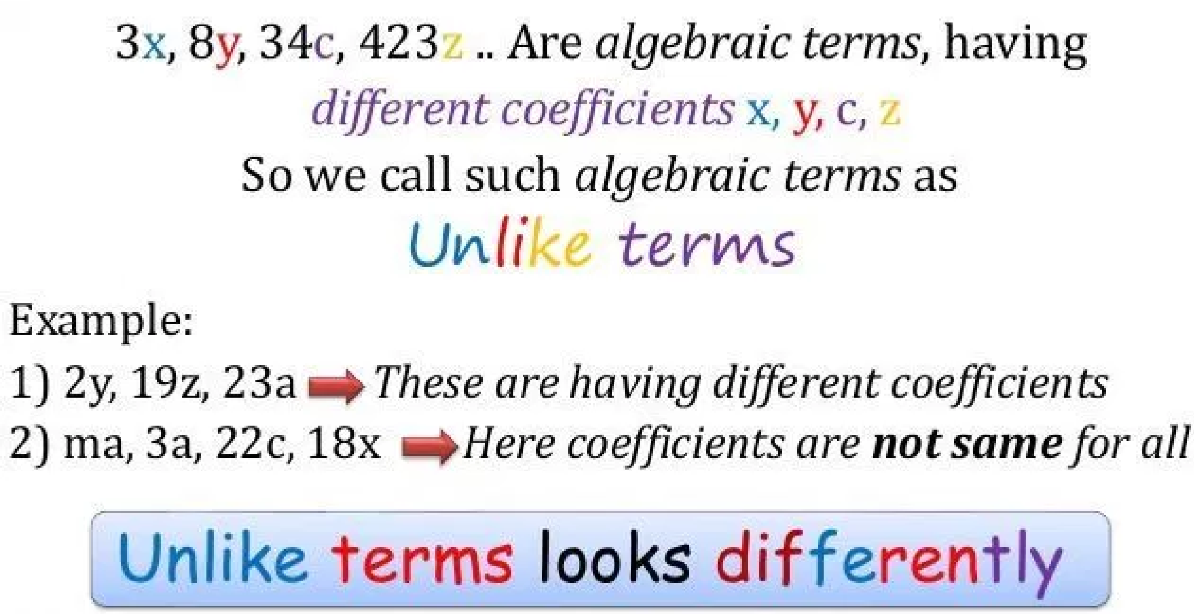
There are two types of coefficients:
1. Numerical coefficient or simply coefficient
2. Literal coefficient



For Example:
(i) In 7x ; 7 is coefficient of x
(ii) In 7xy, the numerical coefficient of the term 7xy is 7 and the literal coefficient is xy.
In a more general way,
Coefficient of xy = 7
Coefficient of 7x = y
Coefficient of 7y = x
(iii) In ($-mn^2$), the numerical coefficient of the term is (- 1) and the literal coefficient is mn^2.
In a more general way,
Coefficient of $mn^2 = -1$
Coefficient of $(-n^2) = m$
Coefficient of $m = (-n^2)$
(iv) In $-5x^2y$; 5 is coefficient of $-x^2y$; -5 is coefficient of x^2y.

Like and unlike terms: Two or more terms having the same algebraic factors are called like terms, and two or more terms having different algebraic factors are called unlike terms.



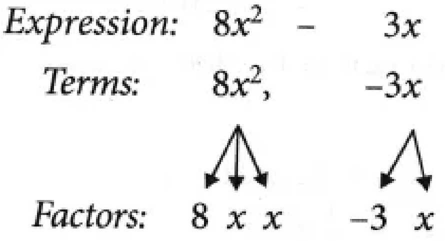


Example: In the expression $5x^2 + 7xy - 7y - 5xy$, look at the terms 7xy and (- 5xy). The factors of 7xy are 7, x, and y and the factors of (- 5xy) are (- 5), x, and y. The algebraic factors (which contain variables) of both terms are x and y. Hence, they are like terms. Other terms $5x^2$ and (- 7y) have different algebraic factors [5 × x × x and (- 7y)]. Hence, they are unlike terms.

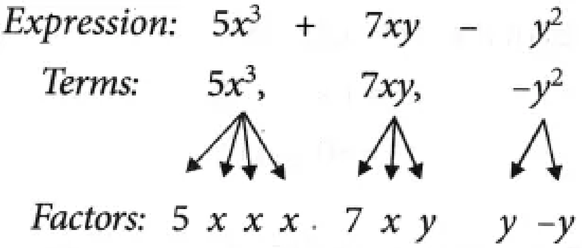
Factors And Coefficients Of A Polynomial With Examples

Example 1: Write the coefficient of:
(i) x^2 in $3x^3 - 5x^2 + 7$
(ii) xy in 8xyz
(iii) −y in $2y^2 - 6y + 2$
(iv) x^0 in 3x + 7
Solution:
(i) −5
(ii) 8z
(iii) 6
(iv) Since $x^0 = 1$,
Therefore $3x + 7 = 3x + 7x^0$
coefficient of x^0 is 7.

Example 2: Find the terms and factors of algebraic expression $8x^2 - 3x$.
Solution:



Example 3: Find the terms and factors of algebraic expression $5x^3 + 7xy - y^2$.
Solution:



This is called the tree diagram and it is the best way to represent expression, terms, and factors.

Example 4: Identify like terms in the following:
$2xy$, $-xy^2$, x^2y, $5y$, $8yx$, $12yx^2$, $-11xy$
Solution: $2xy$, $8yx$, $-11xy$ are like terms having the same algebraic factors x and y.
x^2y and $12yx^2$ are also like terms having the same algebraic factors x, x and y.

Example 5: State whether the given pairs of terms are like or unlike terms:
(a) $19x$, $19y$ (b) $4m^2p$, $7pm^2$
Solution:
(a) $19x$ and $19y$ are unlike terms having different algebraic factors, i.e., x and y.
(b) $4m^2p$, $7pm^2$ are like terms having the same algebraic factors, i.e., m, m, p.

Relationship Between Zeros And Coefficients Of A Polynomial

Consider quadratic polynomial
$P(x) = 2x^2 - 16x + 30$.
Now, $2x^2 - 16x + 30 = (2x - 6)(x - 3)$
$= 2(x - 3)(x - 5)$
The zeros of P(x) are 3 and 5.
Sum of the zeros $= 3 + 5 = 8 = \frac{-(-16)}{2} = -\left[\frac{\text{coefficient of x}}{\text{coefficient of } x^2}\right]$

Product of the zeros $= 3 \times 5 = 15 = \frac{30}{2} = \left[\frac{\text{constant term}}{\text{coefficient of } x^2}\right]$
So if $ax^2 + bx + c$, $a \neq 0$ is a quadratic polynomial and α, β are two zeros of polynomial then
$\alpha + \beta = -\frac{b}{a}$
$\alpha\beta = \frac{c}{a}$
In general, it can be proved that if α, β, γ are the zeros of a cubic polynomial $ax^3 + bx^2 + cx + d$, then
$\alpha + \beta + \gamma = \frac{-b}{a}$
$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$
$\alpha\beta\gamma = \frac{-d}{a}$
Note: $\frac{b}{a}$, $\frac{c}{a}$ and $\frac{d}{a}$ are meaningful because $a \neq 0$.

Relationship Between Zeros And Coefficients Of A Polynomial Example Problems With Solutions

Example 1: Find the zeros of the quadratic polynomial $6x^2 - 13x + 6$ and verify the relation between the zeros and its coefficients.

Sol. We have, $6x^2 - 13x + 6 = 6\times2 - 4x - 9x + 6$
$= 2x(3x - 2) - 3(3x - 2)$
$= (3x - 2)(2x - 3)$
So, the value of $6x^2 - 13x + 6$ is 0, when
$(3x - 2) = 0$ or $(2x - 3) = 0$ i.e.,
When $x = \frac{2}{3}$ or $\frac{3}{2}$
Therefore, the zeros of $6x^2 - 13x + 6$ are
$\frac{2}{3}$ and $\frac{3}{2}$
Sum of the zeros
$= \frac{2}{3} + \frac{3}{2} = \frac{13}{6} = -\frac{(-13)}{6} = -\left[\frac{\text{coefficient of x}}{\text{coefficient of x}^2}\right]$
Product of the zeros
$= \frac{2}{3} \times \frac{3}{2} = \frac{6}{6} = \left[\frac{\text{constant term}}{\text{coefficient of x}^2}\right]$

Example 2: Find the zeros of the quadratic polynomial $4\times2 - 9$ and verify the relation between the zeros and its coefficients.

Sol. We have,
$4x^2 - 9 = (2x)^2 - 3^2 = (2x - 3)(2x + 3)$
So, the value of $4x^2 - 9$ is 0, when
$2x - 3 = 0$ or $2x + 3 = 0$
i.e., when $x = \frac{3}{2}$ or $x = \frac{-3}{2}$.
Therefore, the zeros of $4x^2 - 9$ are $\frac{3}{2}$ & $\frac{-3}{2}$.
Sum of the zeros
$= \frac{3}{2} - \frac{3}{2} = 0 = -\frac{(0)}{4} = -\left[\frac{\text{coefficient of x}}{\text{coefficient of x}^2}\right]$
Product of the zeros
$= \frac{3}{2} \times \frac{-3}{2} = \frac{-9}{4} = \left[\frac{\text{constant term}}{\text{coefficient of x}^2}\right]$

Example 3: Find the zeros of the quadratic polynomial $9x^2 - 5$ and verify the relation between the zeros and its coefficients.
Sol. We have,
$9x^2 - 5 = (3x)^2 - (\sqrt{5})^2 = (3x - \sqrt{5})(3x + \sqrt{5})$
So, the value of $9x^2 - 5$ is 0,
when $3x - \sqrt{5} = 0$ or $3x + \sqrt{5} = 0$
i.e., when $x = \frac{\sqrt{5}}{3}$ or $x = \frac{-\sqrt{5}}{3}$.
Sum of the zeros
$= \frac{\sqrt{5}}{3} - \frac{\sqrt{5}}{3} = 0 = -\frac{(0)}{9} = -\left[\frac{\text{coefficient of x}}{\text{coefficient of x}^2}\right]$
Product of the zeros
$= \left(\frac{\sqrt{5}}{3}\right) \times \left(\frac{-\sqrt{5}}{3}\right) = \frac{-5}{9} = \left[\frac{\text{constant term}}{\text{coefficient of x}^2}\right]$

Example 4: If α and β are the zeros of ax2 + bx + c, a ≠ 0 then verify the relation between the zeros and its coefficients.
Sol. Since a and b are the zeros of polynomial $ax^2 + bx + c$.
Therefore, $(x - \alpha)$, $(x - \beta)$ are the factors of the polynomial $ax^2 + bx + c$.
$\Rightarrow ax^2 + bx + c = k(x - \alpha)(x - \beta)$
$\Rightarrow ax^2 + bx + c = k\{x^2 - (\alpha + \beta)x + \alpha\beta\}$
$\Rightarrow ax^2 + bx + c = kx^2 - k(\alpha + \beta)x + k\alpha\beta$ …(1)
Comparing the coefficients of x^2, x and constant terms of (1) on both sides, we get
$a = k$, $b = -k(\alpha + \beta)$ and $c = k\alpha\beta$
$\Rightarrow \alpha + \beta = \frac{-b}{k}$ and $\alpha\beta = \frac{c}{k}$
$\alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$ [∵ k = a]

Sum of the zeros $= \frac{-b}{a} = \frac{-\text{ coefficient of x}}{\text{coefficient of x}^2}$
Product of the zeros $= \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of x}^2}$

Example 5: Prove relation between the zeros and the coefficient of the quadratic polynomial $ax^2 + bx + c$.

Sol. Let a and b be the zeros of the polynomial $ax^2 + bx + c$

$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$ ....(1)

$\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$ ....(2)

By adding (1) and (2), we get

$\alpha + \beta = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$

$= \frac{-2b}{2a} = \frac{-b}{a} = \frac{\text{- coefficient of x}}{\text{coefficient of x}^2}$

Hence, sum of the zeros of the polynomial

$ax^2 + bx + c$ is $\frac{-b}{a}$

By multiplying (1) and (2), we get

$\alpha\beta = \frac{-b+\sqrt{b^2-4ac}}{2a} \times \frac{-b-\sqrt{b^2-4ac}}{2a}$

$= \frac{b^2-b^2+4ac}{4a^2}$

$= \frac{4ac}{4a^2} = \frac{c}{a}$

$= \frac{\text{constant term}}{\text{coefficient of x}^2}$

Hence, product of zeros $= \frac{c}{a}$

Example 6: find the zeroes of the quadratic polynomial $x^2 - 2x - 8$ and verify a relationship between zeroes and its coefficients.

Sol. $x^2 - 2x - 8 = x^2 - 4x + 2x - 8$

$= x (x - 4) + 2 (x - 4) = (x - 4) (x + 2)$

So, the value of $x^2 - 2x - 8$ is zero when

$x - 4 = 0$ or $x + 2 = 0$ i.e., when $x = 4$ or $x = - 2$.

So, the zeroes of $x^2 - 2x - 8$ are 4, – 2.

Sum of the zeroes

$= 4 - 2 = 2 = -\frac{(-2)}{1} = \frac{\text{- coefficient of x}}{\text{coefficient of x}^2}$

Product of the zeroes

$= 4 (-2) = -8 = \frac{-8}{1} = \frac{\text{constant term}}{\text{coefficient of x}^2}$

Example 7: Verify that the numbers given along side of the cubic polynomials are their zeroes. Also verify the relationship between the zeroes and the coefficients. $2x^3 + x^2 - 5x + 2$; , 1, – 2

Sol. Here, the polynomial p(x) is $2x^3 + x^2 - 5x + 2$

Value of the polynomial $2x^3 + x^2 - 5x + 2$

when x = 1/2

$= 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 - 5\left(\frac{1}{2}\right) + 2 = \frac{1}{4} + \frac{1}{4} - \frac{5}{2} + 2 = 0$

So, 1/2 is a zero of p(x).

On putting x = 1 in the cubic polynomial

$2x^3 + x^2 - 5x + 2$

= 2(1)3 + (1)2 −¬ 5(1) + 2 = 2 + 1 − 5 + 2 = 0

On putting x = – 2 in the cubic polynomial

$2x^3 + x^2 - 5x + 2$

= 2(−2)3 + (−2)2 − 5 (−2) + 2

= − 16 + 4 + 10 + 2 = 0

Hence, $\frac{1}{2}$, 1, – 2 are the zeroes of the given polynomial.

Sum of the zeroes of p(x)

$= \frac{1}{2} + 1 - 2 = \frac{-1}{2} = \frac{-\text{ coefficient of } x^2}{\text{coefficient of } x^3}$

Sum of the products of two zeroes taken at a time

$= \frac{1}{2} \times 1 + \frac{1}{2} \times (-2) + 1 \times (-2)$

$= \frac{1}{2} - 1 - 2 = \frac{-5}{2} = \frac{\text{coefficient of } x}{\text{coefficient of } x^3}$

Product of all the three zeroes

$= \frac{1}{2} \times (1) \times (-2) = -1$

$= \frac{-2}{2} = \frac{-\text{ constant term}}{\text{coefficient of } x^3}$

Review Factoring Polynomials

This lesson will review the process of factoring, which is used when solving equations and simplifying rational expressions.

To factor polynomial expressions, there are several approaches that can be used to simplify the process. While all of these approaches are not used for each problem, it is best to examine your expression for the possible existence of these situations. Ask yourself the following questions:

Are there Common Factors?

Factor out the Greatest Common Factor (GCF) of the expression, if one exists. This will make it simpler to factor the remaining expression.

Take care NOT to drop this GCF, as it is still part of the expression's answer.

Example 1:

$$2x^2y - 6xy^2$$

$$2xy(x - 3y)$$

Example 2:

$$9x^2 - 27x$$

$$9x(x - 3)$$

Does the expression have only 2 terms?

If it does, is the expression a DIFFERENCE of PERFECT SQUARES?
If so, you should be able to write the expression as a product of the sum and difference of the square roots of the terms.

Sometimes, as in Example 2 below, it is best to write the terms in square notation so you can see what the terms will be in factored form. Be sure to use parentheses!

This process is also called Factoring with DOTS (Difference of Two Squares).

Example 1:

$$x^2 - 25$$

$$(x + 5)(x - 5)$$

Example 2:

$$16x^2 - 25y^2$$

$$(4x)^2 - (5y)^2$$

$$(4x + 5y)(4x - 5y)$$

Does the expression have exactly 3 terms?

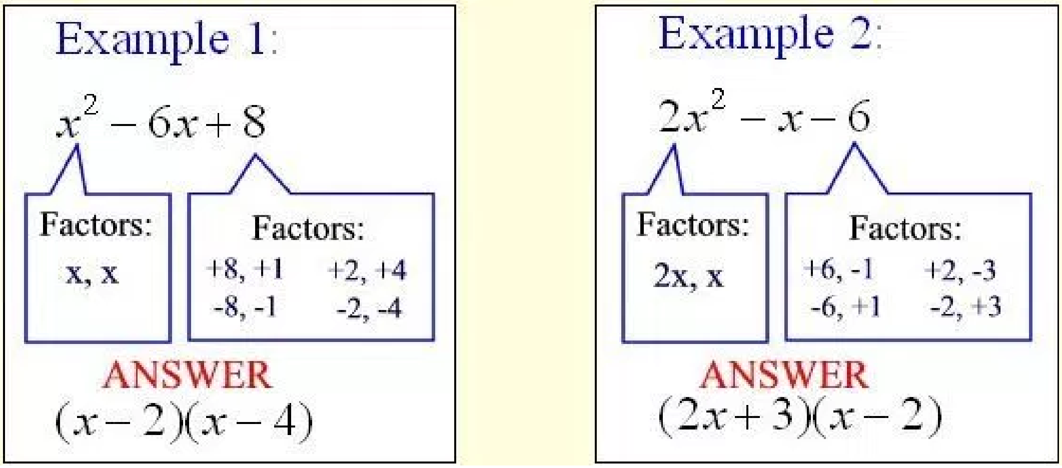
If yes, then the expression may factor into the product of two binomials. One way to solve this type of problem is to use trial and error, keeping certain "hints" in mind.

Hints:
With the trinomial arranged in proper order (highest to lowest powers):
• if the leading coefficient is 1, you are looking for two numbers that multiply to the last term and add to the coefficient of the middle term.

• if the leading coefficient is not 1, you will have to look more carefully to find the answer. See Factoring Trinomials () – Set Up, Guess and Check Method and Factoring by Grouping Method.

For the examples below, use the hint above for factoring when the leading coefficient is 1, and the trial and error (guess and check) method when the leading coefficient is 2.

Always check your work by multiplying the binomials to see if your center term matches the original problem.



Special trinomial: Perfect Square

Consider what happens when a binomial is squared:

where the center term is twice the product of a and b.

If you can recognize this pattern, it is very easy to factor a trinomial that is the perfect square of a binomial.

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+(-b))^2 = (a-b)^2 = a^2 - 2ab + b^2$$

where the center term is twice the product of a and b.

Does the first term have a Negative Coefficient?

If yes, then factor out the negative sign first, using the common factor method at the top of this page.
Remember, if the leading term has a coefficient of (- 1), and there are
NO other common terms, then the GCF is = -1.

Example 1:

$$-2x^2 + 18$$
$$-2(x^2 - 9)$$
$$-2(x+3)(x-3)$$

Example 2:

$$-x^2 + 8x + 20$$
$$-(x^2 - 8x - 20)$$
$$-(x-10)(x+2)$$

Solving Polynomials Equations of Higher Degree

If 'x' is a variable, 'n' is a positive integer and $a_0, a_1, a_2, \ldots, a_n$ are constants, then a polynomial in variable x is $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$

Coefficients

$$f(x) = \underbrace{\overbrace{a_n}x^n} + \underbrace{\overbrace{a_{n-1}}x^{n-1}} + \ldots + \underbrace{\overbrace{a_1}x} + \underbrace{a_0}$$

Terms

Degree of a Polynomial: The power of the highest degree term

Zero of a Polynomial: A real number α is a zero of a polynomial $f(x)$, iff $f(\alpha) = 0$.
Finding the zero of a polynomial $f(x)$ means solving the polynomial equation $f(x) = 0$

When the powers in polynomial equations increase, it becomes more difficult to find their solutions (roots).

Consider an equation such as

$$x^5 - 11x^4 + 43x^3 - 73x^2 + 56x - 16 = 0$$

Finding the roots of an equation such as this can prove to be quite a task. In this course, we will just be touching the surface on techniques for solving higher degree polynomial equations.

Let's be sure that we understand the vocabulary associated with this type of task.

The following statements are different ways of asking the same thing!!

- Solve the polynomial equation P(x) = 0.
- Find the roots of the polynomial equation P(x) = 0.
- Find the zeroes of the polynomial function P(x) (P(x) = 0).
- Factor the polynomial function P(x) = 0 and express the roots.

How many roots should we expect to find? A polynomial of degree n will have n roots, some of which may be multiple roots (they repeat). For example,

$$x^3 - 9x^2 + 24x - 16 = 0$$

is a polynomial of degree 3 (highest power) and as such will have 3 roots. This equation is really (x-1)(x-4)(x-4) = 0 giving solutions of x = 1 and x = 4 (repeated).

Examples:

1. Solve the following polynomial equation: $x^4 = 13x^2 - 36$

Solution Method: Recognize a pattern within the problem.

$$x^4 = 13x^2 - 36$$

- We are looking for **4** roots.

$$x^4 - 13x^2 + 36 = 0$$

- Set the equation equal to 0.

$$\left(x^2\right)^2 - 13\left(x^2\right) + 36 = 0$$

- Notice that this problem is really the variable x^2 being squared and being used to a power of one. Get in the habit of looking for this pattern.

$$\text{Letting } a = x^2$$

$$(a)^2 - 13(a) + 36 = 0$$

- Letting $x^2 = a$ may help you to see the rest of the solution more easily. Make the substitutions.

$$a^2 - 13a + 36 = 0$$

$$(a-9)(a-4) = 0$$

- Now, we have a nice quadratic equation that we know how to solve. This one factors nicely.

$$a = 9; \quad a = 4$$

- Be careful NOT to STOP when you solve for a. Remember that a really represents x^2.

$$x^2 = 9; \quad x^2 = 4$$

$$x = \pm 3; \quad x = \pm 2$$

- Replace a with x^2 and solve for the answers to the original equation.

2. Find the roots of the polynomial equation $t^5 - 10t^3 + 21t = 0$.

Solution method: Find common factor first then recognize a pattern.

$$t^5 - 10t^3 + 21t = 0$$

- We are looking for **5** roots.

$$t(t^4 - 10t^2 + 21) = 0$$

- There is a common factor of t. Factor it out.

$$t\left(\left(t^2\right)^2 - 10\left(t^2\right) + 21\right) = 0$$

- This problem now contains the same pattern we saw in example 1. It contains the variable t^2 being squared and being used to a power of one. Substitution of another letter is not being used is this example, but could be used if you wish.

$$t\left((t^2 - 7)(t^2 - 3)\right) = 0$$

- Factor the quadratic.

$$t = 0; \quad t^2 - 7 = 0; \quad t^2 - 3 = 0$$

$$t = 0; \quad t^2 = 7; \quad t^2 = 3$$

$$t = 0; \quad t = \pm\sqrt{7}; \quad t = \pm\sqrt{3}$$

- Set each factor equal to zero and solve. Be sure to list both the plus and minus versions when solving the t^2 equations.

3. Find the zeroes of the polynomial function ($P(x) = 0$) when

$$P(x) = (x^2 + 5x - 7)(x + 2)$$

Solution method: Use the quadratic formula.

$$(x^2 + 5x - 7)(x + 2) = 0$$

- We are looking for **3** roots. Think about the highest power of x if the problem were multiplied out.

$$x^2 + 5x - 7 = 0; \quad x + 2 = 0$$

- Set each factor equal to zero. The factor on the left needs to be factored further. Unfortunately, this cannot be done easily.

Use quadratic formula:

$$x = \frac{-5 \pm \sqrt{5^2 - 4(1)(-7)}}{2(1)}$$

- Use the quadratic formula to find the roots from the first factor.

$$x = \frac{-5 \pm \sqrt{53}}{2}; \quad x = -2$$

- Solve for x.

Examining Graphs of Polynomial Equations of Higher Degree

Graphing polynomial functions of higher degree can be quite tedious if done by hand. Fortunately, the graphing calculator can be very helpful in providing us with graphs of these functions very quickly.

Examples:

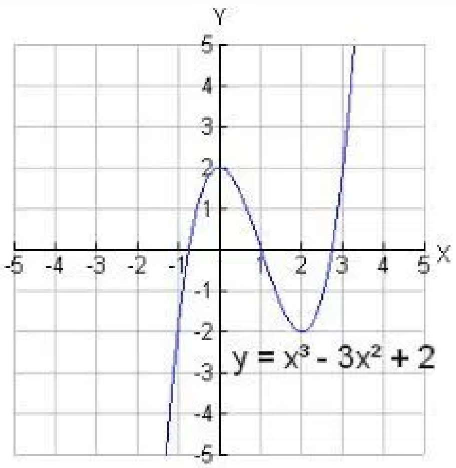
1. Given the graph at the right,

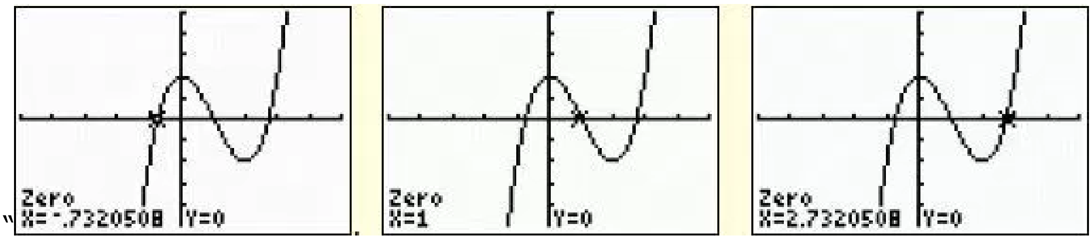
$$P(x) = x^3 - 3x^2 + 2,$$

estimate the zeroes of the function.

(Remember, the zeroes are the locations where the graph crosses the x-axis.)
This graph crosses the x-axis between -1 and 0, at 1, and between 2 and 3.

By observation, one estimated answer may be:
-0.75, 1, and 2.75





Now, let's use the zero option (2nd Calc) on the graphing calculator to get a more accurate estimate of the zeroes and check out our observed answers.

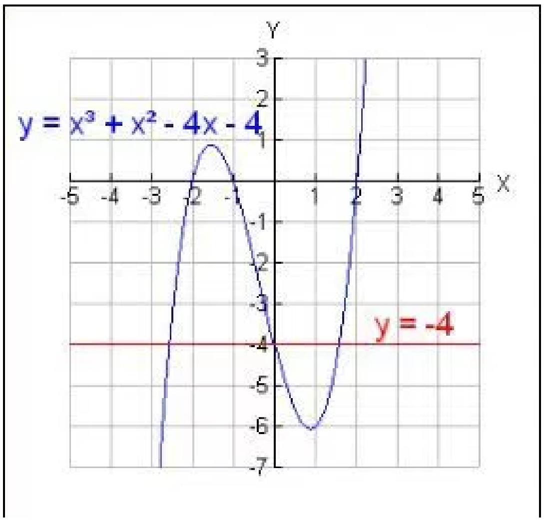
2. Use the graph at the right to estimate the solutions of the equation

$$x^3 + x^2 - 4x - 4 = -4.$$

Look carefully at this question. This equation was NOT set equal to zero and then graphed. Instead, the expression on each side of the equal sign was graphed separately. You are not looking to find where the blue graph crosses the x-axis. You are looking to find where the blue graph and the red graph intersect.

By observation, one estimated answer may be:
-2.5, 0, and 1.5

This problem could also be solved by setting the original equation equal to zero.



This problem could also be solved by setting the original equation equal to zero.

Now, let’s use the intersect option (2nd Calc) on the graphing calculator to get a more accurate estimate of the zeroes and check out our observed answers.