Exercise 16.8

Chapter 16 Vector Calculus Exercise 16.8 1E

Since the base of both the hemisphere H and the parabolic P is the circle $x^2+y^2=4$, then

$$\int_{x^2+y^2=4} \vec{F}.d\vec{r} = \int_{x^2+y^2=4} \vec{F}.d\vec{s}$$

By Stoke's theorem

$$\int_{x^2+y^2=4} \vec{F}.d\vec{r} = \int_{H} curl\,\vec{F}.d\vec{s}$$

And $\displaystyle\int_{x^2+y^2=4} \vec{F}.d\vec{r} = \int_{P} curl\,\vec{F}.d\vec{s}$

And hence $\displaystyle\iint_{H} curl\,\vec{F}.d\vec{s} = \iint_{P} curl\,\vec{F}.d\vec{s}$

Where $\mathbf{r}(t)$ is the vector function for C, the curve around the base of the hemisphere (and therefore the curve around the boundary of the surface S). Since the hemisphere has radius 3 and circular base in the xy-plane, we can write a vector function for the circle of radius 3 around its base as
$\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + 0\mathbf{k}$

The curve traverses exactly one entire circle, so the limits in t should be 0 and \$2\pi\$.
We can now plug (3) and (4) into (1) to get

$$\iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt$$

$$= \int_0^{2\pi}\left[(6\sin t)\mathbf{i} + 0\mathbf{j} + (3\cos t)e^{3\sin t}\mathbf{k}\right]\cdot\left[(-3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 0\mathbf{k}\right]dt$$

To simplify enough to integrate, find the dot product in the integrand by multiplying component-wise and adding the sums:

$$\iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\left[(-18\sin^2 t) + 0 + 0\right]dt$$

Now replace $\sin^2 t$ by using the trig identity $\sin^2 t = \dfrac{1-\cos(2t)}{2}$:

$$\iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\left[-18\left(\frac{1-\cos(2t)}{2}\right)\right]dt$$

$$= -9\int_0^{2\pi}(1-\cos(2t))dt$$

$$= -9\left(t - \frac{\sin(2t)}{2}\right)\Bigg|_0^{2\pi}$$

$$= -9\left(2\pi - \frac{\sin(4\pi)}{2} - 0\right)$$

$$= -9(2\pi - 0)$$

$$= \boxed{-18\pi}$$

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Chapter 16 Vector Calculus Exercise 16.8 3E

Consider the vector field,

$\mathbf{F}(x,y,z)=x^2y^2\ \mathbf{i}+y^2z^2\ \mathbf{j}+xyz\ \mathbf{k}$

Claim: Use Stokes' Theorem to evaluate $\iint_S \text{curl}\ \mathbf{F}\cdot d\mathbf{S}$

Here, S is part of the paraboloid $z=x^2+y^2$ that lies inside the cylinder $x^2+y^2=4$, oriented upward.

Recall the Stokes Theorem:

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in R^3 that contains S. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} \quad \ldots\ldots (1)$$

The intersection of the paraboloid $z=x^2+y^2$ and the cylinder $x^2+y^2=4$ is bounded by the curve $C: x^2+y^2=4, z=4$.

Therefore, the parametric description of C is $\mathbf{r}(t)=\langle 2\cos t, 2\sin t, 4\rangle$

Then, $\mathbf{r}'(t)=\langle -2\sin t, 2\cos t, 0\rangle$, where $0\le t\le 2\pi$

Given vector field $\mathbf{F}(x,y,z)=x^2y^2\ \mathbf{i}+y^2z^2\ \mathbf{j}+xyz\ \mathbf{k}$

This yields the following calculations

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t))&=\left\langle (2\cos t)^2(4)^2,(2\sin t)^2(4)^2,(2\cos t)(2\sin t)(4)\right\rangle\\&=\left\langle 64\cos^2 t, 64\sin^2 t, 16\sin t\cos t\right\rangle\end{aligned}$$

Use Stokes' theorem to evaluate $\iint_S \text{curl}\mathbf{F}.d\mathbf{s}$

$$\begin{aligned}\iint_S \text{curl}\mathbf{F}.d\mathbf{s}&=\int_C \mathbf{F}.d\mathbf{r}\\&=\int_0^{2\pi}\mathbf{F}(r(t))\cdot r'(t)dt\\&=\int_0^{2\pi}\left\langle 64\cos^2 t, 64\sin^2 t, 16\sin t\cos t\right\rangle\cdot\left\langle -2\sin t, 2\cos t, 0\right\rangle dt\\&=\int_0^{2\pi}\left(-128\sin t\cos^2 t+128\cos t\sin^2 t\right)dt\\&=\int_0^{2\pi}\left(128\cos t\sin^2 t\right)dt+\int_0^{2\pi}\left(-128\sin t\cos^2 t\right)dt \qquad \ldots\ldots(1)\end{aligned}$$

Consider the first integral

$$\int 128\sin^2 t\cos t\,dt$$

Take $\sin t = z$

Differentiate both sides with respect to t

$$\cos t\,dt = dz$$

So that

$$\begin{aligned}128\int \sin^2 t\cos t\,dt &= 128\int z^2 dz\\ &= \frac{128}{3}z^3\\ &= \frac{128}{3}\sin^3 t \qquad (z=\sin t)\end{aligned}$$

Now consider the second integral

$$128\int \sin t\cos^2 t\,dt$$

Take $\cos t = z$

Differentiate both sides with respect to t

$$-\sin t\,dt = dz$$

So that

$$\begin{aligned}128\int \sin t\cos^2 t\,dt &= -128\int z^2 dz\\ &= -\frac{128}{3}z^3\\ &= -\frac{128}{3}\cos^3 t \qquad (z=\cos t)\end{aligned}$$

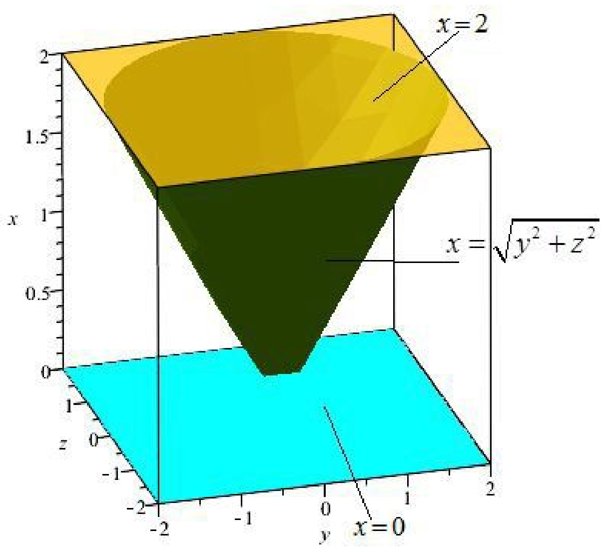
Substitute both the integrals in (1),

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} -128\sin t\cos^2 t + 128\sin^2 t\cos t\;dt\\ &= 128\left[\frac{1}{3}\cos^3 t + \frac{1}{3}\sin^3 t\right]_{t=0}^{t=2\pi}\\ &= 128\left[\frac{1}{3}+0-\left(\frac{1}{3}+0\right)\right]\\ &= \boxed{0}\end{aligned}$$

Therefore $\iint_S \text{curl}\,\mathbf{F}.d\mathbf{s} = \boxed{0}$

Chapter 16 Vector Calculus Exercise 16.8 4E

Sketch of the cone $x=\sqrt{y^2+z^2}$ in $0\le x\le 2$



The equation of the cone $x = \sqrt{y^2 + z^2}$ can be written as $x^2 = y^2 + z^2$

The cone touch the plane $x = 2$, so that the equation of the base of the cone is
$x^2 = y^2 + z^2, x = 2$

That is, $y^2 + z^2 = 4$

Substitute the parametric equations $y = 2\cos v$ and $z = 2\sin v$ in this equation.

The vector equation for the *C* is

$$\mathbf{r}(v) = 2\mathbf{i} + 2\cos v\mathbf{j} + 2\sin v\mathbf{k},\ 0 \le v \le 2\pi$$

Then, the derivative of $\mathbf{r}(v)$ is

$$\mathbf{r}'(v) = -2\sin v\mathbf{j} + 2\cos v\mathbf{k}$$

Now find the value of vector function at $\mathbf{r}(v)$

$$\mathbf{F}(x, y, z) = \tan^{-1}(x^2yz^2)\mathbf{i} + x^2y\mathbf{j} + x^2z^2\mathbf{k}$$

$$\begin{aligned}\mathbf{F}(\mathbf{r}(v)) &= \tan^{-1}\left(2^2(\cos v)(\sin^2 v)\right)\mathbf{i} + 2^2(2\cos v)\mathbf{j} + 2^2(2\sin v)^2\mathbf{k} \\ &= \tan^{-1}\left(4\cos v\sin^2 v\right)\mathbf{i} + 8\cos v\mathbf{j} + 16\sin^2 v\mathbf{k}\end{aligned}$$

Evaluate $\mathbf{F}(\mathbf{r}(v)) \cdot \mathbf{r}'(v)$

$$\begin{aligned}\mathbf{F}(\mathbf{r}(v)) \cdot \mathbf{r}'(v) &= \left(\tan^{-1}\left(4\cos v\sin^2 v\right)\mathbf{i} + 8\cos v\mathbf{j} + 16\sin^2 v\mathbf{k}\right)\cdot(-2\sin v\mathbf{j} + 2\cos v\mathbf{k}) \\ &= -16\sin v\cos v + 32\sin^2 v\cos v \\ &= -8\sin 2v + 32\sin^2 v\cos v\end{aligned}$$

By Stroke's theorem, evaluate $\iint_S \operatorname{curl}\mathbf{F}\,ds$

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F}\,ds &= \int_C \mathbf{F}\cdot d\mathbf{r} \\ &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(v)) \cdot \mathbf{r}'(v)\,dv\end{aligned}$$

$$= \int_0^{2\pi} \left(-8\sin 2v + 32\sin^2 v\cos v\right) dv$$

$$= \int_0^{2\pi} -8\sin 2v\,dv + \int_0^{2\pi} 32\sin^2 v\cos v\,dv$$ Since $\int \sin 2v\,dv = \frac{\cos 2v}{2} + C$

$$\int [f(x)]^n \cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C$$

$$= 8\left[\frac{\cos 2v}{2}\right]_0^{2\pi} + 32\left[\frac{(\sin v)^3}{3}\right]_0^{2\pi}$$

$$= 8\left[\frac{1}{2} - \frac{1}{2}\right] + 32[0]$$

$$= 0$$

Thus, $\boxed{\iint_S \operatorname{curl}\mathbf{F}\,ds = 0}$

Chapter 16 Vector Calculus Exercise 16.8 5E

Consider the following vector field:

$$\mathbf{F}(x, y, z) = xyz\,\mathbf{i} + xy\,\mathbf{j} + x^2yz\,\mathbf{k}$$

Stokes Theorem states a relationship relation between a surface integral and a line integral where *S* is an oriented, smooth surface with a simple, closed, smooth, positively oriented boundary curve *C*:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl}\,\mathbf{F} \cdot d\mathbf{S}$$

Recall that the curl of a vector field **F** is

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Let S_1 be the surfaces of the cube without the bottom $z = -1$, and let C be the corresponding boundary curve. The same boundary curve can be treated as the boundary of the missing square bottom of the cube at $z = -1$, S_2 with same outward orientation.

Apply the Stokes Theorem for both surfaces.

$$\iint_{S_1} \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

Calculate the partial derivatives to find the $\text{curl } \mathbf{F}$.

$$P = xyz,\ \frac{\partial P}{\partial y} = xz,\ \frac{\partial P}{\partial z} = xy$$
$$Q = xy,\ \frac{\partial Q}{\partial x} = y,\ \frac{\partial Q}{\partial z} = 0$$
$$R = x^2yz,\ \frac{\partial R}{\partial x} = 2xyz,\ \frac{\partial R}{\partial y} = x^2z$$

which yields

$$\begin{aligned}\text{curl } \mathbf{F} &= \left(x^2z - 0\right)\mathbf{i} + \left(xy - 2xyz\right)\mathbf{j} + \left(y - xz\right)\mathbf{k} \\ &= \left\langle x^2z, xy - 2xyz, y - xz\right\rangle\end{aligned}$$

On the surface S_2 with $z = -1$ and $\mathbf{n} = \mathbf{k}$, then

$$\begin{aligned}\text{curl } \mathbf{F} \cdot \mathbf{n} &= \left\langle -x^2, xy + 2xy, y + x\right\rangle \cdot \langle 0,0,1\rangle \\ &= x + y\end{aligned}$$

Now, calculate the surface integral over the square $\{(x, y, z) \mid -1 \le x \le 1, -1 \le y \le 1\}$ at $z = -1$.

$$\begin{aligned}\iint_{S_2} \text{curl } \mathbf{F} \cdot d\mathbf{S} &= \iint_{S_2} \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS \\ &= \int_{-1}^{1}\int_{-1}^{1} (x + y)\, dx\, dy \\ &= \int_{-1}^{1}\left[\frac{1}{2}x^2 + xy\right]_{x=-1}^{x=1} dy \\ &= \int_{-1}^{1} 2y\, dy\end{aligned}$$

$$\begin{aligned}&= \left[y^2\right]_{y=-1}^{y=1} \\ &= 1 - 1 \\ &= \boxed{0}\end{aligned}$$

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Consider the vector field

$$\mathbf{F}(x, y, z) = e^{xy}\mathbf{i} + e^{xz}\mathbf{j} + x^2z\mathbf{k}$$

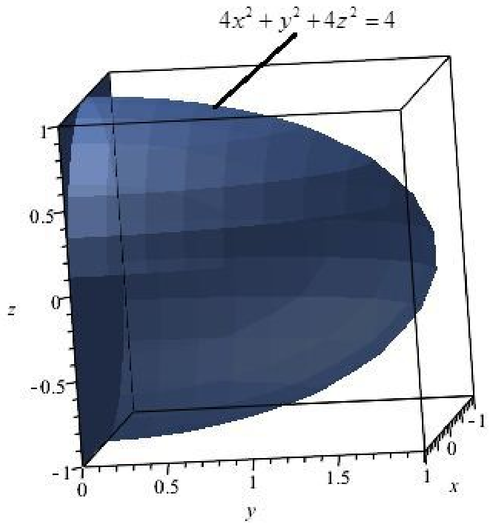
Recall the Stokes Theorem:

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in R^3 that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} \quad \ldots\ldots (1)$$

The surface S is half of the ellipsoid $4x^2+y^2+4z^2=4$ that lies to the right of the xz-plane, oriented in the direction of the positive y-axis.

Sketch of the surface of the ellipsoid $4x^2+y^2+4z^2=4$ right of the xz-plane, oriented in the direction of the positive y-axis is



The ellipsoid $4x^2+y^2+4z^2=4$ that lies to the right of the *xz*-plane, the equation of the *xz*-plane is *y* = 0. The boundary *C* of the given surface is $4x^2+4z^2=4$ or $x^2+z^2=1^2$.

Now substitute $x=\cos t$ and $z=\sin t$ and $y=0$

The vector equation for the C is

$$\mathbf{r}(t) = \cos t\mathbf{i}+\sin t\mathbf{k},\ 0\le t\le 2\pi$$

Then, the derivative of $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{k}$$

Now find the value of vector function at $\mathbf{r}(t)$

$$\mathbf{F}(x,y,z)=e^{xy}\mathbf{i}+e^{xz}\mathbf{j}+x^2z\mathbf{k}$$

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) &= e^{x(0)}\mathbf{i}+e^{(\cos t)(\sin t)}\mathbf{j}+\cos^2 t\sin t\mathbf{k}\\ &= \mathbf{i}+e^{(\cos t)(\sin t)}\mathbf{j}+\cos^2 t\sin t\mathbf{k}\end{aligned}$$

Evaluate $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)$ and

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) &= \left(\mathbf{i}+e^{(\cos t)(\sin t)}\mathbf{j}+\cos^2 t\sin t\mathbf{k}\right)\cdot\left(-\sin t\mathbf{i}+\cos t\mathbf{k}\right)\\ &= -\sin t+\cos^3 t\sin t\end{aligned}$$

Evaluate $\iint_S \text{curl}\,\mathbf{F}\,ds$

$$\iint_S \text{curl}\,\mathbf{F}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dv$$

$$= \int_0^{2\pi} \left(-\sin t + \cos^3 t \sin t\right)\,dt$$

$$= \int_0^{2\pi} -\sin t dt + \int_0^{2\pi} \cos^3 t \sin t dt$$

$$= [\cos t]_0^{2\pi} + \left[-\frac{(\cos t)^4}{4}\right]_0^{2\pi}$$

$$= [1-1] + \left[-\frac{1}{4}+\frac{1}{4}\right]$$

$$= 0$$

Hence the integral

$$\boxed{\iint_S \text{curl}\,\mathbf{F}\,ds \;=\; 0}$$

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Consider the following vector field.

$$\mathbf{F}(x,y,z) = \left(x+y^2\right)\,\mathbf{i} + \left(y+z^2\right)\,\mathbf{j} + \left(z+x^2\right)\,\mathbf{k}$$

The objective is to evaluate $\int_C \mathbf{F}\bullet d\mathbf{r}$. In each case C is oriented counterclockwise as viewed from above using Stokes' Theorem.

$$\mathbf{F}(x,y,z) = \left(x+y^2\right)\,\mathbf{i} + \left(y+z^2\right)\,\mathbf{j} + \left(z+x^2\right)\,\mathbf{k}$$

C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

Stokes Theorem states a relationship relation between a surface integral and a line integral where *S* is an oriented, smooth surface with a simple, closed, smooth, positively oriented boundary curve *C*:

$$\int_C \mathbf{F}\bullet d\mathbf{r} = \iint_S \text{curl}\ \mathbf{F}\bullet d\mathbf{S}$$

Recall that the curl of a vector field is

$$\text{curl}\ \mathbf{F} = \nabla\times\mathbf{F} = \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\,\mathbf{i} + \left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\,\mathbf{j} + \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\,\mathbf{k}$$

If **F** is a continuous defined on an oriented surface *S* with unit normal vector **n**, then the surface integral (flux) of **F** over *S* is

$$\iint_S \mathbf{F}\bullet d\mathbf{S} = \iint_S \mathbf{F}\bullet\mathbf{n}\ dS$$

Also, recall that a scalar line integral $\int_C P\ dx + Q\ dy + R\ dz$ can be interpreted as a vector line integral $\int_C \langle P,\ Q,\ R\rangle\bullet d\mathbf{r}$

To calculate $\textbf{curl F}$, we need to calculate the following partial derivatives

$$P = x + y^2,\ \frac{\partial P}{\partial y} = 2y,\ \frac{\partial P}{\partial z} = 0$$

$$Q = y + z^2,\ \frac{\partial Q}{\partial x} = 0,\ \frac{\partial Q}{\partial z} = 2z$$

$$R = z + x^2,\ \frac{\partial R}{\partial x} = 2x,\ \frac{\partial R}{\partial y} = 0$$

which yields

$$\begin{aligned}\text{curl }\mathbf{F} &= (0-2z)\ \mathbf{i} + (0-2x)\ \mathbf{j} + (0-2y)\ \mathbf{k} \\ &= \langle -2z, -2x, -2y \rangle\end{aligned}$$

Next, we determine the normal vector of the plane and the equation of the plane. Let

$$\mathbf{a} = (0,1,0) - (1,0,0) = (-1,1,0)$$
$$\mathbf{b} = (0,0,1) - (1,0,0) = (-1,0,1)$$

then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \langle 1,1,1 \rangle$$

and

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) &= 0 \\ \langle 1,1,1 \rangle \cdot \langle x-1, y-0, z-0 \rangle &= 0 \\ x - 1 + y + z &= 0 \\ x + y + z &= 1 \end{aligned}$$

From the equation the plane the normal vector we shall use is $\langle 1,1,1 \rangle$.

From $x + y + z = 1$ and the given triangle vertices, the projection of surface on the *xy*-plane is the triangle bounded by the line $x + y = 1$ and the coordinate axis in the first quadrant, $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1 - x\}$.

Now we calculate the line integral using Stokes' Theorem

$$\begin{aligned}\iint_S \text{curl }\mathbf{F} \cdot d\mathbf{S} &= \iint_S \langle -2z, -2y, -2x \rangle \cdot \langle 1,1,1 \rangle\ dS \\ &= \iint_S -2z - 2y - 2x\ dS \\ &= \iint_S -2(1-x-y) - 2y - 2x\ dS \\ &= \iint_S -2\ dS\end{aligned}$$

$$\begin{aligned}\iint_S -2\ dS &= \int_0^1 \int_0^{1-x} -2\ dy\ dx \\ &= -2\int_0^1 (1-x)\ dx\end{aligned}$$

Continue to the above step.

$$\begin{aligned}-2\int_0^1 (1-x)\ dx &= -2\left[x - \frac{1}{2}x^2\right]_{x=0}^{x=1} \\ &= -2\left[\frac{1}{2}\right] \\ &= \boxed{-1}\end{aligned}$$

Thus, integral is $\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = -1}$.

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Consider the vector field

$$\mathbf{F}(x,y,z)=\mathbf{i}+(x+yz)\mathbf{j}+(xy-\sqrt{z})\mathbf{k}$$

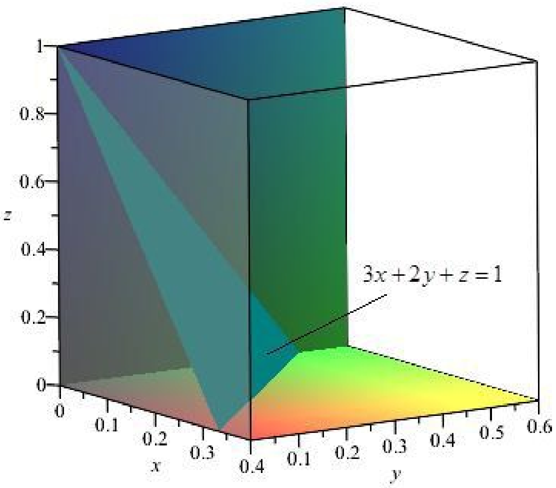
Recall the Stokes Theorem:

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in R^3 that contains S. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} \quad \text{......} (1)$$

Curve C is boundary part of the plane $3x+2y+z=1$ in the first octant

Observe the sketch of the curve



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The vector function for the curve is

$$\mathbf{r}(u,v)=u\mathbf{i}+v\mathbf{j}+(1-3u-2v)\mathbf{k}$$

The domain of the parameter is

$$D=\left\{(u,v):0\le u\le\frac{1}{3},0\le v\le\frac{1}{2}\right\}$$

From equation (1)

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$$
$$=\iint_S \operatorname{curl}\mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA$$

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Evaluate $\operatorname{curl}\mathbf{F}$

$$\operatorname{curl}\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 1 & x+yz & xy-\sqrt{z}\end{vmatrix}$$
$$=\mathbf{i}\left(\frac{\partial}{\partial y}(xy-\sqrt{z})-\frac{\partial}{\partial z}(x+yz)\right)-\mathbf{j}\left(\frac{\partial}{\partial x}(xy-\sqrt{z})-\frac{\partial}{\partial z}(1)\right)$$
$$+\mathbf{k}\left(\frac{\partial}{\partial x}(x+yz)-\frac{\partial}{\partial y}(1)\right)$$
$$=\mathbf{i}(x-y)-\mathbf{j}(y)+\mathbf{k}(1)$$
$$=(x-y)\mathbf{i}-y\mathbf{j}+\mathbf{k}$$

Therefore

$$\operatorname{curl}\mathbf{F}=(x-y)\mathbf{i}-y\mathbf{j}+\mathbf{k}$$

In parametric equation

$$\operatorname{curl}\mathbf{F}=(u-v)\mathbf{i}-v\mathbf{j}+\mathbf{k}$$

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Now, evaluate $\mathbf{r}_u \times \mathbf{r}_v$

Frist differentiate $\mathbf{r}(u,v)$ with respect to u

$$\frac{\partial}{\partial u}\mathbf{r}(u,v) = \frac{\partial}{\partial u}(u)\mathbf{i} + \frac{\partial}{\partial u}(v)\mathbf{j} + \frac{\partial}{\partial u}(1-3u-2v)\mathbf{k}$$
$$= \mathbf{i} + (0)\mathbf{j} + (-3)\mathbf{k}$$
$$= \mathbf{i} - 3\mathbf{k}$$

Therefore,

> $\mathbf{r}_u = \mathbf{i} - 3\mathbf{k}$

Differentiate $\mathbf{r}(u,v)$ with respect to v

$$\frac{\partial}{\partial v}\mathbf{r}(u,v) = \frac{\partial}{\partial v}(u)\mathbf{i} + \frac{\partial}{\partial v}(v)\mathbf{j} + \frac{\partial}{\partial v}(1-3u-2v)\mathbf{k}$$
$$= (0)\mathbf{i} + (1)\mathbf{j} + (-2)\mathbf{k}$$
$$= \mathbf{j} - 2\mathbf{k}$$

Therefore,

$\mathbf{r}_v = \mathbf{j} - 2\mathbf{k}$

Now find the vector $\mathbf{r}_u \times \mathbf{r}_v$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{vmatrix}$$
$$= \mathbf{i}(0+3) - \mathbf{j}(-2) + \mathbf{k}(1)$$
$$= 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

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Now evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$
$$= \iint_S ((u-v)\mathbf{i} - v\mathbf{j} + \mathbf{k}) \cdot (3\mathbf{i} + 2\mathbf{j} + \mathbf{k})\, dA$$
$$= \iint_S (3(u-v) - 2v + 1)\, dA$$

$$= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{3}} (3(u-v) - 2v + 1)\, du dv$$
$$= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{3}} (3u - 5v + 1)\, du dv$$

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Continuation to above calculation

$$=\int_0^{\frac{1}{2}}\left(3\left(\frac{u^2}{2}\right)-5vu+u\right)_0^{\frac{1}{3}}dv$$

$$=\int_0^{\frac{1}{2}}\left(\frac{3}{2}\left(\frac{1}{9}\right)-\frac{5}{3}v+\frac{1}{3}\right)dv$$

$$=\left(\frac{1}{6}v-\frac{5}{3}\left(\frac{v^2}{2}\right)+\frac{1}{3}v\right)_0^{\frac{1}{2}}$$

$$=\frac{1}{6}\left(\frac{1}{2}\right)-\frac{5}{3}\left(\frac{1}{8}\right)+\frac{1}{3}\left(\frac{1}{2}\right)$$

$$=\frac{1}{12}-\frac{5}{24}+\frac{1}{6}$$

$$=\frac{1}{24}$$

Therefore,

$$\boxed{\int_C \mathbf{F}\cdot d\mathbf{r}=\frac{1}{24}}$$

>

Chapter 16 Vector Calculus Exercise 16.8 9E

Consider the vector

$$\mathbf{F}(x,y,z)=yz\mathbf{i}+2xz\mathbf{j}+e^{xy}\mathbf{k}$$

Use strokes 'theorem to evaluate $\int_C \mathbf{F}.d\mathbf{r}$, Where C is the circle $x^2+y^2=16$, $z=5$

Stokes' theorem: Let S be an oriented piecewise-smooth surface that is bounded by simple, closed, piecewise-smooth boundary curve C with positive orientation.

Let $\mathbf{F}$ be a vector field whose components have continuous partial derivative on an open region in $\mathbf{R}^3$ that contains S.Then

$$\int_C \mathbf{F}.d\mathbf{r}=\iint_S \text{curl }\mathbf{F}.\,d\mathbf{s}$$

Now C is the circle in *xy*-plane and for the surface S, whose boundary is C, the unit normal vector is $\mathbf{k}$ (Since C is oriented counterclockwise, orient the surface $x^2+y^2\le 16$, z = 5 upward, then $n=\mathbf{k}$)

First find the

$$\text{curl }\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ yz & 2xz & e^{xy}\end{vmatrix}$$

$$=\mathbf{i}\left(\frac{\partial}{\partial y}\left(e^{xy}\right)-\frac{\partial}{\partial z}(2xz)\right)-\mathbf{j}\left(\frac{\partial}{\partial x}\left(e^{xy}\right)-\frac{\partial}{\partial z}(yz)\right)+\mathbf{k}\left(\frac{\partial}{\partial x}(2xz)-\frac{\partial}{\partial y}(yz)\right)$$

$$=\mathbf{i}\left(xe^{xy}-2x\right)-\mathbf{j}\left(ye^{xy}-y\right)+\mathbf{k}(2z-z)$$

$$=\left(xe^{xy}-2x\right)\mathbf{i}-\left(ye^{xy}-y\right)\mathbf{j}+z\mathbf{k}$$

Now,

$$\iint_S \text{curl }\mathbf{F}.d\mathbf{s}=\iint_S \text{curl }\mathbf{F}.\mathbf{n}\,ds$$

$$=\iint_S \langle xe^{xy}-2x,\ y-ye^{xy},\ z\rangle\cdot\langle 0,0,1\rangle ds$$

$$=\iint_S z\,ds$$

The projection of the circle $x^2+y^2=16,\ z=5$ on the *xy*-plane is the disk $x^2+y^2\le 16$, or (Changing to polar co – ordinates the region under surface S is given by

$$D=\{(r,\theta)\,|\,0\le r\le 4, 0\le\theta\le 2\pi\}$$

The above integral becomes

$$\iint_S \text{curl}\,\mathbf{F}.d\mathbf{s} = \iint_S \text{curl}\,\mathbf{F}.\mathbf{n}\,ds$$

$$= \int_0^{2\pi}\int_0^4 5r\,dr\,d\theta \qquad \text{Since } ds = rdrd\theta$$

$$= 5\left(\frac{r^2}{2}\right)_0^4 (\theta)_0^{2\pi}$$

$$= 5(8)(2\pi)$$

$$= 80\pi$$

Hence $\int_C \mathbf{F}.d\mathbf{r} = \boxed{80\pi}$

Chapter 16 Vector Calculus Exercise 16.8 10E

Use Stokes' Theorem to convert to a surface integral and evaluate.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} \quad \text{...... (1)}$$

Use the formula shown below to calculate surface integrals:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)dA \quad \text{...... (2)}$$

In the above equation $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ and the surface *S* is the graph $z = g(x, y)$ over the region *D*.

Determine the curl **F**.

Take the cross product $\nabla\times\mathbf{F}$ to determine the curl.

Use a determinant to calculate cross product.

The cross product of two vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ is:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Since in this case $\mathbf{F}(x, y, z) = xy\mathbf{i} + 2z\mathbf{j} + 3y\mathbf{k}$, the curl of **F** is:

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2z & 3y \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(3y) - \frac{\partial}{\partial z}(2z)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(3y) - \frac{\partial}{\partial z}(xy)\right]\mathbf{j} \quad \text{...... (3)}$$

$$+\left[\frac{\partial}{\partial x}(2z) - \frac{\partial}{\partial y}(xy)\right]\mathbf{k}$$

$$= (3-2)\mathbf{i} - (0-0)\mathbf{j} + (0-x)\mathbf{k}$$

$$= \mathbf{i} - x\mathbf{k}$$

Substitute the value from equation (3) in the equation (1):

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S}$$
$$= \iint_S (\mathbf{i} - x\mathbf{k})\cdot d\mathbf{S}$$

Rewrite the above surface integral according to equation (2).

In order to do so, determine a function which has a graph that gives the surface *S*.

The surface *S* is a portion of the plane $x+z=5$, so use $z=5-x$ as the equation $g(x,y)$ for the surface.

Consider the partial derivatives:

$$\frac{\partial g}{\partial x} = -1$$
$$\frac{\partial g}{\partial y} = 0$$

Evaluate the above surface integral further:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\mathbf{i} - x\mathbf{k})\cdot d\mathbf{S}$$
$$= \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)dA$$
$$= \iint_D (-1(-1) - 0(0) + x)dA$$
$$= \iint_D (1+x)dA$$

The region *D* is the projection in the *xy*-plane of the cylinder $x^2+y^2=9$, which "cuts out" a part of the surface.

Since $x^2+y^2=9$ has cross-sections that are circles centered at the origin of radius 3, use polar coordinates for the limits, with *r* ranging from the origin to 3 and θ going around the entire circle, from 0 to 2π.

To convert the integrand to polar coordinates, replace *x* using the conversion:

$$x = r\cos\theta$$

Multiply in a factor of *r* to complete the conversion.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D (1+x)dA$$
$$= \int_0^{2\pi}\int_0^3 \left[(1+r\cos\theta)r\right]drd\theta$$

Evaluate the above integral:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\int_0^3 (r + r^2\cos\theta)drd\theta$$
$$= \int_0^{2\pi}\left(\frac{r^2}{2} + \frac{r^2\cos\theta}{2}\right)\Bigg|_0^3 d\theta$$
$$= \int_0^{2\pi}\left(\frac{3^2}{2} + \frac{3^2\cos\theta}{2} - 0\right)d\theta$$
$$= \frac{9}{2}\int_0^{2\pi}(1+\cos\theta)d\theta$$

Simplify the above integral further to obtain the final value:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \frac{9}{2}(\theta + \sin\theta)\Big|_0^{2\pi}$$
$$= \frac{9}{2}(2\pi + \sin(2\pi) - 0)$$
$$= 9\pi$$

Hence, the final value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ is $\boxed{9\pi}$.

Chapter 16 Vector Calculus Exercise 16.8 11E

(A)

$$\vec{F}(x,y,z) = x^2 z\hat{i} + xy^2\hat{j} + z^2\hat{k}$$

Where C is the curve of intersection of the plane $x+y+z=1$ and the cylinder $x^2+y^2=9$

By Stoke's theorem

$$\int_C \vec{F}.d\vec{r} = \iint_S curl\,\vec{F}.d\vec{s}$$

Now $curl\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 z & xy^2 & z^2 \end{vmatrix}$

$$= <0, x^2, y^2>$$

And S is the region under the plane $x+y+z=1$ and above the circler $x^2+y^2=9$

Now the equation of the plane is $x+y+z=1$ or $z=1-x-y$

Then $\dfrac{\partial z}{\partial x} = -1$

And $\dfrac{\partial z}{\partial y} = -1$

Let $f(x,y,z) = x+y+z-1$, then S is a level curve of $f(x,y,z)=0$. As we know $\vec{\nabla} f(x,y,z)$ is normal to S at point (x,y,z), then the unit normal vector is

$$\hat{n} = \frac{\vec{\nabla} f}{\|\vec{\nabla} f\|}$$

Now $\vec{\nabla} f = <1,1,1>$

And $\|\vec{\nabla} f\| = \sqrt{3}$

Then $\iint_S curl\,\vec{F}.d\vec{s} = \iint_S curl\,\vec{F}.\hat{n}\,ds$

$$= \iint_D curl\,\vec{F}.\frac{\vec{\nabla} f}{\|\vec{\nabla} f\|}\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA$$

$$= \iint_D <0, x^2, y^2>.\frac{<1,1,1>}{\sqrt{3}} \times \sqrt{3}dA$$

$$= \iint_D (x^2+y^2)\,dA$$

$$= \int_0^{2\pi}\int_0^3 r^2.r\,dr\,d\theta$$

(Changing to polar co – ordinates)

i.e. $\iint_S curl\,\vec{F}.d\vec{s} = \int_0^{2\pi} d\theta \int_0^3 r^3\,dr$

$$= (\theta)_0^{2\pi}\left(\frac{r^4}{4}\right)_0^3$$

$$= 2\pi \times \frac{81}{4}$$

$$= 81\frac{\pi}{2}$$

Hence $\boxed{\int_C \vec{F}.d\vec{r} = 81\pi/2}$

Chapter 16 Vector Calculus Exercise 16.8 12E

(a)

Consider a vector filed whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S,

$$\mathbf{F}(x,y,z) = x^2y\mathbf{i} + \frac{1}{3}x^3\mathbf{j} + xy\mathbf{k}$$

where C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$ oriented counterclockwise as viewed from above.

The objective is to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ using Stokes' Theorem.

The Stokes' Theorem:

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector filed whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$$

To calculate $\operatorname{curl}\mathbf{F}$:

$$\begin{aligned}
\operatorname{curl}\mathbf{F} &= \nabla\times\mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & \frac{1}{3}x^3 & xy \end{vmatrix} \\
&= \mathbf{i}(x-0) - \mathbf{j}(y-0) + \mathbf{k}(x^2 - x^2) \\
&= x\mathbf{i} - y\mathbf{j}
\end{aligned}$$

From the given curves $z = y^2 - x^2, x^2 + y^2 = 1$.

$$\begin{aligned}
z &= y^2 - x^2 \\
z &= y^2 - (1 - y^2) \\
z &= y^2 - 1 + y^2 \\
&= 2y^2 - 1
\end{aligned}$$

Since the projection of S on yz − plane, $x = 0$

$$\begin{aligned}
z &= y^2 - x^2, x^2 + y^2 = 1 \\
z &= y^2 - 0^2, 0^2 + y^2 = 1 \\
z &= y^2, y^2 = 1 \\
z &= y^2, y = \pm 1
\end{aligned}$$

Recall that the projection of S on yz − plane is $d\mathbf{S} = \frac{dydz}{|\mathbf{n}\cdot\mathbf{i}|}$, where $\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|}$

If we take $\phi \equiv x^2 - y^2 + z$, then,

$$\begin{aligned}
\nabla\phi &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(x^2 - y^2 + z) \\
&= \frac{\partial}{\partial x}(x^2 - y^2 + z)\mathbf{i} + \frac{\partial}{\partial y}(x^2 - y^2 + z)\mathbf{j} + \frac{\partial}{\partial z}(x^2 - y^2 + z)\mathbf{k} \\
&= 2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}
\end{aligned}$$

So,

$$\begin{aligned}
|\nabla\phi| &= \sqrt{(2x)^2 + (-2y)^2 + 1^2} \\
&= \sqrt{4(x^2 + y^2) + 1} \\
&= \sqrt{4(1) + 1} \qquad \text{Since } x^2 + y^2 = 1. \\
&= \sqrt{5}
\end{aligned}$$

Find **n**.

$$\mathbf{n} = \frac{\nabla \phi}{|\nabla \phi|}$$
$$= \frac{2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}}{\sqrt{5}}$$

Then,

$$\mathbf{n} \cdot \mathbf{i} = \left(\frac{2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}}{\sqrt{5}} \right) \cdot \mathbf{i}$$
$$= \frac{2}{\sqrt{5}}$$

So, $|\mathbf{n} \cdot \mathbf{i}| = \frac{2}{\sqrt{5}}$

Thus,

$$d\mathbf{S} = \frac{dx\,dy}{|\mathbf{n} \cdot \mathbf{i}|}$$
$$= \frac{dx\,dy}{2/\sqrt{5}}$$
$$= \frac{\sqrt{5}}{2} dxdy$$

And,

$$\text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \text{curl}\,\mathbf{F} \cdot \mathbf{n}\, d\mathbf{S}$$
$$= (x\mathbf{i} - y\mathbf{j}) \cdot \left(\frac{2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}}{\sqrt{5}} \right) \frac{\sqrt{5}}{2} dx\,dy$$
$$= (x\mathbf{i} - y\mathbf{j}) \cdot (2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}) \frac{1}{2} dx\,dy$$
$$= (2x^2 + 2y^2) \frac{1}{2} dxdy$$
$$= dx\,dy \qquad \text{Since } x^2 + y^2 = 1.$$

Find $\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$.

$$\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \int_{y=-1}^{1} \int_{z=y^2}^{2y^2-1} dydz$$
$$= \int_{y=-1}^{1} (z)_{y^2}^{2y^2-1} dy \qquad \text{Integrate with respect to } z.$$
$$= \int_{y=-1}^{1} (2y^2 - 1 - y^2) dy$$
$$= \int_{y=-1}^{1} (y^2 - 1) dy$$
$$= \left(\frac{y^3}{3} - y \right)_{-1}^{1} \qquad \text{Integrate with respect to } y.$$
$$= \left(\frac{1}{3} - 1 \right) - \left(-\frac{1}{3} + 1 \right)$$
$$= -\frac{2}{3} - \frac{2}{3}$$
$$= -\frac{4}{3}$$

By Stokes' Theorem,

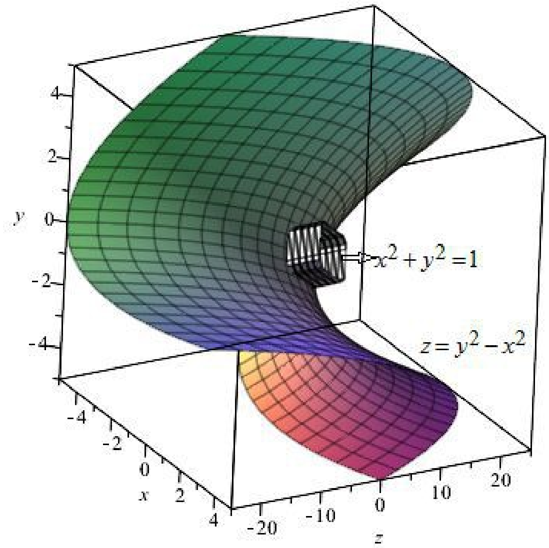
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$$
$$= -\frac{4}{3}$$

Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = \boxed{-\frac{4}{3}}$.

(b)

The objective is to graph both the hyperbolic paraboloid and the cylinder with domain.

Graph of both the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$ is as follows:



(c)

The objective is to find the parametric equations for *C* and use them to graph *C*.

Since $x^2 + y^2 = 1$, so $x = \cos t, y = \sin t$

Since $z = y^2 - x^2$, so,

$$\begin{aligned} z &= y^2 - x^2 \\ &= \sin^2 t - \cos^2 t \\ &= -\left(\cos^2 t - \sin^2 t\right) \\ &= -\cos 2t \end{aligned}$$

So, the parametric equations of the curve of intersection of $z = y^2 - x^2$ and $x^2 + y^2 = 1$ is

$$\boxed{x = \cos t, y = \sin t, z = -\cos 2t}, 0 \le t \le 2\pi.$$

Chapter 16 Vector Calculus Exercise 16.8 13E

Consider the vector field

$$\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} - 2\mathbf{k}$$

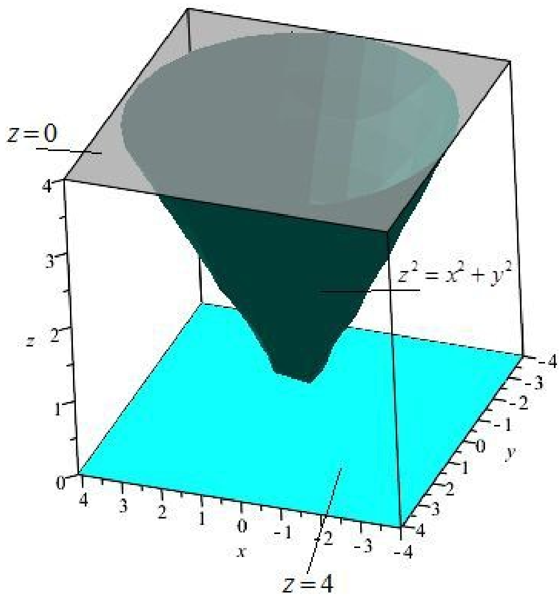
Recall the Stokes Theorem:

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in R^3 that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} \quad \ldots\ldots (1)$$

The surface S is a cone $z^2 = x^2 + y^2, 0 \le z \le 4$, oriented down ward

Sketch of the cone $z^2 = x^2 + y^2, 0 \le z \le 4$



Now substitute $x = 4\cos t$ and $y = 4\sin t$ and $z = 4$

The vector equation for the C is

$$\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + 4\mathbf{k},\ 0 \le t \le 2\pi$$

Then, the derivative of $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j}$$

Now find the value of vector function at $\mathbf{r}(t)$

$$\mathbf{F}(x,y,z) = -y\mathbf{i} + x\mathbf{j} - 2\mathbf{k}$$

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) &= -(4\sin t)\mathbf{i} + (4\cos t)\mathbf{j} - 2\mathbf{k}\\ &= -4\sin t\mathbf{i} + 4\cos t\mathbf{j} - 2\mathbf{k}\end{aligned}$$

Evaluate $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$ and

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= (-4\sin t\mathbf{i} + 4\cos t\mathbf{j} - 2\mathbf{k}) \cdot (-4\sin t\mathbf{i} + 4\cos t\mathbf{j})\\ &= 16\sin^2 t + 16\cos^2 t\\ &= 16(\sin^2 t + \cos^2 t)\\ &= 16\end{aligned}$$

Evaluate $\iint_S \operatorname{curl}\mathbf{F}\, ds$

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F}\, ds &= \int_C \mathbf{F} \cdot d\mathbf{r}\\ &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dv\end{aligned}$$

$$= \int_0^{2\pi} 16\, dt$$

$$= 16(t)_0^{2\pi}$$

$$= 16(2\pi - 0)$$

$$= 32\pi$$

Hence the integral

$$\boxed{\iint_S \operatorname{curl}\mathbf{F}\, ds = 32\pi} \quad \ldots\ldots (2)$$

Now evaluate the integral $\int_C \mathbf{F}\cdot d\mathbf{r}$

The vector function for the curve is

$$\mathbf{r}(t) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + u\mathbf{k}$$

The domain of the parameter is

$$D = \{(u,v): 0 \le u \le 4, 0 \le v \le 2\pi\}$$

From equation (1)

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} \\ &= \iint_S \text{curl}\mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA\end{aligned}$$

Evaluate $\text{curl}\,\mathbf{F}$

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & -2 \end{vmatrix}$$

$$= \mathbf{i}\left(\frac{\partial}{\partial y}(-2) - \frac{\partial}{\partial z}(x)\right) - \mathbf{j}\left(\frac{\partial}{\partial x}(-2) - \frac{\partial}{\partial z}(-y)\right) + \mathbf{k}\left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right)$$

$$\begin{aligned}&= \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(2) \\ &= 2\mathbf{k}\end{aligned}$$

Therefore

$$\text{curl}\,\mathbf{F} = 2\mathbf{k}$$

Now, evaluate $\mathbf{r}_u \times \mathbf{r}_v$

Frist differentiate $\mathbf{r}(u,v)$ with respect to u

$$\begin{aligned}\frac{\partial}{\partial u}\mathbf{r}(u,v) &= \frac{\partial}{\partial u}(u\cos v)\mathbf{i} + \frac{\partial}{\partial u}(u\sin v)\mathbf{j} + \frac{\partial}{\partial u}(u)\mathbf{k} \\ &= \cos v\mathbf{i} + \sin v\mathbf{j} + 1\mathbf{k} \\ &= \cos v\mathbf{i} + \sin v\mathbf{j} + \mathbf{k}\end{aligned}$$

Therefore,

$$\mathbf{r}_u = \cos v\mathbf{i} + \sin v\mathbf{j} + \mathbf{k}$$

Differentiate $\mathbf{r}(u,v)$ with respect to v

$$\begin{aligned}\frac{\partial}{\partial v}\mathbf{r}(u,v) &= \frac{\partial}{\partial v}(u\cos v)\mathbf{i} + \frac{\partial}{\partial v}(u\sin v)\mathbf{j} + \frac{\partial}{\partial v}(u)\mathbf{k} \\ &= (-u\sin v)\mathbf{i} + (u\cos v)\mathbf{j} + (0)\mathbf{k} \\ &= -u\sin v\mathbf{i} + u\cos v\mathbf{j}\end{aligned}$$

Therefore,

$$\mathbf{r}_v = -u\sin v\mathbf{i} + u\cos v\mathbf{j}$$

Now find the vector $\mathbf{r}_u \times \mathbf{r}_v$

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} \\ &= \mathbf{i}(-u\cos v) - \mathbf{j}(u\sin v) + \mathbf{k}(u\cos^2 v + u\sin^2 v) \\ &= -u\cos v\mathbf{i} - u\sin v\mathbf{j} + u\mathbf{k}\end{aligned}$$

Now evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$$
$$= \iint_S (2\mathbf{k}) \cdot (-u\cos v\mathbf{i} - u\sin v\mathbf{j} + u\mathbf{k}) dA$$
$$= \iint_S 2u dA$$

$$= \int_0^{2\pi}\int_0^4 2u du dv$$
$$= \int_0^{2\pi} \left(\frac{2u^2}{2}\right)_0^4 dv$$

Continuation to above calculation

$$= \int_0^{2\pi} \left(\frac{2(16)}{2}\right) dv$$
$$= 16(v)_0^{2\pi}$$
$$= 16(2\pi)$$
$$= 32\pi$$

Therefore,

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = 32\pi} \quad \ldots\ldots (3)$$

Thus, from (2) and (3), we can see that

$$\boxed{\iint_S \text{curl}\,\mathbf{F} \cdot ds = \int_C \mathbf{F} \cdot d\mathbf{r}}.$$

Hence the Stockes' Theorem is verified.

Chapter 16 Vector Calculus Exercise 16.8 14E

We have $F = -2yz\mathrm{i} + y\mathrm{j} + 3x\mathrm{k}$ and the surface S as $z = 5 - x^2 - y^2$ that lies above $z = 1$ oriented upward.

Let $x = u\cos t, y = u\sin t$, and $z = 5 - u^2$ be the parametric equation for $0 \le u \le 2$

And $0 \le t \le 2\pi$. Then, $\mathbf{r}_u = \langle \cos t, \sin t, -2u \rangle$ and $\mathbf{r}_t = \langle -u\sin t, u\cos t, 0 \rangle$.

Also, $\mathbf{r}_u \times \mathbf{r}_t = \langle 2u^2\cos t, 2u^2\sin t, u \rangle$.

Find curl F.

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2yz & y & 3x \end{vmatrix}$$
$$= 0\mathbf{i} - (3 + 2y)\mathbf{j} + 2z\mathbf{k}$$

Evaluate $\iint_S \text{curl}\,\mathbf{F} \cdot ds$.

$$\iint_S \text{curl}\,\mathbf{F} \cdot ds = \int_0^2\int_0^{2\pi} \left[-2u^2\sin t(3 + 2u\sin t) + 2u(5 - u^2)\right] dt du$$
$$= \int_0^2\int_0^{2\pi} \left[-6u^2\sin t - 4u^3\sin^2 t + 10u - 2u^3\right] dt du$$
$$= \int_0^2\int_0^{2\pi} \left[-4u^3\left(\frac{1 - \cos 2t}{2}\right) + 10u - 2u^3\right] dt du$$
$$= 2\pi\int_0^2 (-4u^3 + 10u)\, du$$
$$= 8\pi \quad \ldots(1)$$

The boundary of the surface S is the circle $x^2+y^2=4$. Then, $x=2\cos t, y=2\sin t$, and $z=1$, for $0 \le t \le 2\pi$. Also, $\mathbf{r} = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + \mathbf{k}$ and $\mathbf{r}' = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$.

Now, $\mathbf{F}(\mathbf{r}(t)) = -4\sin t\mathbf{i} + 2\sin t\mathbf{j} + 6\cos t\mathbf{k}$

and $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 4 - 4\cos 2t + 2\sin 2t$.

Find $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (4 - 4\cos 2t + 2\sin 2t)\, dt \\ &= 4(2\pi) \\ &= 8\pi \qquad \text{....(2)}\end{aligned}$$

Thus, from (1) and (2), we can see that $\iint_S \text{curl}\,\mathbf{F} \cdot ds = \int_C \mathbf{F} \cdot d\mathbf{r}$.

Stokes' Theorem:

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let **F** be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S.

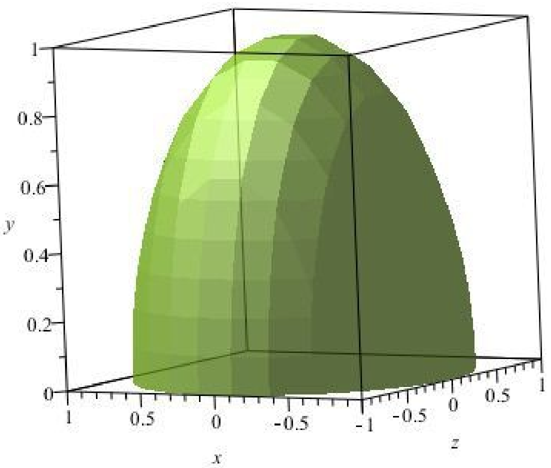
Then,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} \quad \text{...... (1)}$$

To verify that Stokes' Theorem is true for the vector field $\mathbf{F}(x,y,z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$:

Here S is the hemisphere $x^2+y^2+z^2=1, y \ge 0$ oriented in the direction of the positive y-axis.

The hemisphere S: $x^2+y^2+z^2=1, y \ge 0$ is shown below:



The parametric notation for S is

$$\mathbf{r}(\phi,\theta) = \sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k}, \text{ where } 0 \le \theta \le \pi,\ 0 \le \phi \le \pi$$

Then the partial derivative of **r** with respect to ϕ,

$$\begin{aligned}\mathbf{r}_\phi &= \frac{\partial \mathbf{r}}{\partial \phi} \\ &= \frac{\partial}{\partial \phi}(\sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k}) \\ &= \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi \rangle\end{aligned}$$

And, the partial derivative of **r** with respect to θ,

$$\begin{aligned}\mathbf{r}_\theta &= \frac{\partial \mathbf{r}}{\partial \theta} \\ &= \frac{\partial}{\partial \theta}(\sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k}) \\ &= \langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0 \rangle\end{aligned}$$

Their cross product is,

$$\mathbf{r}_\theta \times \mathbf{r}_\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \end{vmatrix}$$

$$= \mathbf{i}\left(-\sin^2\phi\cos\theta - 0\right) - \mathbf{j}\left(\sin^2\phi\sin\theta - 0\right)$$

$$+ \mathbf{k}\left(-\sin^2\theta\sin\phi\cos\phi - \cos^2\theta\sin\phi\cos\phi\right)$$

$$= -\left(\sin^2\phi\cos\theta\mathbf{i} + \sin^2\phi\sin\theta\mathbf{j} + \sin\phi\cos\phi\left(\sin^2\theta + \cos^2\theta\right)\mathbf{k}\right)$$

$$= -\left\langle \sin^2\phi\cos\theta\mathbf{i} + \sin^2\phi\sin\theta\mathbf{j} + \sin\phi\cos\phi\mathbf{k} \right\rangle$$

Now find $\text{curl}\mathbf{F}$ as shown below.

$$\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$$

$$\text{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= \mathbf{i}\left(\frac{\partial x}{\partial y} - \frac{\partial z}{\partial z}\right) - \mathbf{j}\left(\frac{\partial x}{\partial x} - \frac{\partial y}{\partial z}\right) + \mathbf{k}\left(\frac{\partial z}{\partial x} - \frac{\partial y}{\partial y}\right)$$

$$= \mathbf{i}(0-1) - \mathbf{j}(1-0) + \mathbf{k}(0-1)$$

$$= -\mathbf{i} - \mathbf{j} - \mathbf{k}$$

Then we have,

$$\text{curl}\mathbf{F} \cdot \left(\mathbf{r}_\theta \times \mathbf{r}_\phi\right) = -\left\langle -\mathbf{i} - \mathbf{j} - \mathbf{k} \right\rangle \cdot \left\langle \sin^2\phi\cos\theta\mathbf{i} + \sin^2\phi\sin\theta\mathbf{j} + \sin\phi\cos\phi\mathbf{k} \right\rangle$$

Therefore

$$\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_D \text{curl}\mathbf{F} \cdot \left(\mathbf{r}_\theta \times \mathbf{r}_\phi\right) dA$$

$$= \iint_D -\left\langle -\mathbf{i} - \mathbf{j} - \mathbf{k} \right\rangle \cdot \left\langle \sin^2\phi\cos\theta\mathbf{i} + \sin^2\phi\sin\theta\mathbf{j} + \sin\phi\cos\phi\mathbf{k} \right\rangle dA$$

$$= \int_0^\pi \int_0^\pi \left(\sin^2\phi\cos\theta + \sin^2\phi\sin\theta + \sin\phi\cos\phi\right) d\theta\, d\phi$$

$$= \int_0^\pi \left(\int_0^\pi \left(\sin^2\phi\left(\cos\theta + \sin\theta\right) + \cos\phi\sin\phi\right) d\theta \right) d\phi$$

Continuation of the above,

$$\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \int_0^\pi \left[\sin^2\phi\left(\sin\theta - \cos\theta\right) + \cos\phi\sin\phi \cdot \theta\right]_{\theta=0}^{\theta=\pi} d\phi$$

$$= \int_0^\pi \left[\left(\sin^2\phi\left(\sin\pi - \cos\pi\right) + \pi\cos\phi\sin\phi\right) - \left(\sin^2\phi\left(\sin 0 - \cos 0\right) + 0\right)\right] d\phi$$

$$= \int_0^\pi \left[\left(\sin^2\phi\left(0 - (-1)\right) + \pi\cos\phi\sin\phi\right) - \left(\sin^2\phi\right)(0-1)\right] d\phi$$

$$= \int_0^\pi \left[2\sin^2\phi + \pi\cos\phi\sin\phi\right] d\phi$$

$$= \int_0^\pi \left[2\left(\frac{1-\cos 2\phi}{2}\right) + \pi\left(\frac{1}{2}\sin 2\phi\right)\right] d\phi$$

$$= \int_0^\pi \left[1 - \cos 2\phi + \frac{\pi}{2}\left(\sin 2\phi\right)\right] d\phi$$

Continuation of the above,

$$\iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} = \left[\phi - \frac{1}{2}\sin 2\phi - \frac{\pi}{2}\cos 2\phi\right]_0^{\pi}$$

$$= \left(\pi - \frac{1}{2}\sin 2\pi - \frac{\pi}{2}\cos 2\pi\right) - \left(0 - \frac{1}{2}\sin 0 - \frac{\pi}{2}\cos 0\right)$$ Therefore, we have

$$= \left(\pi - 0 - \frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right)$$

$$= \pi$$

$$\iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} = \pi$$ ...... (1)

Now from the figure observe that, the boundary of the surface is lies on xz -plane.

For any point on xz -plane, $y = 0$

So the boundary of the hemisphere *S* is given by,

$$x^2 + z^2 = 1,\ y = 0$$

Consider the parametric equation of this boundary of the hemisphere *S*,

$$x = \cos t, z = \sin t,\ 0 \le t \le 2\pi$$

So the equation of the boundary curve is given as,

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$$

$$\mathbf{r}(t) = \langle \cos t, 0, \sin t\rangle,\ 0 \le t \le 2\pi$$

$$\mathbf{r}'(t) = \langle -\sin t, 0, \cos t\rangle$$ ...... (2)

Also, $x(t) = \cos t, y(t) = 0, z(t) = \sin t$

Then the vector field becomes,

$$\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) = 0\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) = \langle 0, \sin t, \cos t\rangle$$ ...... (3)

So by the equations (2), and (3), we have the line integral as

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{t=0}^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$$

$$= \int_0^{2\pi} \langle 0, \sin t, \cos t\rangle \cdot \langle -\sin t, 0, \cos t\rangle\,dt$$

$$= \int_0^{2\pi} \left(0 + 0 + \cos^2 t\right)dt$$

$$= \int_0^{2\pi} \cos^2 t\,dt$$

$$= \int_0^{2\pi} \left(\frac{1 + \cos 2t}{2}\right)dt$$

$$= \frac{1}{2}\int_0^{2\pi} (1 + \cos 2t)\,dt$$

Continuation of the above,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{2}\left[t+\frac{1}{2}\sin 2t\right]_0^{2\pi}$$
$$= \frac{1}{2}\left(\left(2\pi+\frac{1}{2}\sin 4\pi\right)-(0+0)\right)$$
$$= \frac{1}{2}(2\pi)$$
$$= \pi$$

Therefore, we have $\int_C \mathbf{F}\cdot d\mathbf{r} = \pi$ ...... (4)

So by the equations (1) and (2), observe that

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \pi$$

Hence Stokes' Theorem is verified for the vector field $\mathbf{F}(x,y,z) = y\mathbf{i}+z\mathbf{j}+x\mathbf{k}$, and for the given surface S: $x^2+y^2+z^2=1, y\geq 0$.

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We use Stokes' Theorem to write the line integral as a surface integral and then simplify until we can prove that the eventual solution will not depend on the shape or location of C.

Stokes' Theorem:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} \quad ...... (1)$$

We will also need the following formula for equating different forms of line integrals:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx+Q\,dy+R\,dz \quad ...... (2)$$

When $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$. Since we can apply Stokes' Theorem to a line integral in the form of the left side of (2), we can also apply it to a line integral of the form of the right side of (2).

We will also need the following formula for calculating surface integrals:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D\left(-P\frac{\partial g}{\partial x}-Q\frac{\partial g}{\partial y}+R\right)dA \quad ...... (3)$$

Where $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ and the surface S is given by the graph $z=g(x,y)$ over the region D.

We start with the given line integral and convert it as follows:

$$\int_C z\,dx-2x\,dy+3y\,dz = \iint_S \operatorname{curl}(z\mathbf{i}-2x\mathbf{j}+3y\mathbf{k})\cdot d\mathbf{S} \quad ...... (4$$

By recognizing it has the form in (2) and applying Stokes' Theorem (equation (1)) to convert to the surface integral.

Find the curl in the integrand. The easiest way to find the curl of a vector $\mathbf{F}$ is to take the cross product $\nabla\times\mathbf{F}$. Use a determinant to calculate cross product: the cross product of two vectors $\mathbf{a}=\langle a_1,a_2,a_3\rangle$ and $\mathbf{b}=\langle b_1,b_2,b_3\rangle$ can be written as

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Since in this case $\mathbf{F}(x,y,z) = z\mathbf{i}-2x\mathbf{j}+3y\mathbf{k}$, the curl of $\mathbf{F}$ is

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -2x & 3y \end{vmatrix}$$
$$= \left[\frac{\partial}{\partial y}(3y)-\frac{\partial}{\partial z}(-2x)\right]\mathbf{i}-\left[\frac{\partial}{\partial x}(3y)-\frac{\partial}{\partial z}(z)\right]\mathbf{j}+\left[\frac{\partial}{\partial x}(-2x)-\frac{\partial}{\partial y}(z)\right]\mathbf{k}$$
$$= (3-0)\mathbf{i}-(0-1)\mathbf{j}+(-2-0)\mathbf{k}$$
$$= 3\mathbf{i}+1\mathbf{j}-2\mathbf{k}$$

We can now plug this vector in for the curl in (4):

$$\int_C zdx - 2xdy + 3ydz = \iint_S (3\mathbf{i} + 1\mathbf{j} - 2\mathbf{k}) \cdot d\mathbf{S}$$

We wish to apply (3) to solve the surface integral. We need an equation for the surface S; we know C is in plane $x + y + z = 1$ so we rewrite this as $z = 1 - x - y$ and use that as the $g(x, y)$ function that generates the surface S. Find the necessary partial derivatives:

$$\frac{\partial g}{\partial x} = -1$$

$$\frac{\partial g}{\partial y} = -1$$

And plug into (3):

$$\begin{aligned}\int_C zdx - 2xdy + 3ydz &= \iint_S (3\mathbf{i} + 1\mathbf{j} - 2\mathbf{k}) \cdot d\mathbf{S} \\ &= \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA \\ &= \iint_D (-3(-1) - 1(-1) - 2) dA \\ &= \boxed{\iint_D (2) dA}\end{aligned}$$

We have successfully simplified our original line integral down to a double integral over a region D. This double integral can be thought of as a volume with D as the base and the integrand as the height. Since the integrand is 2, a constant, the value of the integral will equal the area of D times 2, just as we can find the volume of any solid with a flat base and perpendicular height by multiplying the area of the base times the height. Therefore, the only quantity that will cause this integral's solution to vary is the area of the region D, and since D is the region under the surface that was originally enclosed by C, D is also enclosed by C. Therefore, this integral varies only according to the area enclosed by C, and other factors, such as the location and shape of C, will not affect the solution to the integral.

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The work is the line integral of the force along the given path C. We calculate this integral by converting to a surface area via Stokes' Theorem.
Stokes' Theorem:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} \quad \ldots\ldots (1)$$

We will also need the following formula for calculating surface integrals:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA \quad \ldots\ldots (2)$$

Where $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ and the surface S is given by the graph $z = g(x, y)$ over the region D.

We know

$$Work = \int_C \mathbf{F} \cdot d\mathbf{r}$$

So by (1) we know that

$$Work = \iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} \quad \ldots\ldots (3$$

Also. We calculate curl **F**. The easiest way to find the curl of a vector **F** is to take the cross product $\nabla \times \mathbf{F}$. Use a determinant to calculate cross product: the cross product of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ can be written as

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Since in this case $\mathbf{F}(x, y, z) = z^2\mathbf{i} + 2xy\mathbf{j} + 4y^2\mathbf{k}$, the curl of **F** is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & 2xy & 4y^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(4y^2) - \frac{\partial}{\partial z}(2xy)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(4y^2) - \frac{\partial}{\partial z}(z^2)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(z^2)\right]\mathbf{k}$$

$$= (8y - 0)\mathbf{i} - (0 - 2z)\mathbf{j} + (2y - 0)\mathbf{k}$$

$$= 8y\mathbf{i} + 2z\mathbf{j} + 2y\mathbf{k}$$

We can now plug this vector in for the curl in equation (3):

$$Work = \iint_S (8y\mathbf{i} + 2z\mathbf{j} + 2y\mathbf{k}) \cdot d\mathbf{S}$$

We wish to apply (2) to solve the surface integral. We need an equation for a surface S containing the path of the particle. The four points the particle traverses form a rectangle with one side along the x-axis, one side along the line such that $y = 2, z = 1$, and two parallel lines that slant upward from the x-axis to meet the other line. Since the slanted sides of the rectangle go from $(x, 0, 0)$ to $(x, 2, 1)$, this plane's projection in the yz-plane is a line of slope 1/2. Therefore, the equation for this plane is $z = y/2$, as that is true no matter what x is.

The only other thing we need to plug into (2) is the region of integration D that bounds the surface. The boundaries of the region in x and y are $0 \le x \le 1$ and $0 \le 2 \le y$, so those are our limits of integration in x and y when plugging into (2).

Solve $Work = \iint_S (8y\mathbf{i} + 2z\mathbf{j} + 2y\mathbf{k}) \cdot d\mathbf{S}$ by plugging into (2)

$$Work = \iint_S (8y\mathbf{i} + 2z\mathbf{j} + 2y\mathbf{k}) \cdot d\mathbf{S}$$

$$= \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

$$= \int_0^1 \int_0^2 (-8y(0) - 2z(1/2) + 2y)\, dy\, dx$$

$$= \int_0^1 \int_0^2 (-z + 2y)\, dy\, dx$$

We fill in $z = y/2$ from the surface equation:

$$Work = \int_0^1 \int_0^2 \left(-\frac{y}{2} + 2y\right) dy\, dx$$

$$= \int_0^1 \left(-\frac{y^2}{4} + y^2\right)\Bigg|_0^2 dx$$

$$= \int_0^1 \left(-\frac{2^2}{4} + 2^2 - 0\right) dx$$

$$= \int_0^1 (3)\, dx$$

$$= 3x\Big|_0^1$$

$$= 3(1) - 0$$

$$= \boxed{3}$$

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Consider the following to be evaluated:

$$\int_C (y+\sin x)\,dx + (z^2+\cos y)\,dy + x^3\,dz$$

Where C is the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle, 0 \le t \le 2\pi$.

Rewrite the given integral as:

$$\int_C < y+\sin x, z^2+\cos y, x^3 > . < dx, dy, dz >$$

This is in the form $\int_C \vec{F}.d\vec{r}$, where $\vec{F}(x,y,z) = < y+\sin x, z^2+\cos y, x^3 >$

By stokes theorem:

$$\int_C \vec{F}.d\vec{r} = -\iint_S curl\vec{F}.d\vec{s}$$

Here the curve C has parametric representation:

$$\vec{r}(t) = < \sin t, \cos t, \sin 2t >$$

That is:

$$x = \sin t, y = \cos t, z = \sin 2t$$

Now, since $2\sin t\cos t = \sin 2t$, then clearly the curve C lies on the surface $z = 2xy$.

Then let

$$f(x,y,z) = z - 2xy$$

Then S is a level surface of $f(x,y,z) = 0$. Hence, $\vec{\nabla} f(x,y,z)$ is normal to the surface S.

Now, the unit normal vector is:

$$\hat{n} = \frac{\vec{\nabla} f}{\|\vec{\nabla} f\|}$$

Note that

$$\vec{\nabla} f = < -2y, -2x, 1 >$$

And

$$\|\vec{\nabla} f\| = \sqrt{4x^2 + 4y^2 + 1}$$

Also, $\frac{\partial z}{\partial x} = 2y \; and \; \frac{\partial z}{\partial y} = 2x$

Then evaluate:

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{4x^2+4y^2+1} = \|\vec{\nabla} f\|$$

Therefore the integral is:

$$\begin{aligned}\iint_C \text{curl}\,\vec{F}.d\vec{s} &= \iint_S \text{curl}\vec{F}.\vec{n}ds \\ &= \iint_D \text{curl}\,\vec{F}.\frac{\vec{\nabla} f}{\|\vec{\nabla} f\|}\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}dA \\ &= \iint_D < -2z, -3x^2, -1 > < -2y, -2x, 1 > dA \\ &= \iint_D (4yz + 6x^3 - 1)\,dA\end{aligned}$$

Further, take $\frac{\partial Q}{\partial x}=8xy^2$ and $\frac{\partial P}{\partial y}=6x^3-1$

Now, apply Green's Theorem:

$$\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA=\int_C Pdx+Qdy$$

Where,

$Q=4x^2y^2$ and $P=6x^3y-y$

Then the integral is:

$$\begin{aligned}\iint_S \text{curl}\vec{F}.d\vec{s}&=\iint_D\left(8xy^2+6x^3-1\right)dA\\&=\int_C\left(6x^3y-y\right)dx+4x^2y^2dy\\&=\int_0^{2\pi}\left(6\sin^3 t\cos t-\cos t\right)\cos t dt+4\sin^2 t\cos^2 t\left(-\sin t\right)dt\\&=\int_0^{2\pi}6\sin^3 t\cos^2 t-\cos^2 t-4\sin^3 t\cos^2 tdt\end{aligned}$$

Further evaluate:

$$\begin{aligned}\iint_S \text{curl}\vec{F}.d\vec{s}&=\int_0^{2\pi}2\sin^3 t\cos^2 t-\cos^2 tdt\\&=2\left[\frac{-\sin^2 t\cos^3 t}{5}-\frac{2}{15}\cos^3 t\right]_0^{2\pi}-\left[\frac{t}{2}+\frac{1}{4}\sin 2t\right]_0^{2\pi}\\&=2(0)-\pi\\&=-\pi\end{aligned}$$

Therefore the value of the given integral is:

$$\boxed{\int_C\left(y+\sin x\right)dx+\left(z^2+\cos y\right)dy+x^3dz=-\pi}$$

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S is a sphere and $\vec{F}$ satisfies Stoke's theorem,
Take a simple closed curve T on S. it separates S into two parts S_1 and S_2

Then
$$\begin{aligned}\iint_S curl\,\vec{F}.d\vec{s}&=\iint_{S_1} curl\,\vec{F}.d\vec{s}+\iint_{S_2} curl\,\vec{F}.d\vec{s}\\&=\int_T\vec{F}.d\vec{r}+\int_{-T}\vec{F}.d\vec{r}\\&\text{(By Stoke's theorem)}\\&=\int_T\vec{F}.d\vec{r}-\int_T\vec{F}.d\vec{r}\\&=0\end{aligned}$$

Hence $\iint_S curl\,\vec{F}.d\vec{s}=0$

Chapter 16 Vector Calculus Exercise 16.820E

(A)

$$\begin{aligned}\int_C\left(f\vec{\nabla}g\right).d\vec{r}&=\iint_S curl\left(f\vec{\nabla}g\right).d\vec{s}\\&\text{(By Stoke's theorem)}\\&=\iint_S\left[f\,curl\left(\vec{\nabla}g\right)+\vec{\nabla}f\times\vec{\nabla}g\right].d\vec{s}\\&\left(\text{As } curl\left(f\vec{F}\right)=f\,curl\,\vec{F}+\vec{\nabla}f\times\vec{F}\right)\\&=\iint_S\left[f\,curl\left(grad\ g\right)+\left(\vec{\nabla}f\times\vec{\nabla}g\right)\right].d\vec{s}\\&\left(\text{Since } curl\left(grad\ f\right)=0\right)\\&=\iint_S\left(\vec{\nabla}f\times\vec{\nabla}g\right).d\vec{s}\end{aligned}$$

(B)

$$\int_C \left(f\,\vec{\nabla}f\right).d\vec{r} = \iint_S curl\left(f\,\vec{\nabla}f\right).d\vec{s}$$

(By Stoke's theorem)

$$= \iint_S \left[f\,curl\left(\vec{\nabla}f\right) + \left(\vec{\nabla}f\right)\times\left(\vec{\nabla}f\right)\right].d\vec{s}$$

(As $curl\left(f\,\vec{F}\right) = f\,curl\,\vec{F} + \vec{\nabla}f\times\vec{F}$)

$$= \iint_S \left[f\left(\vec{0}\right) + \vec{0}\right].d\vec{s}$$

(As $curl\left(grad\ f\right) = \vec{0}$ and $\vec{a}\times\vec{a} = \vec{0}$)

$$= \iint_S \vec{0}.d\vec{s}$$

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$$= 0$$

(C)

$$\int_C \left(f\,\vec{\nabla}g + g\vec{\nabla}f\right).d\vec{r} = \iint_S curl\left(f\,\vec{\nabla}g + g\,\vec{\nabla}f\right).d\vec{s}$$

(By Stokes theorem)

$$= \iint_S \left[curl\left(f\,\vec{\nabla}g\right) + curl\left(g\vec{\nabla}f\right)\right].d\vec{s}$$

(As $curl\left(\vec{F}+\vec{G}\right) = curl\,\vec{F} + curl\,\vec{G}$)

$$= \iint_S \left[f\,curl\left(\vec{\nabla}g\right) + \vec{\nabla}f\times\vec{\nabla}g + g\,curl\left(\vec{\nabla}f\right) + \vec{\nabla}g\times\vec{\nabla}f\right].d\vec{s}$$

(As $curl\left(f\,\vec{F}\right) = f\,curl\,\vec{F} + \vec{\nabla}f\times\vec{F}$)

$$= \iint_S \left[f\vec{0} + \vec{\nabla}f\times\vec{\nabla}g + g\vec{0} - \vec{\nabla}f\times\vec{\nabla}g\right].d\vec{s}$$

(As $curl\left(grad\ f\right) = 0$ and $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$)

$$= \iint_S \vec{0}.\ d\vec{s}$$

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$$= 0$$

(C)

$$\int_C \left(f\,\vec{\nabla}g + g\vec{\nabla}f\right).d\vec{r} = \iint_S curl\left(f\,\vec{\nabla}g + g\,\vec{\nabla}f\right).d\vec{s}$$

(By Stokes theorem)

$$= \iint_S \left[curl\left(f\,\vec{\nabla}g\right) + curl\left(g\vec{\nabla}f\right)\right].d\vec{s}$$

(As $curl\left(\vec{F}+\vec{G}\right) = curl\,\vec{F} + curl\,\vec{G}$)

$$= \iint_S \left[f\,curl\left(\vec{\nabla}g\right) + \vec{\nabla}f\times\vec{\nabla}g + g\,curl\left(\vec{\nabla}f\right) + \vec{\nabla}g\times\vec{\nabla}f\right].d\vec{s}$$

(As $curl\left(f\,\vec{F}\right) = f\,curl\,\vec{F} + \vec{\nabla}f\times\vec{F}$)

$$= \iint_S \left[f\vec{0} + \vec{\nabla}f\times\vec{\nabla}g + g\vec{0} - \vec{\nabla}f\times\vec{\nabla}g\right].d\vec{s}$$

(As $curl\left(grad\ f\right) = 0$ and $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$)

$$= \iint_S \vec{0}.\ d\vec{s}$$

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$$= 0$$