10

Fourier Methods

Fourier Series

In this chapter we are going to look in more detail at the implications of the principles of superposition which we met at the beginning of the book when we added the two separate solutions of the simple harmonic motion equation. Our discussion of monochromatic waves has led to the idea of repetitive behaviour in a simple form. Now we consider more complicated forms of repetition which arise from superposition.

Any function which repeats itself regularly over a given interval of space or time is called a periodic function. This may be expressed by writing it as $f(x) = f(x \pm \alpha)$ where α is the interval or period.

The simplest examples of a periodic function are sines and cosines of fixed frequency and wavelength, where α represents the period τ, the wavelength λ or the phase angle 2π rad, according to the form of x. Most periodic functions for example the square wave system of Figure 10.1, although quite simple to visualize are more complicated to represent mathematically. Fortunately this can be done for almost all periodic functions of interest in physics using the method of Fourier Series, which states that any periodic function may be represented by the series

$$\begin{aligned} f(x) = \tfrac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x \ldots + a_n \cos nx \\ + b_1 \sin x + b_2 \sin 2x \ldots + b_n \sin nx, \end{aligned} \tag{10.1}$$

that is, a constant $\frac{1}{2}a_0$ plus sine and cosine terms of different amplitudes, having frequencies which increase in discrete steps. Such a series must of course, satisfy certain conditions, chiefly those of convergence. These convergence criteria are met for a function with discontinuities which are not too severe and with first and second differential coefficients which are well behaved. At such discontinuities, for instance in the square wave where $f(x) = \pm h$ at $x = 0, \pm 2\pi$, etc. the series represents the mean of the values of the function just to the left and just to the right of the discontinuity.

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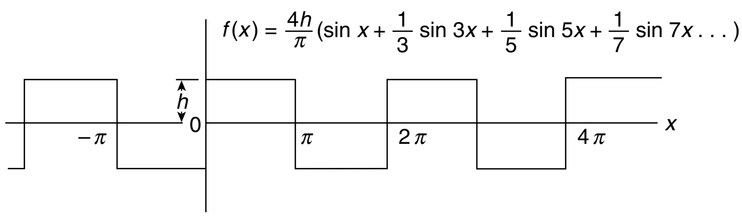


Figure 10.1 Square wave of height h and its Fourier sine series representation (odd function)

We may write the series in several equivalent forms:

$$\begin{aligned} f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} c_n \cos(nx - \theta_n) \end{aligned}$$

where

$$c_n^2 = a_n^2 + b_n^2$$

and

$$\tan\theta_n = b_n/a_n$$

or

$$f(x) = \sum_{n=-\infty}^{\infty} d_n \, \mathrm{e}^{\mathrm{i}nx}$$

where

$$2d_n = a_n - \mathrm{i}b_n (n \geq 0)$$

and

$$2d_n = a_{-n} + \mathrm{i}b_{-n} (n < 0)$$

To find the values of the coefficients a_n and b_n let us multiply both sides of equation (10.1) by $\cos nx$ and integrate with respect to x over the period 0 to 2π (say).

Every term

$$\int_0^{2\pi} \cos mx \cos nx \, \mathrm{d}x = \begin{cases} 0 \text{ if } m \neq n \\ \pi \text{ if } m = n \end{cases}$$

whilst every term

$$\int_0^{2\pi} \sin mx \cos nx \, \mathrm{d}x = 0 \text{ for all } m \text{ and } n.$$

Thus for $m = n$,

$$a_n \int_0^{2\pi} \cos^2 nx\,\mathrm{d}x = \pi a_n$$

so that

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\,\mathrm{d}x$$

Similarly, by multiplying both sides of equation (10.1) by $\sin nx$ and integrating from 0 to 2π we have, since

$$\int_0^{2\pi} \sin mx \sin nx\,\mathrm{d}x = \begin{cases} 0 \text{ if } m \neq n \\ \pi \text{ if } m = n \end{cases}$$

that

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\,\mathrm{d}x$$

Immediately we see that the constant $(n = 0)$, given by $\frac{1}{2}a_0 = \ 1/2\pi \int_0^{2\pi} f(x)\,\mathrm{d}x$, is just the average of the function over the interval 2π. It is, therefore, the steady or 'd.c.' level on which the alternating sine and cosine components of the series are superimposed, and the constant can be varied by moving the function with respect to the x-axis. When a periodic function is symmetric about the x-axis its average value, that is, its steady or d.c. base level, $\frac{1}{2}a_0$, is zero, as in the square wave system of Figure 10.1. If we raise the square waves so that they stand as pulses of height $2h$ on the x-axis, the value of $\frac{1}{2}a_0$ is $h\pi$ (average value over 2π). The values of a_n represent twice the average value of the product $f(x) \cos nx$ over the interval 2π; b_n can be interpreted in a similar way.

We see also that the series representation of the function is the sum of cosine terms which are even functions $[\cos x = \cos(-x)]$ and of sine terms which are odd functions $[\sin x = -\sin(-x)]$. Now every function $f(x) = \frac{1}{2}[f(x) + f(-x)] \ + \frac{1}{2}[f(x) - f(-x)]$, in which the first bracket is even and the second bracket is odd. Thus, the cosine part of a Fourier series represents the even part of the function and the sine terms represent the odd part of the function. Taking the argument one stage further, a function $f(x)$ which is an even function is represented by a Fourier series having only cosine terms; if $f(x)$ is odd it will have only sine terms in its Fourier representation. Whether a function is completely even or completely odd can often be determined by the position of the y-axis. Our square wave of Figure 10.1 is an odd function $[f(x) = -f(-x)]$; it has no constant and is represented by $f(x) = 4h/\pi(\sin x + 1/3 \sin 3x \ + 1/5 \sin 5x$, etc. but if we now move the y-axis a half period to the right as in Figure 10.2, then $f(x) = f(-x)$, an even function, and the square wave is represented by

$$f(x) = \frac{4h}{\pi}\left(\cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \tfrac{1}{7}\cos 7x + \cdots\right)$$

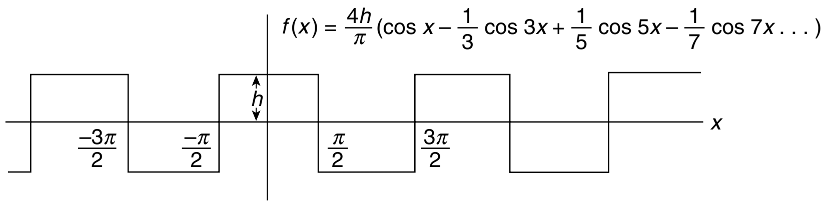


Figure 10.2 The wave of Figure 10.1 is now symmetric about the y axis and becomes a cosine series (even function)

If we take the first three or four terms of the series representing the square wave of Figure 10.1 and add them together, the result is Figure 10.3. The fundamental, or first harmonic, has the frequency of the square wave and the higher frequencies build up the squareness of the wave. The highest frequencies are responsible for the sharpness of the vertical sides of the waves; this type of square wave is commonly used to test the frequency response of amplifiers. An amplifier with a square wave input effectively 'Fourier analyses' the input and responds to the individual frequency components. It then puts them together again at its output, and if a perfect square wave emerges from the amplifier it proves that the amplifier can handle the whole range of the frequency components equally well. Loss of sharpness at the edges of the waves shows that the amplifier response is limited at the higher frequency range.

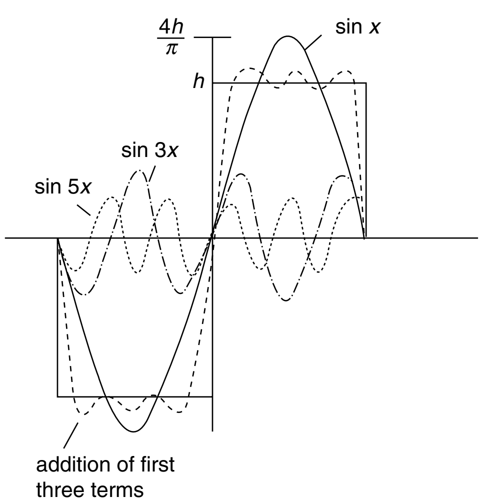


Figure 10.3 Addition of the first three terms of the Fourier series for the square wave of Figure 10.1 shows that the higher frequencies are responsible for sharpening the edges of the pulse

Example of Fourier Series

Consider the square wave of height h in Figure 10.1. The value of the function is given by

$$f(x) = h \quad \text{for} \quad 0 < x < \pi$$

and

$$f(x) = -h \quad \text{for} \quad \pi < x < 2\pi$$

The coefficients of the series representation are given by

$$a_n = \frac{1}{\pi}\left[h \int_0^{\pi} \cos nx \,\mathrm{d}x - h \int_{\pi}^{2\pi} \cos nx \,\mathrm{d}x\right] = 0$$

because

$$\int_0^{\pi} \cos nx \,\mathrm{d}x = \int_{\pi}^{2\pi} \cos nx \,\mathrm{d}x = 0$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\left[h \int_0^{\pi} \sin\, nx \,\mathrm{d}x - h \int_{\pi}^{2\pi} \sin nx \,\mathrm{d}x\right] \\ &= \frac{h}{n\pi}\left[[\cos nx]_{\pi}^{0} + [\cos nx]_{\pi}^{2\pi}\right] \\ &= \frac{h}{n\pi}\left[(1 - \cos n\pi) + (1 - \cos n\pi)\right] \end{aligned}$$

giving $b_n = 0$ for n even and $b_n = 4h/n\pi$ for n odd. Thus, the Fourier series representation of the square wave is given by

$$f(x) = \frac{4h}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\right)$$

Fourier Series for any Interval

Although we have discussed the Fourier representation in terms of a periodic function its application is much more fundamental, for any section or interval of a well behaved function may be chosen and expressed in terms of a Fourier series. This series will accurately represent the function *only within the chosen interval*. If applied outside that interval it will not follow the function but will periodically repeat the value of the function within the chosen interval. If we represent this interval by a Fourier cosine series the repetition will be that of an even function, if the representation is a Fourier sine series an odd function repetition will follow.

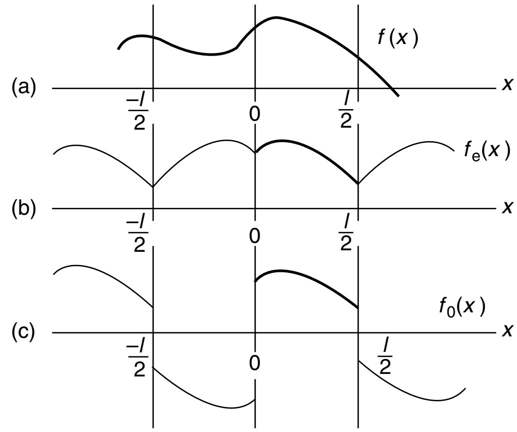


Figure 10.4 A Fourier series may represent a function over a selected half-interval. The general function in (a) is represented in the half-interval $0 < x < l/2$ by f_e, an even function cosine series in (b), and by f_o, an odd function sine series in (c). These representations are valid only in the specified half-interval. Their behaviour outside that half-interval is purely repetitive and departs from the original function

Suppose now that we are interested in the behaviour of a function over only one-half of its full interval and have no interest in its representation outside this restricted region. In Figure 10.4a the function $f(x)$ is shown over its full space interval $-l/2$ to $+l/2$, but $f(x)$ can be represented completely in the interval 0 to $+l/2$ by either a cosine function (which will repeat itself each half-interval as an even function) or it can be represented completely by a sine function, in which case it will repeat itself each half-interval as an odd function. Neither representation will match $f(x)$ outside the region 0 to $+l/2$, but in the half-interval 0 to $+l/2$ we can write

$$f(x) = f_e(x) = f_o(x)$$

where the subscripts e and o are the even (cosine) or odd (sine) Fourier representations, respectively.

The arguments of sines and cosines must, of course, be phase angles, and so far the variables x has been measured in radians. Now, however, the interval is specified as a distance and the variable becomes $2\pi x/l$, so that each time x changes by l the phase angle changes by 2π.

Thus

$$f_e(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi n x}{l}$$

where

$$\begin{aligned}
a_n &= \frac{1}{\frac{1}{2}\,\text{interval}} \int_{-l/2}^{l/2} f(x)\cos\frac{2\pi n x}{l}\,\mathrm{d}x \\
&= \frac{2}{l}\left[\int_{-l/2}^{0} f_{\mathrm{e}}(x)\cos\frac{2\pi n x}{l}\,\mathrm{d}x + \int_{0}^{l/2} f_{\mathrm{e}}(x)\cos\frac{2\pi n x}{l}\,\mathrm{d}x\right] \\
&= \frac{4}{l}\int_{0}^{l/2} f(x)\cos\frac{2\pi n x}{l}\,\mathrm{d}x
\end{aligned}$$

because

$$f(x) = f_{\mathrm{e}}(x) \quad \text{from } x = 0 \text{ to } l/2$$

and

$$f(x) = f(-x) = f_{\mathrm{e}}(x) \quad \text{from } x = 0 \text{ to } -l/2$$

Similarly we can represent $f(x)$ by the sine series

$$f(x) = f_{\mathrm{o}}(x) = \sum_{n=1}^{\infty} b_n \sin\frac{2\pi n x}{l}$$

in the range $x = 0$ to $l/2$ with

$$\begin{aligned}
b_n &= \frac{1}{\frac{1}{2}\,\text{interval}} \int_{-l/2}^{l/2} f(x)\sin\frac{2\pi n x}{l}\,\mathrm{d}x \\
&= \frac{2}{l}\left[\int_{-l/2}^{0} f_{\mathrm{o}}(x)\sin\frac{2\pi n x}{l}\,\mathrm{d}x + \int_{0}^{l/2} f_{\mathrm{o}}(x)\sin\frac{2\pi n x}{l}\,\mathrm{d}x\right]
\end{aligned}$$

In the second integral $f_{\mathrm{o}}(x) = f(x)$ in the interval 0 to $l/2$ whilst

$$\begin{aligned}
\int_{-l/2}^{0} f_{\mathrm{o}}(x)\sin\frac{2\pi n x}{l}\,\mathrm{d}x &= \int_{l/2}^{0} f_{\mathrm{o}}(-x)\sin\frac{2\pi n x}{l}\,\mathrm{d}x = -\int_{l/2}^{0} f_{\mathrm{o}}(x)\sin\frac{2\pi n x}{l}\,\mathrm{d}x \\
&= \int_{0}^{l/2} f_{\mathrm{o}}(x)\sin\frac{2\pi n x}{l}\,\mathrm{d}x = \int_{0}^{l/2} f(x)\sin\frac{2\pi n x}{l}\,\mathrm{d}x
\end{aligned}$$

Hence

$$b_n = \frac{4}{l}\int_{0}^{l/2} f(x)\sin\frac{2\pi n x}{l}\,\mathrm{d}x$$

If we follow the behaviour of $f_e(x)$ and $f_o(x)$ outside the half-interval 0 to $l/2$ (Figure 10.4a, b) we see that they no longer represent $f(x)$.

Application of Fourier Sine Series to a Triangular Function

Figure 10.5 shows a function which we are going to describe by a sine series in the half-interval 0 to π. The function is

$$f(x) = x \quad \left(0 < x < \frac{\pi}{2}\right)$$

and

$$f(x) = \pi - x \quad \left(\frac{\pi}{2} < x < \pi\right)$$

Writing $f(x) = \sum b_n \sin nx$ gives

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi/2} x \sin nx \, \mathrm{d}x + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\sin nx \, \mathrm{d}x \\ &= \frac{4}{n^2\pi}\sin\frac{n\pi}{2} \end{aligned}$$

When n is even $\sin n\pi/2 = 0$, so that only terms with odd values of n are present and

$$f(x) = \frac{4}{\pi}\left(\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + \cdots\right)$$

Note that at $x = \pi/2$, $f(x) = \pi/2$, giving

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

We shall use this result a little later.

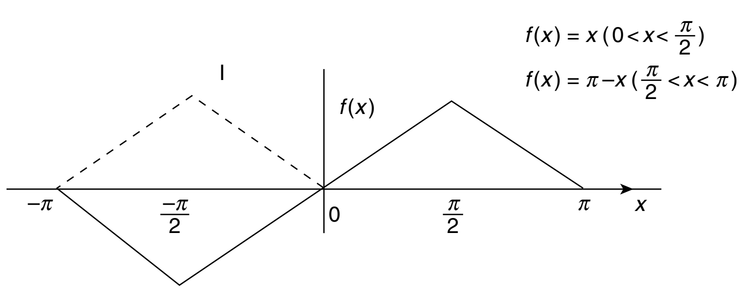


Figure 10.5 Function representing a plucked string and defined over a limited interval. When the string vibrates all the permitted harmonics contribute to the initial configuration

Note that the solid line in the interval 0 to $-\pi$ in Figure 10.5 is the Fourier sine representation for $f(x)$ repeated outside the interval 0 to π whilst the dotted line would result if we had represented $f(x)$ in the interval 0 to π by an even cosine series.

(Problems 10.1, 10.2, 10.3, 10.4, 10.5, 10.6, 10.7, 10.8, 10.9)

Application to the Energy in the Normal Modes of a Vibrating String

If we take a string of length l with fixed ends and pluck its centre a distance d we have the configuration of the half interval 0 to π of Figure 10.5 which we represented as a Fourier sine series. Releasing the string will set up its normal mode or standing wave vibrations, each of which we have shown on p. 126 to have the displacement

$$y_n = (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin \frac{\omega_n x}{c} \tag{5.10}$$

where $\omega_n = n\pi c/l$ is the normal mode frequency.

The total displacement, which represents the shape of the plucked string at $t = 0$ is given by summing the normal modes

$$y = \sum y_n = \sum (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin \frac{\omega_n x}{c}$$

Note that this sum resembles a Fourier series where the fixed ends of the string, $y = 0$ at $x = 0$ and $x = l$ allow only the sine terms in x in the series expansion. If the string remains plucked *at rest* only the terms in x with appropriate coefficients are required to describe it, but its vibrational motion after release has a time dependence which is expressed in each harmonic coefficient as

$$A_n \cos \omega_n t + B_n \sin \omega_n t$$

The significance of these coefficients emerges when we consider the initial or boundary conditions in time.

Let us write the total displacement of the string at time $t = 0$ as

$$\begin{aligned} y_0(x) = \sum y_n(x) &= \sum (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin \frac{\omega_n x}{c} \\ &= \sum A_n \sin \frac{\omega_n x}{c} \quad \text{at} \quad t = 0 \end{aligned}$$

Similarly we write the velocity of the string at time $t = 0$ as

$$\begin{aligned} v_0(x) &= \frac{\partial}{\partial t} y_0(x) = \sum \dot{y}_n(x) \\ &= \sum (-\omega_n A_n \sin \omega_n t + \omega_n B_n \cos \omega_n t) \sin \frac{\omega_n x}{c} \\ &= \sum \omega_n B_n \sin \frac{\omega_n x}{c} \quad \text{at} \quad t = 0 \end{aligned}$$

Both $y_0(x)$ and $v_0(x)$ are thus expressed as Fourier sine series, but if the string is at rest at $t = 0$, then $v_0(x) = 0$ and all the B_n coefficients are zero, leaving only the A_n's. If the displacement of the string $y_0(x) = 0$ at time $t = 0$ whilst the string is moving, then all the A_n's are zero and the Fourier coefficients are the $\omega_n B_n$'s.

We can solve for both A_n and $\omega_n B_n$ in the usual way for if

$$y_0(x) = \sum A_n \sin \frac{\omega_n x}{c}$$

and

$$v_0(x) = \sum \omega_n B_n \sin \frac{\omega_n x}{c}$$

for a string of length l then

$$A_n = \frac{2}{l} \int_0^l y_0(x) \sin \frac{\omega_n x}{c} \, \mathrm{d}x$$

and

$$\omega_n B_n = \frac{2}{l} \int_0^l v_0(x) \sin \frac{\omega_n x}{c} \, \mathrm{d}x$$

If the plucked string of mass m (linear density ρ) is released from rest at $t = 0$ $(v_0(x) = 0)$ the energy in each of its normal modes of vibration, given on p. 134 as

$$E_n = \tfrac{1}{4} m \omega_n^2 (A_n^2 + B_n^2)$$

is simply

$$E_n = \tfrac{1}{4} m \omega_n^2 A_n^2$$

because all B_n's are zero.

The total vibrational energy of the released string will be the sum $\sum E_n$ over all the modes present in the vibration.

Let us now solve the problem of the plucked string released from rest. The configuration of Figure 10.5 (string length l, centre plucked a distance d) is given by

$$\begin{aligned} y_0(x) &= \frac{2dx}{l} \qquad && 0 \le x \le \frac{l}{2} \\ &= \frac{2d(l-x)}{l} && \frac{l}{2} \le x \le l \end{aligned}$$

so

$$\begin{aligned} A_n &= \frac{2}{l} \left[\int_0^{l/2} \frac{2dx}{l} \sin \frac{\omega_n x}{c} \, \mathrm{d}x + \int_{l/2}^{l} \frac{2d(l-x)}{l} \sin \frac{\omega_n x}{c} \, \mathrm{d}x \right] \\ &= \frac{8d}{n^2 \pi^2} \sin \frac{n\pi}{2} \left(\text{for } \omega_n = \frac{n\pi c}{l} \right) \end{aligned}$$

We see at once that $A_n = 0$ for n even (when the sine term is zero) so that all even harmonic modes are missing. The physical explanation for this is that the even harmonics would require a node at the centre of the string which is always moving after release.

The displacement of our plucked string is therefore given by the addition of all the permitted (odd) modes as

$$y_0(x) = \sum_{n\,\text{odd}} y_n(x) = \sum_{n\,\text{odd}} A_n \sin\frac{\omega_n x}{c}$$

where

$$A_n = \frac{8d}{n^2\pi^2}\sin\frac{n\pi}{2}$$

The energy of the *n*th mode of oscillation is

$$E_n = \frac{1}{4} m\omega_n^2 A_n^2 = \frac{64d^2 m\omega_n^2}{4(n^2\pi^2)^2}$$

and the total vibrational energy of the string is given by

$$E = \sum_{n\,\text{odd}} E_n = \frac{16d^2 m}{\pi^4}\sum_{n\,\text{odd}}\frac{\omega_n^2}{n^4} = \frac{16d^2c^2m}{\pi^2 l^2}\sum_{n\,\text{odd}}\frac{1}{n^2}$$

for

$$\omega_n = \frac{n\pi c}{l}$$

But we saw in the last section that

$$\sum_{n\,\text{odd}}\frac{1}{n^2} = \frac{\pi^2}{8}$$

so

$$E = \sum E_n = \frac{2mc^2d^2}{l^2} = \frac{2Td^2}{l}$$

where $T = \rho c^2$ is the constant tension in the string.

This vibrational energy, in the absence of dissipation, must be equal to the potential energy of the plucked string before release and the reader should prove this by calculating the work done in plucking the centre of the string a small distance d, where $d \ll l$.

To summarize, our plucked string can be represented as a sine series of Fourier components, each giving an allowed normal mode of vibration when it is released. The concept of normal modes allows the energies of each mode to be added to give the total energy of vibration which must equal the potential energy of the plucked string before release. The energy of the *n*th mode is proportional to n^{-2} and therefore decreases with increasing frequency. Even modes are forbidden by the initial boundary conditions.

The boundary conditions determine which modes are allowed. If the string were struck by a hammer those harmonics having a node at the point of impact would be absent, as in the case of the plucked string. Pianos are commonly designed with the hammer striking a point one seventh of the way along the string, thus eliminating the seventh harmonic which combines to produce discordant effects.

Fourier Series Analysis of a Rectangular Velocity Pulse on a String

Let us now consider a problem similar to that of the last section except that now the displacement $y_0(x)$ of the string is zero at time $t = 0$ whilst the velocity $v_0(x)$ is non-zero. A string of length l, fixed at both ends, is struck by a mallet of width a about its centre point. At the moment of impact the displacement

$$y_0(x) = 0$$

but the velocity

$$\begin{aligned} v_0(x) = \frac{\partial y_0(x)}{\partial t} &= 0 \quad \text{for} \quad \left| x - \frac{l}{2} \right| \geq \frac{a}{2} \\ &= v \quad \text{for} \quad \left| x - \frac{l}{2} \right| < \frac{a}{2} \end{aligned}$$

This situation is shown in Figure 10.6.
The Fourier series is given by

$$v_0(x) = \sum_n \dot{y}_n = \sum_n \omega_n B_n \sin \frac{\omega_n x}{c}$$

where

$$\begin{aligned} \omega_n B_n &= \frac{2}{l} \int_{+l/2-a/2}^{l/2+a/2} v \sin \frac{\omega_n x}{c} \, \mathrm{d}x \\ &= \frac{4v}{n\pi} \sin \frac{n\pi}{2} \sin \frac{n\pi a}{2l} \end{aligned}$$

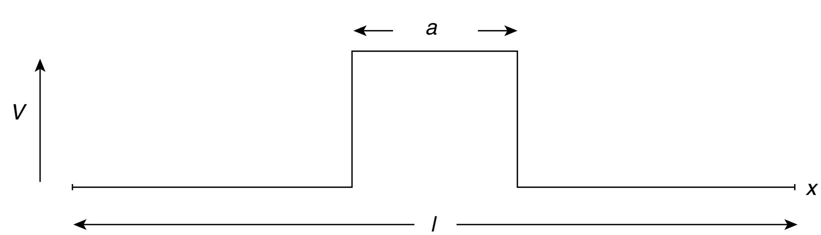


Figure 10.6 Velocity distribution at time $t = 0$ of a string length l, fixed at both ends and struck about its centre point by a mallet of width a. Displacement $y_0(x) = 0$; velocity $v_0(x) = v$ for $|x - l/2| < a/2$ and zero outside this region

Again we see that $\omega_n B_n = 0$ for n even $(\sin n\pi/2 = 0)$ because the centre point of the string is never stationary, as is required in an even harmonic.

Thus

$$v_0(x) = \sum_{n\,\mathrm{odd}} \frac{4v}{n\pi} \sin\frac{n\pi a}{2l} \sin\frac{\omega_n x}{c}$$

The energy per mode of oscillation

$$\begin{aligned} E_n &= \tfrac{1}{4} m\omega_n^2 (A_n^2 + B_n^2) \\ &= \tfrac{1}{4} m\omega_n^2 B_n^2 \quad (\text{All } A_n\text{'s} = 0) \\ &= \tfrac{1}{4} m \frac{16v^2}{n^2\pi^2} \sin^2\frac{n\pi a}{2l} \\ &= \frac{4mv^2}{n^2\pi^2} \sin^2\frac{n\pi a}{2l} \end{aligned}$$

Now

$$n = \frac{\omega_n}{\omega_1} = \frac{\omega_n l}{\pi c}$$

for the fundamental frequency

$$\omega_1 = \frac{\pi c}{l}$$

So

$$E_n = \frac{4mv^2c^2}{l^2\omega_n^2} \sin^2\frac{\omega_n a}{2c}$$

Again we see, since $\omega_n \propto n$ that the energy of the nth mode $\propto n^{-2}$ and decreases with increasing harmonic frequency. We may show this by rewriting

$$\begin{aligned} E_n(\omega) &= \frac{mv^2a^2}{l^2} \frac{\sin^2(\omega_n a/2c)}{(\omega_n a/2c)^2} \\ &= \frac{mv^2a^2}{l^2} \frac{\sin^2\alpha}{\alpha^2} \end{aligned}$$

where

$$\alpha = \omega_n a/2c$$

and plotting this expression as an energy-frequency spectrum in Figure 10.7.

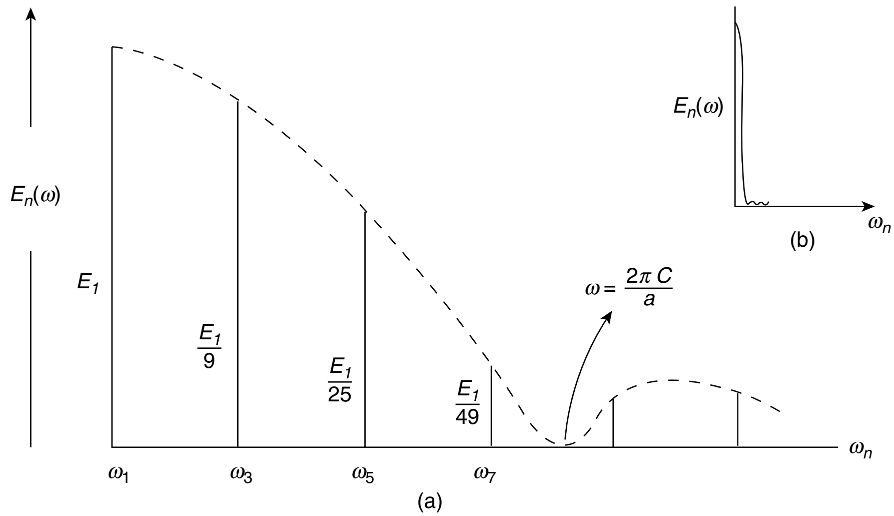


Figure 10.7 (a) Distribution of the energy in the harmonics ω_n of the string of Figure 10.6. The spectrum $E_n(\omega) \propto \sin^2\alpha/\alpha^2$ where $\alpha = \omega_n a/2c$. Most of the energy in the string is contained in the frequency range $\Delta\omega \approx 2\pi c/a$, and for $a = \Delta x$ (the spatial width of the pulse), $\Delta x/c = \Delta t$ and $\Delta\omega\Delta t \approx 2\pi$ (Bandwidth Theorem). Note that the values of $E_n(\omega)$ for $\omega_3, \omega_5, \omega_7$, etc. are magnified for clarity. (b) The true shape of the pulse

The familiar curve of $\sin^2\alpha/\alpha^2$ again appears as the envelope of the energy values for each ω_n.

If the energy at ω_1 is E_1 then $E_3 = E_1/9$ and $E_5 = E_1/25$ so the major portion of the energy in the velocity pulse is to be found in the low frequencies. The first zero of the envelope $\sin^2\alpha/\alpha^2$ occurs when

$$\alpha = \frac{\omega a}{2c} = \pi$$

so the width of the central frequency pulse containing most of the energy is given by

$$\omega \approx \frac{2\pi c}{a}$$

This range of energy-bearing harmonics is known as the 'spectral width' of the pulse written

$$\Delta\omega \approx \frac{2\pi c}{a}$$

The 'spatial width' a of the pulse may be written as Δx so we have

$$\Delta x \Delta\omega \approx 2\pi c$$

Reducing the width Δx of the mallet will increase the range of frequencies $\Delta\omega$ required to take up the energy in the rectangular velocity pulse. Now c is the velocity of waves on the string so a wave travels a distance Δx along the string in a time

$$\Delta t = \Delta x / c$$

which defines the duration of the pulse giving

$$\Delta\omega\Delta t \approx 2\pi$$

or

$$\Delta\nu\Delta t \approx 1$$

the Bandwidth Theorem we first met on p. 134.

Note that the harmonics have frequencies

$$\omega_n = \frac{n\pi c}{l}$$

so $\pi c/l$ is the harmonic interval. When the length l of the string becomes very long and $l \to \infty$ so that the pulse is isolated and non-periodic, the harmonic interval becomes so small that it becomes differential and the Fourier series summation becomes the Fourier Integral discussed on p. 283.

The Spectrum of a Fourier Series

The Fourier series can always be represented as a frequency spectrum. In Figure 10.8 a the relative amplitudes of the frequency components of the square wave of Figure 10.1 are plotted, each sine term giving a single spectral line. In a similar manner, the distribution of energy with frequency may be displayed for the plucked string of the earlier section. The frequency of the r th mode of vibration is given by $\omega_r = r\pi c/l$, and the energy in each mode varies inversely with r^2, where r is odd. The spectrum of energy distribution is therefore given by Figure 10.8 b.

Suppose now that the length of this string is halved but that the total energy remains constant. The frequency of the fundamental is now increased to $\omega'_r = 2r\pi c/l$ and the frequency interval between consecutive spectral lines is doubled (Figure 10.8 c). Again, the smaller the region in which a given amount of energy is concentrated the wider the frequency spectrum required to represent it.

Frequently, as in the next section, a Fourier series is expressed in its complex or exponential form

$$f(t) = \sum_{n=-\infty}^{\infty} d_n \, e^{in\omega t}$$

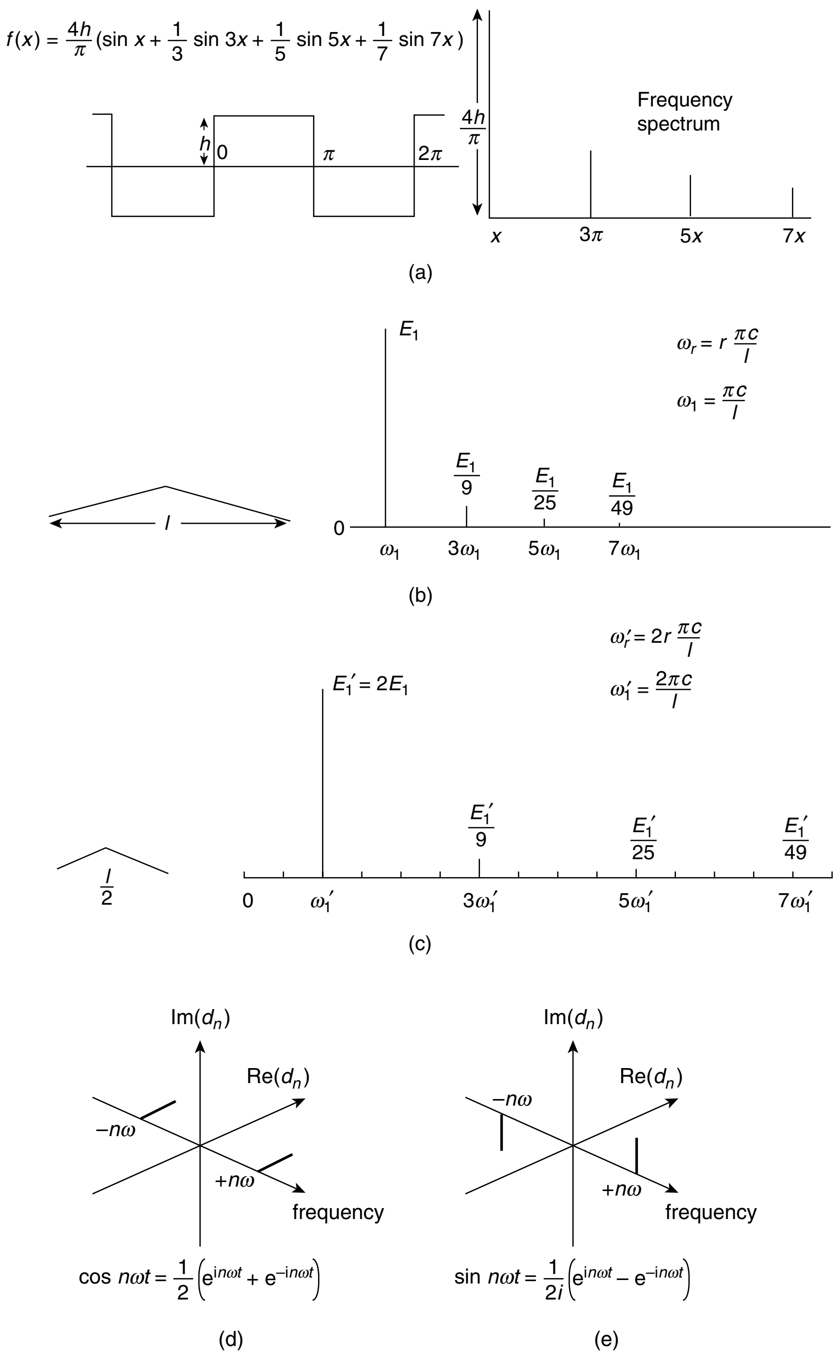


Figure 10.8 (a) Fourier sine series of a square wave represented as a frequency spectrum; (b) energy spectrum of a plucked string of length l; and (c) the energy spectrum of a plucked string of length $l/2$ with the same total energy as (b), demonstrating the Bandwidth Theorem that the greater the concentration of the energy in space or time the wider its frequency spectrum. Complex exponential frequency spectrum of (d) $\cos\omega t$ and (e) $\sin\omega t$

where $2d_n = a_n - \mathrm{i}b_n (n \geq 0)$ and $2d_n = a_{-n} + \mathrm{i}b_{-n} (n < 0)$.
Because

$$\cos n\omega t = \tfrac{1}{2}(\mathrm{e}^{\mathrm{i}n\omega t} + \mathrm{e}^{-\mathrm{i}n\omega t})$$

and

$$\sin n\omega t = \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}n\omega t} - \mathrm{e}^{-\mathrm{i}n\omega t})$$

a frequency spectrum in the complex plane produces two spectral lines for each frequency component $n\omega$, one at $+n\omega$ and the other at $-n\omega$. Figure 10.8 d shows the cosine representation, which lies wholly in the real plane, and Figure 10.8 e shows the sine representation, which is wholly imaginary. The amplitudes of the lines in the positive and negative frequency ranges are, of course, complex conjugates, and the modulus of their product gives the square of the true amplitude. The concept of a negative frequency is seen to arise because the $\mathrm{e}^{-\mathrm{i}n\omega t}$ term increases its phase in the opposite sense to that of the positive term $\mathrm{e}^{\mathrm{i}n\omega t}$. The negative amplitude of the negative frequency in the sine representation indicates that it is in antiphase with respect to that of the positive term.

Fourier Integral

At the beginning of this chapter we saw that one Fourier representation of the function could be written

$$f(x) = \sum_{n=-\infty}^{\infty} d_n \,\mathrm{e}^{\mathrm{i}nx}$$

where $2d_n = a_n - \mathrm{i}b_n (n \geq 0)$ and $2d_n = a_{-n} + \mathrm{i}b_{-n} (n < 0)$.

If we use the time as a variable we may rewrite this as

$$f(t) = \sum_{n=-\infty}^{\infty} d_n \,\mathrm{e}^{\mathrm{i}n\omega t}$$

where, if T is the period,

$$d_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{e}^{-\mathrm{i}n\omega t}\,\mathrm{d}t$$

(for $n = -2, -1, 0, 1, 2$, etc.).

If we write $\omega = 2\pi\nu_1$, where ν_1 is the fundamental frequency, we can write

$$f(t) = \sum_{n=-\infty}^{\infty}\left[\int_{-T/2}^{T/2} f(t')\,\mathrm{e}^{-\mathrm{i}2\pi n\nu_1 t'}\,\mathrm{d}t'\right]\mathrm{e}^{\mathrm{i}2\pi n\nu_1 t}\cdot\frac{1}{T}$$

If we now let the period T approach infinity we are isolating a single pulse by saying that it will not be repeated for an infinite period; the frequency $\nu_1 = 1/T \to 0$, and $1/T$ becomes infinitesimal and may be written dν.

Furthermore, n times ν_1, when n becomes as large as we please and $1/T = \nu_1 \to 0$, may be written as $n\nu_1 = \nu$, and the sum over n now becomes an integral, since unit change in n produces an infinitesimal change in $n/T = n\nu_1$.

Hence, for an infinite period, that is for a single non-periodic pulse, we may write

$$f(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t')\,\mathrm{e}^{-\mathrm{i}2\pi\nu t'}\,\mathrm{d}t' \right] \mathrm{e}^{\mathrm{i}2\pi\nu t}\,\mathrm{d}\nu$$

which is called the *Fourier Integral*.

We may express this as

$$f(t) = \int_{-\infty}^{\infty} F(\nu)\,\mathrm{e}^{\mathrm{i}2\pi\nu t}\,\mathrm{d}\nu$$

where

$$F(\nu) = \int_{-\infty}^{\infty} f(t')\,\mathrm{e}^{-\mathrm{i}2\pi\nu t'}\,\mathrm{d}t'$$

is called the *Fourier Transform* of $f(t)$. We shall discuss the transform in more detail in a later section of this chapter.

We see that when the period is finite and $f(t)$ is periodic, the expression

$$f(t) = \sum_{n=-\infty}^{\infty} d_n\,\mathrm{e}^{\mathrm{i}n\omega t}$$

tells us that the representation is in terms of an infinite number of different frequencies, each frequency separated by a finite amount from its nearest neighbour, but when $f(t)$ is not periodic and has an infinite period then

$$f(t) = \int_{-\infty}^{\infty} F(\nu)\,\mathrm{e}^{\mathrm{i}2\pi\nu t}\,\mathrm{d}\nu$$

and this expression is the integral (not the sum) of an infinite number of frequency components of amplitude $F(\nu)\,\mathrm{d}\nu$ infinitely close together, since ν varies continuously instead of in discrete steps.

For a periodic function the amplitude of the Fourier series coefficient

$$d_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t)\,\mathrm{e}^{-\mathrm{i}n\omega t}\,\mathrm{d}t$$

whereas the corresponding amplitude in the Fourier integral is

$$F(\nu)\,\mathrm{d}\nu = \left(\frac{1}{T}\right) \int_{-\infty}^{\infty} f(t')\,\mathrm{e}^{-\mathrm{i}n\omega t'}\,\mathrm{d}t'$$

This corroborates the statement we made when discussing the frequency spectrum that the narrower or less extended the pulse the wider the range of frequency components required to represent it. A truly monochromatic wave of one frequency and wavelength (or wave number) requires a wave train of infinite length before it is properly defined.

No wave train of finite length can be defined in terms of *one* unique wavelength.

Since a monochromatic wave, infinitely long, of single frequency and constant amplitude transmits no information, its amplitude must be modified by adding other frequencies (as we have seen in Chapter 5) before the variation in amplitude can convey information. These ideas are expressed in terms of the Bandwidth Theorem.

Fourier Transforms

We have just seen that the Fourier integral representing a non-periodic wave group can be written

$$f(t) = \int_{-\infty}^{\infty} F(\nu)\,\mathrm{e}^{\mathrm{i}2\pi\nu t}\,\mathrm{d}\nu$$

where its Fourier transform

$$F(\nu) = \int_{-\infty}^{\infty} f(t')\,\mathrm{e}^{-\mathrm{i}2\pi\nu t'}\,\mathrm{d}t'$$

so that integration with respect to one variable produces a function of the other. Both variables appear as a product in the index of an exponential, and this product must be non-dimensional. Any pair of variables which satisfy this criterion forms a Fourier pair of transforms, since from the symmetry of the expressions we see immediately that if

$$F(\nu) \text{ is the Fourier transform of } f(t)$$

then

$$f(-\nu) \text{ is the Fourier transform of } F(t)$$

If we are given the distribution in time of a function we can immediately express it as a spectrum of frequency, and vice versa. In the same way, a given distribution in space can be expressed as a function of wave numbers (this merely involves a factor, $1/2\pi$, in front of the transform because $k = 2\pi/\lambda$).

A similar factor appears if ω is used instead of ν. If the function of $f(t)$ is even only the cosine of the exponential is operative, and we have a Fourier cosine transform

$$f(t) = \int_{0}^{\infty} F(\nu)\cos 2\pi\nu t\,\mathrm{d}\nu$$

and

$$F(\nu) = \int_0^\infty f(t)\cos 2\pi\nu t\,\mathrm{d}t$$

If $f(t)$ is odd only the sine terms operate, and sine terms replace the cosines above. Note that only positive frequencies appear. The Fourier transform of an even function is real and even, whilst that of an odd function is imaginary and odd.

Examples of Fourier Transforms

The two examples of Fourier transforms chosen to illustrate the method are of great physical significance. They are

1. The 'slit' function of Figure 10.9a,
2. The Gaussian function of Figure 10.11.

As shown, they are both even functions and their transforms are therefore real; the physical significance of this is that all the frequency components have the same phase at zero time.

The Slit Function

This is a function having height h over the time range $\pm d/2$. Thus, $f(t) = h$ for $|t| < d/2$ and zero for $|t| > d/2$, so that

$$F(\nu) = \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-\mathrm{i}2\pi\nu t}\,\mathrm{d}t = \int_{-d/2}^{d/2} h\,\mathrm{e}^{-\mathrm{i}2\pi\nu t}\,\mathrm{d}t$$
$$= \frac{-h}{\mathrm{i}2\pi\nu}[\mathrm{e}^{-\mathrm{i}2\pi\nu d/2} - \mathrm{e}^{+\mathrm{i}2\pi\nu d/2}] = hd\frac{\sin\alpha}{\alpha}$$

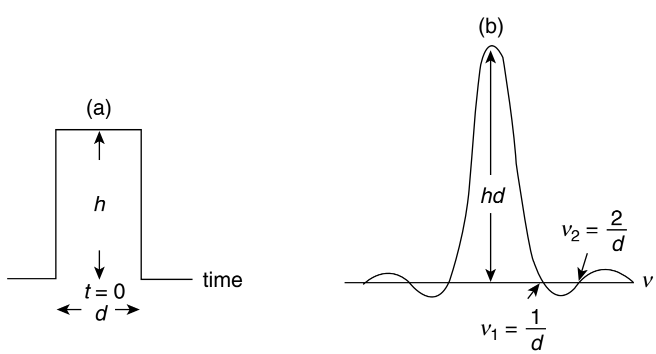


Figure 10.9 (a) Narrow slit function of extent d in time and of height h, and (b) its Fourier transform

where

$$\alpha = \frac{2\pi\nu d}{2}$$

Again we see the Fourier transformation of a rectangular pulse in time to a $\sin\alpha/\alpha$ pattern in frequency. The Fourier transform of the same pulse in space will give the same distribution as a function of wavelength. Figure 10.9b shows that as the pulse width decreases in time the separation between the zeros of the transform is increased. The negative values in the spectrum of the transform indicate a phase reversal for the amplitude of the corresponding frequency component.

The Fourier Transform Applied to Optical Diffraction from a Single Slit

This topic belongs more properly to the next chapter where it will be treated by another method, but here we derive the fundamental result as an example of the Fourier Transform. The elegance of this method is seen in problems more complicated than the one-dimensional example considered here. We shall see its extension to two dimensions in Chapter 12 when we consider the diffraction patterns produced by rectangular and circular apertures.

The amplitude of light passing through a single slit may be represented in space by the rectangular pulse of Figure 10.9a where d is now the width of the slit. A plane wave of monochromatic light, wavelength λ, falling normally on a screen which contains the narrow slit of width $d \sim \lambda$, forms a secondary system of plane waves diffracted in all directions with respect to the screen. When these diffracted waves are focused on to a second screen the intensity distribution (square of the amplitude) may be determined in terms of the aperture dimension d, the wavelength λ and the angle of diffraction θ.

In Figure 10.10 the light diffracted through an angle θ is brought to focus at a point P on the screen PP_0. Finding the amplitude of the light at P is the simple problem of adding all the small contributions in the diffracted wavefront taking account of all the phase differences which arise with variation of path length from P to the points in the slit aperture from which the contributions originate. The diffraction amplitude in k or wave number space is the Fourier transform of the pulse, width d, in x space in Figure 10.9b. The conjugate parameters ν and t are exactly reciprocal but the product of x and k involves the term 2π which requires either a constant factor $1/2\pi$ in front of one of the transform integrals or a common factor $1/\sqrt{2\pi}$ in front of each. This factor is however absorbed into the constant value of the maximum intensity and all other intensities are measured relative to it.

The constant pulse height now measures the amplitude h of the small wave sources across the slit width d and the Fourier transform method is the addition by integration of their contributions.

In Figure 10.10 we see that the path difference between the contribution at the centre of the slit and that at a point x in the slit is given by $x\sin\theta$, so that the phase difference is

$$\phi = \frac{2\pi}{\lambda}x\sin\theta = kx\sin\theta$$

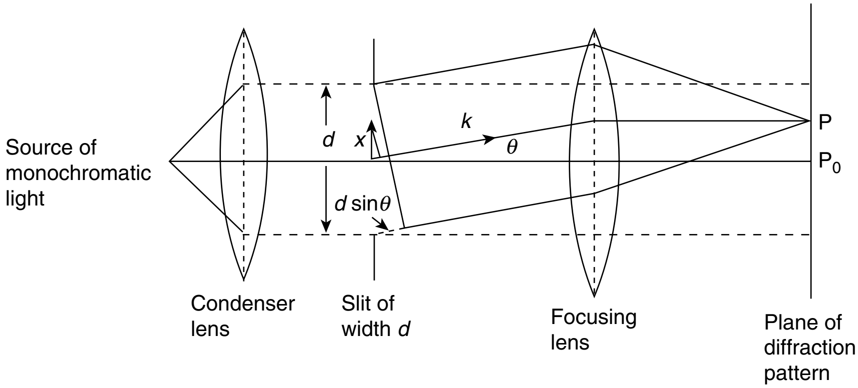


Figure 10.10 A monochromatic plane wave normally incident on a narrow slit of width d is diffracted an angle θ, and the light in this direction is focused at a point P. The amplitude at P is the superposition of all contributions with their appropriate phases with respect to the central point in the slit. The contribution from a point x in the slit has phase $\phi = 2\pi x \sin\theta/\lambda$ with respect to the central contribution. The phase difference from contributing points on opposite edges of the slit is $\phi = 2\pi d \sin\theta/\lambda = 2\alpha$

The product $kx\sin\theta$ can, however be expressed in a form more suitable for extension to two- and three-dimensional examples by writing it as $\mathbf{k}\cdot\mathbf{x} = klx$, the scalar product of the vector $\mathbf{k}$, giving the wave propagation direction, and the vector $\mathbf{x}$, l being the direction cosine

$$\begin{aligned} l &= \cos(\pi/2 - \theta) \\ &= \sin\theta \end{aligned}$$

of $\mathbf{k}$ with respect to the x-axis.

Adding all the small contributions across the slit to obtain the amplitude at P by the Fourier transform method gives

$$\begin{aligned} F(k) &= \frac{1}{2\pi}\int f(x)\,\mathrm{e}^{-\mathrm{i}\phi}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-d/2}^{+d/2} h\,\mathrm{e}^{-\mathrm{i}klx}\,\mathrm{d}x \\ &= \frac{h}{-\mathrm{i}kl}\frac{1}{2\pi}(\mathrm{e}^{-\mathrm{i}kld/2} - \mathrm{e}^{+\mathrm{i}kld/2}) \\ &= \frac{-2\mathrm{i}h}{-\mathrm{i}kl2\pi}\sin\frac{kld}{2} \\ &= \frac{dh}{2\pi}\frac{\sin\alpha}{\alpha} \end{aligned}$$

where

$$\alpha = \frac{kld}{2} = \frac{\pi}{\lambda} d \sin\theta$$

The intensity I at P is given by the square of the amplitude; that is, by the product of $F(k)$ and its complex conjugate $F^*(k)$, so that

$$I = \frac{d^2h^2}{4\pi^2}\frac{\sin^2\alpha}{\alpha^2}$$

where I_0, the principal maximum intensity at $\alpha = 0$, (P_0 in Figure 10.10) is now

$$I_0 = \frac{d^2h^2}{4\pi^2}$$

The Gaussian Curve

This curve often appears as the wave group description of a particle in wave mechanics. The Fourier transform of a Guassian distribution is another Gaussian distribution.

In Figure 10.11a the Gaussian function of height h is symmetrically centred at time $t = 0$, and is given by $f(t) = h\,\mathrm{e}^{-t^2/\sigma^2}$, where the width parameter or standard deviation σ is that value of t at which the height of the curve has a value equal to e^{-1} of its maximum.

Its transform is

$$\begin{aligned} F(\nu) &= \int_{-\infty}^{\infty} h\,\mathrm{e}^{-t/\sigma^2}\,\mathrm{e}^{-\mathrm{i}2\pi\nu t}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} h\,\mathrm{e}^{(-t/\sigma^2-\mathrm{i}2\pi\nu t+\pi^2\nu^2\sigma^2)}\,\mathrm{e}^{-\pi^2\nu^2\sigma^2}\,\mathrm{d}t \\ &= h\,\mathrm{e}^{(-\pi^2\nu^2\sigma^2)}\int_{-\infty}^{\infty}\mathrm{e}^{-(t/\sigma+\mathrm{i}\pi\nu\sigma)^2}\,\mathrm{d}t \end{aligned}$$

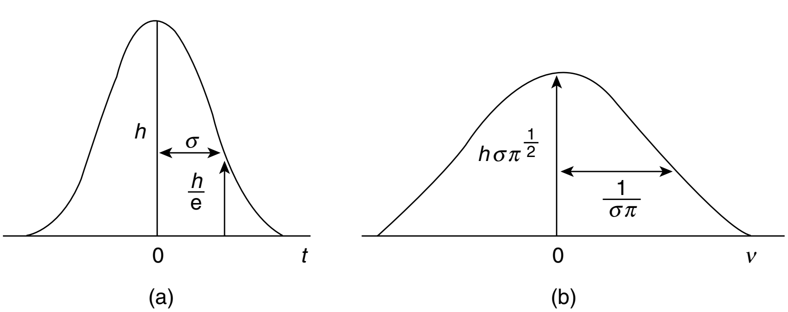


Figure 10.11 (a) A Gaussian function Fourier transforms (b) into another Gaussian function

The integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

and substituting, with $x = (t/\sigma + i\pi\nu\sigma)$ and $dt = \sigma\,dx$, gives

$$F(\nu) = h\sigma\pi^{1/2}e^{-\pi^2\nu^2\sigma^2}$$

another Gaussian distribution in frequency space (Figure 10.11b) with a new height $h\sigma\pi^{1/2}$ and a new width parameter $(\sigma\pi)^{-1}$.

As in the case of the slit and the diffraction pattern, we see again that a narrow pulse in time (width σ) leads to a wide frequency distribution [width $(\sigma\pi)^{-1}$].

When the curve is normalized so that the area under it is unity, h takes the value $(\sigma\pi)^{1/2}$ because

$$\frac{1}{(\sigma\pi^{1/2})}\int_{-\infty}^{\infty} e^{-t^2/\sigma^2}\,dt = 1$$

Thus, the height of a normalized curve transforms into a pulse of unit height whereas a pulse of unit height transforms to a pulse of width $(\sigma\pi)^{-1}$.

If we consider a family of functions with progressively increasing h values and decreasing σ values, each satisfying the condition of unit area under their curves, we are led in the limit as the height $h \to \infty$ and the width $\sigma \to 0$ to an infinitely narrow pulse of finite area unity which defines the Dirac delta (δ) function. The transform of such a function is the constant unity, and Figures. 10.12a and b show the family of normalized Gaussian distributions and their transforms. Figure 10.13 shows a number of common Fourier transform pairs.

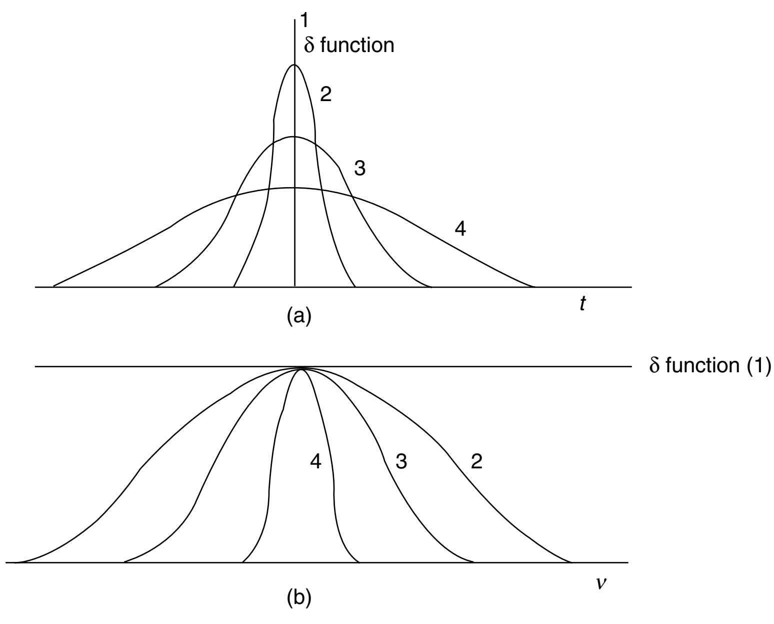


Figure 10.12 (a) A family of normalized Gaussian functions narrowed in the limit to Dirac's delta function; (b) the family of their Fourier transforms

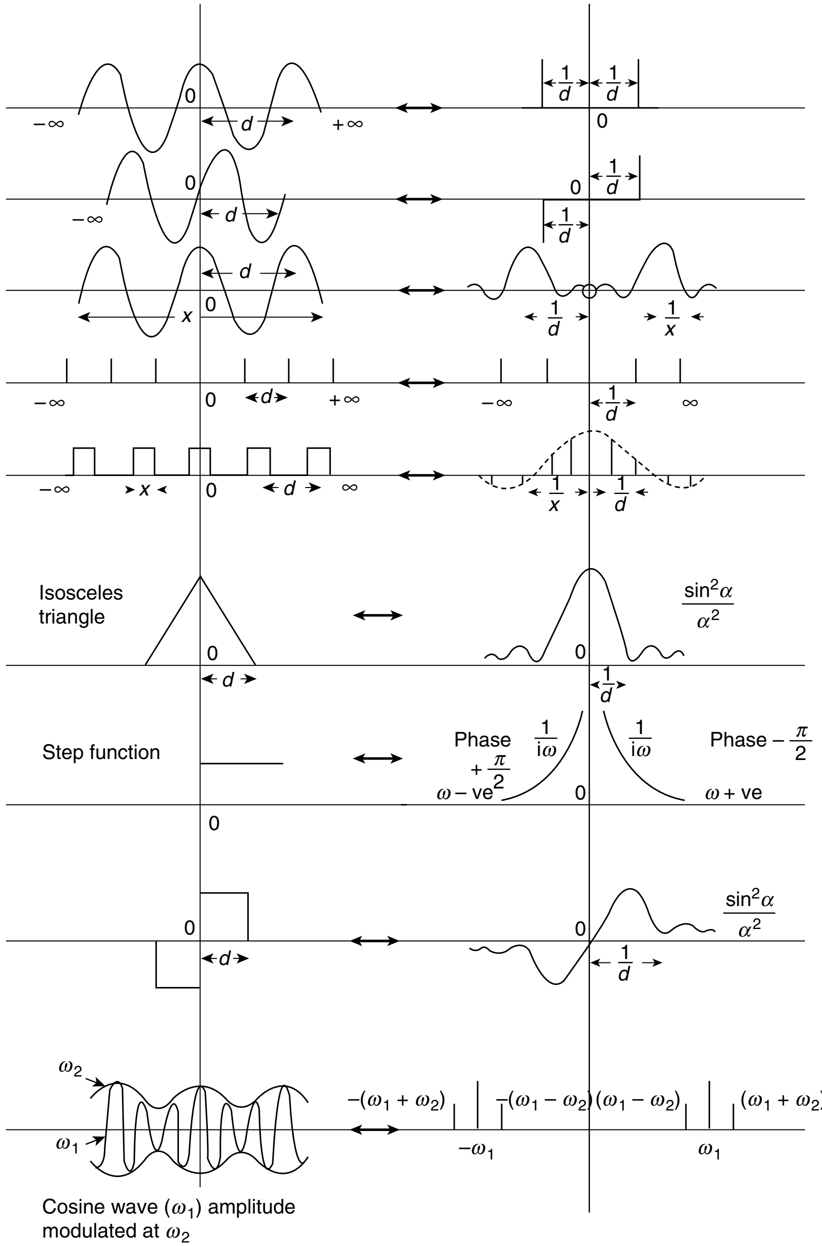


Figure 10.13 Some common Fourier transform pairs

In wave mechanics the position x of a particle and its momentum p_x are conjugate parameters and its Gaussian wave group representation may be Fourier transformed from x to p_x space and vice versa. The Fourier Transform gives the amplitude of the wave function but the probability of finding the particle at x or its having a given momentum p_x is proportional to the square of the amplitude.

The Dirac Delta Function, its Sifting Property and its Fourier Transform

The Dirac δ function is defined by

$$\begin{aligned}\delta(x) &= 0 \text{ at } x \neq 0\\ &= \infty \text{ at } x = 0\end{aligned}$$

and

$$\int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1$$

i.e., an infinitely narrow pulse centred on $x = 0$. It is also known as the unit impulse function.

A valuable characteristic is its sifting property, that is

$$\int_{-\infty}^{\infty} \delta(x - x_0)f(x)\mathrm{d}x = f(x_0)$$

The Fourier Transform of $\delta(x - x_0) = \mathrm{e}^{-ikx_0}$ because by definition

$$F(\delta(x - x_0)) = \int_{-\infty}^{\infty} \delta(x - x_0)\mathrm{e}^{-ikx}\mathrm{d}x$$

so writing $f(x) = \mathrm{e}^{-ikx}$ and applying the sifting property gives $f(x_0) = \mathrm{e}^{-ikx_0}$. Note that $\mathrm{e}^{-ikx_0} = \mathrm{e}^{ikx_0} = 1$ for $x_0 = 0$.

From the form of the transform we see that if a function $f(x)$ is a sum of individual functions then the Fourier Transform $F(f(x))$ is the sum of their individual transforms. Thus, if

$$f(x) = \sum_j \delta(x - x_j)$$

then

$$Ff(x) = \sum_j \mathrm{e}^{-ikx_j}$$

Figure 10.14 shows two Dirac δ functions situated at $x = \pm\frac{a}{2}$ so that $f(x) = \delta(x - \frac{a}{2}) + \delta(x + \frac{a}{2})$ giving $F(f(x)) = \mathrm{e}^{ik\frac{a}{2}} + \mathrm{e}^{-ik\frac{a}{2}} = 2\cos ka/2$.

Convolution

Given two functions $f(x)$ and $h(x)$, their convolution, written

$$f(x) \otimes h(x) = \int_{-\infty}^{\infty} f(x)h(x)\mathrm{d}x$$

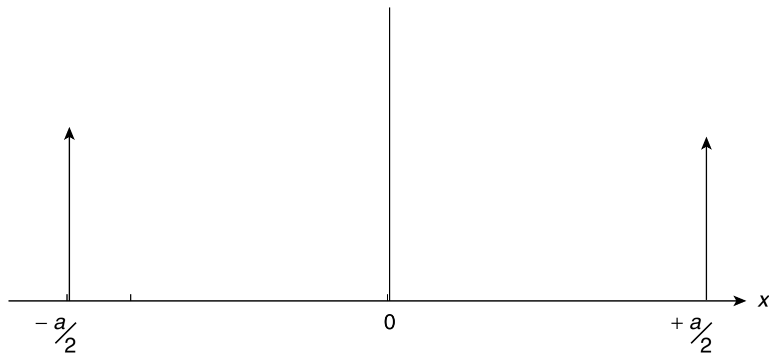


Figure 10.14 The Fourier transform of two Dirac δ functions located at $x = \pm a/2$ is 2 cos $ka/2$

is the overlap area under the product of the two functions as one function scans across the other. It the functions are two dimensional, $f(x, y)$ and $h(x, y)$, their convolution is the volume overlap under their product.

To illustrate a one-dimensional convolution consider the rectangular pulse of length D in Figure 10.15 convolved with an identical pulse. This is known as self-convolution. The convolution will be the sum of the shaded areas such as that of Figure 10.15a as one pulse slides over the other. We can see that the base length of the resulting convolved pulse will be $2D$ and that it will be symmetric about its peak, that is, when the two pulses completely overlap. If we consider the left-hand pulse as an infinite series of δ functions, of which we show a few, then Figure 10.15b shows that the integrated sum is an isosceles triangle of base length $2D$.

Another example is the convolution of a small triangular pulse with a rectangular pulse length D, Figure 10.16. Again, we use the series of δ functions to show the sum of the components of the resulting convolution and its integrated form for an infinite series of δ functions. The length of the final pulse is again the sum of the lengths of the two pulses.

Such a pulse would result in the convolution of a rectangular pulse with an exponential time function, for example, when a rectangular pulse is passed into an integrating network formed by a series resistance and parallel condenser, Figure 10.17. Here, the exponential time function of the network may be considered as fixed in time while the pulse performs the scanning operation. Note in Figures 10.15, 10.16 and 10.17 that the component contributions of the left hand pulses are summed in reverse order. This is explained in the discussion following eq. 10.2.

A convolution $f(x) \otimes h(x)$ is generally written in the form

$$g(x') = \int_{-\infty}^{\infty} f(x)h(x' - x)\mathrm{d}x \tag{10.2}$$

This a particularly relevant form when we consider the Optical Transfer Function on page 391. There, x is an object space coordinate and x' is an image space coordinate so the convolution relates image to object. If the function $h(x' - x)$ is a localized pulse in the object space and x' lies within it on the object axis x then the pulse $h(x' - x)$ is reversed

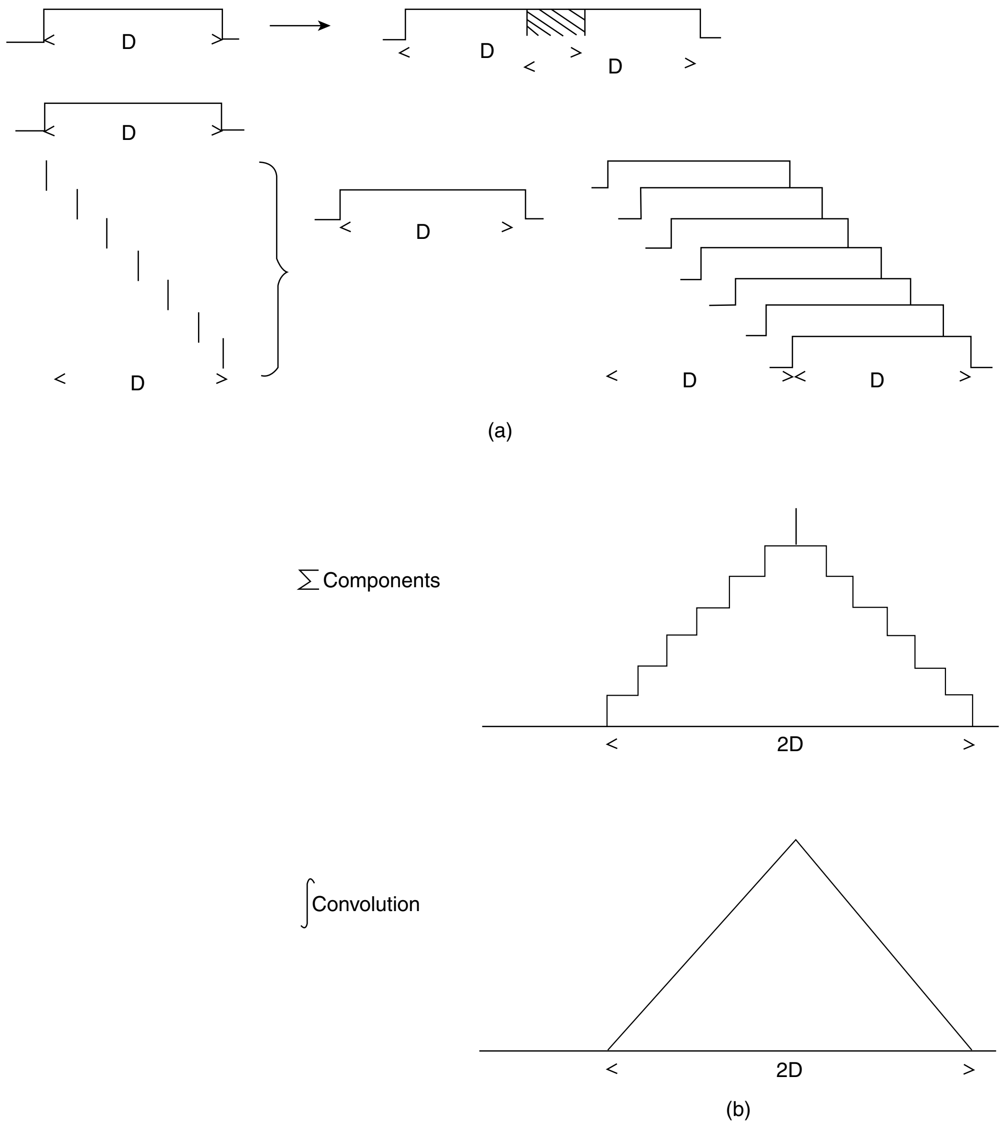


Figure 10.15 (a) A convolution is the integral of all overlapping areas as one function scans another. A rectangular pulse length *D* scans an identical pulse and the overlap area is shaded at one point of the scanning. (b) The scanning pulse is represented by several Dirac δ (impulse) functions and the component overlap areas are summed. When the number of impulse functions is large the sum of the components is integrated to become the triangular pulse

in image space (axis x') so that its trailing edge becomes its leading edge. Figure 10.18(a) shows the pulse on the object axis and Figure 10.18(b) shows the reversed pulse on the image axis.

The product $f(x)\,h(x'-x)$ exists only where the functions overlap and in Figure 10.18(b) $g(x_1')$ is the superposition of all the individual overlapping contributions that

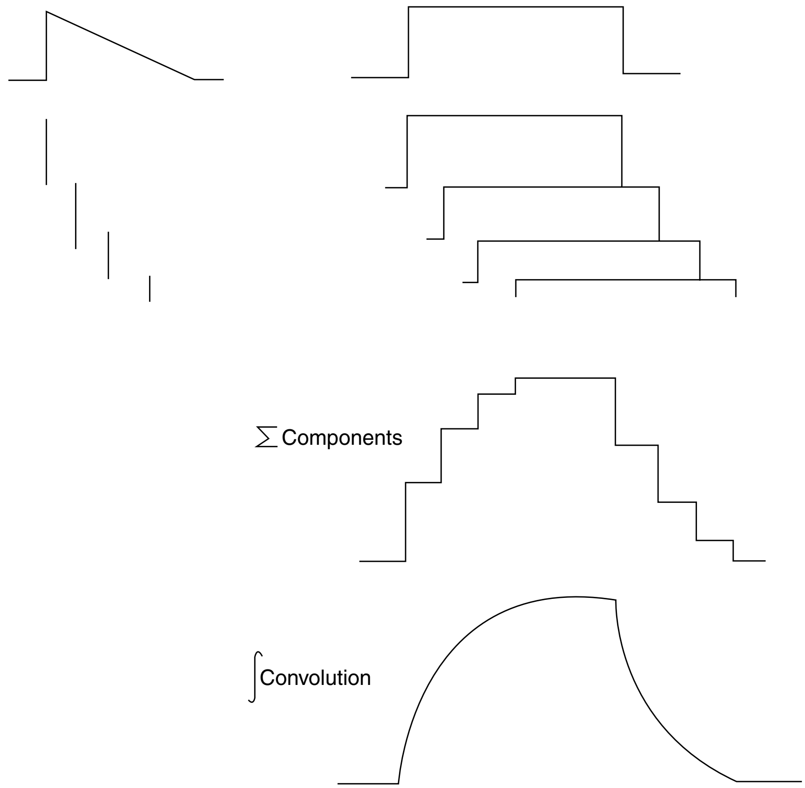


Figure 10.16 The convolution of a triangular with a rectangular pulse using the method of Figure 10.15

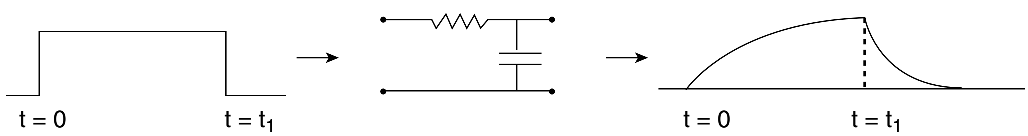


Figure 10.17 The convolution of Figure 10.16 is the same as that of a rectangular electrical pulse passing through an integrating circuit formed by a series resistance and a parallel condenser

exist at x'_1. The contribution to $g(x'_1)$ at x'_1 by x_1 and dx at x_1 is $f(x_1)h(x'_1 - x_1)\mathrm{d}x$ where $f(x_1)$ is a number which magnifies the pulse of Figure 10.18(b) to become the pulse of Figure 10.18(c). Each value of x in the overlap region makes a contribution to $g(x'_1)$; x values beyond the overlap make no contribution. The contributions begin when the leading edge of $h(x' - x)$ reaches x'_1 and they cease when its trailing edge passes x'_1.

Note that by changing the variable $x'' = x' - x$ in Equation (10.2).

$$f \otimes h = h \otimes f$$

This result is also evident when we consider the Convolution Theorem in the next section.

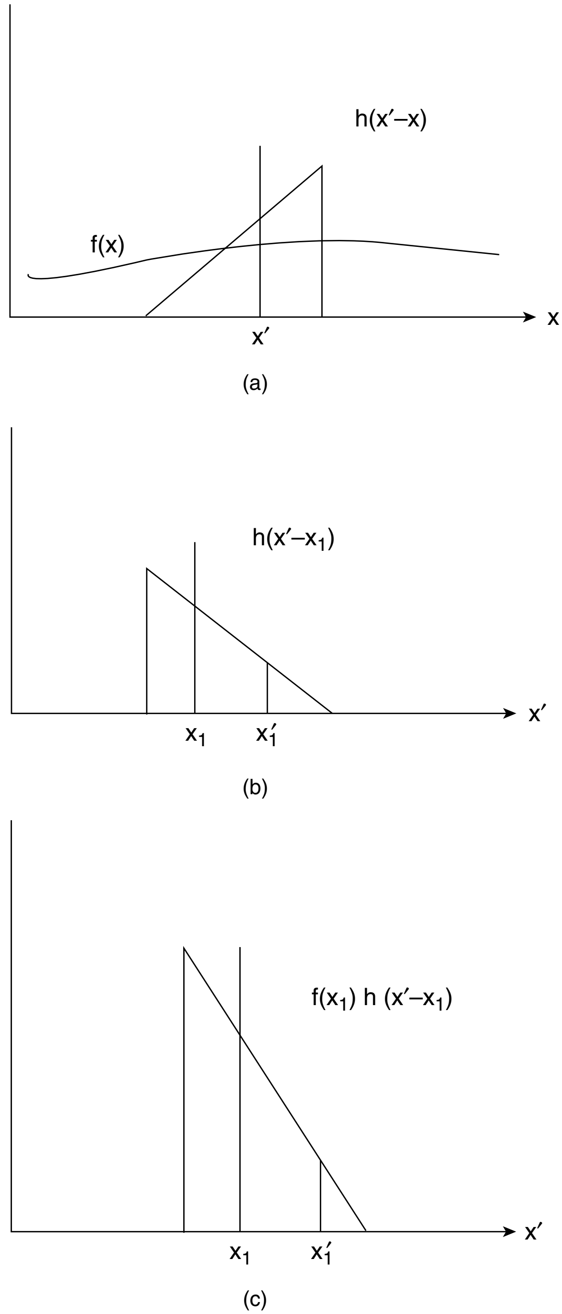


Figure 10.18 The function $h(x' - x)$ in the object space is reversed in the image space in Figure 10.18(b). (b) The convolution $g(x_1')$ is the superposition of all individual overlapping contributions to $f(x)h(x' - x)$ that exist at x_1'. (c) The contribution made by $f(x_1)\mathrm{d}\,x$ to $g(x_1')$ where $f(x_1)$ is a number which magnifies $h(x_1' - x)$

Returning to the convolution of the rectangular pulses in Figure 10.15 and taking the left-hand pulse as $f(x)$ each impulse x_i of the infinite series sweeps across the right-hand pulse $h(x'-x)$ to give the triangular convolution $g(x')$. If the left-hand pulse is now $h(x'-x)$ sweeping across the right-hand pulse $f(x)$ with x'_i as a fixed location in $h(x'-x)$, the series of overlaps, as x'_i moves across $f(x)$, gives the same triangular convolution.

The Convolution Theorem

The importance of the convolution process may be seen by considering the following.

When a signal, electrical or optical, passes through a system such as an amplifier or a lens, the resulting output is a function of the original signal and the system response. We have seen that a slit, in passing light from an optical source, may act as an angular filter, restricting the amount of information it passes and superimposing its own transform on the radiation passing through. An electrical filter can behave in a similar fashion.

Effectively there are two transformations, one into the intermediate system and one out again.

A convolution reduces this to a single transformation. The transform of the intermediate system is applied to the orginal function or signal and the resulting output is the integrated product of each point operating on the transformed response.

The convolution theorem states that the Fourier transform of the convolution of two functions is the product of the Fourier transforms of the individual functions, that is, if

$$g(x') = f(x) \otimes h(x)$$

then

$$F(g) = F(f \otimes h) = F(f) \cdot F(h)$$

The proof is straightforward.

The convolution $g(x')$ is a function of k, so its transform is

$$\begin{aligned} F(g) = G(k) &= \int_{-\infty}^{\infty} g(x')\mathrm{e}^{-ihx'}\mathrm{d}x' \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x)h(x'-x)\mathrm{d}x\right]\mathrm{e}^{-ikx'}\mathrm{d}x' \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} h(x'-x)\mathrm{e}^{-ikx'}\mathrm{d}x'\right] f(x)\mathrm{d}x \end{aligned}$$

Putting $x' - x = y$ gives $\mathrm{d}y = \mathrm{d}x'$ and $\mathrm{e}^{-ikx'} = \mathrm{e}^{-iky}\mathrm{e}^{-ikx}$ and so

$$\begin{aligned} F(g) = G(k) &= \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-ikx}\mathrm{d}x \int_{-\infty}^{\infty} h(y)\mathrm{e}^{-iky}\mathrm{d}y \\ &= F(f) \cdot F(h) = F(h) \cdot F(f) \end{aligned}$$

We can use this result to find the Fourier Transform of the resulting triangular pulse in Figure 10.15(b). The slit may be seen as a rectangular pulse of width d and its Fourier

Transform on page 288 gave its diffraction pattern as $\propto \sin\alpha/\alpha$ where $\alpha = kld/2$. Each of the pulses in Figure 10.15(b) contributes a Fourier Transform $\propto \sin\beta/\beta$ where

$$\beta = \frac{klD}{2}$$

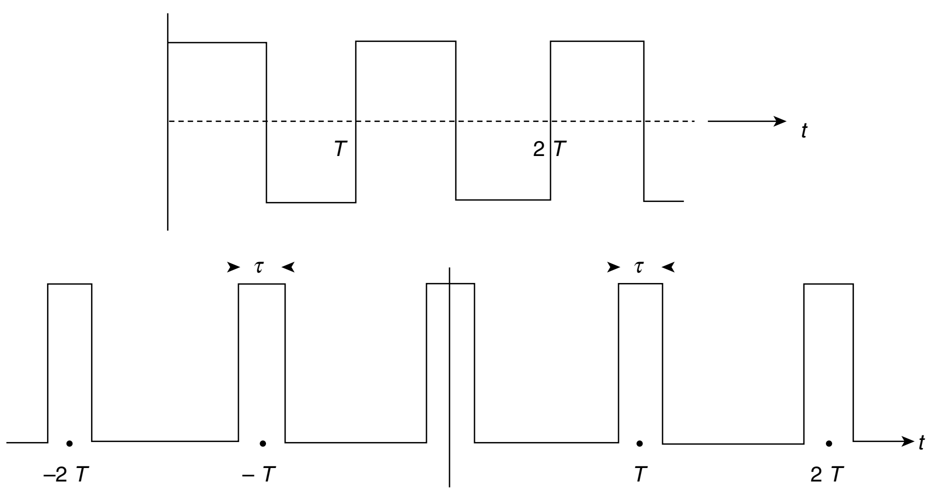
so the Fourier Transform of the isosceles triangular pulse is $\propto \sin^2\beta/\beta^2$.

Note that the analysis above is equally true if the arguments of the two functions are exchanged under the convolution process so that we have $f(x' - x)$ and $h(x)$. We use this in the discussion on the Optical Transfer Function on page 393.

(Problems 10.10, 10.11, 10.12, 10.13, 10.14, 10.15, 10.16, 10.17, 10.18, 10.19)

Problem 10.1

After inspection of the two wave forms in the diagram what can you say about the values of the constant, absence or presence of sine terms, cosine terms, odd or even harmonics, and range of harmonics required in their Fourier series representation? (Do not use any mathematics.)



Problem 10.2

Show that if a periodic waveform is such that each half-cycle is identical except in sign with the previous one, its Fourier spectrum contains no even order frequency components. Examine the result physically.

Problem 10.3

A half-wave rectifier removes the negative half-cycles of a pure sinusoidal wave $y = h\sin x$. Show that the Fourier series is given by

$$y = \frac{h}{\pi}\left(1 + \frac{\pi}{1\cdot 2}\sin x - \frac{2}{1\cdot 3}\cos 2x - \frac{2}{3\cdot 5}\cos 4x - \frac{2}{5\cdot 7}\cos 6x \ldots\right)$$

Problem 10.4

A full-wave rectifier merely inverts the negative half-cycle in Problem 10.3. Show that this doubles the output and removes the undesirable modulating ripple of the first harmonic.

Problem 10.5

Show that $f(x) = x^2$ may be represented in the interval $\pm\pi$ by

$$f(x) = \frac{2}{3}\pi^2 + \sum(-1)^n \frac{4}{n^2}\cos nx$$

Problem 10.6

Use the square wave sine series of unit height $f(x) = 4/\pi(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x)$ to show that

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} = \pi/4$$

Problem 10.7

An infinite train of pulses of unit height, with pulse duration 2τ and a period between pulses of T, is expressed as

$$\begin{aligned} f(t) &= 0 \quad \text{for } -\tfrac{1}{2}T < t < -\tau \\ &= 1 \quad \text{for } -\tau < t < \tau \\ &= 0 \quad \text{for } \tau < t < \tfrac{1}{2}T \end{aligned}$$

and

$$f(t+T) = f(t)$$

Show that this is an even function with the cosine coefficients given by

$$a_n = \frac{2}{n\pi}\sin\frac{2\pi}{T}n\tau$$

Problem 10.8

Show, in Problem 10.7, that as τ becomes very small the values of $a_n \to 4\tau/T$ and are independent of n, so that the spectrum consists of an infinite set of lines of constant height and spacing. The representation now has the same form in both time and frequency; such a function is called 'self reciprocal'. What is the physical significance of the fact that as $\tau \to 0$, $a_n \to 0$?

Problem 10.9

The pulses of Problems 10.7 and 10.8 now have amplitude $1/2\tau$ with unit area under each pulse. Show that as $\tau \to 0$ the infinite series of pulses is given by

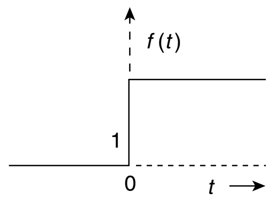
$$f(t) = \frac{1}{T} + \frac{2}{T}\sum_{n=1}^{\infty}\cos 2\pi nt/T$$

Under these conditions the amplitude of the original pulses becomes infinite, the energy per pulse remains finite and for an infinity of pulses in the train the total energy in the waveform is also infinite. The amplitude of the individual components in the frequency representation is finite, representing finite energy, but again, an infinity of components gives an infinite energy.

Problem 10.10

The unit step function is defined by the relation

$$\begin{aligned} f(t) &= 1 \ (t > 0) \\ &= 0 \ (t < 0) \end{aligned}$$



This is a very important function in physics and engineering, but it does not satisfy the criteria for Fourier representation because its integral is not finite. A similar function of finite period will satisfy the criteria. If this function is defined

$$\begin{aligned} f(t) &= 1(0 < t < T) \\ &= 0 \text{ elsewhere} \end{aligned}$$

show that if the transform

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \int_0^T \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \\ &= \frac{1}{\mathrm{i}\omega}[1 - \mathrm{e}^{\mathrm{i}\omega T}] \end{aligned}$$

then

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega \\ &= \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\mathrm{i}\omega}\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega \end{aligned}$$

(use the fact that for T very large

$$\int_{-\infty}^{\infty} \frac{1}{\mathrm{i}\omega}\,\mathrm{e}^{\mathrm{i}\omega(t-T)}\,\mathrm{d}\omega = \int_{-\infty}^{\infty} \frac{1}{\mathrm{i}\omega}\,\mathrm{e}^{-\mathrm{i}\omega T}\,\mathrm{d}\omega = -\pi$$

Note that the integral for the second term of $f(t)$ gives $-\pi$ for $t < 0$ and $+\pi$ for $t > 0$. This spectral representation is shown in Figure 10.13.)

Problem 10.11

Optical wave trains emitted by radiating atoms are of finite length and only an infinite wave train may be defined in terms of one frequency. The radiation from atoms therefore has a frequency bandwidth which contributes to the spectral linewidth. The random phase relationships between these wave trains create incoherence and produce the difficulties in obtaining interference effects from separate sources.

Let a finite length monochromatic wave train of wavelength λ_0 be represented by

$$f(t) = f_0 \,\mathrm{e}^{\mathrm{i}2\pi\nu_0 t}$$

and be a cosine of constant amplitude f_0 extending in time between $\pm\tau/2$. The distance $l = c\tau$ is called the coherence length. This finite train is the superposition of frequency components of amplitude $F(\nu)$ where the transform gives

$$f(t) = \int_{-\infty}^{\infty} F(\nu)\,\mathrm{e}^{\mathrm{i}2\pi\nu t}\,\mathrm{d}\nu$$

so that

$$\begin{aligned} F(\nu) &= \int_{-\infty}^{\infty} f(t')\,\mathrm{e}^{-\mathrm{i}2\pi\nu t'}\,\mathrm{d}t' \\ &= \int_{-\tau/2}^{+\tau/2} f_0\,\mathrm{e}^{-\mathrm{i}2\pi(\nu-\nu_0)t'}\,\mathrm{d}t' \end{aligned}$$

Show that

$$F(\nu) = f_0\tau \frac{\sin[\pi(\nu-\nu_0)\tau]}{\pi(\nu-\nu_0)\tau}$$

and that the relative energy distribution in the spectrum follows the intensity distribution curve in a single slit diffraction pattern.

Problem 10.12

Show that the total width of the first maximum of the energy spectrum of Problem 10.11 has a frequency range $2\Delta\nu$ which defines the coherence length l of Problem 10.11 as $\lambda_0^2/\Delta\lambda$.

Problem 10.13

For a ruby beam the value of $\Delta\nu$ in Problem 10.12 is found to be 10^4 Hz and $\lambda_0 = 6.936 \times 10^{-7}$ m. Show that $\Delta\lambda = 1.6 \times 10^{-17}$ m and that the coherence length l of the beam is 3×10^4 m.

Problem 10.14

The energy of the finite wave train of the damped simple harmonic vibrations of the radiating atom in Chapter 2 was described by $E = E_0\,\mathrm{e}^{-\omega_0 t/Q}$. Show from physical arguments that this defines a frequency bandwidth in this train of $\Delta\omega$ about the frequency ω_0, where the quality factor $Q = \omega_0/\Delta\omega$. (Suggested line of argument—at the maximum amplitude all frequency components are in phase. After a time τ the frequency component ω_0 has changed phase by $\omega_0\tau$. Other components have a phase change which interfere destructively. What bandwidth and phase change is acceptable?)

Problem 10.15

Consider Problem 10.14 more formally. Let the damped wave be represented as a function of time by

$$f(t) = f_0\,\mathrm{e}^{\mathrm{i}2\pi\nu_0 t}\,\mathrm{e}^{-t/\tau}$$

where f_0 is constant and τ is the decay constant.

Use the Fourier transform to show that the amplitudes in the frequency spectrum are given by

$$F(\nu) = \frac{f_0}{1/\tau + \mathrm{i}2\pi(\nu - \nu_0)}$$

Write the denominator of $F(\nu)$ as $r\,\mathrm{e}^{\mathrm{i}\theta}$ to show that the energy distribution of frequencies in the region of $\nu - \nu_0$ is given by

$$|F(\nu)|^2 = \frac{f_0^2}{r^2} = \frac{f_0^2}{(1/\tau)^2 + (\omega - \omega_0)^2}$$

Problem 10.16

Show that the expression $|F(\nu)|^2$ of Problem 10.15 is the resonance power curve of Chapter 3; show that it has a width at half the maximum value $(f_0\tau)^2$ which gives $\Delta\nu = 1/\pi\tau$, and show that a spectral line which has a value of $\Delta\lambda$ in Problem 10.12 equal to 3×10^{-9} m has a finite wave train of coherence length equal to 32×10^{-6} m (32 μm) if $\lambda_0 = 5.46 \times 10^{-7}$ m.

Problem 10.17

Sketch the self-convolution of the double slit function shown in Figure Q 10.17.

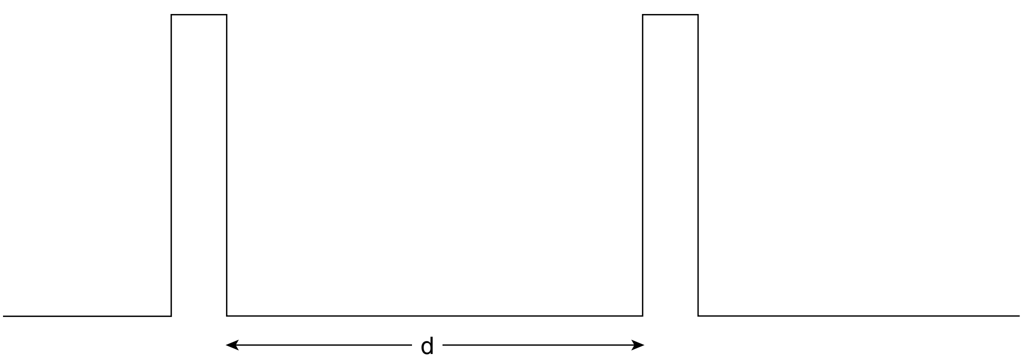


Figure Q.10.17

Problem 10.18

Sketch the convolution of the two functions in Figure Q 10.18 and use the convolution theorem to find its Fourier transform.

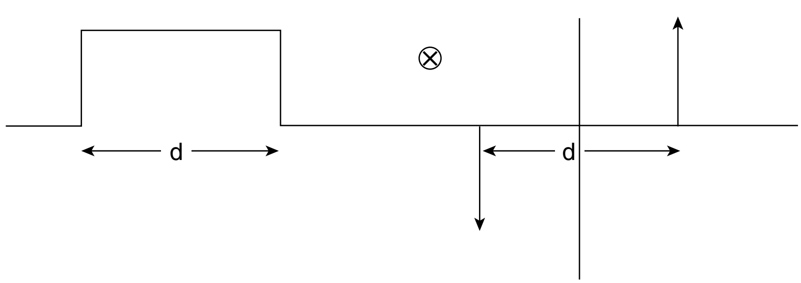


Figure Q.10.18

Problem 10.19

The convolution of two identical circles of radius r is very important in the modern method of testing lenses against an ideal diffraction limited criterion.

In Figure Q 10.19 show that the area of overlap is

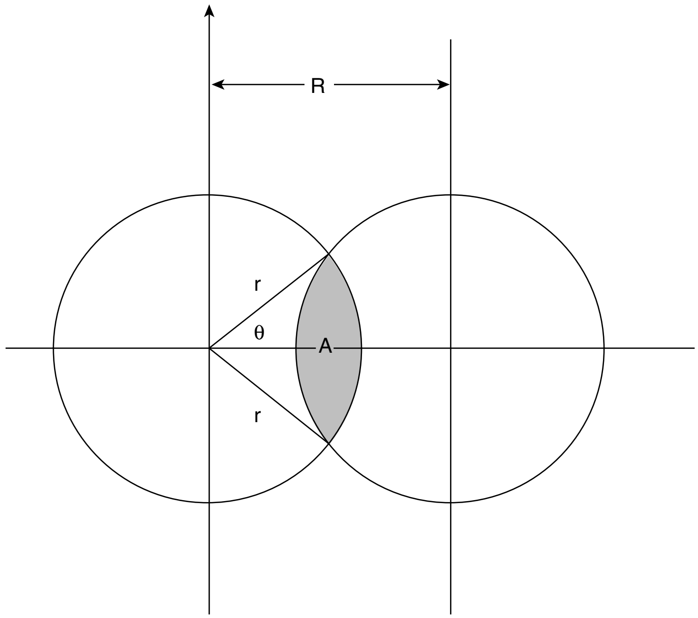


Figure Q.10.19

$$A = r^2(2\theta - 2\sin\theta\cos\theta)$$

and show for

$$R \leq 2r$$

that the convolution

$$O(R) = r^2\left[2\cos^{-1}\frac{R}{2r} - 2\left(1 - \frac{R^2}{4r^2}\right)^{\frac{1}{2}}\frac{R}{2r}\right]$$

Sketch $O(R)$ for $O \leq R \leq 2r$

Apart from a constant the linear operator $\hat{O}$ is known as the modulation factor of the optical transfer function.

Summary of Important Results

Fourier Series

Any function may be represented in the interval $\pm\pi$ by

$$f(x) = \frac{1}{2}a_0 + \sum_1^n a_n\cos nx + \sum_1^n b_n\sin nx$$

where

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, \mathrm{d}x$$

and

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, \mathrm{d}x$$

Fourier Integral

A single non-periodic pulse may be represented as

$$f(t) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(t') \, \mathrm{e}^{-\mathrm{i}2\pi\nu t'} \, \mathrm{d}t' \right] \mathrm{e}^{\mathrm{i}2\pi\nu t} \, \mathrm{d}\nu$$

or as

$$f(t) = \int_{-\infty}^{+\infty} F(\nu) \, \mathrm{e}^{\mathrm{i}2\pi\nu t} \, \mathrm{d}\nu$$

where

$$F(\nu) = \int_{-\infty}^{+\infty} f(t') \, \mathrm{e}^{-\mathrm{i}2\pi\nu t} \, \mathrm{d}t'$$

$f(t)$ and $F(\nu)$ are *Fourier Transforms* of each other. When t is replaced by x and ν by k the right hand side of each transform has a factor $1/\sqrt{2\pi}$. The Fourier Transform of a rectangular pulse has the shape of $\sin\alpha/\alpha$. (Important in optical diffraction.)