Linear Differential Equation

Q.1. Solve the differential equation : $(x + 1)\ dy/dx - y = e^{3x} (x + 1)^2$.

Solution : 1

We have $(x + 1)\ dy/dx - y = e^{\mathbf{3x}} (x + 1)2$.

Dividing both sides by $(x + 1)$, we get

$dy/dx - 1/(x + 1) = e^{\mathbf{3x}} (x + 1)$.

Therefore, I. F. = $e - \int 1/(x + 1)dx = e - \log(x + 1) = 1/(x + 1)$. Multiplying by I. F. we get $1/(x + 1)\ [dy/dx - 1/(x + 1)] = e^{\mathbf{3x}}.\ (x + 1)\ .\ 1/(x + 1)$.

ntegrating both sides , we get $y.1/(x + 1) = \int e^{\mathbf{3x}} dx$ Or, $y/(x + 1) = (e^{\mathbf{3x}})/3 + c$ Or, $y = \{(e^{\mathbf{3x}})\ /3\}(x + 1) + c\ (x + 1)$.

Q.2. Solve the differential equation : $ydx - (x +2y^2)dy = 0$.

Solution : 2

We have $ydx - (x + 2y^{\mathbf{2}})dy = 0$

Or, $ydx/dy - x - 2y^{\mathbf{2}} = 0$

Or, $dx/dy - x/y - 2y = 0$

Or, $dx/dy - x/y = 2y$

I.F. = $e^{\int pdy} = e - \int^{(1/y)dy} = e^{-\log y} = e^{\log y\ -1} = y^{-1} = 1/y$.

Multiplying both sides by $1/y$, we get

$1/y[dx/dy = x/y] = 1/y \times 2y$

Integrating , we get

$x.1/y = \int 2dy$

Or, $x/y = 2y + c \Rightarrow x = 2y^{\mathbf{2}} + cy$.

Q.3. Solve the differential equation : $\tan x \, dy/dx + 2y = \sec x$.

Solution : 3

We have $\tan x \, dy/dx + 2y = \sec x$

Or, $dy/dx + (2/\tan x)y = 1/\sin x$ [$dy/dx + Py = Q$]

Here we have $P = 2/\tan x$, $Q = 1/\sin x$

I . F . = $e^{\int P dx} = e^{2\int \cot x dx} = e^{2\log \sin x} = e^{\log \sin^2 x} = \sin^2 x$

Integrating, we get $y.\sin^2 x = \int (1/\sin x).\sin^2 x \, dx + c$

Or, $(\sin^2 x)y = \int \sin x dx + c = -\cos x + c$

Or, $(\sin^2 x) y + \cos x = c$.

Q.4. Solve the differential equation : $(1 - x^2) dy/dx - xy = x$, given $y = 2$ when $x = 0$.

Solution : 4

We have, $(1 - x^2) dy/dx - xy = x$

Or, $dy/dx - [x /(1 - x^2)]y = x/(1 - x^2)$

The I.F. = P = $e^{\int -x /(1 - x^2)dx}$

Then in $\int -x /(1 - x^2)dx$ [put $1 - x^2 = t \Rightarrow dt = -2xdx$

$= 1/2\int dt/t = 1/2 \log t$

Then $P = e^{1/2\log t} = e^{\log\sqrt{t}} = \sqrt{t} = \sqrt{(1 - x^2)}$

Therefore, $dy/dx.\sqrt{(1 - x^2)} - [x/(1 - x^2)]y\sqrt{(1 - x^2)} = x/(1 - x^2)\times\sqrt{(1 - x^2)}$

Or, $d/dx[y.\sqrt{(1 - x^2)}] = x/\sqrt{(1 - x^2)}$

Integrating we get $\int d/dx[y.\sqrt{(1 - x^2)}].dx = \int [x/\sqrt{(1 - x^2)}]dx + c$

Or, $y.\sqrt{(1 - x^2)} = 1/2 \int [2x/\sqrt{(1 - x^2)}]dx + c$

$= -1/2 \int[-2x/(1 - x^2)]dx + c$

$= -1/2\int dt/\sqrt{t} + c$ [where, $t = 1 - x^2 \Rightarrow dt = -2xdx$]

$= -1/2[\ t-1/2] + c$

$= -\sqrt{(1 - x^2)} + c$

Or, $y\sqrt{(1 - x^2)} + \sqrt{(1 - x^2)} = c$

Putting $x = 0$ and $y = 2$ we get $c = 3$

Hence, $y\sqrt{(1 - x^2)} + \sqrt{(1 - x^2)} = 3$.

Q.5. Solve : $(1 + y + x^2y)dx + (x + x^3)dy = 0$.

Solution : 5

We have $(1 + y + x^2y)dx + (x + x^3)dy = 0$

Or, $1 + y + x^2y + (x + x^3).dy/dx = 0$

Or, $1 + y(1 + x^2) + x(1 + x^2).dy/dx = 0$

Or, $dy/dx + [y(1 + x^2)]/[x(1 + x^2)] + 1/[x(1 + x^2)] = 0$

Or, $dy/dx + y/x = -1/[x(1 + x^2)]$

Which is of the form : $dy/dx + P.y = Q$; P & Q are function of x.

Integrating factor $= e^{\int Pdx} = e^{\int(1/x)dx}$ [$P = 1/x$]

$= e^{\log x} = x$.

Multiplying both sides by I.F., we get

$x.[dy/dx + y/x] = x .\{-1/[x(1 + x^2)]\}$

Or, $x.dy/dx + y = -1 /(1 + x^2)$

Or, $d(x.y)/dx = -1/(1 + x^2)$

Integrating both sides, we get

$\int d/dx(x.y).dx = \int[(-1)/(1 + x2)]dx$

Or, $x.y = \tan -1 x + c$

Q.6. Solve : $\cos^2 x\, dy/dx + y = \tan x$.

Solution : 6

We are given, $\cos^{\mathbf{2}} x\, dy/dx + y = \tan x$

Or, $dy/dx + y.\sec^{\mathbf{2}} x = \tan x. \sec^{\mathbf{2}} x$ [dividing by $\cos^{\mathbf{2}} x$]

This is of the form , $dy/dx + Py = Q$, here $P = \sec^{\mathbf{2}} x$

$\int P.dx = \int \sec^{\mathbf{2}} x\, dx = \tan x$.

Integrating factor $= e^{\int P.dx} = e^{\tan x}$

Multiplying both sides by I.F., we get

$etan\ x. dy/dx + y.^{\tan x}.\sec^{\mathbf{2}} x = e^{\tan x}. \tan x. \sec^{\mathbf{2}} x$

Or, $d/dx(y.etan\ x) = e^{\tan x}. \tan x. \sec^{\mathbf{2}} x$.

Integrating both sides we get,

$\int d/dx(y.\ e^{\tan x}).dx = \int e^{\tan x}.\tan x.\sec^{\mathbf{2}} x.\ dx$

Or, $y.^{\tan x} = \int e^{\tan x}. \tan x. \sec^{\mathbf{2}} x.\ dx$

$= \int e^{\mathbf{t}}.t.dt$ [put $\tan x = t \Rightarrow \sec^{\mathbf{2}} x\, dx = dt$]

$= t\int e^{\mathbf{t}}t.d^{\mathbf{t}} - \int e^{\mathbf{t}}.d^{\mathbf{t}}$ [Integrating by parts]

$= t.e^{\mathbf{t}} - e^{\mathbf{t}} + c$

$= e^{\tan x}(\tan x -1)$

Therefore, the solution is $y.^{\tan x} = etan\ x(\tan x - 1) + c$.

Q.7. Solve : $dy/dx + \{2x/(1 + x^2)\}y = 1/(1 + x^2)^2$.

Solution : 7

We are given,

$dy/dx + \{2x/(1 + x^2)\}y = 1/(1 + x^2)^2$

This is of the form

$dy/dx + Py + Q$, where P and Q are functions of x.

$P = 2x/(1 + x^2)$, then $\int Pdx = \int 2x/(1 + x^2)dx = \log(1 + x^2)$

Therefore, I.F. $= e\int Pdx = e^{\log(1 + x^2)} = 1 + x^2$

Multiplying both sides by $(1 + x^2)$, we get

$(1 + x^2).dy/dx + 2x.y = 1/(1 + x^2)$

Or, $d/dx[y.(1 + x^2)] = 1/(1 + x^2)$

Integrating both sides we get

$\int d/dx[y.(1 + x^2)].dx = \int dx/(1 + x^2) + c$

Or, $y.(1 + x^2) = \tan -1 (x) + c$.

Q.8. Solve : $(1 + y^2)\, dx/dy = \tan^{-1} y - x$.

Solution : 8

We are given , $(1 + y^2)\, dx/dy = \tan^{-1} y - x$

Or, $dx/dy = (\tan^{-1} y)/(1 + y^2) - x/(1 + y^2)$

Or, $dx/dy + x.\{1/(1 + y^2)\} = \tan^{-1} y /(1 + y^2)$

This is of the form :

$dx/dy + Px = Q$

$\int P.dy = \int dy/(1 + y^2) = \tan^{-1} y$

I.F. $= e\int Pdy = e\tan^{-1}y$

Multiplying by I.F.,

we get $e^{\tan^{-1} y} [dx/dy + \{1/(1 + y^2)\}.x] = e^{\tan^{-1} y} [\tan^{-1} y/(1 + y^2)]$

Or, d/dy[etan $^{-1}$ y. x] = e^tan $^{-1}$ y [tan $^{-1}$ y/(1 + y^2)]

Integrating ,

we get e^tan $^{-1}$ y . x = ∫e^tan $^{-1}$ y [tan^{-1} y/(1 + y^2)] dy

Now ∫ e^tan^{-1} y [tan^{-1} y/(1 + y^2)]dy

[put y = tan θ => dy = sec^2 θ dθ and tan $^{-1}$ y = θ]

= ∫ eθ .θ. (1/sec^{2}θ). sec^{2}θ. dθ

= ∫θ.eθ dθ = θ. eθ – ∫1.eθ. dθ

= θ.eθ – eθ = eθ.(θ^{-1})

= e^tan^{-1} y .(tan^{-1} y $^{-1}$)

Therefore, x.e^tan $^{-1}$ y = e^tan $^{-1}$ y.(tan $^{-1}$ y $^{-1}$) + c .

Q.9. Solve the following differential equation : sin x dy/dx – y = cos^2 x sin x tan (x/2).

Solution : 9

We are given,

sin x dy/dx – y = cos^2 x sin x tan (x/2)

Or, dy/dx – y/sin x = cos^2 x tan (x/2) [Dividing by sin x]

It is of the form, dy/dx + Py = Q ,

∫Pdx = ∫(–1/sin x) dx = ∫(–cosec x) dx = –log tan (x/2) = log cot (x/2)

I.F. = e∫Pdx = elog cot (x/2) = cot (x/2)

Multiplying by cot(x/2), we get

(dy/dx – y/sin x) cot(x/2) = cos^2 x tan (x/2)cot (x/2)

integrating , we get y cot(x/2) = ∫cos^2 x dx + c

= ∫[(1 + cos ^{2}x)/2]dx + c

y cot(x/2) = (1/2)x + (sin 2x)/4 = c .

Q.10. Solve the following differential equation for a particular solution : dy = (5x – 4y)dx ; when y = 0 and x = 0.

Solution : 10

We have, dy = (5x – 4y)dx

Or, dy/dx + 4y = 5x

I.F. = $e\int 4dx = e^{4x}$

Multiplying by e4x and integrating we get,

$y.e^{4x} = \int 5x.e^{4x}\ dx$

$= 5x.e^{4x}/4 - 5/4\int e^{4x}\ dx + c$

$= 5/^{4x}.e^{4x} - 5/16\ e^{4x} + c.$