Sample Paper-02
Mathematics
Class – XII

Time allowed: 3 hours | Maximum Marks: 100

General Instructions:

a) All questions are compulsory.
b) The question paper consists of 26 questions divided into three sections A, B and C. Section A comprises of 6 questions of one mark each, Section B comprises of 13 questions of four marks each and Section C comprises of 7 questions of six marks each.
c) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
d) Use of calculators is not permitted.

Section A

1. Give example of a function which is neither one-one nor onto .
2. Calculate the direction cosines of the vector $\vec{a} = 3i - 2j + 5k$.
3. Let L be the set of all lines in a plane and R be the relation in L defined as R={(L_1,L_2):L_1 is parallel to L_2}. Is L reflexive?
4. Suppose X is a 2x3 matrix, Z is a 5x3 matrix. Find the order of Y such that both XY and YZ are well defined.
5. Find the area of the triangle with vertices at the points (1,0),(6,0),(4,3).
6. Find x such that $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix}$

Section B

7. Evaluate $\sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$
8. Show that the points A,B and C with position vectors $\vec{a} = 3i - 4j - 4k, \vec{b} = 2i - j + k, \vec{c} = i - 3j - 5k$ form the vertices of a right angled triangle.
9. The probability of solving a specific problem independently by A and B are ½ and 1/3 respectively. If both try to solve the problem independently, find the probability that (a) problem is solved (b) exactly one of them solves the problem.
10. Find all points of discontinuity of the function f where f is defined by:

$$f(x) = \begin{cases} x^3 - x + 1, x \le -3 \\ -2x, -3 < x < 3 \\ 3x + 2, x \ge 3 \end{cases}$$

11. Solve the differential equation $x\frac{dy}{dx} - y + x\cos ec\left(\frac{y}{x}\right) = 0,\ y(1) = 0$

12. Using properties of determinants prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

13. A poisonous substance is dropped in a lake next to a village. The waves move is circles at a speed of 2cm per second. At the instant when radius of the circular wave is 14cm, evaluate how fast the enclosed area is increasing. Discuss two harmful consequences of polluting water bodies.
14. Show that if $f : A \rightarrow B \text{ and } g : B \rightarrow C$ are onto, then $g \circ f : A \rightarrow C$ is onto.
15. Verify Rolle's theorem for $f(x)=x^2+2x-8$, $x\in[-4,2]$.
16. Show that $\left(\left|\vec{a}\right|\vec{b}+\left|\vec{b}\right|\vec{a}\right).\left(\left|\vec{a}\right|\vec{b}-\left|\vec{b}\right|\vec{a}\right)=0$
17. Integrate $\int \frac{dx}{x(x^4-1)}$.
18. Find the distance between the lines l_1 and l_2 given by :

    $\vec{r}=(i+3j-2k)+\lambda(2i-3j+k)$

    $\vec{r}=(2i+4j-k)+\mu(2i-3j+k)$
19. Find the vector equation of the plane passing through the intersection of the planes

    $\vec{r}.(2i+2j-3k)=7, \vec{r}.(2i+5j+3k)=9 \text{ and the point } (2,1,3)$

Section C

20. A factory can hire two tailors A and B in order to stich pants and shirts. Tailor A can stich 6 shirts and 4 pants in a day. Tailor B can stich 10 shirts and 4 pants in a day. Tailor A charges 15 per day and tailor B charges 20 per day. The factory has to produce minimum 60 shirts and 32 pants. State as a linear programming problem and minimize the labour cost.
21. Find the area of the region included between the two parabolas $y^2=4ax$ and $x^2=4ay$, $a>0$.
22. Bag X contains 2 white and 3 red balls. Bag Y contains 5 white and 4 red balls. Bag Z contains 2 white and 3 red balls. A ball is drawn at random from one of the bags and it is found to be red. What is the probability that it is drawn from bag Y?
23. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.
24. Solve the following system of equations using matrix method

    $\frac{3}{x}-\frac{2}{y}+\frac{3}{z}=8$

    $\frac{2}{x}+\frac{1}{y}-\frac{1}{z}=1$

    $\frac{4}{x}-\frac{3}{y}+\frac{3}{z}=4$
25. If $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$, find $\frac{d^2y}{dx^2}$
26. $\int_0^\pi \frac{x\,dx}{4\cos^2 x+9\sin^2 x}$

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Section A

1. Solution: $f(x)=x^2$, is neither one-one nor onto.
   f(3)=f(-3)=9, hence not one-one. Also f(x) does not assume any negative values, hence it is not onto.
2. Solution:
   $$\left|\vec{a}\right|=\sqrt{(3)^2+(-2)^2+(5)^2}=\sqrt{38}$$
   $$\therefore l=\frac{3}{\sqrt{38}},m=\frac{-2}{\sqrt{38}},n=\frac{5}{\sqrt{38}}$$
3. Solution: Yes, since every line is parallel to itself, thus the above relation is reflexive.
4. Solution: Let the order of Y be nxp.
   $\because$ XY is defined $\Rightarrow$ n=3, $\because$ YZ is defined $\Rightarrow$ p=5
   $\because$ Y has order 3x5.
5. Solution:
   $$Area=\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=\frac{1}{2}\begin{vmatrix}1 & 0 & 1\\ 6 & 0 & 1\\ 4 & 3 & 1\end{vmatrix}=\frac{15}{2}sq.units$$
6. Solution:
   $$\begin{vmatrix}3 & x\\ x & 1\end{vmatrix}=\begin{vmatrix}3 & 4\\ 2 & 1\end{vmatrix}\Rightarrow 3-x^2=3-8\Rightarrow x^2=8\Rightarrow x=\pm 2\sqrt{2}$$

Section B

7. Solution:
   $$Let\ x=\sin^{-1}\left(\frac{5}{13}\right),y=\cos^{-1}\left(\frac{3}{5}\right)$$
   $$\Rightarrow \sin x=\frac{5}{13},\cos x=\sqrt{1-\sin^2 x}=\frac{12}{13},\tan x=\frac{5}{12}$$
   $$\cos y=\frac{3}{5},\sin y=\sqrt{1-\cos^2 y}=\frac{4}{5},\tan y=\frac{4}{3}$$
   $$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{5/12+4/3}{1-(5/12)(4/3)}=\frac{63}{16}$$
   $$x+y=\tan^{-1}\left(\frac{63}{16}\right)$$
8. Solution:

$\vec{a} = 3i - 4j - 4k, \vec{b} = 2i - j + k, \vec{c} = i - 3j - 5k$

$\therefore \overrightarrow{AB} = (2i - j + k) - (3i - 4j - 4k) = -i - 3j - 5k$

$\overrightarrow{BC} = (i - 3j - 5k) - (2i - j + k) = -i - 2j - 6k$

$\overrightarrow{CA} = (3i - 4j - 4k) - (i - 3j - 5k) = 2i - j + k$

$\left|\overrightarrow{AB}\right|^2 = 35, \left|\overrightarrow{BC}\right|^2 = 41, \left|\overrightarrow{CA}\right|^2 = 6$

$\therefore \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{CA}\right|^2 = \left|\overrightarrow{BC}\right|^2$

Thus, A, B,C form the vertices of a right angled triangle.

9. Solution:

(A) Let A denote the event that problem is solved by A and let B denote the event that problem is solved by B.

$\therefore P(A) = 1/2, P(B) = 1/3, P(\overline{A}) = 1 - 1/2 = 1/2, P(\overline{B}) = 2/3$

P(Problem is solved)= 1- P(Problem is not solved)=1- $P(\overline{A}\overline{B}) = 1 - (1/2)(2/3) = 2/3$

(B) P(exactly one of them solves the problem)= $P(\overline{A}B or A\overline{B}) = (1/2)(2/3) + (1/2)(1/3) = 1/2$

10. Solution:

The function is defined for all points of the real line.

Case I: If c < -3, f(c)=c^3-c+1,

$\lim_{x\to c}(f(x)) = \lim_{x\to c}(x^3 - x + 1) = c^3 - c + 1 = f(c)$

$\therefore f \text{ is continuous } \forall\ x < -3$

Case II:If c > -3

$f(c) = 3c + 2$

$\lim_{x\to c}(f(x)) = \lim_{x\to c}(3x + 2) = 3c + 2 = f(c)$

$\therefore f \text{ is continuous } \forall\ x > 3$

Case II:If c = -3

$\lim_{x\to -3^-}(f(x)) = \lim_{x\to -3^-}(x^3 - x + 1) = -23$

$\lim_{x\to -3^+}(f(x)) = \lim_{x\to -3^+}(-2x) = 6$

Since, L.H.L ≠ R.H.L at x=-3, f(x) is not continuous at x=-3.

Similarly if c=3, L.H.L=-6, R.H.L=11. Thus f is not continuous at x=3

11. Solution:

$x\frac{dy}{dx} - y + x\cos ec\left(\frac{y}{x}\right) = 0 \Rightarrow \frac{dy}{dx} - \frac{y}{x} + \cos ec\left(\frac{y}{x}\right) = 0$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \cos ec\left(\frac{y}{x}\right)$

$Let\ \frac{y}{x} = v \Rightarrow y = vx$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$
$$\therefore v + x\frac{dv}{dx} = v - \cos ec(v)$$

$$\frac{-dv}{\cos ecv} = \frac{dx}{x}$$
$$\int \frac{-dv}{\cos ecv} = \int \frac{dx}{x}$$
$$\int -\sin v dv = \int \frac{dx}{x}$$
$$\therefore \cos v = \log x + c \Rightarrow \cos\left(\frac{y}{x}\right) = \log x + c$$
$$At\ x = 1, y = 0 \therefore 1 = 0 + c \Rightarrow c = 1$$
$$\therefore \cos\left(\frac{y}{x}\right) = \log x + 1$$

12. Solution:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix} (C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$$
$$= (a-b)(b^3-c^3) - (b-c)(a^3-b^3)$$
$$= (a-b)(b-c)(b^2+bc+c^2) - (b-c)(a-b)(a^2+ab+b^2)$$
$$= (a-b)(b-c)(\cancel{b^2} + bc + c^2 - a^2 - ab - \cancel{b^2})$$
$$= (a-b)(b-c)(c-a)(a+b+c)$$

13. Solution:

$$A = \pi r^2$$
$$\therefore \frac{dA}{dt} = \left(\frac{dA}{dr}\right)\left(\frac{dr}{dt}\right) = 2\pi r\frac{dr}{dt}$$
$$\frac{dr}{dt} = 2cm/\sec$$
$$\therefore if\ \ r = 14, \frac{dA}{dt} = 2\pi(14)(2) = 176cm^2/\sec$$

Harmful effects of poisoning water bodies:

Spread of epidemic, death of animals drinking water from the water source.

14. Solution:
Let z∈ C
∵ g is onto there exists b∈ B s.t g(b)=z.
Now, ∵ b∈ B and f is onto there exists a∈ As.t.f(a)=b.

$\because g(b)=z \Rightarrow g(f(a))=z \Rightarrow (g\circ f)(a)=z$

$\therefore g\circ f$ *is onto.*

15. Solution: $\because$ f(x) is a polynomial $\Rightarrow$ f(x) is continuous on [-4,2].

$\because$ f(x) is a polynomial $\Rightarrow$ f(x) is differentiable on]-4,2[.

$f(-4)=(-4)^2+2(-4)-8=0$

f(2)=4+4-8=0

$\therefore$ f(-4)=f(2)

$\therefore$ all the conditions of Rolle's theorem are satisfied.

So, $\exists\, c\in\,]-4,2[$ s.t. f '(c)=0.

f '(x)=2x+2, f '(x)=0 $\Rightarrow$ 2x+2=0 $\Rightarrow$ x=-1

$\therefore$ for c=-1 $\in$]-4,2[, f '(c)=0.

Thus, Rolle 's Theorem is verified.

16. Solution:

$$\left(|\vec a|\vec b+|\vec b|\vec a\right).\left(|\vec a|\vec b-|\vec b|\vec a\right)=|\vec a||\vec a|\vec b.\vec b+|\vec b||\vec a|\vec a.\vec b-|\vec a||\vec b|\vec b.\vec a-|\vec b||\vec b|\vec a.\vec a$$

$$=|\vec a|^2|\vec b|^2+|\vec b||\vec a|\vec a.\vec b-|\vec a||\vec b|\vec b.\vec a-|\vec b|^2|\vec a|^2=0(\because \vec a.\vec b=\vec b.\vec a)$$

17. Solution:

$$\int\frac{dx}{x(x^4-1)}=\int\frac{x^3dx}{x^4(x^4-1)}$$

$$Let\ x^4=t\Rightarrow 4x^3dx=dt\Rightarrow x^3dx=dt/4$$

$$\therefore \frac{1}{4}\int\frac{dt}{t(t-1)}=\frac{1}{4}\left[\int\left(\frac{1}{t-1}-\frac{1}{t}\right)dt\right]$$

$$=\frac{1}{4}[\log(t-1)-\log t]+c$$

$$=\frac{1}{4}\log\left(\frac{x^4-1}{x^4}\right)+c$$

18. Solution:

Clearly, the above lines are parallel.

$$\therefore Distance=\left|\frac{\vec b\times(\overrightarrow{a_2}-\overrightarrow{a_1})}{|\vec b|}\right|$$

$\vec{b} = 2i - 3j + k, \overrightarrow{a_1} = i + 3j - 2k, \overrightarrow{a_2} = 2i + 4j - k$

$\therefore \overrightarrow{a_2} - \overrightarrow{a_1} = i + j + k$

$$\therefore \vec{b} \times (\overrightarrow{a_2} - \overrightarrow{a_1}) = \begin{vmatrix} i & j & k \\ 2 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = i(-3-1) - j(2-1) + k(2+3) = -4i - j + 5k$$

$$\therefore dis\tan ce = \frac{\left|\sqrt{16+1+25}\right|}{\left|\sqrt{4+9+1}\right|} = \frac{\sqrt{42}}{\sqrt{14}}$$

19. Solution:

$\overrightarrow{n_1} = 2i + 2j - 3k, d_1 = 7$

$\overrightarrow{n_2} = 2i + 5j + 3k, d_2 = 9$

Equation of plane:

$\vec{r}.(\overrightarrow{n_1} + \lambda \overrightarrow{n_2}) = d_1 + \lambda d_2$

$\vec{r}.(2i + 2j - 3k + \lambda(2i + 5j + 3k)) = 7 + 9\lambda$

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore x(2+2\lambda) + y(2+5\lambda) + z(-3+3\lambda) = 7 + 9\lambda$

Putting $(x, y, z) = (2,1,3)$ *we get* $\lambda = \frac{10}{9}$

Substituting the value of λ *we get,* $\vec{r}.(38i + 68j + 3k) = 153$

Section C

20. Solution:

Suppose tailor A works for x days and tailor B works for y days.

Then, Cost Z= 15x + 20y

	Tailor A	Tailor B	Min Requirement
Shirts	6	10	60
Pants	4	4	32
Cost per day	15	20	

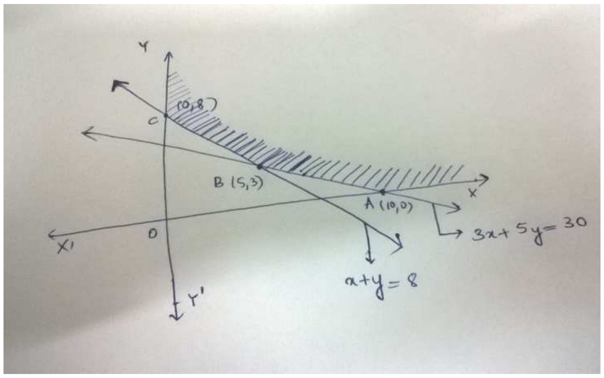
The mathematical formulation of the problem is as follows:

Min Z= 15x+20y

$6x + 10y \geq 60 \Rightarrow 3x + 5y \geq 30$

s.t $4x + 4y \geq 32 \Rightarrow x + y \geq 8$

$x \geq 0, y \geq 0$



We graph the above inequalities. The feasible region is as shown in the figure. We observe the feasible region is unbounded and the corner points are A,Band C. The co-ordinates of the corner points are (10,0), (5,3),(0,8).

Corner Point	Z=15x +20y
(10,0)	150
(5,3)	**135**
(0,8)	160

Thus cost is minimized by hiring A for 5 days and hiring B for 3 days.

21. Solution:

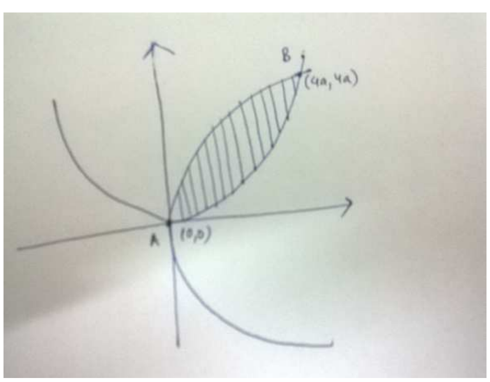
The point of intersection of the two curves:

$$x^2 = \frac{y^4}{16a^2} \Rightarrow 4ay = \frac{y^4}{16a^2}$$

$$\Rightarrow y(y^3 - 64a^3) = 0 \Rightarrow y = 0, y = 4a$$

$$If\ \ y = 0, x = 0; y = 4a \Rightarrow x = 4a$$

$\therefore$ points of intersection are A(0,0) and B(4a,4a)



$$Area = \int_0^{4a} (y_2 - y_1)dx = \int_0^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a} \right) dx$$

$$= \sqrt{4a} \frac{x^{3/2}}{3/2} - \frac{x^3}{12a} \Bigg|_0^{4a} = \frac{16a^2}{3}$$

22. Solution:

Let E denote the event that the ball drawn is red.

Let E_1 denote the event that the ball is drawn from bag X, $P(E_1)=1/3$.

Let E_2 denote the event that the ball is drawn from bag Y, $P(E_2)=1/3$

Let E_3 denote the event that the ball is drawn from bag Z, $P(E_3)=1/3$

$P(E/E_1)=3/5$,$P(E/E_2)=4/9$, $P(E/E_3)=3/5$, $P(E_2/E)=?$

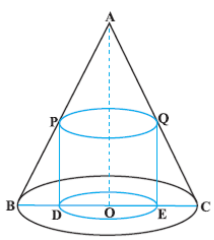
By Baye's theorem,

$$P(E_2 / E) = \frac{P(E / E_2)P(E_2)}{P(E / E_1)P(E_1) + P(E / E_2)P(E_2) + P(E / E_3)P(E_3)}$$

$$= \frac{(4/9)(1/3)}{(3/5)(1/3) + (4/9)(1/3) + (3/5)(1/3)}$$

$$= \frac{10}{37}$$

23. Solution:



Let OC=r be the radius of the cone and OA=h be its height.

Let a cylinder with radius OE = x and height h' be inscribed in the cone.

Surface Area= $2\pi xh'$

$\because \triangle QEC \sim \triangle AOC$,

$$\frac{QE}{AO} = \frac{CE}{CO} \Rightarrow \frac{h'}{h} = \frac{r-x}{r} \Rightarrow h' = h\left(\frac{r-x}{r}\right)$$

$$\therefore S = S(x) = 2\pi xh' = 2\pi xh\left(\frac{r-x}{r}\right) = \frac{2\pi h}{r}\left(rx - x^2\right)$$

$$S'(x) = \frac{2\pi h}{r}\left(r - 2x\right)$$

$$S''(x) = \frac{2\pi h}{r}(-2)$$
$$S'(x) = 0 \Rightarrow x = r/2$$
$$Also,\ S''(r/2) = \frac{-4\pi h}{r} < 0$$

Hence, x=r/2 is a point of maxima.

Thus, the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

24. Solution:

Let $\frac{1}{x} = u, \frac{1}{y} = v, \frac{1}{z} = w$

$\therefore$ the system of equations becomes,

$$3u - 2v + 3w = 8$$
$$2u + v - w = 1$$
$$4u - 3v + 2w = 4$$
$$Let\ A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}, b = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$
$$|A| = -17 \neq 0, A^{-1} = \frac{-1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}, U = A^{-1}b = \frac{-1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$
$$\therefore u = 1 \Rightarrow x = 1, v = 2 \Rightarrow y = \frac{1}{2}, w = 3 \Rightarrow z = \frac{1}{3}$$

25. Solution:

$$x = a(\cos t + t\sin t), y = a(\sin t - t\cos t)$$
$$\frac{dx}{dt} = a(\cancel{-\sin t} + \cancel{\sin t} + t\cos t) = at\cos t \Rightarrow \frac{dt}{dx} = \frac{1}{at\cos t}$$
$$\frac{dy}{dt} = a(\cancel{\cos t} - \cancel{\cos t} - t(-\sin t)) = at\sin t$$
$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) = (at\sin t)\frac{1}{at\cos t} = \tan t$$
$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}(\tan t)\frac{dt}{dx} = (\sec^2 t)\left(\frac{1}{at\cos t}\right) = \frac{1}{at\cos^3 t}$$

26. Solution :

$$I=\int_0^{\pi}\frac{xdx}{4\cos^2 x+9\sin^2 x}=\int_0^{\pi}\frac{(\pi-x)dx}{4\cos^2 x+9\sin^2 x}$$

$$\therefore 2I=\pi\int_0^{\pi}\frac{dx}{4\cos^2 x+9\sin^2 x}=2\pi\int_0^{\frac{\pi}{2}}\frac{dx}{4\cos^2 x+9\sin^2 x}$$

$$=2\pi\left[\int_0^{\frac{\pi}{4}}\frac{dx}{4\cos^2 x+9\sin^2 x}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{dx}{4\cos^2 x+9\sin^2 x}\right]$$

$$=2\pi\left[\int_0^{\frac{\pi}{4}}\frac{\sec^2 xdx}{4+9\tan^2 x}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos ec^2xdx}{4\cot^2 x+9}\right]$$

Putting tanx=t and cotx=u, we get

$$2I=2\pi\left[\int_0^{1}\frac{dt}{4+9t^2}-\int_1^{0}\frac{du}{4u^2+9}\right]=2\pi\left[\frac{1}{9}\left(\frac{3}{2}\right)\tan^{-1}\frac{t}{2/3}\Big|_0^1-\frac{1}{4}\left(\frac{2}{3}\right)\tan^{-1}\frac{u}{3/2}\Big|_1^0\right]$$

$$=2\pi\left[\frac{1}{6}\tan^{-1}\left(\frac{3}{2}\right)+\frac{1}{6}\tan^{-1}\left(\frac{2}{3}\right)\right]=\frac{2\pi}{6}\left(\frac{\pi}{2}\right)=\frac{\pi^2}{6}$$

$$\therefore I=\frac{\pi^2}{12}$$