12. Systems of Linear Equations

Solving Systems Of Equations By Elimination Method

Step I: Let the two equations obtained be

$a_1x + b_1y + c_1 = 0$ ....(1)
$a_2x + b_2y + c_2 = 0$ ....(2)

Step II: Multiplying the given equation so as to make the co-efficients of the variable to be eliminated equal.

Step III: Add or subtract the equations so obtained in Step II, as the terms having the same coefficients may be either of opposite or the same sign.

Step IV : Solve the equations in one varibale so obtained in Step III.

Step V: Substitute the value found in Step IV in any one of the given equations and then copmpute the value of the other variable.

Elimination Method Examples

Example 1: Solve the following system of linear equations by applying the method of elimination by equating the coefficients :

(i) $4x - 3y = 4$ (ii) $5x - 6y = 8$
$2x + 4y = 3$ $3x + 2y = 6$

Sol. (i) We have,

$4x - 3y = 4$ ....(1)
$2x + 4y = 3$ ....(2)
Let us decide to eliminate x from the given equation. Here, the co-efficients of x are 4 and 2 respectively. We find the L.C.M. of 4 and 2 is 4. Then, make the co-efficients of x equal to 4 in the two equations.
Multiplying equation (1) with 1 and equation (2) with 2, we get
$4x - 3y = 4$ ....(3)
$4x + 8y = 6$ ....(4)
Subtracting equation (4) from (3), we get
$-11y = -2 \Rightarrow y = \frac{2}{11}$
Substituting y = 2/11 in equation (1), we get
$\Rightarrow 4x - 3 \times \frac{2}{11} = 4$
$\Rightarrow 4x - \frac{6}{11} = 4$
$\Rightarrow 4x = 4 + \frac{6}{11}$
$\Rightarrow 4x = \frac{50}{11}$
$\Rightarrow x = \frac{50}{44} = \frac{25}{22}$
Hence, solution of the given system of equation is :
$x = \frac{25}{22}, \; y = \frac{2}{11}$

(ii) We have;

$5x - 6y = 8$ ....(1)
$3x + 2y = 6$ ....(2)
Let us eliminate y from the given system of equations. The co-efficients of y in the given equations are 6 and 2 respectively. The L.C.M. of 6 and 2 is 6. We have to make the both coefficients equal to 6. So, multiplying both sides of equation (1) with 1 and equation (2) with 3, we get
$5x - 6y = 8$ ....(3)
$9x + 6y = 18$ ....(4)
Adding equation (3) and (4), we get
$14x = 26 \Rightarrow x = \frac{13}{7}$
Putting $x = \frac{13}{7}$ in equation (1), we get

$5 \times \frac{13}{7} - 6y = 8 \quad \Rightarrow \frac{65}{7} - 6y = 8$

$\Rightarrow 6y = \frac{65}{7} - 8 = \frac{65-56}{7} = \frac{9}{7}$

$\Rightarrow y = \frac{9}{42} = \frac{3}{14}$

Hence, the solution of the system of equations is $x = \frac{13}{7}$, $y = \frac{3}{14}$

Example 2: Solve the following system of linear equations by usnig the method of elimination by equating the coefficients:

$3x + 4y = 25$; $5x - 6y = -9$

Sol. The given system of equation is

$3x + 4y = 25$ ….(1)

$5x - 6y = -9$ ….(2)

Let us eliminate y. The coefficients of y are 4 and – 6. The LCM of 4 and 6 is 12. So, we make the coefficients of y as 12 and – 12.

Multiplying equation (1) by 3 and equation (2) by 2, we get

$9x + 12y = 75$ ….(3)

$10x - 12y = -18$ …(4)

Adding equation (3) and equation (4), we get

$19x = 57 \Rightarrow x = 3.$

Putting x = 3 in (1), we get,

$3 \times 3 + 4y = 25$

$\Rightarrow 4y = 25 - 9 = 16 \Rightarrow y = 4$

Hence, the solution is x = 3, y = 4.

Verification: Both the equations are satisfied by x = 3 and y = 4, which shows that the solution is correct.

Example 3: Solve the following system of equations:

$15x + 4y = 61$; $4x + 15y = 72$

Sol. The given system of equation is

$15x + 4y = 61$ ….(1)

$4x + 15y = 72$ ….(2)

Let us eliminate y. The coefficients of y are 4 and 15. The L.C.M. of 4 and 15 is 60. So, we make the coefficients of y as 60. Multiplying (1) by 15 and (2) by 4, we get

$225x + 60y = 915$ ….(3)

$16x + 60y = 288$ ….(4)

Substracting (4) from (3), we get

$209x = 627 \Rightarrow x = 3$

Putting x = 3 in (1), we get

$15 \times 3 + 4y = 61$ $45 + 4y = 61$

$4y = 61 - 45 = 16 \Rightarrow y = 4$

Hence, the solution is x = 3, y = 4.

Verification: On putting x = 3 and y = 4 in the given equations, they are satisfied. Hence, the solution is correct.

Example 4: Solve the following system of equations by using the method of elimination by equating the co-efficients.

$\frac{x}{y} + \frac{2y}{5} + 2 = 10$; $\frac{2x}{7} - \frac{5}{2} + 1 = 9$

Sol. The given system of equation is

$\frac{x}{y} + \frac{2y}{5} + 2 = 10 \Rightarrow \frac{x}{y} + \frac{2y}{5} = 8$ …(1)

$\frac{2x}{7} - \frac{5}{2} + 1 = 9 \Rightarrow \frac{2x}{7} - \frac{5}{2} = 8$ ….(2)

The equation (1) can be expressed as :

$\frac{5x+4y}{10} = 8 \Rightarrow 5x + 4y = 80$ ….(3)

Similarly, the equation (2) can be expressed as :

$\frac{4x-7y}{14} = 8 \Rightarrow 4x - 7y = 112$ ….(4)

Now the new system of equations is

$5x + 4y = 80$ ….(5)

$4x - 7y = 112$ ….(6)

Now multiplying equation (5) by 4 and equation (6) by 5, we get
20x – 16y = 320 ….(7)
20x + 35y = 560 ….(8)
Subtracting equation (7) from (8), we get ;
y = $-\frac{240}{51}$
Putting y = $-\frac{240}{51}$ in equation (5), we get ;
5x + 4 × $\frac{-240}{51}$ = 80 ⇒ 5x – $\frac{960}{51}$ = 80
⇒ 5x = 80 + $\frac{960}{51}$ = $\frac{4080+960}{51}$ = $\frac{5040}{51}$
⇒ x = $\frac{5040}{255}$ = $\frac{1008}{51}$ = $\frac{336}{17}$ ⇒ x = $\frac{336}{17}$
Hence, the solution of the system of equations is, x = $\frac{336}{17}$, y = $-\frac{80}{17}$.

Example 5: Solve the following system of linear equatoins by using the method of elimination by equating the coefficients
√3x – √2y = √3 = ; √5x – √3y = √2

Sol. The given equations are

√3x – √2y = √3 ….(1)
√5x – √3y = √2 ….(2)
Let us eliminate y. To make the coefficients of equal, we multiply the equation (1) by √3 and equation (2) by √2 to get
3x – √6y = 3 ….(3)
√10x + √6y = 2 ….(4)
Adding equation (3) and equation (4), we get
3x + √10x = 5 ⇒ (3 + √10) x = 5
$\Rightarrow x = \frac{5}{3+\sqrt{10}} = \left(\frac{5}{\sqrt{10}+3}\right) \times \left(\frac{\sqrt{10}-3}{\sqrt{10}-3}\right)$
$= 5\left(\sqrt{10}-3\right)$
Putting x = 5(√10– 3) in (1) we get
√3 × 5(√10 – 3) –√2 y = √3
⇒ 5√30 – 15√3 – √2y = √3
⇒ √2y = 5√30 – 15√3 – √3
⇒ √2y = 5√30 – 16√3
⇒ $y = \frac{5\sqrt{30}}{\sqrt{2}} - \frac{16\sqrt{3}}{\sqrt{2}}$
⇒ y = 5√15 – 8√6
Hence, the solution is x = 5(√10– 3) and y = 5√15 – 8√6

Example 6: Solve for x and y :
$\frac{ax}{b}$ – $\frac{by}{a}$ = a + b ; ax – by = 2ab

Sol. The given system of equations is

$\frac{ax}{b}$ – $\frac{by}{a}$ = a + b ….(1)
ax – by = 2ab ….(2)
Dividing (2) by a, we get
x – $\frac{by}{a}$ = 2b ….(3)
On subtracting (3) from (1), we get
$\frac{ax}{b}$ – x = a – b ⇒ $x\left(\frac{a}{b}-1\right)$ = a – b
⇒ x = $\frac{(a-b)b}{a-b}$ = b ⇒ x = b
On substituting the value of x in (3), we get
b – $\frac{by}{a}$ = 2b ⇒ $b\left(1-\frac{y}{a}\right)$ = 2b
⇒ 1 – $\frac{y}{a}$ = 2 ⇒ $\frac{y}{a}$ = 1 – 2
⇒ $\frac{y}{a}$ = –1 ⇒ y = –a
Hence, the solution of the equations is
x = b, y = – a

Example 7: Solve the following system of linear equations :

2(ax – by) + (a + 4b) = 0
2(bx + ay) + (b – 4a) = 0

Sol. $2ax - 2by + a + 4b = 0$ …. (1)
$2bx + 2ay + b - 4a = 0$ …. (2)
Multiplyng (1) by b and (2) by a and subtracting, we get
$2(b^2 + a^2)\,y = 4\,(a^2 + b^2) \Rightarrow\ y = 2$
Multiplying (1) by a and (2) by b and adding, we get
$2(a^2 + b^2)\,x + a^2 + b^2 = 0$
$2(a^2 + b^2)\,x = -\,(a^2 + b^2) \Rightarrow\ x = -\,1/2$
Hence $x = -1/2$, and $y = 2$

Example 8: Solve $(a - b)\,x + (a + b)\,y = a^2 - 2ab - b^2$
$(a + b)\,(x + y) = a^2 + b^2$

Sol. The given system of equation is
$(a - b)\,x + (a + b)\,y = a^2 - 2ab - b^2$ ….(1)
$(a + b)\,(x + y) = a^2 + b^2$ ….(2)
$\Rightarrow\ (a + b)\,x + (a + b)\,y = a^2 + b^2$ ….(3)

Subtracting equation (3) from equation (1), we get
$(a - b)\,x - (a + b)\,x = (a^2 - 2ab - b^2) - (a^2 + b^2)$
$\Rightarrow\ -2bx = -\,2ab - 2b^2$
$\Rightarrow\ x = \frac{-2ab}{-2b} - \frac{2b^2}{-2b} = a + b$
Putting the value of x in (1), we get
$\Rightarrow\ (a - b)\,(a + b) + (a + b)\,y = a^2 - 2ab - b^2$
$\Rightarrow\ (a + b)\,y = a^2 - 2ab - b^2 - (a^2 - b^2)$
$\Rightarrow\ (a + b)\,y = -\,2ab$
$\Rightarrow\ y = \frac{-2ab}{a+b}$
Hence, the solution is $x = a + b$,
$y = \frac{-2ab}{a+b}$

Solving Systems Of Equations By Substitution Method

In this method, we first find the value of one variable (y) in terms of another variable (x) from one equation. Substitute this value of y in the second equation. Second equation becomes a linear equation in x only and it can be solved for x.
Putting the value of x in the first equation, we can find the value of y.
This method of solving a system of linear equations is known as the method of **elimination by substitution.** 'Elimination', because we get rid of y or 'eliminate' y from the second equation. 'Substitution', because we 'substitute' the value of y in the second equation.

Working rule:

Let the two equations be
$a_1x + b_1y + c_1 = 0$ ….(1)
$a_2x + b_2y + c_2 = 0$ ….(2)

Step I: Find the value of one variable, say y, in terms of the other i.e., x from any equation, say (1).

Step II: Substitute the value of y obtained in step 1 in the other equation i.e., equation (2). This equation becomes equation in one variable x only.

Step III: Solve the equation obtained in step II to get the value of x.

Step IV: Substitute the value of x from step II to the equation obtained in step I. From this equation, we get the value of y. In this way, we get the solution i.e. values of x and y.

Remark : Verification is a must to check the answer.

Substitution Method Examples

Example 1: Solve each of the following system of equations by eliminating x (by substitution) :

(i) $x + y = 7$
$2x - 3y = 11$

(ii) $x + y = 7$
$12x + 5y = 7$

(iii) $2x - 7y = 1$
$4x + 3y = 15$

(iv) $3x - 5y = 1$ (v) $5x + 8y = 9$
$5x + 2y = 19$ $2x + 3y = 4$

Sol. **(i)** We have,

$x + y = 7$ ....(1)
$2x - 3y = 11$ ....(2)
We shall eliminate x by substituting its value from one equation into the other. from equaton (1), we get
$x + y = 7 \Rightarrow x = 7 - y$
Substituting the value of x in equation (2), we get
$2 \times (7 - y) - 3y = 11$
$\Rightarrow 14 - 2y - 3y = 11$
$\Rightarrow -5y = -3$ or, $y = 3/5$
Now, substituting the value of y in equation (1), we get
$x + 3/5 = 7 \Rightarrow x = 32/5$.
Hence, $x = 32/5$ and $y = 3/5$

(ii) We have,
$x + y = 7$ ....(1)
$12x + 5y = 7$ ....(2)
From equation (1), we have
$x + y = 7 \Rightarrow x = 7 - y$
Substituting the value of y in equation (2), we get
$\Rightarrow 12(7 - y) + 5y = 7$
$\Rightarrow 84 - 12y + 5y = 7$
$\Rightarrow -7y = -77$
$\Rightarrow y = 11$
Now, Substituting the value of y in equation (1), we get
$x + 11 = 7 \Rightarrow x = -4$
Hence, $x = -4$, $y = 11$.

(iii) We have,
$2x - 7y = 1$ ....(1)
$4x + 3y = 15$ ....(2)
From equation (1), we get
$2x - 7y = 1 \Rightarrow x = \frac{7y+1}{2}$
Substituting the value of x in equation (2), we get ;
$\Rightarrow 4 \times \frac{7y+1}{2} + 3y = 15$
$\Rightarrow \frac{28y+4}{2} + 3y = 15$
$\Rightarrow 28y + 4 + 6y = 30$
$\Rightarrow 34y = 26 \Rightarrow y = \frac{13}{17}$
Now, substituting the value of y in equation (1), we get
$2x - 7 \times \frac{13}{17} = 1$
$\Rightarrow 2x = 1 + \frac{91}{17} = \frac{108}{17} \Rightarrow x = \frac{108}{34} = \frac{54}{17}$
Hence, $x = \frac{54}{17}$, $y = \frac{13}{17}$

(iv) We have,
$3x - 5y = 1$ .... (1)
$5x + 2y = 19$ .... (2)
From equation (1), we get;
$3x - 5y = 1 \Rightarrow x = \frac{5y+1}{3}$
Substituing the value of x in equation (2), we get
$\Rightarrow 5 \times \frac{5y+1}{3} + 2y = 19$
$\Rightarrow 25y + 5 + 6y = 57 \Rightarrow 31y = 52$
Thus, $y = \frac{52}{31}$
Now, substituting the value of y in equation (1), we get
$3x - 5 \times \frac{52}{31} = 1$
$\Rightarrow 3x - \frac{260}{31} = 1 \Rightarrow 3x = \frac{291}{31}$
$\Rightarrow x = \frac{97}{31}$
Hence, $x = \frac{97}{31}$, $y = \frac{52}{31}$

(v) We have,
5x + 8y = 9 ….(1)
2x + 3y = 4 ….(2)
From equation (1), we get
5x + 8y = 9 ⇒ x = $\frac{9-8y}{5}$

Substituting the value of x in equation (2), we get
⇒ 2 × $\frac{9-8y}{5}$ + 3y = 4
⇒ 18 − 16y + 15y = 20
⇒ −y = 2 or y = − 2

Now, substituting the value of y in equation (1), we get
5x + 8 (−2) = 9
5x = 25 ⇒ x = 5
Hence, x = 5, y = − 2.

Example 2: Solve the following systems of equations by eliminating 'y' (by substitution) :
(i) 3x − y = 3 (ii) 7x + 11y − 3 = 0 (iii) 2x + y − 17 = 0
7x + 2y = 20 8x + y − 15 = 0 17x − 11y − 8 = 0

Sol. **(i)** We have,
3x − y = 3 ….(1)
7x + 2y = 20 ….(2)
From equation (1), we get ;
3x − y = 3 ⇒ y = 3x − 3
Substituting the value of 'y' in equation (2), we get
⇒ 7x + 2 × (3x − 3) = 20
⇒ 7x + 6x − 6 = 20
⇒ 13x = 26 ⇒ x = 2
Now, substituting x = 2 in equation (1), we get;
3 × 2 − y = 3
⇒ y = 3
Hence, x = 2, y = 3.

(ii) We have,
7x + 11y − 3 = 0 ….(1)
8x + y − 15 = 0 …..(2)
From equation (1), we get;
7x + 11y = 3
⇒ y = $\frac{3-7x}{11}$
Substituting the value of 'y' in equation (2), we get
⇒ 8x + $\frac{3-7x}{11}$ = 15
⇒ 88x + 3 − 7x = 165
⇒ 81x = 162
⇒ x = 2
Now, substituting, x = 2 in the equation (2), we get
8 × 2 + y = 15
⇒ y = − 1
Hence, x = 2, y = − 1.

(iii) We have,
2x + y = 17 ….(1)
17x − 11y = 8 ….(2)
From equation (1), we get;
2x + y = 17 ⇒ y = 17 − 2x
Substituting the value of 'y' in equation (2), we get
⇒ 17x − 11 (17 − 2x) = 8
⇒ 17x − 187 + 22x = 8
⇒ 39x = 195
⇒ x = 5
Now, substituting the value of 'x' in equation (1), we get
2 × 5 + y = 17
⇒ y = 7
Hence, x = 5, y = 7.

Example 3: Solve the following systems of equations,

(i) $\frac{15}{u} + \frac{2}{v} = 17$ (ii) $\frac{11}{v} - \frac{7}{u} = 1$

$\frac{1}{u} + \frac{1}{v} = \frac{36}{5}$ $\frac{9}{v} + \frac{4}{u} = 6$

Sol. (i) The given system of equation is

$\frac{15}{u} + \frac{2}{v} = 17$ ....(1)

$\frac{1}{u} + \frac{1}{v} = \frac{36}{5}$ ....(2)

Considering 1/u = x, 1/v = y, the above system of linear equations can be written as

15x + 2y = 17 ....(3)

$x + y = \frac{36}{5}$ ....(4)

Multiplying (4) by 15 and (iii) by 1, we get

15x + 2y = 17 ....(5)

$15x + 15y = \frac{36}{5} \times 15 = 108$ ....(6)

Subtracting (6) form (5), we get

−13y = − 91 ⇒ y = 7

Substituting y = 7 in (4), we get

$x + 7 = \frac{36}{5}$ ⇒ $x = \frac{36}{5} - 7 = \frac{1}{5}$

But, $y = \frac{1}{v} = 7$ ⇒ $v = \frac{1}{7}$

and, $x = \frac{1}{u} = \frac{1}{5}$ ⇒ u = 5

Hence, the required solution of the given system is u = 5, v = 1/7.

(ii) The given system of equation is

$\frac{11}{v} - \frac{7}{u} = 1$; $\frac{9}{v} + \frac{4}{u} = 6$

Taking 1/n = x and 1/u = y, the above system of equations can be written as

11x − 7y = 1 ....(1)

9x − 4y = 6 ....(2)

Multiplying (1) by 4 and (2) by 7, we get,

44x − 28y = 4 ....(3)

63x − 28y = 42 ....(4)

Subtracting (4) from (3) we get,

− 19x = −38 ⇒ x = 2

Substituting the above value of x in (2), we get;

9 × 2 − 4y = 6 ⇒ −4y = − 12

⇒ y = 3

But, $x = \frac{1}{v} = 2$ ⇒ $v = \frac{1}{2}$

and, $y = \frac{1}{u} = 3$

⇒ $u = \frac{1}{3}$

Hence, the required solution of the given system of the equation is

$v = \frac{1}{2}$, $u = \frac{1}{3}$

Example 4: Solve 2x + 3y = 11 and 2x − 4y = − 24 and hence find the value of 'm' for which y = mx + 3.

Sol. We have,

2x + 3y = 11 ....(1)

2x − 4y = − 24 ....(2)

From (1), we have 2x = 11 − 3y

Substituting 2x = 11 − 3y in (2), we get

11 − 3y − 4y = −24

⇒ −7y = − 24 − 11

⇒ −7y = − 35

⇒ y = 5

Putting y = 5 in (1), we get

2x + 3 × 5 = 11

2x = 11 − 15

⇒ x = −4/2 = − 2

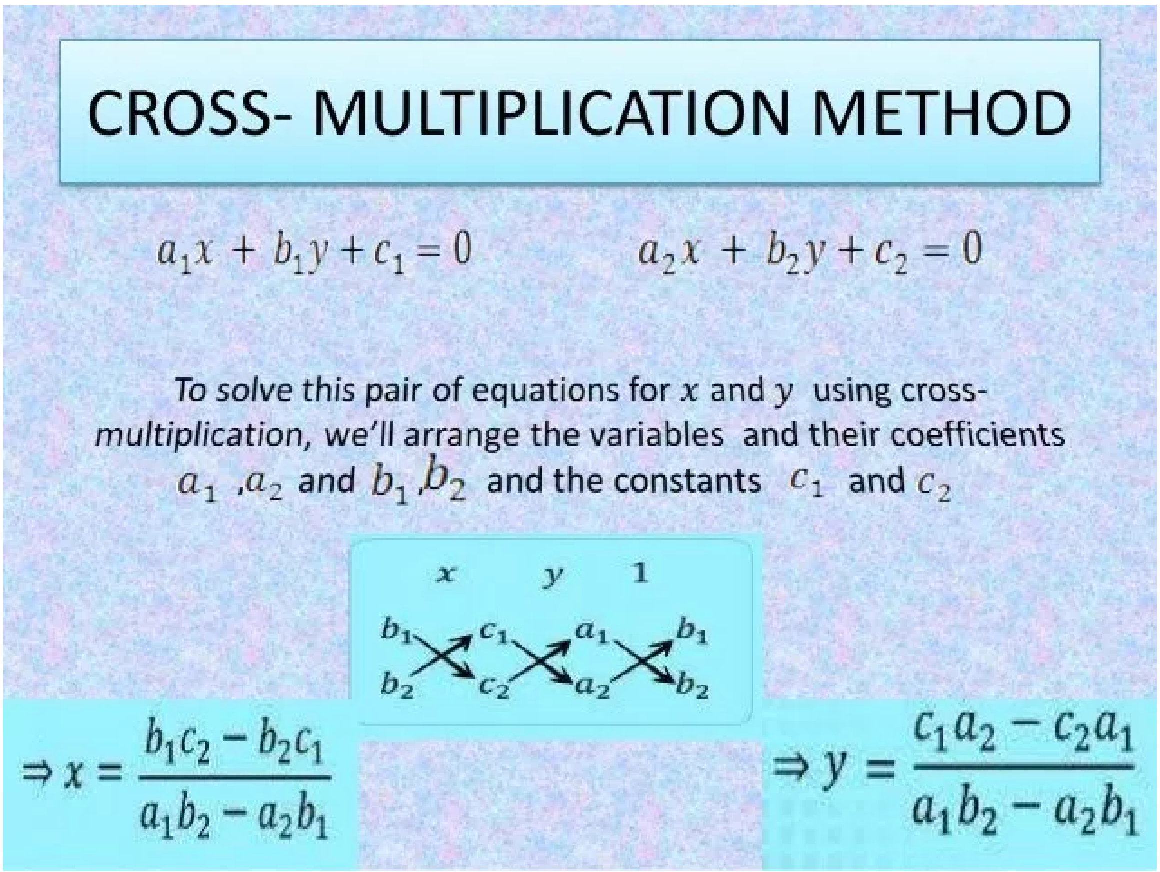
Hence, x = − 2 and y = 5

Again putting x = − 2 and y = 5 in y = mx + 3, we get

5x = m(−2) + 3

⇒ −2m = 5 − 3
⇒ m = − 1

Cross Multiplication Method For Solving Equations



By the method of elimination by substitution, only those equations can be solved, which have unique solution. But the method of cross multiplication discussed below is applicable in all the cases; whether the system has a unique solution, no solution or infinitely many solutions.
Let us solve the following system of equations
$a_1x + b_1y + c_1 = 0$ ….(1)
$a_2x + b_2y + c_2 = 0$ ….(2)
Multiplying equation (1) by b_2 and equation (2) by b_1, we get
$a_1b_2x + b_1b_2y + b_2c_1 = 0$ ….(3)
$a_2b_1x + b_1b_2y + b_1c_2 = 0$ ….(4)
Subtracting equation (4) from equation (3), we get
$(a_1b_2 - a_2b_1)\, x + (b_2c_1 - b_1c_2) = 0$

$$\Rightarrow x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

$$\left[a_1b_2 - a_2b_1 \neq 0 \text{ and } \frac{a_1}{a_2} \neq \frac{b_1}{b_2}\right]$$

$$\text{Similarly, } y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

These values of x and y can also be written as

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{-y}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1}$$

Cross Multiplication Method Examples

Example 1: Solve the following system of equations by cross-multiplication method.
2x + 3y + 8 = 0
4x + 5y + 14 = 0

Sol. The given system of equations is
2x + 3y + 8 = 0
4x + 5y + 14 = 0
By cross-multiplication, we get

$$\frac{x}{\begin{matrix}3 & 8\\ 5 & 14\end{matrix}} = \frac{-y}{\begin{matrix}2 & 8\\ 4 & 14\end{matrix}} = \frac{1}{\begin{matrix}2 & 3\\ 4 & 5\end{matrix}}$$

$$\Rightarrow \frac{x}{3\times14-5\times8} = \frac{x}{3\times14-5\times8} = \frac{1}{2\times5-4\times3}$$

$$\Rightarrow \frac{x}{42-40} = \frac{-y}{28-32} = \frac{1}{10-12}$$

$$\Rightarrow \frac{x}{2} = \frac{-y}{-4} = \frac{1}{-2}$$

$$\Rightarrow \frac{x}{2} = \frac{-1}{2}$$

⇒ x = – 1

$$\Rightarrow \frac{-y}{-4} = \frac{-1}{2}$$

⇒ y = – 2

Hence, the solution is x = – 1, y = – 2

We can verify the solution.

Example 2: Solve the followning system of equations by the method of cross-multiplication.

2x – 6y + 10 = 0

3x – 7y + 13 = 0

Sol. The given system of equations is

2x – 6y + 10 = 0 ….(1)

3x – 7y + 13 = 0 ….(2)

By cross-multiplication, we have

$$\frac{x}{\begin{matrix}-6 & 10\\ -7 & 13\end{matrix}} = \frac{-y}{\begin{matrix}2 & 10\\ 3 & 13\end{matrix}} = \frac{1}{\begin{matrix}2 & -6\\ 3 & -7\end{matrix}}$$

$$\Rightarrow \frac{x}{-6\times13-(-7)\times10} = \frac{-y}{2\times13-3\times10} = \frac{1}{2\times(-7)-3\times(-6)}$$

$$\Rightarrow \frac{x}{78+70} = \frac{-y}{26-30} = \frac{1}{-14+18}$$

$$\Rightarrow \frac{x}{-8} = \frac{-y}{-4} = \frac{1}{4}$$

$$\Rightarrow \frac{x}{-8} = \frac{1}{4}$$

⇒ x = – 2

$$\Rightarrow \frac{-y}{-4} = \frac{1}{4}$$

⇒ y = 1

Hence, the solution is x = – 2, y = 1

Example 3: Solve the following system of equations by the method of cross-multiplication.

11x + 15y = – 23; 7x – 2y = 20

Sol. The given system of equations is

11x + 15y + 23 = 0

7x – 2y – 20 = 0

Now, by cross-multiplication method, we have

$$\frac{x}{\begin{matrix}15 & 23\\ -2 & -20\end{matrix}} = \frac{-y}{\begin{matrix}11 & 23\\ 7 & -20\end{matrix}} = \frac{1}{\begin{matrix}11 & 15\\ 7 & -2\end{matrix}}$$

$$\Rightarrow \frac{x}{15\times(-20)-(-2)\times23} = \frac{-y}{11\times(-20)-7\times23} = \frac{1}{11\times(-2)-7\times15}$$

$$\Rightarrow \frac{x}{-300+46} = \frac{-y}{-220-161} = \frac{1}{-22-105}$$

$$\Rightarrow \frac{x}{-254} = \frac{-y}{-381} = \frac{1}{-127}$$

$$\Rightarrow \frac{x}{-254} = \frac{1}{-127} \Rightarrow x = 2$$

$$\text{and} \frac{-y}{-381} = \frac{1}{-127} \Rightarrow y = -3$$

Hence, x = 2, y = – 3 is the required solution.

Example 4: Solve the following system of equations by cross-multiplication method.

ax + by = a – b; bx – ay = a + b

Sol. Rewriting the given system of equations, we get
ax + by − (a − b) = 0
bx − ay − (a + b) = 0
By cross-multiplication method, we have

$$\frac{x}{\begin{matrix} b & -(a-b) \\ -a & -(a+b) \end{matrix}} = \frac{-y}{\begin{matrix} a & -(a-b) \\ b & -(a+b) \end{matrix}} = \frac{1}{\begin{matrix} a & b \\ b & -a \end{matrix}}$$

$$\Rightarrow \frac{x}{b\times\{-(a+b)\}-(-a)\times\{-(a-b)\}} = \frac{-y}{-a(a+b)+b(a-b)} = \frac{1}{-a^2-b^2}$$

$$\Rightarrow \frac{x}{-ab-b^2-a^2+ab} = \frac{-y}{-a^2-ab+ab-b^2} = \frac{1}{-(a^2+b^2)}$$

$$\Rightarrow \frac{x}{-(a^2+b^2)} = \frac{-y}{-(a^2+b^2)} = \frac{1}{-(a^2+b^2)}$$

$$\Rightarrow \frac{x}{-(a^2+b^2)}\frac{1}{-(a^2+b^2)} \Rightarrow x = 1$$

$$and\ \frac{-y}{-(a^2+b^2)}\frac{1}{-(a^2+b^2)} \Rightarrow y = -1$$

Example 5: Solve the following system of equations by cross-multiplication method.
x + y = a − b; ax − by = $a^2 + b^2$

Sol. The given system of equations can be rewritten as:
x + y − (a − b) = 0
ax − by − ($a^2 + b^2$) = 0
By cross-multiplication method, we have

$$\frac{x}{\begin{matrix} 1 & -(a-b) \\ -b & -(a^2+b^2) \end{matrix}} = \frac{-y}{\begin{matrix} 1 & -(a-b) \\ a & -(a^2+b^2) \end{matrix}} = \frac{1}{\begin{matrix} 1 & 1 \\ a & -b \end{matrix}}$$

$$\Rightarrow \frac{x}{-(a^2+b^2)-(-b)\times\{-(a-b)\}} = \frac{-y}{-(a^2+b^2)-a\times\{-(a-b)\}} = \frac{1}{-b-a}$$

$$\Rightarrow \frac{x}{-(a^2+b^2)-b(a-b)} = \frac{-y}{-(a^2+b^2)+a(a-b)} = \frac{1}{-(b+a)}$$

$$\Rightarrow \frac{x}{-a^2-b^2-ab+b^2} = \frac{-y}{-a^2-b^2+a^2-ab} = \frac{1}{-(a+b)}$$

$$\Rightarrow \frac{x}{-a(a+b)} = \frac{-y}{-b(a+b)} = \frac{1}{-(a+b)}$$

$$\Rightarrow \frac{x}{-a(a+b)} = \frac{1}{-(a+b)} \Rightarrow x = a$$

$$and\ \frac{-y}{-b(a+b)} = \frac{1}{-(a+b)} \Rightarrow y = -b$$

Example 6: Solve the following system of equations by the method of cross-multiplication:
$\frac{x}{a} + \frac{y}{b} = a + b$; $\frac{x}{a^2} + \frac{y}{b^2} = 2$

Sol: The given system of equations is rewritten as:
$\frac{x}{a} + \frac{y}{b} - (a + b)$ ….(1)
$\frac{x}{a^2} + \frac{y}{b^2} - 2$ ….(2)
Multiplying equation (1) by ab, we get
bx + ay − ab (a + b) = 0 ….(3)
Multiplying equation (2) by a^2 b^2, we get
$b^2x + a^2y - 2a^2b^2 = 0$ ….(4)
By cross multiplication method, we have

$$\frac{x}{\begin{matrix} a & -ab(a+b) \\ a^2 & -2a^2b^2 \end{matrix}} = \frac{-y}{\begin{matrix} b & -ab(a+b) \\ b^2 & -2a^2b^2 \end{matrix}} = \frac{1}{\begin{matrix} b & a \\ b^2 & a^2 \end{matrix}}$$

$$\Rightarrow \frac{x}{-2a^3b^2+a^3b(a+b)} = \frac{-y}{-2a^2b^3+ab^3(a+b)} = \frac{1}{a^2b-ab^2}$$

$$\Rightarrow \frac{x}{-2a^3b^2+a^4b+a^3b^2} = \frac{y}{-2a^2b^3+a^2b^3+ab^4} = \frac{1}{ab(a-b)}$$

$$\Rightarrow \frac{x}{a^4b-a^3b^2} = \frac{-y}{ab^4-a^2b^3} = \frac{1}{ab(a-b)}$$

$$\Rightarrow \frac{x}{a^3b(a-b)} = \frac{y}{ab^3(a-b)} = \frac{1}{ab(a-b)}$$

$$\Rightarrow \frac{x}{a^3b(a-b)} = \frac{1}{ab(a-b)}$$

$$\Rightarrow x = \frac{a^3b(a-b)}{ab(a-b)} = a^2$$

$$And\ \frac{y}{ab^3(a-b)} = \frac{1}{ab(a-b)}$$

$$\Rightarrow y = \frac{ab^3(a-b)}{ab(a-b)} = b^2$$

Hence, the solution x = a^2, y = b^2

Example 7: Solve the following system of equations by cross-multiplication method –

ax + by = 1; bx + ay = $\frac{(a+b)^2}{a^2+b^2} - 1$

Sol: The given system of equations can be written as

ax + by – 1 = 0 ....(1)

$$bx + ay = \frac{(a+b)^2}{a^2+b^2} - 1$$

$$\Rightarrow bx + ay = \frac{a^2+2ab+b^2-a^2-b^2}{a^2+b^2}$$

$$\Rightarrow bx + ay = \frac{2ab}{a^2+b^2}$$

$$\Rightarrow bx + ay - \frac{2ab}{a^2+b^2} = 0 \qquad ..... (2)$$

Rewritting the equations (1) and (2), we have

ax + by – 1 = 0

$\Rightarrow bx + ay - \frac{2ab}{a^2+b^2} = 0$

Now, by cross-multiplication method, we have

$$\frac{x}{\begin{array}{cc} b & -1 \\ a & -\frac{2ab}{a^2+b^2} \end{array}} = \frac{-y}{\begin{array}{cc} a & -1 \\ b & -\frac{2ab}{a^2+b^2} \end{array}} = \frac{1}{\begin{array}{cc} a & b \\ b & a \end{array}}$$

$$\Rightarrow \frac{x}{b\times\left(\frac{-2ab}{a^2+b^2}\right)-a\times(-1)} = \frac{-y}{a\times\left(\frac{-2ab}{a^2+b^2}\right)-b\times(-1)} = \frac{1}{a\times a-b\times b}$$

$$\Rightarrow \frac{x}{-\frac{2ab^2}{a^2+b^2}+a} = \frac{-y}{\frac{-2a^2b}{a^2+b^2}+b} = \frac{1}{a^2-b^2}$$

$$\Rightarrow \frac{x}{\frac{-2ab^2+a^3+ab^2}{a^2+b^2}} = \frac{-y}{\frac{-2a^2b+a^2b+b^3}{a^2+b^2}} = \frac{1}{a^2-b^2}$$

$$\Rightarrow \frac{x}{\frac{a(a^2-b^2)}{a^2+b^2}} = \frac{-y}{\frac{b(b^2-a^2)}{a^2+b^2}} = \frac{1}{a^2-b^2}$$

$$\Rightarrow \frac{x}{\frac{a(a^2-b^2)}{a^2+b^2}} = \frac{1}{a^2-b^2} \Rightarrow x = \frac{a}{a^2+b^2}$$

$$and\ \frac{-y}{\frac{b(b^2-a^2)}{a^2+b^2}} = \frac{1}{a^2-b^2} \Rightarrow y = \frac{b}{a^2+b^2}$$

Hence, the solution is $x = \frac{a}{a^2+b^2}, y = \frac{b}{a^2+b^2}$

Example 8: Solve the following system of equations in x and y by cross-multiplication method

(a – b) x + (a + b) y = a^2 – 2ab – b^2

(a + b) (x + y) = a^2 + b^2

Sol: The given system of equations can be rewritten as :
(a – b) x + (a +b) y – (a^2 – 2ab – b^2) = 0
(a + b) x + (a + b) y – (a^2 + b^2) = 0
By cross-multiplication method, we have

$$\frac{x}{\begin{array}{cc}(a+b) & -(a^2-2ab-b^2) \\ (a+b) & -(a^2+b^2)\end{array}} = \frac{-y}{\begin{array}{cc}(a-b) & -(a^2-2ab-b^2) \\ (a+b) & -(a^2+b^2)\end{array}} = \frac{1}{\begin{array}{cc}(a-b) & (a+b) \\ (a+b) & (a+b)\end{array}}$$

$$\Rightarrow \frac{x}{(a+b)\times\{-(a^2+b^2)\}-(a+b)\times\{-(a^2-2ab-b^2)\}} = \frac{-y}{(a-b)\times\{-(a^2+b^2)\}-(a+b)\times\{-(a^2-2ab-b^2)\}} = \frac{1}{(a-b)\times(a+b)-(a+b)\times(a+b)}$$

$$\Rightarrow \frac{x}{-(a+b)(a^2+b^2)+(a+b)(a^2-2ab-b^2)} = \frac{-y}{-(a-b)(a^2+b^2)+(a+b)(a^2-2ab-b^2)} = \frac{1}{(a-b)(a+b)-(a+b)^2}$$

$$\Rightarrow \frac{x}{(a+b)[-(a^2+b^2)+(a+b)(a^2-2ab-b^2)]} = \frac{-y}{(a+b)(a^2-2ab-b^2)-(a-b)(a^2+b^2)} = \frac{1}{(a+b)(a-b-a-b)}$$

$$\Rightarrow \frac{x}{(a+b)(-2ab-2b^2)} = \frac{-y}{a^3-a^2b-3ab^2-b^3-a^3-ab^2+a^2b+b^3} = \frac{1}{(a+b)(-2b)}$$

$$\Rightarrow \frac{x}{-(a+b)(2a+2b)b} = \frac{-y}{-4ab^2} = \frac{1}{-2b(a+b)}$$

$$\Rightarrow \frac{x}{-2(a+b)(a+b)b} = \frac{1}{-2b(a+b)} \Rightarrow x = a+b$$

$$and\ \frac{-y}{-4ab^2} = \frac{1}{-2b(a+b)} \Rightarrow y = \frac{2ab}{a+b}$$

Hence, the solution of the given system of equations is x = a + b, $y = \frac{2ab}{a+b}$

Example 9: Solve the following system of equations by cross-multiplications method.
a(x + y) + b (x – y) = a^2 – ab + b^2
a(x + y) – b (x – y) = a^2 + ab + b^2

Sol: The given system of equations can be rewritten as
ax + bx + ay – by – (a^2 – ab + b^2) = 0
⇒ (a + b) x + (a – b) y – (a^2 – ab + b^2) = 0 ….(1)
And ax – bx + ay + by – (a^2 + ab + b^2) = 0
⇒ (a – b) x + (a + b) y – (a^2 + ab + b^2) = 0 …(2)
Now, by cross-multiplication method, we have

$$\frac{x}{\begin{array}{cc}(a-b) & -(a^2-ab+b^2) \\ (a+b) & -(a^2+ab+b^2)\end{array}} = \frac{-y}{\begin{array}{cc}(a+b) & -(a^2-ab+b^2) \\ (a-b) & -(a^2+ab+b^2)\end{array}} = \frac{1}{\begin{array}{cc}(a+b) & (a-b) \\ (a-b) & (a+b)\end{array}}$$

$$\Rightarrow \frac{x}{(a-b)\times\{-(a^2+ab+b^2)\}-(a+b)\times\{-(a^2-ab+b^2)\}} = \frac{-y}{(a+b)\times\{-(a^2+ab+b^2)\}-(a-b)\times\{-(a^2-ab+b^2)\}} = \frac{1}{(a+b)\times(a+b)-(a-b)(a-b)}$$

$$\Rightarrow \frac{x}{-(a-b)(a^2+ab+b^2)+(a+b)(a^2-ab+b^2)} = \frac{-y}{-(a+b)(a^2+ab+b^2)+(a-b)(a^2-ab+b^2)} = \frac{1}{(a+b)^2-(a-b)^2}$$

$$\Rightarrow \frac{x}{-(a^3-b^3)+(a^3+b^2)} = \frac{-y}{-a^3-2a^2b-2ab^2-b^3+a^3-2a^2b+2ab^2-b^3} = \frac{1}{a^2+2ab+b^2-a^2+2ab-b^2}$$

$$\Rightarrow \frac{x}{2b^3} = \frac{-y}{-4a^2b-2b^3} = \frac{1}{4ab}$$

$$\Rightarrow \frac{x}{2b^3} = \frac{-y}{-2b(2a^2+b^2)} = \frac{1}{4ab}$$

$$\Rightarrow \frac{x}{2b^3} = \frac{1}{4ab} \Rightarrow x = \frac{b^2}{2a}$$

$$and\ \frac{-y}{-2b(2a^2+b^2)} = \frac{1}{4ab} \Rightarrow y = \frac{2a^2+b^2}{2a}$$

Hence, the solution is $x = \frac{b^2}{2a}, y = \frac{2a^2+b^2}{2a}$

Example 10: Solve the following system of equations by the method of cross-multiplication.

$$\frac{a}{x} - \frac{b}{y} = 0;\ \frac{ab^2}{x} + \frac{a^2b}{y} = a^2 + b^2;$$

Where x ≠ 0, y ≠ 0

Sol: The given system of equations is

$$\frac{a}{x} - \frac{b}{y} = 0 \qquad \text{.........(1)}$$

$$\frac{ab^2}{x} + \frac{a^2b}{y} - \left(a^2 + b^2\right) = 0 \qquad \text{.........(2)}$$

Putting $\frac{a}{x} = u$ and $\frac{b}{y} = v$ in equatinos (1) and (2) the system of equations reduces to

$u - v + 0 = 0$

$b^2u + a^2v - (a^2 + b^2) = 0$

By the method of cross-multiplication, we have

$$\frac{u}{\begin{vmatrix} -1 & 0 \\ a^2 & -(a^2+b^2) \end{vmatrix}} = \frac{-v}{\begin{vmatrix} 1 & 0 \\ b^2 & -(a^2+b^2) \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & -1 \\ b^2 & a^2 \end{vmatrix}}$$

$$\Rightarrow \frac{u}{a^2+b^2-a^2\times 0} = \frac{-v}{-(a^2+b^2)-b^2\times 0} = \frac{1}{a^2-(-b^2)}$$

$$\Rightarrow \frac{u}{a^2+b^2} = \frac{-v}{-(a^2+b^2)} = \frac{1}{a^2+b^2}$$

$$\Rightarrow \frac{u}{a^2+b^2} = \frac{1}{a^2+b^2} \Rightarrow u = 1$$

$$and\ \frac{-v}{-(a^2+b^2)} = \frac{1}{a^2+b^2} \Rightarrow v = 1 and\ \mathrm{u} = \frac{a}{x} = 1 \Rightarrow x = a$$

$$v = \frac{b}{y} = 1 \Rightarrow y = b$$

Hence, the solution of the given system of equations is x = a, y = b.

Graphical Method Of Solving Linear Equations In Two Variables

Let the system of pair of linear equations be

$a_1x + b_1y = c_1$ ....(1)

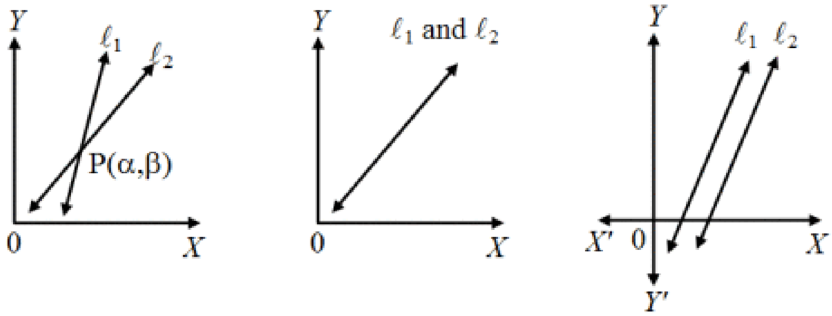
$a_2x + b_2y = c_2$ ....(2)

We know that given two lines in a plane, only one of the following three possibilities can happen –

(i) The two lines will intersect at one point.

(ii) The two lines will not intersect, however far they are extended, i.e., they are parallel.

(iii) The two lines are coincident lines.



Types Of Solutions:

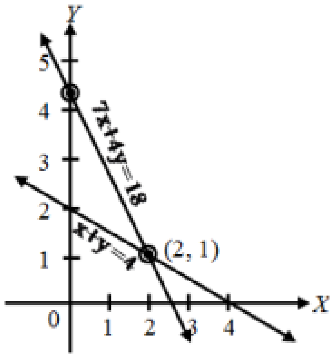
There are three types of solutions

1. Unique solution.
2. Infinitely many solutions
3. No solution.

(A) Consistent: If a system of simultaneous linear equations has at least one solution then the system is said to be consistent.

(i) Consistent equations with unique solution: The graphs of two equations intersect at a unique point.

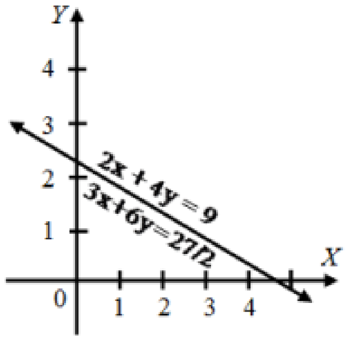
For Example Consider
x + 2y = 4
7x + 4y = 18



The graphs (lines) of these equations intersect each other at the point (2, 1) i.e., x = 2, y = 1.
Hence, the equations are consistent with unique solution.

(ii) Consistent equations with infinitely many solutions: The graphs (lines) of the two equations will be coincident.

For Example Consider 2x + 4y = 9 ⇒ 3x + 6y = 27/2

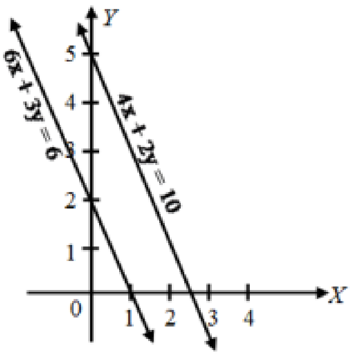


The graphs of the above equations coincide. Coordinates of every point on the lines are the solutions of the equations. Hence, the given equations are consistent with infinitely many solutions.

(B) Inconsistent Equation: If a system of simultaneous linear equations has no solution, then the system is said to be inconsistent.

No Solution: The graph (lines) of the two equations are parallel.

For Example Consider
4x + 2y = 10
6x + 3y = 6



The graphs (lines) of the given equations are parallel. They will never meet at a point. So, there is no solution. Hence, the equations are inconsistent.

S.No	Graph of Two Equations	Types of Equations
1	Intersecting lines	Consistent, with unique solution
2	Coincident	Consistent with infinite solutions
3	Parallel lines	Inconsistent (No solution)

Pair of lines $a_1x+b_1y+c_1=0$ $a_2x+b_2y+c_2=0$	$\frac{a_1}{a_2}$	$\frac{b_1}{b_2}$	$\frac{c_1}{c_2}$	Compare the ratio
$2x+3y+4=0$ $5x+6y+9=0$	$\frac{2}{5}$	$\frac{3}{6}$	$\frac{4}{9}$	$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$
$x+2y+5=0$ $3x+6y+15=0$	$\frac{1}{3}$	$\frac{2}{6}$	$\frac{5}{15}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
$2x-3y+4=0$ $4x-6y+10=0$	$\frac{2}{4}$	$\frac{-3}{-6}$	$\frac{4}{10}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

Graphical representation	Algebraic interpretation
Intersecting lines	Exactly one solution (unique)
Coincident lines	Infinitely many solutions
Parallel lines	No solution

From the table above you can observe that if the line $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

(i)	for the intersecting lines then $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$
(ii)	for the coincident lines then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
(iii)	for the parallel lines then $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

Sl No.	Pair of lines	$\frac{a_1}{a_2}$	$\frac{b_1}{b_2}$	$\frac{c_1}{c_2}$	Compare the ratios	Graphical representation	Algebraic interpretation
1.	$x-2y=0$ $3x+4y-20=0$	$\frac{1}{3}$	$\frac{-2}{4}$	$\frac{0}{-20}$	$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	Intersecting lines	Exactly one solution (unique)
2.	$2x+3y-9=0$ $4x+6y-18=0$	$\frac{2}{4}$	$\frac{3}{6}$	$\frac{-9}{-18}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	Coincident lines	Infinitely many solutions
3.	$x+2y-4=0$ $2x+4y-12=0$	$\frac{1}{2}$	$\frac{2}{4}$	$\frac{-4}{-12}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$	Parallel lines	No solution

Graphical Method Examples

Example 1: The path of highway number 1 is given by the equation x + y = 7 and the highway number 2 is given by the equation 5x + 2y = 20. Represent these equations geometrically.

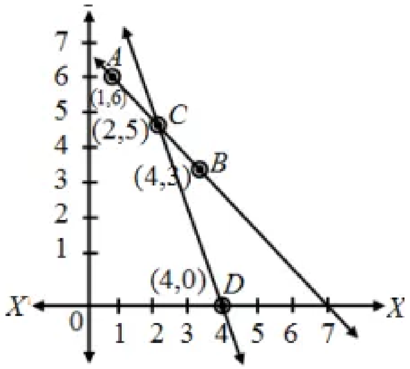
Sol. We have, x + y = 7
⇒ y = 7 – x ….(1)
In tabular form

x	1	4
y	6	3
Points	A	B

and 5x + 2y = 20
⇒ y = $\frac{20-5x}{2}$ ….(2)
In tabular form

x	2	4
y	5	0
Points	C	D

Plot the points A (1, 6), B(4, 3) and join them to form a line AB.
Similarly, plot the points C(2, 5). D (4, 0) and join them to get a line CD. Clearly, the two lines intersect at the point C. Now, every point on the line AB gives us a solution of equation (1). Every point on CD gives us a solution of equation (2).



Example 2: A father tells his daughter, " Seven years ago, I was seven times as old as you were then. Also, three years from now, I shall be three times as old as you will be." Represent this situation algebraically and graphically.

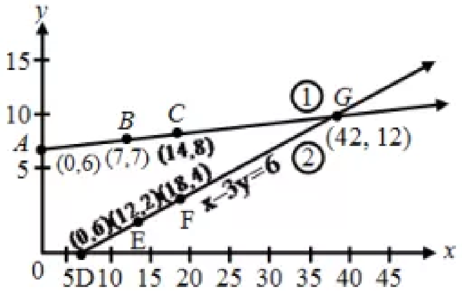
Sol. Let the present age of father be x years and that of daughter = y years
Seven years ago father's age = (x – 7) years
Seven years ago daughter's age = (y – 7) years
According to the problem
(x – 7) = 7(y – 7) or x – 7y = – 42 ….(1)
After 3 years father's age = (x + 3) years
After 3 years daughter's age = (y + 3) years
According to the condition given in the question
x + 3 = 3(y + 3) or x – 3y = 6 ….(2)
x – 7y = –42 ⇒ $y = \frac{x+42}{7}$

x	0	7	14
y	6	7	8
Points	A	B	C

x – 3y = 6 ⇒ $y = \frac{x-6}{3}$

x	6	12	18
y	0	2	4
Points	D	E	F

Plot the points A(0, 6), B(7, 7), C(14, 8) and join them to get a straight line ABC. Similarly plot the points D(6, 0), E(12, 2) and F(18, 4) and join them to get a straight line DEF.



Example 3: 10 students of class X took part in a Mathematics quiz. If the number of girls is 4 more than the number of boys, find the number of boys and girls who took part in the quiz.

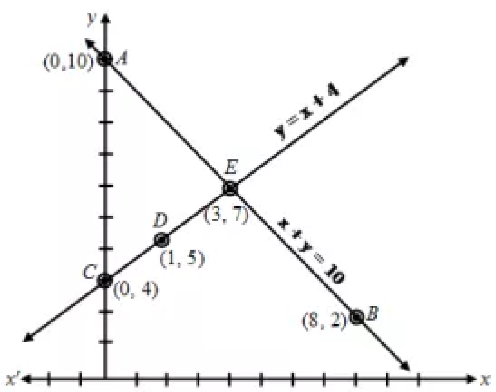
Sol. Let the number of boys be x and the number of girls be y.
Then the equations formed are
x + y = 10 ....(1)
and y = x + 4 ....(2)
Let us draw the graphs of equations (1) and (2) by finding two solutions for each of the equations. The solutions of the equations are given.
x + y = 10 ⇒ y = 10 – x

x	0	8
y	10	2
Points	A	B

y = x + 4

x	0	1	3
y	4	5	7
Points	C	D	E

Plotting these points we draw the lines AB and CE passing through them to represent the equations. The two lines AB and Ce intersect at the point E (3, 7). So, x = 3 and y = 7 is the required solution of the pair of linear equations.



i.e. Number of boys = 3
Number of girls = 7.

Verification:
Putting x = 3 and y = 7 in (1), we get
L.H.S. = 3 + 7 = 10 = R.H.S., (1) is verified.
Putting x = 3 and y = 7 in (2), we get
7 = 3 + 4 = 7, (2) is verified.
Hence, both the equations are satisfied.

Example 4: Half the perimeter of a garden, whose length is 4 more than its width is 36m. Find the dimensions of the garden.

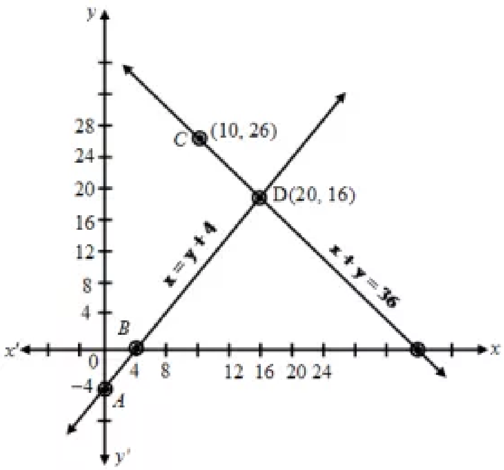
Sol. Let the length of the garden be x and width of the garden be y.
Then the equation formed are
x = y + 4 ….(1)
Half perimeter = 36
x + y = 36 ….(2)
y = x – 4

x	0	4
y	-4	0
Points	A	B

y = 36 – x

x	10	20
y	26	16
Points	C	D

Plotting these points we draw the lines AB and CD passing through them to represent the equations.



The two lines AB and CD intersect at the point (20, 16), So, x = 20 and y = 16 is the required solution of the pair of linear equations i.e. length of the garden is 20 m and width of the garden is 16 m.

Verification:
Putting x = 20 and y = 16 in (1). We get
20 = 16 + 4 = 20, (1) is verified.
Putting x = 20 and y = 16 in (2). we get
20 + 16 = 36
36 = 36, (2) is verified.
Hence, both the equations are satisfied.

Example 5: Draw the graphs of the equations x – y + 1 = 0 and 3x + 2y – 12 = 0. Determine the coordinates of the vertices of the triangle formed by these lines and the x-axis, and shade the triangular region.

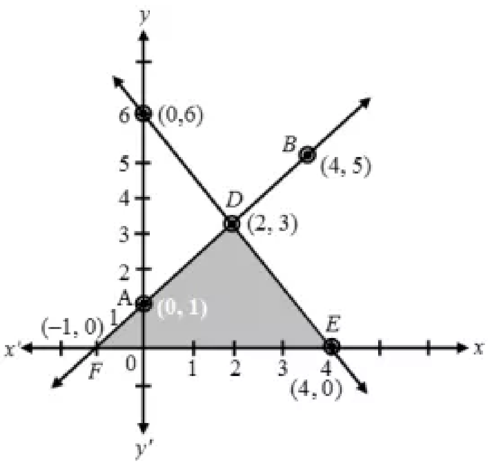
Sol. Pair of linear equations are:
x – y + 1 = 0 ….(1)
3x + 2y – 12 = 0 ….(2)
x – y + 1 = 0 ⇒ y = x + 1

x	0	4
y	1	5
Points	A	B

3x + 2y – 12 = 0 ⇒ y = $\frac{12-3x}{2}$

x	0	2
y	6	3
Points	C	D

Plot the points A(0, 1), B(4, 5) and join them to get a line AB. Similarly, plot the points C(0, 6), D(2, 3) and join them to form a line CD.



Clearly, the two lines intersect each other at the point D(2, 3). Hence x = 2 and y = 3 is the solution of the given pair of equations.
The line CD cuts the x-axis at the point
E (4, 0) and the line AB cuts the x-axis at the point F(–1, 0).
Hence, the coordinates of the vertices of the triangle are; D(2, 3), E(4, 0), F(–1, 0).

Verification:
Both the equations (1) and (2) are satisfied by x = 2 and y = 3. Hence, Verified.

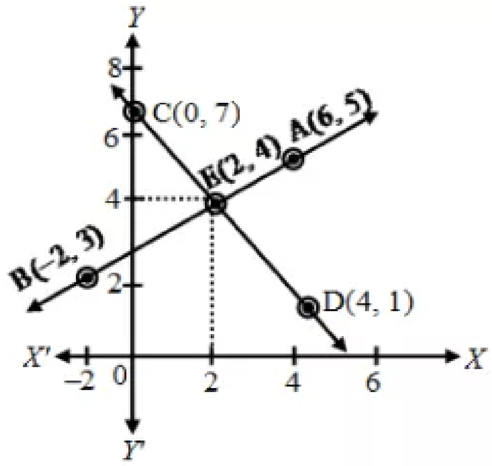
Example 6: Show graphically that the system of equations
x – 4y + 14 = 0; 3x + 2y – 14 = 0 is consistent with unique solution.

Sol. The given system of equations is
x – 4y + 14 = 0 ....(1)
3x + 2y – 14 = 0 ....(2)
x – 4y + 14 = 0 ⇒ y = $\frac{x+14}{4}$

x	6	-2
y	5	3
Points	A	B

3x + 2y – 14 = 0 ⇒ y = $\frac{-3x+14}{2}$

x	0	4
y	7	1
Points	C	D



The given equations representing two lines, intersect each other at a unique point (2, 4). Hence, the equations are consistent with unique solution.

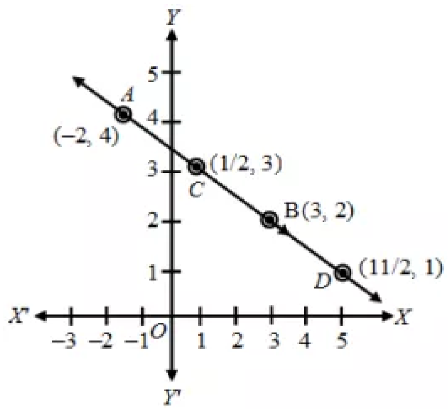
Example 7: Show graphically that the system of equations
2x + 5y = 16; $3x + \frac{15}{2} = 24$ has infinitely many solutions.

Sol. The given system of equations is
2x + 5y = 16 ....(1)
$3x + \frac{15}{2} = 24$ ....(2)
2x + 5y = 16 ⇒ y = $\frac{16-2x}{5}$

x	-2	3
y	4	2
Points	A	B

$3x + \frac{15}{2} = 24$ ⇒ y = $\frac{48-6x}{15}$

x	1/2	11/2
y	3	1
Points	C	D



The lines of two equations are coincident. Coordinates of every point on this line are the solution. Hence, the given equations are consistent with infinitely many solutions.

Example 8: Show graphically that the system of equations
2x + 3y = 10, 4x + 6y = 12 has no solution.

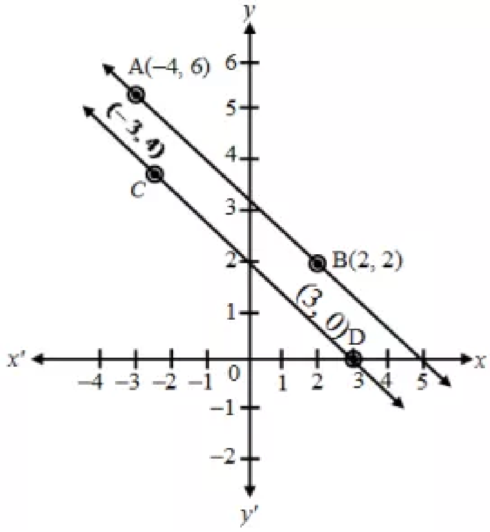
Sol. The given equations are
2x + 3y = 10 ⇒ y = $\frac{10-2x}{3}$

x	-4	2
y	6	2
Points	A	B

4x + 6y = 12 ⇒ y = $\frac{12-4x}{6}$

x	-3	3
y	4	0
Points	C	D

Plot the points A (–4, 6), B(2, 2) and join them to form a line AB. Similarly, plot the points C(–3, 4), D(3, 0) and join them to get a line CD.



Clearly, the graphs of the given equations are parallel lines. As they have no common point, there is no common solution. Hence, the given system of equations has no solution.

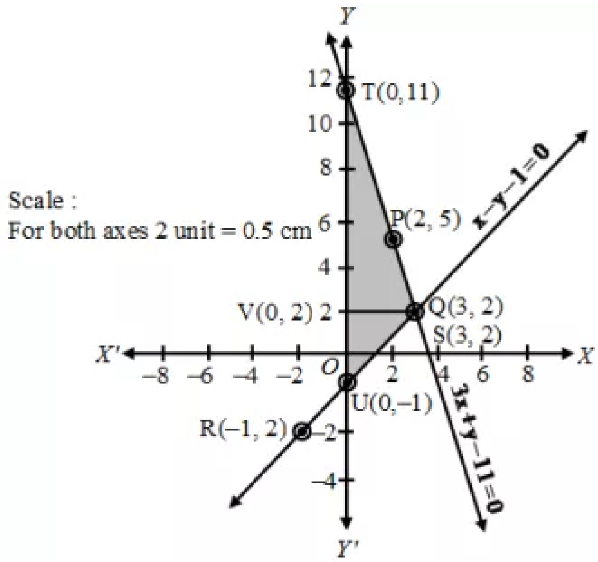
Example 9: Given the linear equation 2x + 3y – 8 = 0, write another linear equation in two variables such that the
geometrical representing of the pair so formed is :
(i) intersecting lines
(ii) parallel lines
(iii) coincident lines

Sol. We have, 2x + 3y – 8 = 0
(i) Another linear equation in two variables such that the geometrical representation of the pair so formed is intersecting lines is
3x – 2y – 8 = 0
(ii) Another parallel lines to above line is
4x + 6y – 22 = 0
(iii) Another coincident line to above line is
6x + 9y – 24 = 0

Example 10: Solve the following system of linear equations graphically;
3x + y – 11 = 0 ; x – y – 1 = 0
Shade the region bounded by these lines and also y-axis. Then, determine the areas of the region bounded by these lines and y-axis.

Sol. We have,
3x + y – 11 = 0 and x – y – 1 = 0
(a) Graph of the equation 3x + y – 11 = 0
We have, 3x + y – 11 = 0
⇒ y = – 3x + 11
When, x = 2, y = –3 × 2 + 11 = 5
When, x = 3, y = – 3 × 3 + 11 = 2
Plotting the points P (2, 5) and Q(3, 2) on the graph paper and drawing a line joining between them, we get the graph of the equation 3x + y – 11 = 0 as shown in fig.

(b) Graph of the equation x – y – 1 = 0
We have,
x – y – 1 = 0
y = x – 1
When, x = – 1, y = –2
When, x = 3, y = 2
Plotting the points R(–1, –2) and S(3, 2) on the same graph paper and drawing a line joining between them, we get the graph of the equation x – y – 1 = 0 as shown in fig.



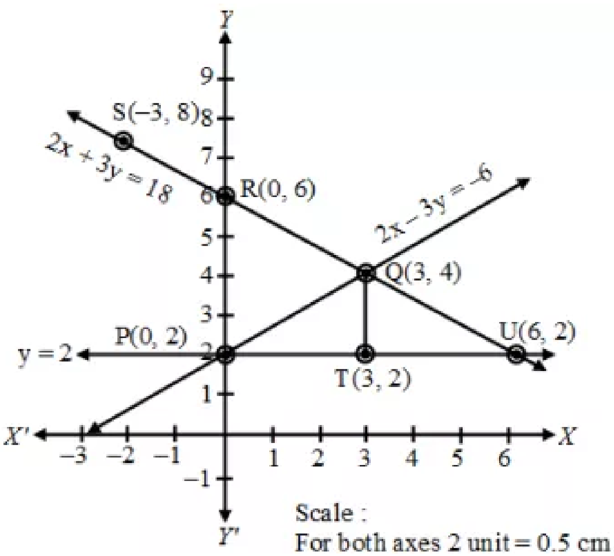
You can observe that two lines intersect at
Q(3, 2). So, x = 3 and y = 2. The area enclosed by the lines represented by the given equations and also the y-axis is shaded.
So, the enclosed area = Area of the shaded portion
= Area of ΔQUT = 1/2 × base × height
= 1/2 × (TU × VQ) = 1/2 × (TO + OU) × VQ
= 1/2 (11 + 1) 3 = 1/2 × 12 × 3 = 18 sq.units.
Hence, required area is 18 sq. units.

Example 11: Draw the graphs of the following equations
2x − 3y = − 6; 2x + 3y = 18; y = 2
Find the vertices of the triangles formed and also find the area of the triangle.

Sol. **(a)** Graph of the equation 2x − 3y = − 6;
We have, 2x − 3y = − 6 ⇒ y = $\frac{2x+6}{3}$
When, x = 0, y = 2
When, x = 3, y = 4
Plotting the points P(0, 2) and Q(3, 4) on the graph paper and drawing a line joining between them we get the graph of the equation 2x − 3y = − 6 as shown in fig.

(b) Graph of the equation 2x + 3y = 18;
We have 2x + 3y = 18 ⇒ y = $\frac{-2x+18}{3}$
When, x = 0, y = 6
When, x = − 3, y = 8
Plotting the points R(0, 6) and S(−3, 8) on the same graph paper and drawing a line joining between them, we get the graph of the equation 2x + 3y = 18 as shown in fig.

(c) Graph of the equation y = 2
It is a clear fact that y = 2 is for every value of x. We may take the points T (3, 2), U(6, 2) or any other values.
Plotting the points T(3, 2) and U(6, 2) on the same graph paper and drawing a line joining between them, we get the graph of the equation y = 2 as shown in fig.



From the fig., we can observe that the lines taken in pairs intersect each other at points Q(3, 4), U (6, 2) and P(0, 2). These form the three vertices of the triangle PQU.

To find area of the triangle so formed

The triangle is so formed is PQU (see fig.)

In the ΔPQU

QT (altitude) = 2 units

and PU (base) = 6 units

so, area of ΔPQU = (base × height)

= 1/2 (PU × QT) = 1/2 × 6 × 2 sq. untis

= 6 sq. units.

Example 12: On comparing the ratios $\frac{a1}{a2}, \frac{b1}{b2} and \frac{c1}{c2}$ and without drawing them, find out whether the lines representing the following pairs of linear equations intersect at a point, are parallel or coincide.

(i) 5x − 4y + 8 = 0, 7x + 6y − 9 = 0

(ii) 9x + 3y + 12 = 0, 18x + 6y + 24 = 0

(iii) 6x − 3y + 10 = 0, 2x − y + 9 = 0

Sol. Comparing the given equations with standard forms of equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ we have,

(i) a1 = 5, b_1 = − 4, c_1 = 8;

a_2 = 7, b_2 = 6, c_2 = − 9

$$\therefore \frac{a_1}{a_2} = \frac{5}{7}, \frac{b_1}{b_2} = \frac{-4}{6}$$

$$\Rightarrow \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

Thus, the lines representing the pair of linear equations are intersecting.

(ii) a1 = 9, b_1 = 3, c_1 = 12;

a_2 = 18, b_2 = 6, c_2 = 24

$$\therefore \frac{a_1}{a_2} = \frac{9}{18} = \frac{1}{2}, \frac{b_1}{b_2} = \frac{3}{6} = \frac{1}{2} and \frac{c_1}{c_2} = \frac{12}{24} = \frac{1}{2}$$

$$\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Thus, the lines representing the pair of linear equation coincide.

(iii) a1 = 6, b_1 = − 3, c_1 = 10;

a_2 = 2, b_2 = − 6, c_2 = 9

$$\therefore \frac{a_1}{a_2} = \frac{6}{2} = 3, \frac{b_1}{b_2} = \frac{-3}{-1} = 3 and \frac{c_1}{c_2} = \frac{10}{9}$$

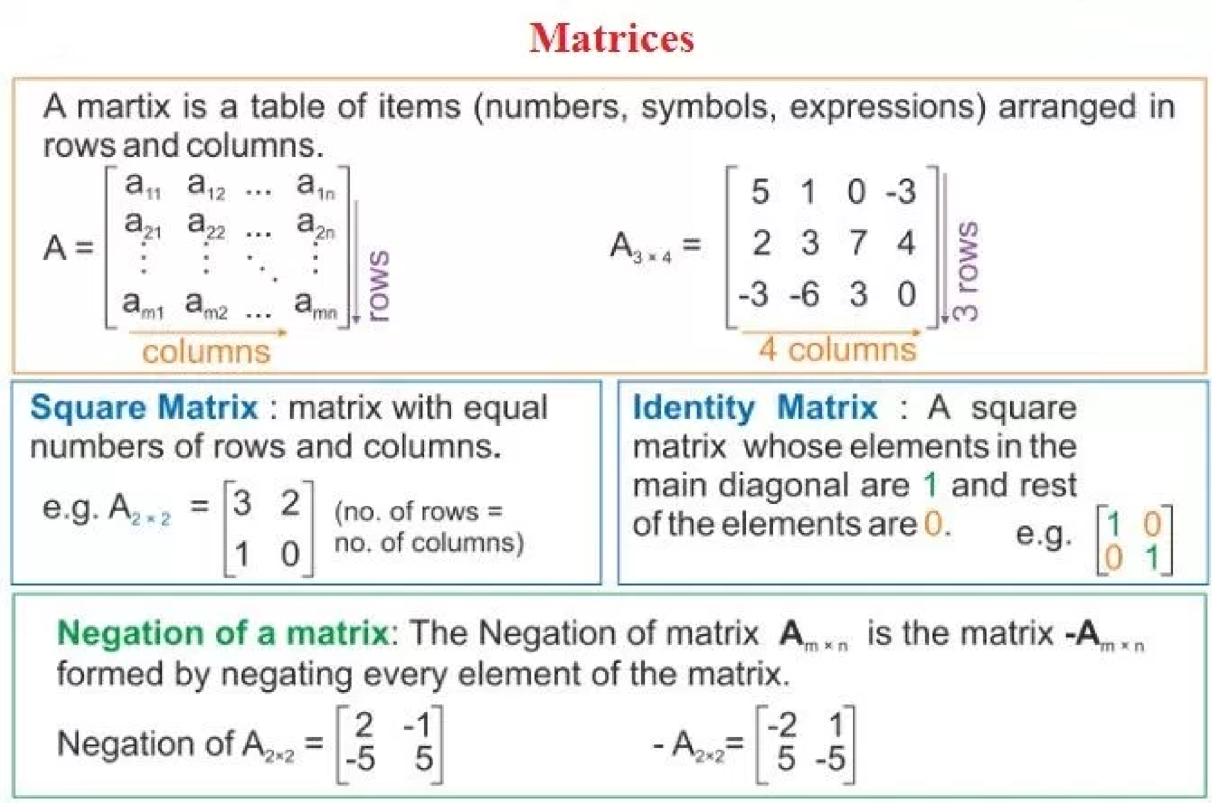
$$\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

Thus, the lines representing the pair of linear equations are parallel.

Matrices

Definition

A rectangular arrangement of numbers (which may be real or complex numbers) in rows and columns, is called a matrix. This arrangement is enclosed by small ( ) or big [ ] brackets. The numbers are called the elements of the matrix or entries in the matrix.



Order of a matrix

A matrix having m rows and n columns is called a matrix of order m×n or simply m×n matrix (read as an m by n matrix). A matrix A of order m×n is usually written in the following manner:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots a_{1j} & \dots a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots a_{2j} & \dots a_{2n} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{i1} & a_{i2} & a_{i3} & \dots a_{ij} & \dots a_{in} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{m1} & a_{m2} & a_{m3} & \dots a_{mj} & \dots a_{mn} \end{bmatrix} \text{ or } A = [a_{ij}]_{m\times n},$$

where $i = 1, 2, \dots\dots m$
$j = 1, 2, \dots\dots n$

Here a_{ij} denotes the element of i^{th} row and j^{th} column.

Example **:** order of matrix $\begin{bmatrix} 3 & -1 & 5 \\ 6 & 2 & -7 \end{bmatrix}$ is 2×3.

A matrix of order m × n contains mn elements. Every row of such a matrix contains n elements and every column contains m elements.

Equality of matrices

Two matrix A and B are said to be equal matrix if they are of same order and their corresponding elements are equal.

Types of matrices

1. **Row matrix:** A matrix is said to be a row matrix or row vector if it has only one row and any number of columns. Example: [5 0 3] is a row matrix of order 1× 3 and [2] is a row matrix of order 1×1.
2. **Column matrix:** A matrix is said to be a column matrix or column vector if it has only one column and any number of rows.
   Example: $\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$ is a column matrix of order 3 × 1 and [2] is a column matrix of order 1 × 1. Observe that [2] is both a row matrix as well as a column matrix.
3. **Singleton matrix:** If in a matrix there is only one element then it is called singleton matrix.
   Thus, A = $[a_{ij}]_{m\times n}$ is a singleton matrix, if m = n =1
   Example: [2], [3], [a], [−3] are singleton matrices.
4. **Null or zero matrix:** If in a matrix all the elements are zero then it is called a zero matrix and it is generally denoted by O. Thus A = $[a_{ij}]_{m\times n}$ is a zero matrix if a_{ij} = 0 for all i and j.
   Example: $[0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \end{bmatrix}$ are all zero matrices, but of different orders.
5. **Square matrix:** If number of rows and number of columns in a matrix are equal, then it is called a square matrix.
   Thus A = $[a_{ij}]_{m\times n}$ is a square matrix if m = n.
   Example: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3×3.
   (i) If m ≠ n then matrix is called a rectangular matrix.
   (ii) The elements of a square matrix A for which i = j, i.e. a_{11}, a_{22}, a_{33}, ….. a_{mn} are called diagonal elements and the line joining these elements is called the principal diagonal or leading diagonal of matrix A.
6. **Diagonal matrix:** If all elements except the principal diagonal in a square matrix are zero, it is called a diagonal matrix. Thus a square matrix A = $[a_{ij}]$ is a diagonal matrix if a_{ij} ≠ 0 when i ≠ j.
   Example: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a diagonal matrix of order 3× 3, which can be denoted by diag [2, 3, 4].
7. **Identity matrix:** A square matrix in which elements in the main diagonal are all '1' and rest are all zero is called an identity matrix or unit matrix. Thus, the square matrix A = $[a_{ij}]$ is an identity matrix, if
   $$a_{ij} = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j \end{cases}$$
   We denote the identity matrix of order n by I_n.
   Example: $[1]$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 1, 2 and 3 respectively.
8. **Scalar matrix:** A square matrix whose all non diagonal elements are zero and diagonal elements are equal is called a scalar matrix. Thus, if A = $[a_{ij}]$ is a square matrix and $a_{ij} = \begin{cases} \alpha, \text{if } i = j \\ 0, \text{ if } i \neq j \end{cases}$, then A is a scalar matrix.
   Unit matrix and null square matrices are also scalar matrices.
9. **Triangular matrix:** A square matrix $[a_{ij}]$ is said to be triangular matrix if each element above or below the principal diagonal is zero. It is of two types
   (i) Upper triangular matrix: A square matrix $[a_{ij}]$ is called the upper triangular matrix, if a_{ij} = 0 when i > j.

Example: $\begin{bmatrix} 3 & 1 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 6 \end{bmatrix}$ is an upper triangular matrix of order 3 × 3.

(ii) Lower triangular matrix: A square matrix $[a_{ij}]$ is called the lower triangular matrix, if $a_{ij} = 0$ when i< j.

Example: $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 2 \end{bmatrix}$ is a lower triangular matrix of order 3 × 3.

Trace of a matrix

The sum of diagonal elements of a square matrix. A is called the trace of matrix A , which is denoted by tr A.

$$tr\ A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \dots a_{nn}$$

Properties of trace of a matrix

Let $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$ and λ be a scalar

1. tr(λA) = λ tr(A)
2. tr(A – B) = tr(A) – tr(B)
3. tr(AB) = tr(BA)
4. tr(A)= tr(A′) or tr (A^T)
5. tr(I_n) = n
6. tr (0)= 0
7. tr(AB) ≠ trA.trB

Addition and subtraction of matrices

If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are two matrices of the same order then their sum A+B is a matrix whose each element is the sum of corresponding elements i.e., $A + B = [a_{ij} + b_{ij}]_{m\times n}$.
Similarly, their subtraction is defined as $A - B = [a_{ij} - b_{ij}]_{m\times n}$

Matrix addition and subtraction can be possible only when matrices are of the same order.
Properties of matrix addition : If A, B and C are matrices of same order, then

1. A + B = B + A (Commutative law)
2. (A + B) + C = A + (B + C) (Associative law)
3. A + O = O + A = A, where O is zero matrix which is additive identity of the matrix.
4. A + (–A) = O = (–A) + A , where (–A) is obtained by changing the sign of every element of A, which is additive inverse of the matrix.
5. $\left.\begin{matrix} A + B = A + C \\ B + A = C + A \end{matrix}\right\} \Rightarrow B = C \qquad \text{(Cancellation law)}$

What are the Special Types of Matrices?

Singular and Non-singular matrix :

Any square matrix *A* is said to be non-singular if |A| ≠ 0, and a square matrix *A* is said to be singular if |A| = 0. Here |A| (or det(A) or simply det |A| means corresponding determinant of square matrix *A*.

Example : $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ then $|A| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 10 - 12 = -2 \Rightarrow A$ is a non-singular matrix.

Hermitian and Skew-hermitian matrix :

A square matrix is said to be hermitian matrix if

$a_{ij} = \bar{a}_{ji}$; $\forall i, j$ i.e., $A = A^\theta$.

Example : $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$, $\begin{bmatrix} 3 & 3-4i & 5+2i \\ 3+4i & 5 & -2+i \\ 5-2i & -2-i & 2 \end{bmatrix}$

are Hermitian matrices.

If A is a Hermitian matrix then $a_{ii} = \bar{a}_{ii} \Rightarrow a_{ii}$ is real $\forall i$, thus every diagonal element of a Hermitian matrix must be real.

A square matrix, $A = |a_{ij}|$ is said to be a Skew-Hermitian if $a_{ij} = -\bar{a}_{ji}$. $\forall i, j$ i.e. $A^\theta = -A$. If A is a skew-Hermitian matrix, then $a_{ii} = -\bar{a}_{ii} \Rightarrow a_{ii} + \bar{a}_{ii} = 0$ i.e. a_{ii} must be purely imaginary or zero.

Example : $\begin{bmatrix} 0 & -2+i \\ 2-i & 0 \end{bmatrix}$, $\begin{bmatrix} 3i & -3+2i & -1-i \\ 3+2i & -2i & -2-4i \\ 1-i & 2-4i & 0 \end{bmatrix}$

are skew-hermitian matrices.

Orthogonal matrix :

A square matrix *A* is called orthogonal if $AA^T = I = A^T A$ *i.e.,* if $A^{-1} = A$.

Example : $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

is orthogonal because $A^{-1} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = A^T$

In fact every unit matrix is orthogonal. Determinant of orthonogal matrix is − 1 or 1.

Idempotent matrix :

A square matrix A is called an idempotent matrix if $A^2 = A$.

Example : $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ is an idempotent matrix, because

$$A^2 = \begin{bmatrix} 1/4+1/4 & 1/4+1/4 \\ 1/4+1/4 & 1/4+1/4 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = A.$$

Also, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are idempotent matrices because $A^2 = A$ and $B^2 = B$.

In fact every unit matrix is indempotent.

Involutory matrix :

A square matrix A is called an involutory matrix if $A^2 = I$ or $A^{-1} = A$.

Example:

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an involutory matrix because $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

In fact every unit matrix is involutory.

Nilpotent matrix :

A square matrix A is called a nilpotent matrix if there exists a $p \in N$ such that $A^p = 0$.

Example: $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

is a nilpotent matrix because $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$, (Here $P = 2$)

Determinant of every nilpotent matrix is 0.

Unitary matrix : A square matrix is said to be unitary, if $\overline{A}'A = I$ since $|\overline{A}'| = |A|$ and $|\overline{A}'A| = |\overline{A}'||A|$ therefore if $\overline{A}'A = I$, we have $|\overline{A}'||A| = 1$.

Thus the determinant of unitary matrix is of unit modulus. For a matrix to be unitary it must be non-singular.

Hence $\overline{A}'A = I \Rightarrow A\overline{A}' = I$

Periodic matrix :

A matrix A will be called a periodic matrix if where k is a positive integer. If $A^{k+1} = A$ however k is the least positive integer for $A^{k+1} = A$, then k is said to be the period of A.
Differentiation of a matrix : If then is a differentiation of matrix A.

Differentiation of a matrix : If $A = \begin{bmatrix} f(x) & g(x) \\ h(x) & l(x) \end{bmatrix}$ then

$\frac{dA}{dx} = \begin{bmatrix} f'(x) & g'(x) \\ h'(x) & l'(x) \end{bmatrix}$ is a differentiation of matrix A.

Example **:** If $A = \begin{bmatrix} x^2 & \sin x \\ 2x & 2 \end{bmatrix}$ then $\frac{dA}{dx} = \begin{bmatrix} 2x & \cos x \\ 2 & 0 \end{bmatrix}$

Conjugate of a matrix :

The matrix obtained from any given matrix A containing complex number as its elements, on replacing its elements by the corresponding conjugate complex numbers is called conjugate of A and is denoted by $\overline{A}$.

$$\textbf{Example: } A = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$$

$$\text{then } \overline{A} = \begin{bmatrix} 1-2i & 2+3i & 3-4i \\ 4+5i & 5-6i & 6+7i \\ 8 & 7-8i & 7 \end{bmatrix}$$

Properties of conjugates

(i) $\left(\overline{\overline{A}}\right) = A$

(ii) $\overline{(A+B)} = \overline{A} + \overline{B}$

(iii) $\overline{(\alpha A)} = \overline{\alpha}\overline{A}$, α being any number

(iv) $\overline{(AB)} = \overline{A}\ \overline{B}$, A and B being conformable for multiplication

Transpose conjugate of a matrix :

The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by A^{θ}. The conjugate of the transpose of A is the same as the transpose of the conjugate of A

i.e. $\overline{(A')} = (\overline{A})' = A^{\theta}$.

If $A = [a_{ij}]_{m\times n}$ then $A^{\theta} = [b_{ji}]_{n\times m}$ where $b_{ji} = \overline{a}_{ij}$

i.e., the $(j, i)^{th}$ element of A^{θ} = the conjugate of $(i, j)^{th}$ element of A.

$$\textbf{Example: If } A = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix},$$

$$\text{then } A^{\theta} = \begin{bmatrix} 1-2i & 4+5i & 8 \\ 2+3i & 5-6i & 7-8i \\ 3-4i & 6+7i & 7 \end{bmatrix}$$

Properties of transpose conjugate

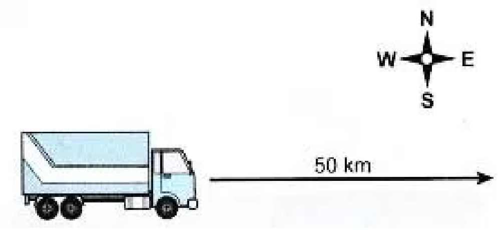
(i) $(A^{\theta})^{\theta} = A$

(ii) $(A+B)^{\theta} = A^{\theta} + B^{\theta}$

(iii) $(kA)^{\theta} = \overline{K}A^{\theta}$, K being any number

(iv) $(AB)^{\theta} = B^{\theta}A^{\theta}$

Understanding Scalar and Vector Quantities



1. Physical quantities can be grouped into scalar quantities and vector quantities.
2. Above Figure shows a truck travelling a distance of 50 km in the eastward direction. We describe the journey of the truck by stating the magnitude and the direction of its travel:
   (a) The magnitude is 50 km.
   (b) The direction is East.
3. **Scalar quantities** are physical quantities that have **magnitude** only.
4. **Vector quantities** are physical quantities that have **magnitude** and **direction**.
5. Some examples of scalar and vector quantities are listed in Table.

Scalar quantities	Vector quantities
Length	Displacement
Time	Velocity
Temperature	Acceleration
Mass	Momentum
Speed	Force

Example 1

Mei is putting up a night at a campsite during her training program. It is a warm night with a temperature of 30°C and she will have only 3 hours of sleep before hiking to the base camp of the mountain located 2 km away.

From the above description, determine whether each of the quantities involved is a scalar or a vector quantity and explain your answer.

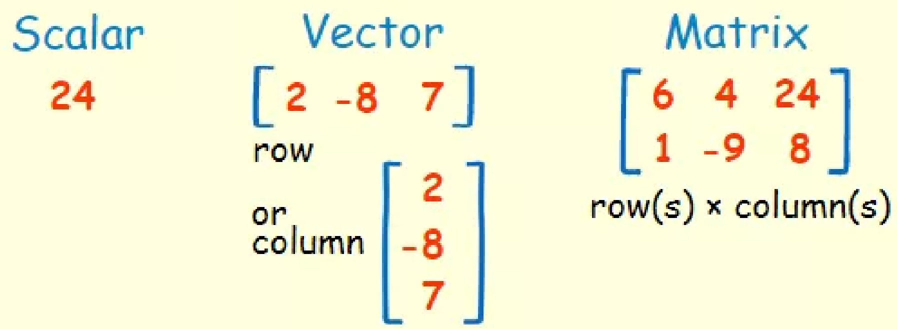
Solution:

Event	Quantity	Explanation
A warm night with a temperature of 30°C	Scalar quantity	Temperature has magnitude only.
3 hours of sleep	Scalar quantity	Time has magnitude only.
Hiking to the base camp 2 km away	Vector quantity	The magnitude of the distance and the direction of the hike are involved.

Scalars, Vectors and Matrices

And when we include matrices we get this interesting pattern:

- A **scalar** is a number, like **3, -5, 0.368, etc**,
- A **vector** is a **list** of numbers (can be in a row or column),
- A **matrix** is an **array** of numbers (one or more rows, one or more columns).



In fact a **vector is also a matrix**! Because a matrix can have just one row or one column.

So the rules that work for matrices also work for vectors.

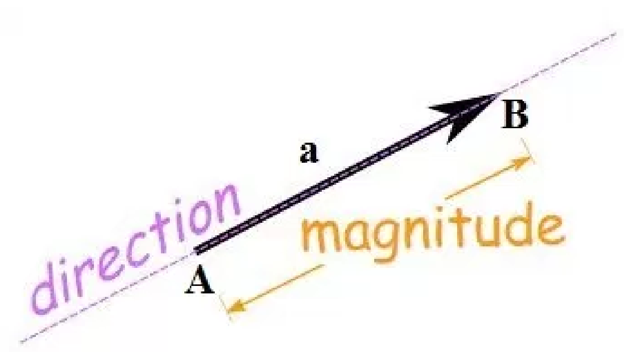
Vectors

Vectors represent one of the most important mathematical systems, which is used to handle certain types of problems in Geometry, Mechanics and other branches of Applied Mathematics, Physics and Engineering.

Scalar and vector quantities: Those quantities which have only magnitude and which are not related to any fixed direction in space are called scalar quantities, or briefly scalars. Examples: Mass, Volume, Density, Work, Temperature etc. Those quantities which have both magnitude and direction, are called vectors. Displacement, velocity, acceleration, momentum, weight, force are examples of vector quantities.

Representation of vectors

Geometrically a vector is represent by a line segment. For example, $a = \overrightarrow{AB}$. Here A is called the initial point and B, the terminal point or tip.



Magnitude or modulus of a is expressed as $|a| = |\overrightarrow{AB}| = |AB|$

Types of vector

1. **Zero or null vector:** A vector whose magnitude is zero is called zero or null vector and it is represented by $\overrightarrow{O}$.

2. **Unit vector:** A vector whose modulus is unity, is called a unit vector. The unit vector in the direction of a vector is denoted by $\hat{a}$, read as "a cap". Thus, $|\hat{a}| = 1$.

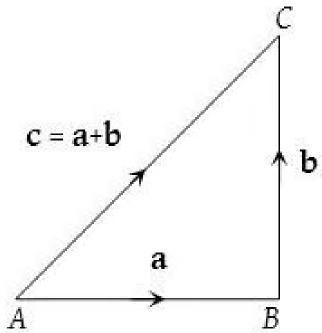
$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\text{Vector } a}{\text{Magnitude of } a}$$

3. **Like and unlike vectors:** Vectors are said to be like when they have the same sense of direction and unlike when they have opposite directions.
4. **Collinear or parallel vectors:** Vectors having the same or parallel supports are called collinear or parallel vectors.
5. **Co-initial vectors:** Vectors having the same initial point are called co-initial vectors.
6. **Coplanar vectors:** A system of vectors is said to be coplanar, if their supports are parallel to the same plane.
   Two vectors having the same initial point are always coplanar but such three or more vectors may or may not be coplanar.
7. **Coterminous vectors:** Vectors having the same terminal point are called coterminous vectors.
8. **Negative of a vector:** The vector which has the same magnitude as the vector a but opposite direction, is called the negative of a and is denoted by −a. Thus, if $\overrightarrow{PQ} = a$, then $\overrightarrow{QP} = -a$.
9. **Reciprocal of a vector:** A vector having the same direction as that of a given vector but magnitude equal to the reciprocal of the given vector a is known as the reciprocal of a and is denoted by a^{-1}. Thus, if |a| = a, $|a^{-1}| = \frac{1}{a}$.
10. **Localized and free vectors:** A vector which is drawn parallel to a given vector through a specified point in space is called a localized vector. For example, a force acting on a rigid body is a localized vector as its effect depends on the line of action of the force. If the value of a vector depends only on its length and direction and is independent of its position in the space, it is called a free vector.
11. **Position vectors:** The vector $\overrightarrow{OA}$ which represents the position of the point A with respect to a fixed point O (called origin) is called position vector of the point A. If (x, y, z) are co-ordinates of the point A, then $\overrightarrow{OA} = xi + yj + zk$.
12. **Equality of vectors:** Two vectors and b are said to be equal, if (i) |a| = |b| (ii) They have the same or parallel support and (iii) The same sense.

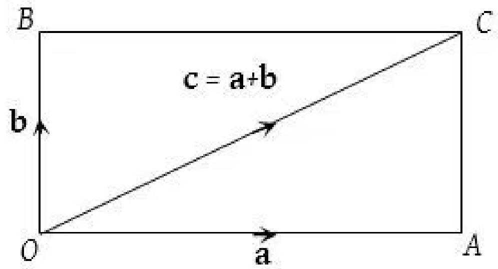
Properties of vectors

(1) Addition of vectors:

(i) **Triangle law of addition :** If in a ΔABC , $\overrightarrow{AB} = \mathbf{a}$ $\overrightarrow{BC} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$, then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ *i.e.,* $\mathbf{a} + \mathbf{b} = \mathbf{c}$.



(ii) **Parallelogram law of addition :** If in a parallelogram $OACB$, $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$



Then $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$ *i.e.,* $\mathbf{a} + \mathbf{b} = \mathbf{c}$, where OC is a diagonal of the parallelogram $OABC$.

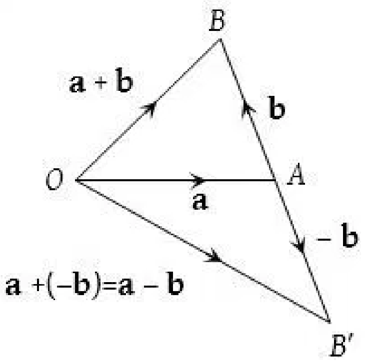
(iii) Addition in component form:
Addition in component form: If the vectors are defined in terms of i, j and k, i.e., if a = a_1i+a_2j+a_3k and b = b_1i+b_2j+b_3k , then their sum is defined as a + b = (a_1 + b_1)i + a_2 (a_2 + b_2)j + (a_3 + b_3)k.

Properties of vector addition: Vector addition has the following properties.

1. **Binary operation:** The sum of two vectors is always a vector.
2. **Commutativity:** For any two vectors a and b, a + b = b + a.
3. **Associativity:** For any three vectors a, b and c, a+(b+c)=(a+b)+c.
4. **Identity:** Zero vector is the identity for addition. For any vector
5. **Additive inverse:** For every vector a its negative vector −a exists such that a +(−a) = (−a) + a = O i.e., (−a) is the additive inverse of the vector a.

(2) Subtraction of vectors:

If a and b are two vectors, then their subtraction a − b is defined as a − b = a + (−b) where −b is the negative of b having magnitude equal to that of b and direction opposite to b. If a = a_1i+a_2j+a_3k and b = b_1i+b_2j+b_3k.
Then a – b = (a_1 – b_1)i + (a_2 – b_2)j + (a_3 – b_3)k.



Properties of vector subtraction:

1. a − b ≠ b − a
2. (a − b) − c ≠ a − (b − c)
3. Since any one side of a triangle is less than the sum and greater than the difference of the other two sides, so for any two vectors a and b, we have
   (a) |a + b| ≤ |a| + |b
   (b) |a + b| ≥ |a| − |b|
   (c) |a − b| ≤ |a| + |b|
   (d) |a − b| ≥ |a| − |b|

(3) Multiplication of a vector by a scalar:
If **a** is a vector and m is a scalar (i.e., a real number) then **ma** is a vector whose magnitude is m times that of **a** and whose direction is the same as that of **a**, if m is positive and opposite to that of , if m is negative.

Properties of Multiplication of vectors by a scalar:
The following are properties of multiplication of vectors by scalars, for vectors and scalars m, n.

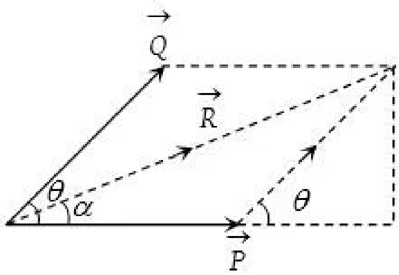
1. m (−a) = (−m) a = −(ma)
2. (−m) (−a) = ma
3. m (na) = (mn) a = n (ma)
4. (m + n) a = ma + na
5. m (a + b) = ma +mb

(4) Resultant of two forces

Let $\vec{P}$ and $\vec{Q}$ be two forces and $\vec{R}$ be the resultant of these two forces then, $\vec{R} = \vec{P} + \vec{Q}$

$$|\vec{R}| = R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

where $|\vec{P}| = P$, $|\vec{Q}| = Q$,



Also, $\tan\alpha = \dfrac{Q\sin\theta}{P + Q\cos\theta}$

Deduction : When $|\vec{P}| \dotplus \vec{Q}|$,

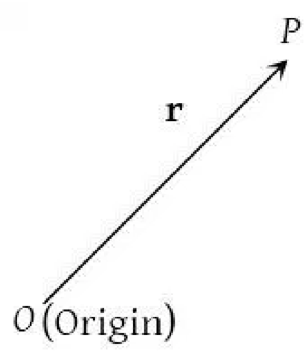
i.e., $P = Q$, $\tan\alpha = \dfrac{P\sin\theta}{P + P\cos\theta} = \dfrac{\sin\theta}{1+\cos\theta} = \tan\dfrac{\theta}{2}$;

$\therefore \quad \alpha = \dfrac{\theta}{2}$

Hence, the angular bisector of two unit vectors a and b is along the vector sum a + b.

Position vector

If a point O is fixed as the origin in space (or plane) and P is any point, then $\overrightarrow{OP}$ is called the position vector of P with respect to O.



If we say that P is the point r, then we mean that the position vector of P is r with respect to some origin O.

(1) $\overrightarrow{AB}$ **in terms of the position vectors of points** A **and** B **:** If **a** and **b** are position vectors of points A and B respectively. Then,

$\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}$

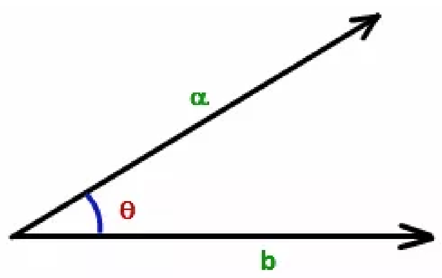
$\therefore \quad \overrightarrow{AB}$ = (Position vector of B) – (Position vector of A) $= \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$

(2) **Position vector of a dividing point :** The position vectors of the points dividing the line AB in the ratio $m : n$ internally or externally are $\frac{m\mathbf{b} + n\mathbf{a}}{m + n}$ or $\frac{m\mathbf{b} - n\mathbf{a}}{m - n}$.

Dot Product

Scalar or Dot product

(1) Scalar or Dot product of two vectors:
If a and b are two non-zero vectors and θ be the angle between them, then their scalar product (or dot product) is denoted by **a.b** and is defined as the scalar |a||b| cos θ , where |a| and |b| are modulii of a and b respectively and 0≤ θ≤π. Dot product of two vectors is a scalar quantity.



a.b = |a| |b| Cosθ

Angle between two vectors:

If a, b be two vectors inclined at an angle θ, then a.b = |a||b| cos θ.

$$\Rightarrow \cos\theta = \frac{\mathbf{a}\,.\,\mathbf{b}}{|\,\mathbf{a}||\,\mathbf{b}|} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{a}\,.\,\mathbf{b}}{|\,\mathbf{a}||\,\mathbf{b}|}\right)$$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$; then

$$\theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}\right).$$

(2) Properties of scalar product:

1. **Commutativity:** The scalar product of two vector is commutative i.e., a . b = b . a.
2. **Distributivity of scalar product over vector addition:** The scalar product of vectors is distributive over vector addition i.e.,
   (a) a.(b + c) = a . b − a . c, (Left distrihutivity)
   (b) (b + c).a = b . a + c . a , (Right distrihutivity)
3. Let a and b be two non-zero vectors a . b = 0 ⇔ a ⊥ b.
   As i, j, k are mutually perpendicular unit vectors along the co-ordinate axes, therefore, i . j = j . i =0 ; j . k = k . j = 0; k . i = i . k = 0.
4. For any vector a, a . a = $|a|^2$.
   As i, j, k are unit vectors along the co-ordinate axes, therefore i . i = $|i|^2$, j . j = $|j|^2$ and k . k = $|k|^2$
5. If m, n are scalars and a, b be two vectors, then ma . nb = mn(a . b) = (mn a).b = a.(mn b)
6. For any vectors a and b, we have
   (a) a. (−b) = − (a.b) = (−a).b
   (b) (−a).(−b) = a.b
7. For any two vectors a and b, we have
    1. $|a + b|^2 = |a|^2 + |b|^2 + 2a.b$
    2. $|a - b|^2 = |a|^2 + |b|^2 + 2a.b$
    3. $(a+b).(a-b) = |a|^2 - |b|^2$
    4. $|a + b| = |a| + |b| \Rightarrow a \parallel b$
    5. $|a + b|^2 = |a|^2 + |b|^2 \Rightarrow a \perp b$
    6. $|a + b| = |a - b| = \Rightarrow a \perp b$

(3) Scalar product in terms of components:
If a = a_1i + a_2j + a_3k and b = b_1i + b_2j + b_3k then a . b = $a_1b_1 + a_2b_2 + a_3b_3$.

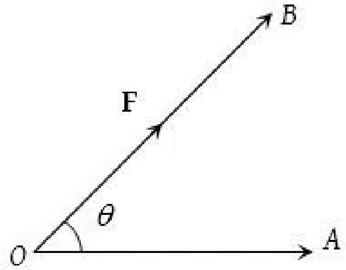
The components of $\mathbf{b}$ along and perpendicular to $\mathbf{a}$ are $\left(\frac{\mathbf{a}\,.\,\mathbf{b}}{|\,\mathbf{a}|^2}\right)\mathbf{a}$

and $\mathbf{b} - \left(\frac{\mathbf{a}\,.\,\mathbf{b}}{|\,\mathbf{a}|^2}\right)\mathbf{a}$ respectively.

The components of b along and perpendicular to are and respectively.

(4) Work done by a force:

Work done = $|\,\mathbf{F}||\,\overrightarrow{OA}\,|\cos\theta = \mathbf{F}\,.\,\overrightarrow{OA} = \mathbf{F}\,.\mathbf{d}$, where $\mathbf{d} = \overrightarrow{OA}$



Work done = (Force). (Displacement)

If a number of forces are acting on a particle, then the sum of the works done by the separate forces is equal to the work done by the resultant force.

Scalar triple product

(1) Scalar triple product of three vectors:

If a, b, c are three vectors, then their scalar triple product is defined as the dot product of two vectors a and b × c. It is generally denoted by a . (b × c) or [a b c].

(2) Properties of scalar triple product:

1. If a, b, c are cyclically permuted, the value of scalar triple product remains the same. i.e., (a × b). c = (b × c). a = (c × a). b or [a b c] = [b c a] = [c a b]
2. The change of cyclic order of vectors in scalar triple product changes the sign of the scalat' triple product but not the magnitude i.e., [a b c] = −[b a c] = −[c b a] = −[a c b]
3. In scalar triple product the positions of dot and cross can be interchanged provided that the cyclic order of the vectors remains same i.e., (a × b). c = a. (b × c)
4. The scalar triple product of three vectors is zero if any two of them are equal.
5. For any three vectors a, b, c and scalar λ, [λ a b c] = λ[a b c]
6. The scalar triple product of three vectors is zero if any two of them are parallel or collinear.
7. If a, b, c, d are four vectors, then [(a + b) c d] = [a c d] + [b c d].
8. The necessary and sufficient condition for three non-zero non-collinear vectors to be coplanar is that [a b c] = 0.
9. Four points with position vectors a, b, c and d will be coplanar, if [a b c] + [d c a] + [d a b] = [a b c].
10. Volume of parallelopiped whose coterminous edges are a, b, c is [a b c] or a(b × c).

(3) Scalar triple product in terms of components:

(i) If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ be three vectors

then, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(ii) If $\mathbf{a} = a_1\mathbf{l} + a_2\mathbf{m} + a_3\mathbf{n}, \mathbf{b} = b_1\mathbf{l} + b_2\mathbf{m} + b_3\mathbf{n}$

and $\mathbf{c} = c_1\mathbf{l} + c_2\mathbf{m} + c_3\mathbf{n}$, then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\mathbf{l}\ \mathbf{m}\ \mathbf{n}]$

(iii) For any three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$

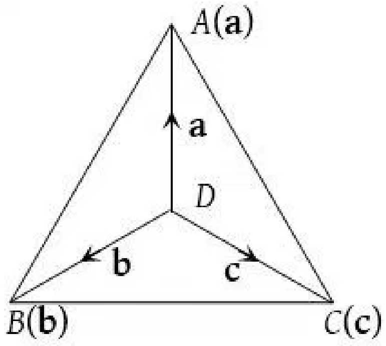
(a) $[\mathbf{a} + \mathbf{b}\ \ \mathbf{b} + \mathbf{c}\ \ \mathbf{c} + \mathbf{a}] = 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

(b) $[\mathbf{a} - \mathbf{b}\ \ \mathbf{b} - \mathbf{c}\ \ \mathbf{c} - \mathbf{a}] = 0$

(c) $[\mathbf{a} \times \mathbf{b}\ \ \mathbf{b} \times \mathbf{c}\ \ \mathbf{c} \times \mathbf{a}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

(4) Tetrahedron:

A tetrahedron is a three-dimensional figure formed by four triangle OABC is a tetrahedron with ΔABC as the base. OA, OB, OC, AB, BC and CA are known as edges of the tetrahedron. OA, BC; OB, CA and OC, AB are known as the pairs of opposite edges. A tetrahedron in which all edges are equal, is called a regular tetrahedron. Any two edges of regular tetrahedron are perpendicular to each other.



Volume of tetrahedron

$$= \frac{1}{3}(\text{area of the base})(\text{corresponding altitude})$$

$$= \frac{1}{6}[\overrightarrow{AB}\ \overrightarrow{BC}\ \overrightarrow{AD}]$$

1. The volume of a tetrahedron
2. If a, b, c are position vectors of vertices A, B and C with respect to O, then volume of tetrahedron OABC $= \frac{1}{6}$ [a b c].
3. If a, b, c, d are position vectors of vertices A, B, C, D of a tetrahedron ABCD, then its volume = $\frac{1}{6}$ [b−a c−a d−a].

(5) Reciprocal system of vectors:

Let be three non-coplanar vectors, and let $\mathbf{a}' = \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]},\ \mathbf{b}' = \frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]},\ \mathbf{c}' = \frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}$. a', b', c' are said to form a reciprocal system of vectors for the vectors a, b, c.
If a, b, c and a', b', c' form a reciprocal system of vectors, then

(i) $\mathbf{a}.\mathbf{a}' = \mathbf{b}.\mathbf{b}' = \mathbf{c}.\mathbf{c}' = 1$

(ii) $\mathbf{a}.\mathbf{b}' = \mathbf{a}.\mathbf{c}' = 0; \mathbf{b}.\mathbf{c}' = \mathbf{b}.\mathbf{a}' = 0; \mathbf{c}.\mathbf{a}' = \mathbf{c}.\mathbf{b}' = 0$

(iii) $[\mathbf{a}'\mathbf{b}'\mathbf{c}'] = \frac{1}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

(iv) $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar *iff* so are $\mathbf{a}', \mathbf{b}', \mathbf{c}'$.

Scalar product of four vectors

(a × b) . (c × d) is a scalar product of four vectors. It is the dot product of the vectors a × b and c × d.
It is a scalar triple product of the vectors a,b and c × d as well as scalar triple product of the vectors a × b, c and d.

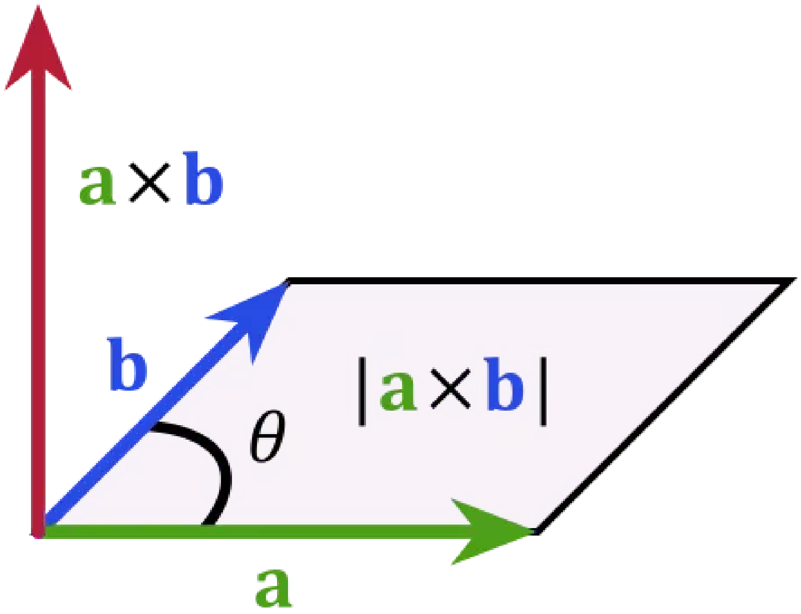
$$(\mathbf{a}\times\mathbf{b}).(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} \mathbf{a}.\mathbf{c} & \mathbf{a}.\mathbf{d} \\ \mathbf{b}.\mathbf{c} & \mathbf{b}.\mathbf{d} \end{vmatrix}$$

Cross Product

Vector or Cross product

(1) Vector product of two vectors:

Let a, b be two non-zero, non-parallel vectors.



Then a × b = |a||b| sin θ $\hat{n}$, and a × b = |a||b| sin θ where θ is the angle between a and b, $\hat{n}$ is a unit vector perpendicular to the plane of a and b such that a, b, $\hat{n}$ form a right-handed system.

(2) Properties of vector product

1. Vector product is not commutative i.e., if and b are any two vectors, then a × b ≠ b × a , however, a × b = −(b × a)
2. If a, b are two vectors and m, n are scalars, then ma × nb = mn(a × b) = m(a × nb) = n(ma × b).
3. Distributivity of vector product over vector addition.
   Let a, b, c be any three vectors. Then
   (a) a × (b + c) = a × b + a × c (Left distrihutivity).
   (b) (b + c) × a = b × a + c × a (Right distributivity).
4. For any three vectors a, b, c we have a×(b − c) = a×b − a×c.
5. The vector product of two non-zero vectors is zero vector iff they are parallel (Collinear) i.e. a × b = 0 ⇔ a ∥ b, a, b are non-zero vectors.
   It follows from the above property that a × a = 0 for every non-zero vector a, which in turn implies that i × i = j × j = k × k = 0.
6. Vector product of orthonormal triad of unit vectors i, j, k using the definition of the vector product, we obtain i × j = k, j × k = i, k × i = j, j × i = −k, k × j = −i, i × k = −j.

(3) Vector product in terms of components:

If a = a_1i + a_2j + a_3k and b = b_1i + b_2j + b_3k.

Let $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\bar{a} \times \bar{b} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) \times \left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$$

$$= (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(4) Angle between two vectors:

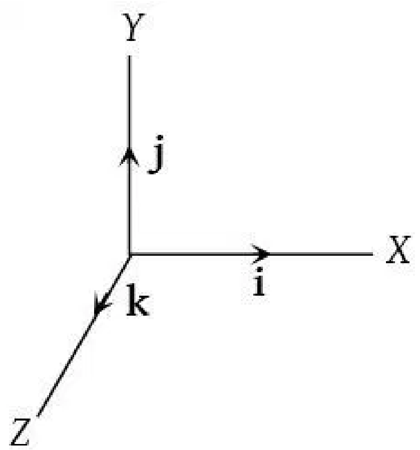
If θ is the angle between a and b,

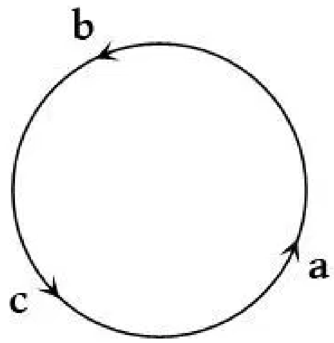
$$\text{Then } \sin\theta = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$$

(5) (i) Right handed system of vectors:

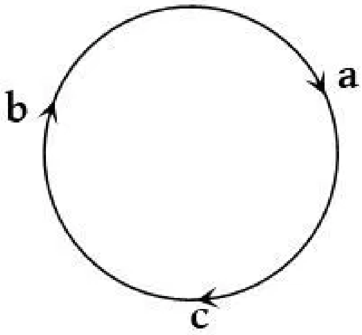
Three mutually perpendicular vectors a, b, c form a right handed system of vector iff a × b = c, b × c = a, c × a = b.

Examples: The unit vectors i, j, k form a right-handed system, i × j = k, j × k = i, k × i = j.





(ii) Left handed system of vectors: The vectors a, b, c mutually perpendicular to one another form a left handed system of vector iff c × b = a, a × c = b, b × a = c.



(6) Vector normal to the plane of two given vectors:
If a, b be two non-zero, nonparallel vectors and let θ be the angle between them. a × b = |a||b| sin θ $\hat{n}$ where $\hat{n}$ is a unit vector perpendicular to the plane of a and b such that a, b, $\hat{n}$ form a right-handed system.

$$\Rightarrow \quad (\mathbf{a}\times\mathbf{b}) = |\mathbf{a}\times\mathbf{b}|\,\hat{\eta} \Rightarrow \hat{\eta} = \frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$$

Thus, $\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$ is a unit vector perpendicular to the plane of **a** and **b**. Note that $-\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$ is also a unit vector perpendicular to the plane of **a** and **b**. Vectors of magnitude 'λ' normal to the plane of **a** and **b** are given by $\pm\frac{\lambda(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}$.

(7) Area of parallelogram and triangle:

1. The area of a parallelogram with adjacent sides a and b is |a × b|.
2. The area of a parallelogram with diagonals d_1 and d_2 is $\frac{1}{2}$ $|d_1 \times d_2|$.
3. The area of a plane quadrilateral ABCD is $\frac{1}{2}\left|\overrightarrow{AC}\times\overrightarrow{BD}\right|$, where AC and BD are its diagonals.
4. The area of a triangle with adjacent sides a and b is $\frac{1}{2}$ |a × b|
5. The area of a triangle ABC is $\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$ or $\frac{1}{2}\left|\overrightarrow{BC}\times\overrightarrow{BA}\right|$ or $\frac{1}{2}\left|\overrightarrow{CB}\times\overrightarrow{CA}\right|$.
6. If a, b, c are position vectors of ΔABC vertices of a then its area = $\frac{1}{2}$ |(a × b) + (b × c) + (c × a)|
7. Three points with position vectors a, b, c are collinear if (a × b) + (b × c) + (c × a) = 0..

(8) Moment of a force:

Moment of a force F about a point O is $\overrightarrow{OP}\times F$, where P is any point on the line of action of the force F.

Vector triple product

Let a, b, c he any three vectors, then the vectors a × (b × c) and (a × b) × c are called vector triple product of a, b, c .
Thus, a ×(b × c) = (a. c) b — (a . b) c

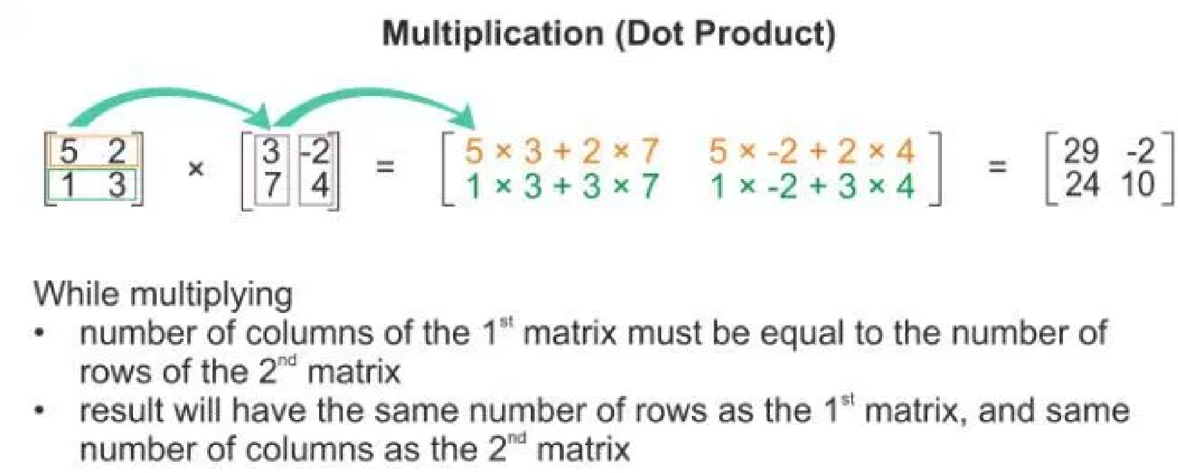
Properties of vector triple product:

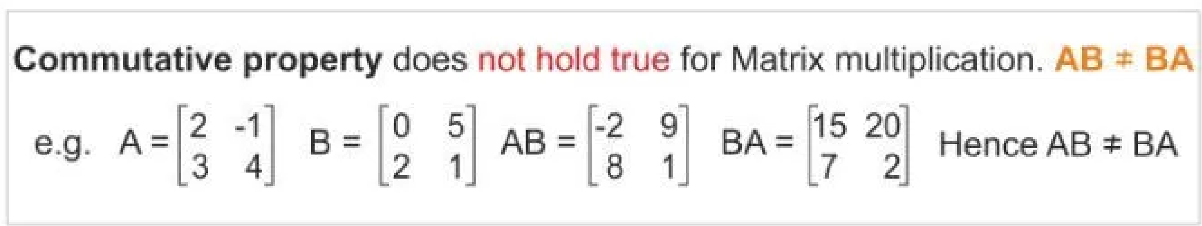
1. The vector triple product a × (b × c) is a linear combination of those two vectors which are within brackets.
2. The vector r = a ×(b × c) is perpendicular to a and lies in the plane of b and c.
3. The formula a ×(b × c) = (a.c)b − (a.b)c is true only when the vector outside the bracket is on the left most side. If it is not, we first shift on left by using the properties of cross product and then apply the same formula.
   Thus, (b × c)× a = − {a ×(b × c)} = − {(a.c)b − (a.b)c} = (a.b)c − (a.c)b
4. Vector triple product is a vector quantity.
5. a × (b × c) ≠ (a × b)× c.

Vector product of four vectors

1. (a × b) × (c × d) is a vector product of four vectors.
   It is the cross product of the vectors a × b and c × d.
2. a×{b ×(c ×d)}, {(a×b)×c}×d are also different products of four vectors a, b, c and d

How to Multiply Matrices





Multiplication of matrices

Two matrices A and B are conformable for the product AB if the number of columns in A (pre-multiplier) is same as the number of rows in B (post multiplier). Thus, if A = $[a_{ij}]_{m\times n}$ and B = $[b_{ij}]_{n\times p}$ are two matrices of order m × n and n × p respectively, then their product AB is of order and is defined as

$$(AB)_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj} = [a_{i1}a_{i2}\ldots a_{in}]\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = (i^{th} \text{ row of A})(j^{th} \text{ column of } B) \quad \ldots\ldots(i)$$

where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots p$

Now we define the product of a row matrix and a column matrix.

Let $A = [a_1 a_2 \ldots. a_n]$ be a row matrix and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ be a column matrix.

Then $AB = [a_1b_1 + a_2b_2 + \ldots. + a_nb_n]$ .....(ii)

Thus, from (i), $(AB)_{ij}$ = Sum of the product of elements of i^{th} row of A with the corresponding elements of j^{th} column of B.

Properties of matrix multiplication

If A,B and C are three matrices such that their product is defined, then

1. AB ≠ BA, (Generally not commutative)
2. (AB)C = A(BC), (Associative Law)
3. IA = A = AI, where I is identity matrix for matrix multiplication.
4. A(B + C) = AB + AC, (Distributive law)
5. If AB = AC ⇏ B = C, (Cancellation law is not applicable)
6. If AB = 0, it does not mean that A = 0 or B = 0, again product of two non zero matrix may be a zero matrix.

Scalar multiplication of matrices

Let $A = [a_{ij}]_{m\times n}$ be a matrix and k be a number, then the matrix which is obtained by multiplying every element of A by k is called scalar multiplication of A by k and it is denoted by kA.
Thus, if $A = [a_{ij}]_{m\times n}$, then $kA = Ak = [ka_{ij}]_{m\times n}$.

Properties of scalar multiplication:

If A, B are matrices of the same order and λ, μ are any two scalars then

1. λ (A + B) = λA + λB
2. (λ + μ)A = λ + μA
3. λ(μA) = (λμA) = μ(λA)
4. (-λA) = -(λA) = λ(-A)

All the laws of ordinary algebra hold for the addition or subtraction of matrices and their multiplication by scalars.

Multiplication of Matrices Problems with Solutions

1.

If $A=[1\ 2\ 3]$ and $B=\begin{bmatrix}-5 & 4 & 0\\ 0 & 2 & -1\\ 1 & -3 & 2\end{bmatrix}$, then $AB=$

(a) $\begin{bmatrix}-5 & 4 & 0\\ 0 & 4 & -2\\ 3 & -9 & 6\end{bmatrix}$ (b) $\begin{bmatrix}3\\ 1\\ 1\end{bmatrix}$

(c) $[-2\ \ -1\ \ 4]$ (d) $\begin{bmatrix}-5 & 8 & 0\\ 0 & 4 & -3\\ 1 & -6 & 6\end{bmatrix}$

Solution:

(c) $AB=[1\ 2\ 3]\begin{bmatrix}-5 & 4 & 0\\ 0 & 2 & -1\\ 1 & -3 & 2\end{bmatrix}=[-2\ -1\ \ 4]$.

2.

If $A=\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}$, then A^4 is equal to

(a) $\begin{bmatrix}1 & a^4\\ 0 & 1\end{bmatrix}$ (b) $\begin{bmatrix}4 & 4a\\ 0 & 4\end{bmatrix}$

(c) $\begin{bmatrix}4 & a^4\\ 0 & 4\end{bmatrix}$ (d) $\begin{bmatrix}1 & 4a\\ 0 & 1\end{bmatrix}$

Solution:

(d) $A^2=A.A=\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 2a\\ 0 & 1\end{bmatrix}$

$A^3=A.A^2=\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2a\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 3a\\ 0 & 1\end{bmatrix}$

$A^4=A.A^3=\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 3a\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 4a\\ 0 & 1\end{bmatrix}$

3.

If $A=\begin{bmatrix}1 & 2\\ -3 & 0\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 0\\ 2 & 3\end{bmatrix}$, then

(a) $A^2=A$ (b) $B^2=B$

(c) $AB\neq BA$ (d) $AB=BA$

Solution:

(c) Since $A^2 = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -3 & -6 \end{bmatrix} \neq A$

$$B^2 = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 9 \end{bmatrix} \neq B$$

Now $AB = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 3 & 0 \end{bmatrix}$

and $BA = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -7 & 4 \end{bmatrix}$

Obviously, $AB \neq BA$.

4.

If $M = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $M^2 - \lambda M - I_2 = 0$, then $\lambda =$

(a) -2 (b) 2

(c) -4 (d) 4

Solution:

(d) $M^2 - \lambda M - I_2 = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 2\lambda \\ 2\lambda & 3\lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = O$$

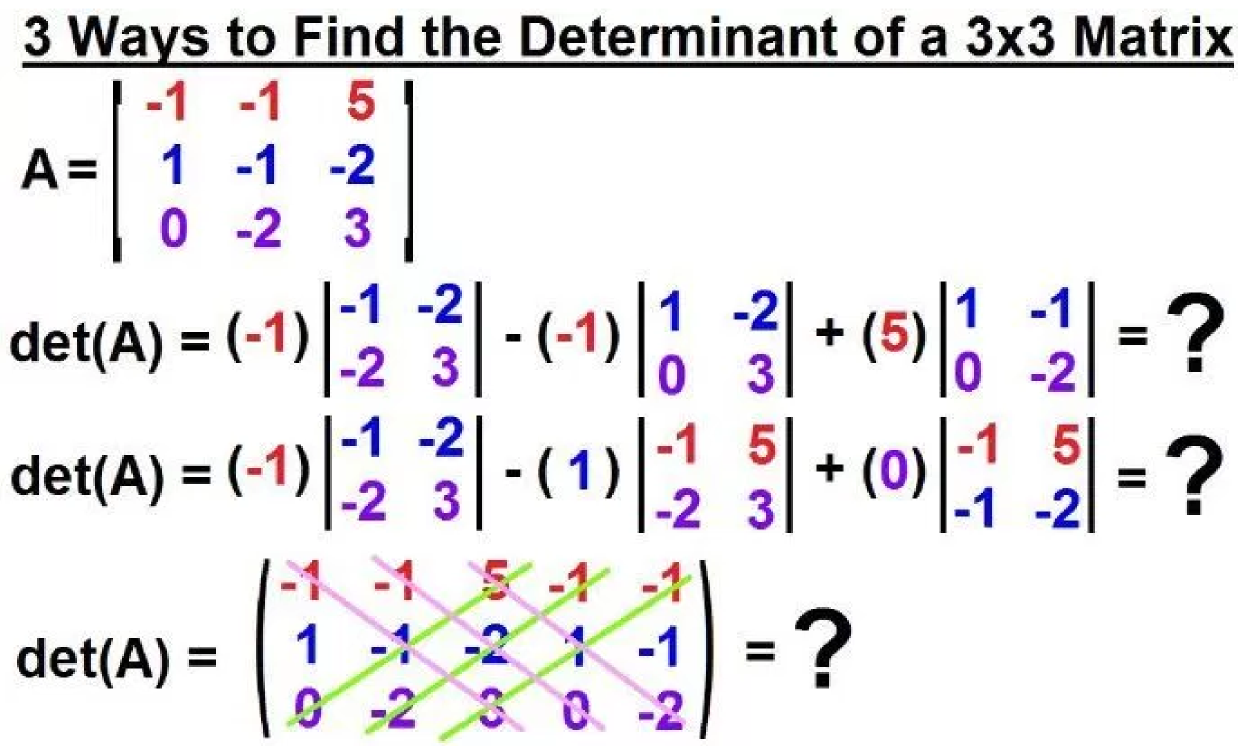
$$\Rightarrow \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix} - \begin{bmatrix} \lambda & 2\lambda \\ 2\lambda & 3\lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = O$$

$$\Rightarrow \begin{bmatrix} 5-\lambda & 8-2\lambda \\ 8-2\lambda & 13-3\lambda \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow 5-\lambda = 1, 8-2\lambda = 0, 13-3\lambda = 1$

$\Rightarrow \lambda = 4$, which satisfies all the three equations.

Determinant of a Matrix



Determinant

Let us consider three homogeneous linear equations $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$ and $a_3x + b_3y + c_3z = 0$

Eliminated x, y, z from above three equations we obtain
$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + (a_2b_3 - a_3b_2) = 0$ …….(i)

The L.H.S. of (i) is represented by $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

Its contains three rows and three columns, it is called a determinant of third order.
The number of elements in a second order is $2^2 = 4$ and the number of elements in a third order determinant is $3^2 = 9$.
Rows and columns of a determinant : In a determinant horizontal lines counting from top 1^{st}, 2^{nd}, 3^{rd},….. respectively known as rows and denoted by R_1, R_2, R_3, ……. and vertical lines counting left to right, 1^{st}, 2^{nd}, 3^{rd},….. respectively known as columns and denoted by C_1, C_2, C_3, …….

Properties of determinants

1. The value of determinant remains unchanged, if the rows and the columns are interchanged.
   Since the determinant remains unchanged when rows and columns are interchanged, it is obvious that any theorem which is true for 'rows' must also be true for 'columns'.
2. If any two rows (or columns) of a determinant be interchanged, the determinant is unaltered in numerical value but is changed in sign only.
3. If a determinant has two rows (or columns) identical, then its value is zero.
4. If all the elements of any row (or column) be multiplied by the same number, then the value of determinant is multiplied by that number.
5. If each element of any row (or column) can be expressed as a sum of two terms, then the determinant can be expressed as the sum of the determinants.

$$e.g., \begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

6. The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column)

$$e.g.,\ D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{and}\ \ D' = \begin{vmatrix} a_1 + ma_2 & b_1 + mb_2 & c_1 + mc_2 \\ a_2 & b_2 & c_2 \\ a_3 - na_1 & b_3 - nb_1 & c_3 - nc_1 \end{vmatrix}$$

Then $D' = D$.

7. If all elements below leading diagonal or above leading diagonal or except leading diagonal elements are zero then the value of the determinant equal to multiplied of all leading diagonal elements.
8. If a determinant D becomes zero on putting x = α, then we say that x − α is factor of determinant.
9. It should be noted that while applying operations on determinants then at least one row (or column) must remain unchanged or, Maximum number of operations = order of determinant −1.
10. It should be noted that if the row (or column) which is changed by multiplied a non-zero number, then the determinant will be divided by that number.

Minors and Cofactors

Minor of an element:

If we take the element of the determinant and delete (remove) the row and column containing that element, the determinant left is called the minor of that element. It is denoted by M_{ij}.

$$\text{Consider the determinant}\ \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

$$\text{then determinant of minors}\ M = \begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix}$$

$$\text{where}\ \ M_{11} = \text{minor of}\ a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{12} = \text{minor of}\ \ a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$M_{13} = \text{minor of}\ a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Similarly, we can find the minors of other elements. Using this concept the value of determinant can be
$\Delta = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$
or, $\Delta = - a_{21}M_{21} + a_{22}M_{22} - a_{23}M_{23}$
or, $\Delta = a_{31}M_{31} - a_{32}M_{32} + a_{33}M_{33}$

Cofactor of an element:

The cofactor of an element a_{ij} (i.e. the element in the i^{th} row and j^{th} column) is defined as $(-1)^{i+j}$ times the minor of that element. It is denoted by C_{ij} or A_{ij} or F_{ij}.
$C_{ij} = (-1)^{i+j} M_{ij}$.

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then determinant of cofactors is $C = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$

where $C_{11} = (-1)^{1+1} M_{11} = +M_{11}$, $C_{12} = (-1)^{1+2} M_{12} = -M_{12}$ and $C_{13} = (-1)^{1+3} M_{13} = +M_{13}$
Similarly, we can find the cofactors of other elements.

Product of two determinants

Let the two determinants of third order be,

$$D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

Let D be their product.

$$\text{Then } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

We can also multiply rows by columns or columns by rows or columns by columns.

Application of determinants in solving a system of linear equations

(1) Solution of system of linear equations in three variables by Cramer's rule:

The solution of the system of linear equations
$a_1x + b_1y + c_1z = d_1$ …..(i)
$a_2x + b_2y + c_2z = d_2$ …..(ii)
$a_3x + b_3y + c_3z = d_3$ …..(iii)

Is given by $x = \dfrac{D_1}{D}$, $y = \dfrac{D_2}{D}$ and $z = \dfrac{D_3}{D}$,

$$\text{where, } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \text{ and } D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Provided that $D \neq 0$
(2) Conditions for consistency:
For a system of 3 simultaneous linear equations in three unknown variable.
(i) If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by $x = D_1/D$, $y = D_2/D$ and $z = D_3/D$.
(ii) If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then the given system of equations is consistent with infinitely many solutions.

(iii) If D = 0 and at least one of the determinants D_1, D_2, D_3 is non-zero, then given of equations is inconsistent.

Some special determinants

(1) Symmetric determinant:
A determinant is called symmetric determinant if for its every element $a_{ij} = -a_{ji}\ \forall$ i, j.

$$e.g., \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

(2) Skew-symmetric determinant:
A determinant is called skew symmetric determinant if for its every element $a_{ij} = -a_{ji}\ \forall$ i, j.

$$e.g., \begin{vmatrix} 0 & 3 & -1 \\ -3 & 0 & 5 \\ 1 & -5 & 0 \end{vmatrix}$$

- Every diagonal element of a skew symmetric determinant is always zero.
- The value of a skew symmetric determinant of even order is always a perfect square and that of odd order is always zero.

(3) Cyclic order:
If elements of the rows (or columns) are in cyclic order. i.e.,

$$\text{(i)} \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

$$\text{(ii)} \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(ab+bc+ca).$$

$$\text{(iii)} \begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$$

$$\text{(iv)} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

$$\text{(v)} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc)$$

Inverse of a Matrix using Minors, Cofactors and Adjugate

Minors and Cofactors

Minor of an element:

If we take the element of the determinant and delete (remove) the row and column containing that element, the determinant left is called the minor of that element. It is denoted by M_{ij}.

Consider the determinant $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$,

then determinant of minors $M = \begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix}$

where M_{11} = minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

M_{12} = minor of $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

M_{13} = minor of $a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Similarly, we can find the minors of other elements. Using this concept the value of determinant can be
$\Delta = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$
or, $\Delta = -a_{21}M_{21} + a_{22}M_{22} - a_{23}M_{23}$
or, $\Delta = a_{31}M_{31} - a_{32}M_{32} + a_{33}M_{33}$

Cofactor of an element:

The cofactor of an element a_{ij} (i.e. the element in the i^{th} row and j^{th} column) is defined as $(-1)^{i+j}$ times the minor of that element. It is denoted by C_{ij} or A_{ij} or F_{ij}.

$C_{ij} = (-1)^{i+j} M_{ij}$.

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then determinant of cofactors is $C = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$

where $C_{11} = (-1)^{1+1} M_{11} = +M_{11}$, $C_{12} = (-1)^{1+2} M_{12} = -M_{12}$ and $C_{13} = (-1)^{1+3} M_{13} = +M_{13}$
Similarly, we can find the cofactors of other elements.

Adjugate (also called Adjoint) of a Square Matrix

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor a_{ij} of in A. Then the transpose of the matrix of cofactors of elements of A is called the adjoint of A and is denoted by adj A

Thus, adj A = $[C_{ij}]^T \Rightarrow (\text{adj } A)_{ij} = C_{ji}$ = cofactor of a_{ij} in A.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$\text{then } adj\, A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix};$$

where C_{ij} denotes the cofactor of a_{ij} in A.

Example : $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, C_{11} = s, C_{12} = -r, C_{21} = -q, C_{22} = p$

$$\therefore adj\, A = \begin{bmatrix} s & -r \\ -q & p \end{bmatrix}^T = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$$

Properties of adjoint matrix:
If A, B are square matrices of order n and is corresponding unit matrix, then

1. A(adj A)=|A|, I_n =(adj A)A
   (Thus A (adj A) is always a scalar matrix)
2. |adj A|= $|A|^{n-1}$
3. adj(adj A) = $|A|^{n-2}$ A
4. adj(adj A) =$|A|^{(n-1)\wedge 2}$
5. adj(A^T) = $(\text{adj } A)^T$
6. adj(AB) = (adj B)(adj A)
7. adj(A^m) = $(\text{adj } A)^m$, m ∈ N
8. adj(kA) = k^{n-1}(adj A), k ∈ R
9. adj(I_n) = I_n
10. adj(0) = 0
11. A is symmetric ⇒ adj A is also symmetric.
12. A is diagonal ⇒ adj A is also diagonal.
13. A is triangular ⇒ adj A is also triangular.
14. A is singular ⇒ |adj A|= 0

Adjoint of a Square Matrix Problems with Solutions

1.

$$\text{If } A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}, \text{ then } adj\, A$$

(a) $\begin{pmatrix} 1 & 4 & -2 \\ -2 & 1 & 4 \\ 4 & -2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -2 & 4 \\ 4 & 1 & -2 \\ -2 & 4 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 4 \\ -4 & 1 & 2 \\ -4 & -2 & 1 \end{pmatrix}$ (d) None of these

Solution:

(b) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$,

$A_{11} = 1, A_{21} = -2, A_{31} = 4$

$A_{12} = 4, A_{22} = 1, A_{32} = -2$

$A_{13} = -2, A_{23} = 4, A_{33} = 1$

$Adj(A) = \begin{bmatrix} 1 & -2 & 4 \\ 4 & 1 & -2 \\ -2 & 4 & 1 \end{bmatrix}$.

2.

The adjoint matrix of $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ is

(a) $\begin{bmatrix} 4 & 8 & 3 \\ 2 & 1 & 6 \\ 0 & 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

(c) $\begin{bmatrix} 11 & 9 & 3 \\ 1 & 2 & 8 \\ 6 & 9 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & 3 \\ -2 & 3 & -3 \end{bmatrix}$

Solution:

(b) Let $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

Then, $A_{11} = 1, A_{12} = -2, A_{13} = -2$

$A_{21} = -1, A_{22} = 3, A_{23} = 3$

$A_{31} = 0, A_{32} = -4, A_{33} = -3$

$$adj(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}.$$

Inverse of a Matrix

A non-singular square matrix of order n is invertible if there exists a square matrix B of the same order such that AB = I_n =BA .

In such a case, we say that the inverse of A is B and we write A^{-1} = B. The inverse of A is given by

$$A^{-1} = \frac{1}{|A|}.adj\,A\ .$$

The necessary and sufficient condition for the existence of the inverse of a square matrix A is that |A| ≠ 0.

Properties of inverse matrix:

If A and B are invertible matrices of the same order, then

1. $(A^{-1})^{-1} = A$
2. $(A^T)^{-1} = (A^{-1})^T$
3. $(AB)^{-1} = B^{-1}A^{-1}$
4. $(A^k)^{-1} = (A^{-1})^k$, $k \in N$ [In particular $(A^2)^{-1} = (A^{-1})^2$]
5. $adj(A^{-1}) = (adj\ A)^{-1}$
6. $$|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$$
7. $A = diag\ (a_1 a_2 \dots a_n) \Rightarrow A^{-1} = diag(a_1^{-1} a_2^{-1} \dots a_n^{-1})$
8. A is symmetric $\Rightarrow A^{-1}$ is also symmetric.
9. A is diagonal, $|A| \neq 0 \Rightarrow A^{-1}$ is also diagonal.
10. A is a scalar matrix $\Rightarrow A^{-1}$ is also a scalar matrix.
11. A is triangular, $|A| \neq 0 \Rightarrow A^{-1}$ is also triangular.
12. Every invertible matrix possesses a unique inverse.
13. Cancellation law with respect to multiplication.
    If A is a non-singular matrix i.e., if $|A| \neq 0$,then A^{-1} exists and $AB = AC \Rightarrow A^{-1}(AB) = A\ \ (AC)$
    $\Rightarrow (A^{-1}A)B = (A^{-1}A)C$
    $\Rightarrow IB = IC \Rightarrow B=C$
    $\therefore AB=AC \Rightarrow B = C \Leftrightarrow |A| \neq 0$. -1

Inverse of a Matrix Problems with Solutions

1.

The inverse matrix of $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$, is

(a) $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{2} & -4 & \frac{5}{2} \\ 1 & -6 & 3 \\ 1 & 2 & -1 \end{bmatrix}$

(c) $\frac{1}{2}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 4 & 2 & 3 \end{bmatrix}$ (d) $\frac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$

Solution:

(a) $$A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{|A|}.adj(A)$$

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}; |A| = 0 - 1(1-9) + 2(1-6) = 8 - 10$$

$$|A| = -2 \neq 0$$

$$Adj\ A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$A_{11} = (-1)^{1+1}[(2)(1) - (3)(1)] = -1$$

$$A_{12} = 8,\ A_{13} = -5,\ A_{21} = 1,\ A_{22} = -6$$

$$A_{23} = 3, A_{31} = -1,\ A_{32} = 2,\ A_{33} = -1$$

$$\therefore A^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}.$$

2.

If $A=\begin{bmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{bmatrix}$, then $A^{-1}=$

(a) A (b) A^2

(c) A^3 (d) A^4

Solution:

(c) Here, $C_{11}=1, C_{12}=-2, C_{13}=-2$

$C_{21}=-1, C_{22}=3, C_{23}=3$

$C_{31}=0, C_{32}=-4, C_{33}=-3$

$$\Rightarrow \det A = |A| = \begin{vmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{vmatrix}=1$$

$$\Rightarrow A^{-1}=\frac{1}{|A|}.(Adj\ A)=\frac{1}{1}.\begin{bmatrix}C_{11} & C_{21} & C_{31}\\ C_{12} & C_{22} & C_{32}\\ C_{13} & C_{23} & C_{33}\end{bmatrix}$$

$$=\begin{bmatrix}1 & -1 & 0\\ -2 & 3 & -4\\ -2 & 3 & -3\end{bmatrix}$$

Now, $A^2=\begin{bmatrix}3 & -4 & 4\\ 0 & -1 & 0\\ -2 & 2 & -3\end{bmatrix}$

and $A^3=A^2.A=\begin{bmatrix}3 & -4 & 4\\ 0 & -1 & 0\\ -2 & 2 & -3\end{bmatrix}\times\begin{bmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{bmatrix}$

$$=\begin{bmatrix}1 & -1 & 0\\ -2 & 3 & -4\\ -2 & 3 & -3\end{bmatrix}=A^{-1}.$$

3.

If matrix $A=\begin{bmatrix}3 & 2 & 4\\ 1 & 2 & -1\\ 0 & 1 & 1\end{bmatrix}$ and $A^{-1}=\frac{1}{K}adj(A)$, then K is

(a) 7 (b) -7

(c) $\frac{1}{7}$ (d) 11

Solution:

(d) $K = |A|; |A| = \begin{vmatrix} 3 & 2 & 4 \\ 1 & 2 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 11$.

Solving Systems of Linear Equations Using Matrices

Homogeneous and non-homogeneous systems of linear equations

A system of equations AX = B is called a homogeneous system if B = O. If B ≠ O, it is called a non-homogeneous system of equations.
e.g., 2x + 5y = 0
3x − 2y = 0
is a homogeneous system of linear equations whereas the system of equations given by

e.g., 2x + 3y = 5
x + y = 2
is a non-homogeneous system of linear equations.

Solution of Non-homogeneous system of linear equations

1. Matrix method: If AX = B, then X = A^{-1}B gives a unique solution, provided A is non-singular. But if A is a singular matrix i.e., if |A| = 0, then the system of equation AX = B may be consistent with infinitely many solutions or it may be inconsistent.
2. Rank method for solution of Non-Homogeneous system AX = B
   1. Write down A, B
   2. Write the augmented matrix [A : B]
   3. Reduce the augmented matrix to Echelon form by using elementary row operations.
   4. Find the number of non-zero rows in A and [A : B] to find the ranks of A and [A : B] respectively.
   5. If ρ(A) ≠ ρ(A : B) then the system is inconsistent.
   6. ρ(A) = ρ(A : B) = the number of unknowns, then the system has a unique solution.
   7. ρ(A) = ρ(A : B) < number of unknowns, then the system has an infinite number of solutions.

Solutions of a homogeneous system of linear equations

Let AX = O be a homogeneous system of 3 linear equations in 3 unknowns.

1. Write the given system of equations in the form AX = O and write A.
2. Find |A|.
3. If |A| ≠ 0, then the system is consistent and x = y = z = 0 is the unique solution.
4. If |A| = 0, then the systems of equations has infinitely many solutions. In order to find that put z = k (any real number) and solve any two equations for x and y so obtained with z = k give a solution of the given system of equations.

Consistency of a system of linear equation AX = B, where A is a square matrix

In system of linear equations AX = B, A = $(a_{ij})_{n\times n}$ is said to be

1. Consistent (with unique solution) if |A| ≠ 0.
   i.e., if A is non-singular matrix.
2. Inconsistent (It has no solution) if |A| = 0 and (adj A)B is a non-null matrix.
3. Consistent (with infinitely m any solutions) if |A| = 0 and (adj A)B is a null matrix.

Rank of matrix

Definition:

Let A be a m×n matrix. If we retain any r rows and r columns of A we shall have a square sub-matrix of order r. The determinant of the square sub-matrix of order r is called a minor of A order r. Consider any matrix A which is of the order of 3×4 say,

$$A = \begin{vmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 1 \end{vmatrix}.$$

It is 3×4 matrix so we can have minors of order 3, 2 or 1. Taking any three rows and three columns minor of order three. Hence minor of order

$$3 = \begin{vmatrix} 1 & 3 & 4 \\ 1 & 2 & 6 \\ 1 & 5 & 0 \end{vmatrix} = 0$$

Making two zeros and expanding above minor is zero. Similarly we can consider any other minor of order 3 and it can be shown to be zero. Minor of order 2 is obtained by taking any two rows and any two columns.

Minor of order $2 = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = 2 - 3 = -1 \neq 0$.

Minor of order 1 is every element of the matrix.

Rank of a matrix: The rank of a given matrix A is said to be r if

1. Every minor of A of order r+1 is zero.
2. There is at least one minor of A of order r which does not vanish. Here we can also say that the rank of a matrix A is said to be r ,if
   - Every square submatrix of order r+1 is singular.
   - There is at least one square submatrix of order r which is non-singular.

The rank r of matrix A is written as ρ(A) = r.

Echelon form of a matrix

A matrix A is said to be in Echelon form if either A is the null matrix or A satisfies the following conditions:

1. Every non- zero row in A precedes every zero row.
2. The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

If can be easily proved that the rank of a matrix in Echelon form is equal to the number of non-zero row of the matrix.

Rank of a matrix in Echelon form: The rank of a matrix in Echelon form is equal to the number of non-zero rows in that matrix.

Solving Systems of Linear Equations Using Matrices Problems with Solutions

1.

If $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 0 & -6 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then $(x, y, z) =$

(a) $(-4, 2, 2)$ (b) $(4, -2, -2)$

(c) $(4, 2, 2)$ (d) $(-4, -2, -2)$

Solution:

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 0 & -6 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} x + 2y + 3z = 6 \\ 3x + y + 2z = -6 \\ 2x + 3y + z = 0 \end{matrix}$

On Simplification the above equation, we get the required result *i.e.,* $x = -4, y = 2, z = 2$.

2.

The solution of the equation $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is

$(x, y, z) =$

(a) $(1, 1, 1)$ (b) $(0, -1\ 2)$

(c) $(-1, 2, 2)$ (d) $(-1, 0, 2)$

Solution:

(d) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow A^{-1} \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & -1 \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$\therefore\ AX = B \Rightarrow X = A^{-1}B \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & -1 \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$

Aliter : $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x + 0y + z \\ -x + y + 0z \\ 0x - y + z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \begin{matrix} x + z = 1 \\ -x + y = 1 \\ z - y = 2 \end{matrix}$

$\Rightarrow (x, y, z) = (-1, 0, 2)$.

3.

The number of solutions of the system of equations $2x + y - z = 7,\ x - 3y + 2z = 1, x + 4y - 3z = 5$ is

(a) 3 (b) 2

(c) 1 (d) 0

Solution:

(d) $\Delta = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -3 & 2 \\ 1 & 4 & -3 \end{vmatrix}$

$= 2(9 - 8) - 1(-3 - 2) - 1(4 + 3) = 7 - 7 = 0$

Hence, number of solutions is zero.

4.

If $A = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 8 & 10 \\ -6 & -12 & -15 \end{bmatrix}$. Then rank of A is equal to

(a) 0 (b) 1

(c) 2 (d) 3

Solution:

(b) $A = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 8 & 10 \\ -6 & -12 & -15 \end{bmatrix}_{3\times 3}$

$|A| = 0$, then rank cannot be 3.

Considering a 2×2 minor, $\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$ its determinant is zero.

Similarly considering

$\begin{bmatrix} 4 & 5 \\ 8 & 10 \end{bmatrix}, \begin{bmatrix} 4 & 8 \\ -6 & -12 \end{bmatrix}, \begin{bmatrix} 8 & 10 \\ -12 & 15 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 4 & 10 \end{bmatrix}, \begin{bmatrix} 4 & 10 \\ -6 & -15 \end{bmatrix}$

their determinants is zero. Each rank can not be 2. Thus rank = 1.

5.

If $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$, then $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is equal to

(a) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

Solution:

(d) We have, $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$

$x + y + z = 0$ ......(i)

$x - 2y - 2z = 3$ ......(ii)

$x + 3y + z = 4$ .....(iii)

On solving $x = 1, y = 2, z = -3$ *i.e.*, $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

Solving Linear Quadratic Systems Algebraically

Let’s look at how to solve a linear quadratic system of equations algebraically.

Example 1:

Solve this system of equations algebraically:

$y = x^2 - x - 6$ (quadratic equation of form $y = ax^2 + bx + c$: parabola)

$y = 2x - 2$ (linear equation of form $y = mx + b$)

First, we solve for one of the variables in the linear equation.	$y = 2x - 2$	Since this is already done for us in this example, we can go to the next step.
Next, we substitute for that variable in the quadratic equation, and solve the resulting equation.	$y = x^2 - x - 6$ $2x - 2 = x^2 - x - 6$ $2x = x^2 - x - 4$ $0 = x^2 - 3x - 4$ $0 = (x - 4)(x + 1)$ $x - 4 = 0$ $x + 1 = 0$ $x = 4$ $x = -1$	Add 2 to both sides. Subtract $2x$ from both sides. Factor. Set each factor = 0 and solve.
We now have two values for x, but we still need to find the corresponding values for y.		
We find the y-values by substituting each value of x into the linear equation.	$y = 2x - 2$ Check 4 $y = 2(4) - 2$ $y = 8 - 2$ $y = 6$ (4, 6)	$y = 2x - 2$ Check -1 $y = 2(-1) - 2$ $y = -2 - 2$ $y = -4$ (-1, -4)
Now we have 2 possible solutions for the system: (4,6) and (-1,-4). We need to check each solution in each equation. 	**Check#1: (4, 6)** $y = x^2 - x - 6$ $6 = (4)^2 - 4 - 6$ $6 = 16 - 4 - 6$ $6 = 6$ it checks ! $y = 2x - 2$ $6 = 2(4) - 2$ $6 = 8 - 2$ $6 = 6$ it also checks !	**Check#2: (-1, -4)** $y = x^2 - x - 6$ $-4 = (-1)^2 - (-1) - 6$ $-4 = 1 + 1 - 6$ $-4 = -4$ it checks ! $y = 2x - 2$ $-4 = 2(-1) - 2$ $-4 = -2 - 2$ $-4 = -4$ it also checks !
We finally have our solution set for this linear quadratic system.	$\{(4, 6), (-1, -4)\}$	

Example 2:

Solve this system of equations algebraically:

$x^2 + y^2 = 26$ (quadratic equation of form $x^2 + y^2 = r^2$: circle)

$x - y = 6$ (linear equation)

First we solve for x in the linear equation.	$x - y = 6$ $x = y + 6$	Add y to both sides
Now substitute this value of x into the quadratic equation replacing the x. Solve the resulting equation.	$x^2 + y^2 = 26$ $(y + 6)^2 + y^2 = 26$ $y^2 + 12y + 36 + y^2 = 26$ $2y^2 + 12y + 36 = 26$ $2y^2 + 12y + 10 = 0$ $y^2 + 6y + 5 = 0$ $(y + 5)(y + 1) = 0$ $y + 5 = 0 \quad y + 1 = 0$ $y = -5 \quad y = -1$	Expand $(y + 6)^2$ Combine similar terms. Divide each term by 2. Factor. Set each factor = 0.

Next, find the values of y by substituting in the linear equation.

$x - y = 6$
$x - (-5) = 6$
$x + 5 = 6$
$x = 1$*
(1,-5)

$x - y = 6$
$x - (-1) = 6$
$x + 1 = 6$
$x = 5$*
(5,-1)

* Remember to write the x-values first in the ordered pairs.

Solution Set

{(1,-5),(5,-1)}

CHECK:

$$x^2 + y^2 = 26$$
$$x - y = 6$$

Check: (1,-5) - plug into the two equations	
$(1)^2 + (-5)^2 = 26$ $1 + 25 = 26$ $26 = 26$ *Check*	$(1) - (-5) = 6$ $1 + 5 = 6$ $6 = 6$ *Check*

Check: (5, -1) - plug into the two equations	
$(5)^2 + (-1)^2 = 26$ $25 + 1 = 26$ $26 = 26$ *Check*	$(5) - (-1) = 6$ $5 + 1 = 6$ $6 = 6$ *Check*

Solving Linear Systems Algebraically Using Substitution

The substitution method is used to eliminate one of the variables by replacement when solving a system of equations.

Think of it as “grabbing” what one variable equals from one equation and “plugging” it into the other equation.

Systems of Equations may also be referred to as “simultaneous equations”.

Let’s look at an example using the substitution method:

Solve this system of equations using substitution. Check.

$$3y - 2x = 11$$
$$y + 2x = 9$$

1. Solve one of the equations for either “x =” or “y =”.
   This example solves the second equation for “y =”.
   $$3y - 2x = 11$$
   $$y = 9 - 2x$$
2. Replace the “y” value in the first equation by what “y” now equals. Grab the “y” value and plug it into the other equation.
   $$3(9 - 2x) - 2x = 11$$
3. Solve this new equation for “x”.
   $$(27 - 6x) - 2x = 11$$
   $$27 - 6x - 2x = 11$$
   $$27 - 8x = 11$$
   $$-8x = -16$$
   $$x = 2$$
4. Place this new “x” value into either of the ORIGINAL equations in order to solve for “y”. Pick the easier one to work with!
   $$y + 2x = 9 \text{ or}$$
   $$y = 9 - 2x$$
   $$y = 9 - 2(2)$$
   $$y = 9 - 4$$
   $$y = 5$$
5. **Check:** substitute x = 2 and y = 5 into BOTH ORIGINAL equations. If these answers are correct, BOTH equations will be TRUE!

$$3y - 2x = 11$$
$$3(5) - 2(2) = 11$$
$$15 - 4 = 11$$
$$11 = 11 \text{ (check!)}$$

$$y + 2x = 9$$
$$5 + 2(2) = 9$$
$$5 + 4 = 9$$
$$9 = 9 \text{ (check!)}$$

Solving a Linear Quadratic System Graphically

Solving a linear-quadratic system of equations graphically involves following a series of steps.

Example:

Solve the following system of equations graphically:

$y = x^2 - 4x - 2$ (quadratic equation of form $y = ax^2 + bx + c$)

$y = x - 2$ (linear equation of form $y = mx + b$)

Step 1: Graph one of the equations. Let's graph the quadratic equation first. By its form, $y = x^2 - 4x - 2$, we know it is a parabola.

Rather than picking numbers at random to form our table of values, let's find the axis of symmetry where the turning point of the parabola will occur.

To find the axis of symmetry, we use the formula $x = -b/2a$

In this example, a = 1, b = -4, and c = -2.

Substituting we get:

$x = -(-4)/2(1)$

$x = 4/2$

$x = 2$ axis of symmetry

Since the x-coordinate of the turning point is 2, let's use this value as the middle value for x in our table. We will also include 3 values above and below 2 in our table.

Substitute each value of x into the quadratic equation to find the corresponding values for y and complete the table.

For example, substituting -1 for x we get

$y = (-1)^2 - 4(-1) - 2 = 1 + 4 - 2 = 3$

x	y
-1	
0	
1	
2	
3	
4	
5	

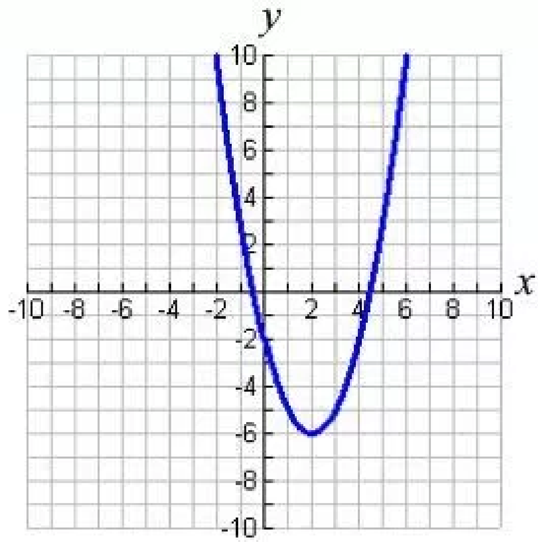
Set up the table.

x	y
-1	3
0	-2
1	-5
2	-6
3	-5
4	-2
5	3

Complete the table.

Next, graph the points from the table to get the graph of the parabola at the right.

Step one done!

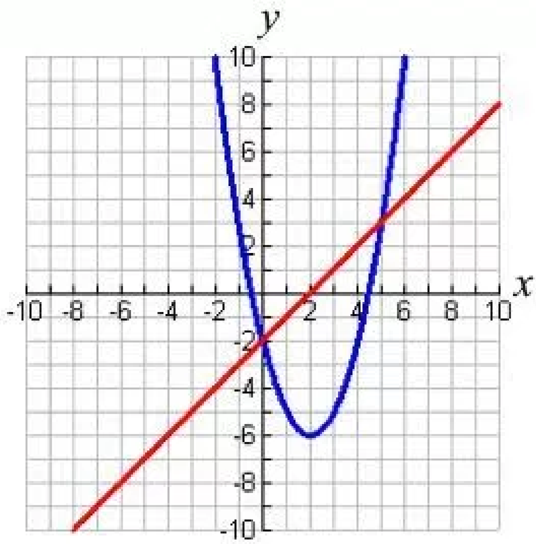


Step 2: Graph the second equation:
Now graph the linear equation, a straight line, $y = x - 2$ on the same set of axes.

To graph the straight line we need to know the slope and the y-intercept. Remember, from the form, $y = mx + b$, m is the slope and b is the y-intercept. For our equation, m = 1, b = -2.

Draw the graph of the line starting at -2 on the y-axis.

Use slope (which is rise over run) to find other points by going up 1 and to the right 1, or down 1 and to the left 1.



$y = x^2 - 4x - 2$
$y = x - 2$

Step 3: Find the intersection points (where they cross).
The last step is to find the point(s) where the two graphs intersect. This is the solution set, the answer, of the system of equations.

Our graphs intersect at 2 points whose coordinates are (0,-2) and (5,3).
The solution set for this problem is: {(0,-2),(5,3)}

Solving Linear Systems Graphically

Solve this system of equations graphically.
$4x - 6y = 12$
$2x + 2y = 6$

If you can graph a straight line, you can solve systems of equations graphically!
The process is very easy. Simply graph the two lines and look for the point where they intersect (cross).

Systems of Equations may also be referred to as "simultaneous equations".

Let's look at an example using a graphical method:

Solve graphically:

$$4x - 6y = 12$$
$$2x + 2y = 6$$

To solve a system of equations graphically, graph both equations and see where they intersect. The intersection point is the solution.

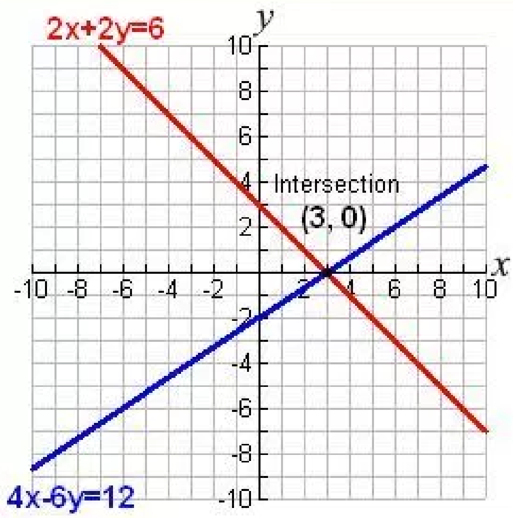
First, solve each equation for "y =".

$$4x - 6y = 12$$
$$4x = 6y + 12$$
$$4x - 12 = 6y$$
$$6y = 4x - 12$$
$$y = \frac{4x}{6} - \frac{12}{6}$$
$$y = \frac{2}{3}x - 2$$

slope = $\frac{2}{3}$
y-intercept = -2

$$2x + 2y = 6$$
$$2x + 2y = 6$$
$$2y = -2x + 6$$
$$y = \frac{-2x}{2} + \frac{6}{2}$$
$$y = -x + 3$$

slope = -1
y-intercept = 3



Graph the lines.

The slope intercept method of graphing was used in this example.

The point of intersection of the two lines, (3,0), is the solution to the system of equations.

This means that (3,0), when substituted into either equation, will make them both true. See the check.

Check: Since the two lines cross at (3,0), the solution is $x = 3$ and $y = 0$. Checking these value shows that this answer is correct. Plug these values into the ORIGINAL equations and get a true result.

$$4x - 6y = 12$$
$$4(3) - 6(0) = 12$$
$$12 - 0 = 12$$
$$12 = 12 \text{ (check)}$$

$$2x + 2y = 6$$
$$2(3) + 2(0) = 6$$
$$6 + 0 = 6$$
$$6 = 6 \text{ (check)}$$