Chapter 3

APPLICATION OF DIFFERENTIAL CALCULUS TO INVESTIGATION OF FUNCTIONS

§ 3.1. Basic Theorems on Differentiable Functions

Fermat's Theorem. Let a function $y=f(x)$ be defined on a certain interval and have a maximum or a minimum value at an interior point x_0 of the interval.

If there exists a derivative $f'(x_0)$ at the point x_0, then $f'(x_0)=0$.

Rolle's Theorem. If a function $f(x)$ is continuous in the interval $[a, b]$, has a finite derivative at all interior points of this interval, and $f(a)=f(b)$, then inside $[a, b]$ there exists a point $\xi \in (a, b)$ such that $f'(\xi)=0$.

Lagrange's Theorem. If a function $f(x)$ is continuous in the interval $[a, b]$ and has a finite derivative at all interior points of the interval, then there exists a point $\xi \in (a, b)$ such that

$$f(b)-f(a)=(b-a)f'(\xi).$$

Test for the Constancy of a Function. If at all points of a certain interval $f'(x)=0$, then the function $f(x)$ preserves a constant value within this interval.

Cauchy's Theorem. Let $\varphi(x)$ and $\psi(x)$ be two functions continuous in the interval $[a, b]$ and have finite derivatives at all interior points of the interval. If these derivatives do not vanish simultaneously and $\varphi(a) \neq \varphi(b)$, then there exists $\xi \in (a, b)$ such that

$$\frac{\psi(b)-\psi(a)}{\varphi(b)-\varphi(a)}=\frac{\psi'(\xi)}{\varphi'(\xi)}.$$

3.1.1. Does the function $f(x)=3x^2-1$ satisfy the condition of the Fermat theorem in the interval $[1, 2]$?

Solution. The given function does not satisfy the condition of the Fermat theorem, since it increases monotonically on the interval $[1, 2]$, and, consequently, takes on the minimum value at $x=1$ and the maximum one at $x=2$, i. e. not at interior points of the interval. Therefore, the Fermat theorem is not applicable; in other words, we cannot assert that $f'(1)=f'(2)=0$. Indeed, $f'(1)=6$, $f'(2)=12$.

3.1.2. Do the following functions satisfy the conditions of the Rolle theorem?

(a) $f(x)=1-\sqrt[3]{x^2}$ in $[-1, 1]$;
(b) $f(x)=\ln\sin x$ in $[\pi/6, 5\pi/6]$;
(c) $f(x)=1-|x|$ in $[-1, 1]$.

If they do not, explain why.

Solution. (a) The function is continuous in the interval $[-1, 1]$; furthermore, $f(-1)=f(1)=0$. Thus, two conditions of the Rolle theorem are satisfied. The derivative $f'(x)=-2/(3\sqrt[3]{x})$ exists at all points except $x=0$. Since this point is an interior one, the third condition of the theorem is not satisfied. Therefore, the Rolle theorem is not applicable to the given function. Indeed, $f'(x)\neq 0$ in $[-1, 1]$.

3.1.3. Prove that the equation

$$3x^5+15x-8=0$$

has only one real root.

Solution. The existence of at least one real root follows from the fact that the polynomial $f(x)=3x^5+15x-8$ is of an odd power. Let us prove the uniqueness of such a root by reductio ad absurdum. Suppose there exist two roots $x_1<x_2$. Then in the interval $[x_1, x_2]$ the function $f(x)=3x^5+15x-8$ satisfies all conditions of the Rolle theorem: it is continuous, vanishes at the end-points and has a derivative at all points. Consequently, at some point ξ, $x_1<\xi<x_2$, $f'(\xi)=0$. But $f'(x)=15(x^4+1)>0$. This contradiction proves that the equation in question has only one real root.

3.1.4. Does the function $f(x)=3x^2-5$ satisfy the conditions of the Lagrange theorem in the interval $[-2, 0]$? If it does, then find the point ξ which figures in the Lagrange formula $f(b)-f(a)=f'(\xi)(b-a)$.

Solution. The function satisfies the conditions of the Lagrange theorem, since it is continuous in the interval $[-2, 0]$ and has a finite derivative at all interior points of the interval. The point ξ is found from the Lagrange formula:

$$f'(\xi)=6\xi=\frac{f(0)-f(-2)}{0-(-2)}=\frac{-5-7}{2}=-6,$$

whence $\xi=-1$.

3.1.5. Apply the Lagrange formula to the function $f(x)=\ln x$ in the interval $[1, e]$ and find the corresponding value of ξ.

3.1.6. Ascertain that the functions $f(x)=x^2-2x+3$ and $g(x)=x^3-7x^2+20x-5$ satisfy the conditions of the Cauchy theorem in the interval $[1, 4]$ and find the corresponding value of ξ.

Solution. The given functions $f(x)$ and $g(x)$ are continuous everywhere, and hence, in the interval $[1, 4]$ as well; their derivatives $f'(x)=2x-2$ and $g'(x)=3x^2-14x+20$ are finite everywhere; in addition, $g'(x)$ does not vanish at any real value of x.

Consequently, the Cauchy formula is applicable to the given functions:

$$\frac{f(4)-f(1)}{g(4)-g(1)}=\frac{f'(\xi)}{g'(\xi)},$$

i. e.

$$\frac{11-2}{27-9}=\frac{2\xi-2}{3\xi^2-14\xi+20}\quad(1<\xi<4).$$

Solving the latter equation, we find two values of ξ: $\xi_1=2$ and $\xi_2=4$.

Of these two values only $\xi_1=2$ is an interior point of the interval.

3.1.7. Do the functions $f(x)=e^x$ and $g(x)=\frac{x^2}{1+x^2}$ satisfy the conditions of the Cauchy theorem in the interval $[-3, 3]$?

3.1.8. On the curve $y=x^3$ find the point at which the tangent line is parallel to the chord through the points $A(-1, -1)$ and $B(2, 8)$.

Solution. In the interval $[-1, 2]$, whose end-points are the abscissas of the points A and B, the function $y=x^3$ is continuous and has a finite derivative; therefore the Lagrange theorem is applicable. According to this theorem there will be, on the arc AB, at least one point M, at which the tangent is parallel to the chord AB. Let us write the Lagrange formula for the given function:

$$f(2)-f(-1)=f'(\xi)[2-(-1)],$$

or

$$8+1=3\xi^2\cdot 3;$$

whence

$$\xi_1=-1,\ \xi_2=1.$$

The obtained values of ξ are the abscissas of the desired points (as we see, there exist two such points). Substituting ξ_1 and ξ_2 in the equation of the curve, we find the corresponding ordinates:

$$y_1=\xi_1^3=1;\ y_2=\xi_2^3=-1.$$

Thus, the required points are: $M_1(1, 1)$ and $M_2(-1, -1)$, of which only the former is an interior point on the arc AB.

Note. This problem can be solved without using the Lagrange theorem; write the equation of the chord as a straight line passing through two given points, and then find the point on the curve at which the tangent is parallel to the chord.

3.1.9. Taking advantage of the test for the constancy of a function, deduce the following formulas known from elementary mathematics:

(a) $\arcsin x+\arccos x=\pi/2$;

(b) $\sin^2 x=(1-\cos 2x)/2$;

(c) $\arccos\frac{1-x^2}{1+x^2}=2\arctan x$ at $0\leqslant x<\infty$;

(d) $\arcsin\frac{2x}{1+x^2}=\begin{cases}\pi-2\arctan x & \text{at } x\geqslant 1,\\ 2\arctan x & \text{at } -1\leqslant x\leqslant 1,\\ -\pi-2\arctan x & \text{at } x\leqslant -1.\end{cases}$

Solution. (a) Let us consider the function

$$f(x)=\arcsin x+\arccos x,$$

defined in the interval $[-1, 1]$. The derivative of the indicated function inside this interval equals zero:

$$f'(x)=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}\equiv 0 \quad (-1<x<1).$$

According to the test for the constancy of a function $f(x)=\text{const}$, i. e. $\arcsin x+\arccos x=C$ $(-1<x<1)$.

To determine the constant C let us put, for instance, $x=0$; then we have $\pi/2=C$, whence

$$\arcsin x+\arccos x=\pi/2 \quad (-1<x<1).$$

The validity of this equality at the points $x=\pm 1$ is verified directly.

(b) Let us take the function

$$f(x)=\sin^2 x+\frac{1}{2}\cos 2x$$

defined throughout the number scale: $-\infty<x<\infty$. The derivative of this function is everywhere equal to zero:

$$f'(x)=2\sin x\cos x-\sin 2x\equiv 0.$$

According to the test for the constancy of a function

$$\sin^2 x+\frac{1}{2}\cos 2x=C.$$

To determine C put, for instance, $x=0$; then we get $1/2=C$.

Wherefrom

$$\sin^2 x+\frac{1}{2}\cos 2x=\frac{1}{2},$$

or

$$\sin^2 x = \frac{1-\cos 2x}{2}.$$

(c) Let us introduce the function

$$f(x) = \arccos\frac{1-x^2}{1+x^2} - 2\arctan x,$$

determined along the entire number scale, since $\left|\frac{1-x^2}{1+x^2}\right| \leqslant 1$.

The derivative of the function $f(x)$ is zero for all $x > 0$:

$$f'(x) = -\frac{1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}}\,\frac{-4x}{(1+x^2)^2} - \frac{2}{1+x^2} = \frac{4x}{2x(1+x^2)} - \frac{2}{1+x^2} \equiv 0.$$

According to the test for the constancy of a function

$$\arccos\frac{1-x^2}{1+x^2} - 2\arctan x = C \quad \text{at} \quad x > 0.$$

To determine C let us put, say, $x=1$, which gives $C = \arccos 0 - 2\arctan 1 = 0$.

The validity of the proved formula at $x=0$ is verified directly.

Note. At $x=0$ the function $\arccos\frac{1-x^2}{1+x^2}$ has no derivative. At $x<0$ its derivative is

$$\left(\arccos\frac{1-x^2}{1+x^2}\right)' = -\frac{2}{1+x^2},$$

which enables us to derive the formula

$$\arccos\frac{1-x^2}{1+x^2} = -2\arctan x \quad (x<0).$$

The latter formula can be obtained on the strength of the fact that $\arccos\frac{1-x^2}{1+x^2}$ is an even function, and $2\arctan x$ is an odd one.

3.1.10. As is known, $(e^x)' = e^x$ for all x. Are there any more functions that coincide with their derivatives everywhere?

Solution. Let the function $f(x)$ be such that $f'(x) = f(x)$ everywhere.

Let us introduce the function

$$\varphi(x) = \frac{f(x)}{e^x} = f(x)e^{-x}.$$

The derivative of this function equals zero everywhere:

$$\varphi'(x) = f'(x)e^{-x} - e^{-x}f(x) \equiv 0.$$

By the test for the constancy of a function $f(x)/e^x = C$, whence $f(x) = Ce^x$.

And so, we have proved that the group of functions for which $f'(x)=f(x)$ is covered by the formula $f(x)=Ce^x$.

3.1.11. Prove the inequality

$$\arctan x_2-\arctan x_1 < x_2-x_1,$$

where $x_2 > x_1$.

Solution. To the function $f(x)=\arctan x$ on the interval $[x_1, x_2]$ apply the Lagrange formula:

$$\arctan x_2-\arctan x_1=\frac{1}{1+\xi^2}(x_2-x_1),$$

where $x_1<\xi<x_2$.

Since

$$0<\frac{1}{1+\xi^2}<1 \text{ and } x_2-x_1>0,$$

then

$$\arctan x_2-\arctan x_1 < x_2-x_1.$$

In particular, putting $x_1=0$ and $x_2=x$, we get

$$\arctan x < x \quad (x>0).$$

3.1.12. Show that the square roots of two successive natural numbers greater than N^2 differ by less than $1/(2N)$.

Solution. To the function $f(x)=\sqrt{x}$ on the interval $[n, n+1]$ apply the Lagrange formula:

$$f(n+1)-f(n)=\sqrt{n+1}-\sqrt{n}=\frac{1}{2\sqrt{\xi}},$$

where $n<\xi<n+1$.

If $n>N^2$, then $\xi>N^2$, hence $1/(2\sqrt{\xi})<1/(2N)$, whence

$$\sqrt{n+1}-\sqrt{n}<1/(2N).$$

3.1.13. Using the Rolle theorem prove that the derivative $f'(x)$ of the function

$$f(x)=\begin{cases} x\sin\frac{\pi}{x} & \text{at } x>0, \\ 0 & \text{at } x=0 \end{cases}$$

vanishes on an infinite set of points of the interval $(0, 1)$.

Solution. The function $f(x)$ vanishes at points where

$$\sin(\pi/x)=0, \quad \pi/x=k\pi, \quad x=1/k,$$
$$k=1, 2, 3, \ldots$$

Since the function $f(x)$ has a derivative at any interior point of the interval $[0, 1]$, the Rolle theorem is applicable to anyone of the intervals $[1/2,\ 1]$, $[1/3,\ 1/2]$, ..., $[1/(k+1),\ 1/k]$, ... Consequently, inside each of the intervals of the sequence, there is a point ξ_k, $1/(k+1) < \xi_k < 1/k$, at which the derivative $f'(\xi_k)=0$. And so we have shown that the derivative vanishes on an infinite set of points (see Fig. 38).

3.1.14. The *Legendre polynomial* is a polynomial defined by the following formula (Rodrigues' formula):

$$P_n(x)=\frac{1}{2^n n!}\cdot\frac{d^n}{dx^n}(x^2-1)^n \quad (n=0,\ 1,\ 2,\ \ldots).$$

Using the Rolle theorem, prove that the Legendre polynomial $P_n(x)$ has n different real roots, all of them found between -1 and $+1$.

Solution. Consider the function

$$f(x)=(x^2-1)^n=(x-1)^n(x+1)^n.$$

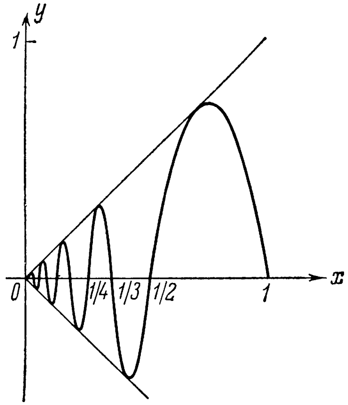


Fig. 38

This function and its $n-1$ successive derivatives vanish at the points $x=\pm 1$ (use the Leibniz formula for higher derivatives of the product of two functions).

It follows from $f(1)=f(-1)=0$ that inside the interval $[-1,\ 1]$ a point ξ_1 can be found at which $f'(\xi_1)=0$, i.e. $x=\xi_1$ will be the root of the first derivative. Now apply the Rolle theorem once again to the function $f'(x)$ on the intervals $[-1,\ \xi_1]$, $[\xi_1,\ 1]$. We find that besides $+1$ and -1 the function $f''(x)$ has two more roots on the interval $[-1,\ 1]$. Reasoning as before, we will show that, apart from $+1$ and -1, the $(n-1)$th derivative has $(n-1)$ more roots on the interval $[-1,\ 1]$, i.e. the function $f^{(n-1)}(x)$ has all in all $n+1$ roots on the interval $[-1,\ 1]$, which divide this interval into n parts. Applying the Rolle theorem once again, we ascertain that the function $f^{(n)}(x)$, and hence, the function $P_n(x)=\frac{1}{2^n n!}f^{(n)}(x)$, has n different roots on the interval $[-1,\ 1]$.

3.1.15. Check whether the Lagrange formula is applicable to the following functions:

(a) $f(x)=x^2$ on $[3,\ 4]$;

(b) $f(x)=\ln x$ on $[1,\ 3]$;

(c) $f(x)=4x^3-5x^2+x-2$ on $[0,\ 1]$;

(d) $f(x)=\sqrt[5]{x^4(x-1)}$ on $[-1/2,\ 1/2]$.

If it is, find the values of ξ appearing in this formula.

3.1.16. Using the Lagrange theorem estimate the value $\ln(1+e)$

3.1.17. Using the Lagrange formula prove the inequality

$$\frac{x}{1+x} < \ln(1+x) < x \quad \text{at } x > 0.$$

§ 3.2. *Evaluation of Indeterminate Forms. L'Hospital's Rule*

I. Indeterminate forms of the type $\frac{0}{0}$, $\frac{\infty}{\infty}$. If the functions $f(x)$ and $g(x)$ are differentiable in a certain neighbourhood of the point a, except, may be, at the point a itself, and $g'(x) \neq 0$, and if

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \quad \text{or} \quad \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty,$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ exists (*L'Hospital's rule*). The point a may be either finite or improper $+\infty$ or $-\infty$.

II. Indeterminate forms of the type $0 \cdot \infty$ or $\infty - \infty$ are reduced to forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by algebraic transformations.

III. Indeterminate forms of the type 1^∞, ∞^0 or 0^0 are reduced to forms of the type $0 \cdot \infty$ by taking logarithms or by the transformation $[f(x)]^{\varphi(x)} = e^{\varphi(x) \ln f(x)}$.

3.2.1. Applying the L'Hospital rule, find the limits of the following functions:

(a) $\lim\limits_{x \to 0} \frac{e^{ax} - e^{-2ax}}{\ln(1+x)}$; (b) $\lim\limits_{x \to -1} \frac{\sqrt[3]{1+2x}+1}{\sqrt{2+x}+x}$;

(c) $\lim\limits_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$; (d) $\lim\limits_{x \to 0} \frac{\ln(1+x^2)}{\cos 3x - e^{-x}}$;

(e) $\lim\limits_{x \to 0} \frac{\sin 3x^2}{\ln \cos(2x^2 - x)}$; (f) $\lim\limits_{x \to \infty} \frac{e^{1/x^2} - 1}{2 \arctan x^2 - \pi}$.

Solution. (a) Here both functions $f(x) = e^{ax} - e^{-2ax}$ and $g(x) = \ln(1+x)$ are infinitesimals in the neighbourhood of zero, since

$$\lim_{x \to 0} f(x) = 1 - 1 = 0; \quad \lim_{x \to 0} g(x) = \ln 1 = 0.$$

Furthermore $f'(x)$ and $g'(x)$ exist in any neighbourhood of the point $x = 0$ that does not contain the point $x = -1$, and

$$g'(x) = \frac{1}{1+x} \neq 0 \quad (x > -1).$$

Finally, there exists a limit of the ratio of the derivatives:

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}\frac{ae^{ax}+2ae^{-2ax}}{1/(1+x)}=3a.$$

Therefore the L'Hospital rule is applicable:

$$\lim_{x\to 0}\frac{e^{ax}-e^{-2ax}}{\ln(1+x)}=\lim_{x\to 0}\frac{ae^{ax}+2ae^{-2ax}}{1/(1+x)}=3a. \qquad (*)$$

Note. When the limit of the ratio is computed according to the L'Hospital rule the result is usually written directly as shown in (*). Whether the desired derivatives and limits exist is ascertained in the course of calculation. In case the ratio of the derivatives $\frac{f'(x)}{g'(x)}$ again represents an indeterminate form, the L'Hospital rule should be applied for a second time, and so on until the indeterminacy is removed or until it becomes clear that the required limits do not exist. Therefore, henceforward we write only the necessary transformations, leaving to the reader the task of checking whether the conditions of their applicability are fulfilled.

(b) $\lim\limits_{x\to -1}\dfrac{\sqrt[3]{1+2x}+1}{\sqrt{2+x}+x}=\lim\limits_{x\to -1}\dfrac{2/(3\sqrt[3]{(1+2x)^2})}{1/(2\sqrt{2+x})+1}=\dfrac{4}{9}$;

(e) $\lim\limits_{x\to 0}\dfrac{\sin 3x^2}{\ln\cos(2x^2-x)}=\lim\limits_{x\to 0}\dfrac{-6x\cos 3x^2\cos(2x^2-x)}{(4x-1)\sin(2x^2-x)}=$

$$=-6\lim_{x\to 0}\frac{\cos 3x^2\cos(2x^2-x)}{4x-1}\lim_{x\to 0}\frac{x}{\sin(2x^2-x)}.$$

The limit of the first factor is computed directly, the limit of the second one, which represents an indeterminate form of the type $\frac{0}{0}$ is found with the aid of the L'Hospital rule:

$$-6\lim_{x\to 0}\frac{\cos 3x^2\cos(2x^2-x)}{4x-1}\lim_{x\to 0}\frac{x}{\sin(2x^2-x)}=$$

$$=-6\cdot\frac{1\cdot 1}{-1}\lim_{x\to 0}\frac{1}{(4x-1)\cos(2x^2-x)}=6\cdot\frac{1}{-1\cdot 1}=-6.$$

3.2.2. It is known that, as $x\to+\infty$, the functions x^k $(k>0)$; $\log_a x$; a^x $(a>1)$ are infinitely large quantities. Applying the L'Hospital rule, compare these quantities.

Solution. 1. $\lim\limits_{x\to+\infty}\dfrac{\log_a x}{x^k}=\lim\limits_{x\to+\infty}\dfrac{\frac{1}{x}\log_a e}{kx^{k-1}}=\log_a e\lim\limits_{x\to+\infty}\dfrac{1}{kx^k}=0$;

2. $\lim\limits_{x\to+\infty}\dfrac{x^m}{a^x}=\lim\limits_{x\to+\infty}\dfrac{mx^{m-1}}{a^x\ln a}=\ldots=\lim\limits_{x\to+\infty}\dfrac{m!}{a^x(\ln a)^m}=0.$

Hence, the power function x^k $(k > 0)$ increases more rapidly than the logarithmic function $\log_a x$ $(a > 1)$, and the exponential function a^x with the base exceeding unity increases more rapidly than the power function x^m.

3.2.3. Find the limits:

(a) $\lim\limits_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$; (b) $\lim\limits_{x \to 0} \left(\cot x - \frac{1}{x}\right)$;

(c) $\lim\limits_{x \to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$.

Solution. (a) We have an indeterminate form of the type $\infty - \infty$. Let us reduce it to an indeterminate form of the type $\frac{0}{0}$ and then apply the L'Hospital rule:

$$\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x \to 1} \frac{x-1-\ln x}{(x-1)\ln x} = \lim_{x \to 1} \frac{1-1/x}{\ln x + 1 - 1/x} =$$

$$= \lim_{x \to 1} \frac{x-1}{x \ln x + x - 1} = \lim_{x \to 1} \frac{1}{\ln x + 2} = \frac{1}{2}.$$

3.2.4. Find the limits:

(a) $\lim\limits_{x \to 0} x^n \ln x$ $(n > 0)$;

(b) $\lim\limits_{x \to 0} [\ln(1 + \sin^2 x) \cot \ln^2(1+x)]$.

Solution. (a) We have an indeterminate form of the type $0 \cdot \infty$. Let us transform it to $\frac{\infty}{\infty}$, and then apply the L'Hospital rule:

$$\lim_{x \to 0} x^n \ln x = \lim_{x \to 0} \frac{\ln x}{x^{-n}} = \lim_{x \to 0} \frac{1/x}{-nx^{-n-1}} = -\frac{1}{n} \lim_{x \to 0} x^n = 0, \text{ since } n > 0.$$

(b) We have an indeterminate form of the type $0 \cdot \infty$:

$$\lim_{x \to 0} [\ln(1 + \sin^2 x) \cot \ln^2(1+x)] = \lim_{x \to 0} \frac{\ln(1+\sin^2 x)}{\tan \ln^2(1+x)} =$$

$$= \lim_{x \to 0} \frac{\frac{1}{1+\sin^2 x} \sin 2x}{2\{1 + \tan^2[\ln^2(1+x)]\} \ln(1+x) \cdot \frac{1}{1+x}} =$$

$$= \lim_{x \to 0} \frac{\sin x}{\ln(1+x)} = \lim_{x \to 0} \frac{\cos x}{\frac{1}{1+x}} = 1.$$

3.2.5. Find the limits:

(a) $\lim\limits_{x \to +0} (1/x)^{\sin x}$; (b) $\lim\limits_{x \to +0} x^{1/\ln(e^x - 1)}$.

Solution. (a) We have an indeterminate form of the type ∞^0. Let $y=(1/x)^{\sin x}$; then

$$\ln y=\sin x \ln(1/x),$$

$$\lim_{x\to+0}\ln y=\lim_{x\to+0}\sin x\ln(1/x) \text{ (indeterminate form of the type } 0\cdot\infty).$$

Let us transform it to $\frac{\infty}{\infty}$ and apply the L'Hospital rule:

$$\lim_{x\to+0}\ln y=\lim_{x\to+0}\frac{-\ln x}{1/\sin x}=\lim_{x\to 0}\frac{-1/x}{-(\cos x)/\sin^2 x}=\lim_{x\to 0}\frac{\sin^2 x}{x\cos x}=0.$$

Hence, $\lim\limits_{x\to+0} y=e^0=1$.

3.2.6. Find the limits:

$$\text{(a)} \lim_{x\to\pi/2}(\sin x)^{\tan x}; \qquad \text{(b)} \lim_{x\to 0} x^x.$$

3.2.7. Compute

$$\lim_{x\to+\pi/2-0}(\tan x)^{\cot x}.$$

Solution. Let us take advantage of the identity

$$(\tan x)^{\cot x}=e^{\cot x\ln\tan x},$$

but

$$\lim_{x\to+\pi/2-0}\cot x\ln\tan x=\lim_{x\to+\pi/2-0}\frac{\ln\tan x}{\tan x}=\lim_{y=\tan x\to+\infty}\frac{\ln y}{y}=0.$$

Whence

$$\lim_{x\to+\pi/2-0}(\tan x)^{\cot x}=e^0=1.$$

3.2.8. Ascertain the existence of the following limits:

(a) $\lim\limits_{x\to 0}\dfrac{x^2\sin(1/x)}{\sin x}$;

(b) $\lim\limits_{x\to\infty}\dfrac{2+2x+\sin 2x}{(2x+\sin 2x)e^{\sin x}}$;

(c) $\lim\limits_{x\to\pi/2}\dfrac{\tan x}{\sec x}$.

Can the L'Hospital rule be applied in computing them? Does its formal application lead to the correct answer?

Solution. (a) The limit exists and equals zero. Indeed,

$$\lim_{x\to 0}\frac{x^2\sin(1/x)}{\sin x}=\lim_{x\to 0}\frac{x}{\sin x}\lim_{x\to 0}x\sin\frac{1}{x}=1\cdot 0=0.$$

But the limit of the ratio of the derivatives does not exist. Indeed,

$$\lim_{x\to 0}\frac{2x\sin(1/x)-\cos(1/x)}{\cos x}=0-\lim_{x\to 0}\cos\frac{1}{x},$$

but $\lim\limits_{x\to 0}\cos(1/x)$ does not exist, hence the L'Hospital rule is not applicable here.

(b) The limit of the ratio of the functions does not exist:

$$\lim_{x\to\infty}\frac{2+2x+\sin 2x}{(2x+\sin 2x)\,e^{\sin x}}=\lim_{x\to\infty}\left(1+\frac{2}{2x+\sin 2x}\right)e^{-\sin x}$$

but $\lim\limits_{x\to\infty} e^{-\sin x}$ does not exist, since the function $e^{-\sin x}$ traverses the values from $1/e$ to e infinitely many times.

Now we will show that the limit of the ratio of derivatives exists:

$$\lim_{x\to\infty}\frac{2+2\cos 2x}{[2+2\cos 2x+(2x+\sin 2x)\cos x]\,e^{\sin x}}=$$

$$=\lim_{x\to\infty}\frac{4\cos^2 x}{4\cos^2 x+(2x+\sin 2x)\cos x}\,e^{-\sin x}=$$

$$=\lim_{x\to\infty}\frac{4\cos x}{2x+4\cos x+\sin 2x}\,e^{-\sin x}=0,$$

since the function $e^{-\sin x}$ is bounded, and $\dfrac{4\cos x}{2x+4\cos x+\sin 2x}\underset{x\to\infty}{\longrightarrow}0$.

Here $\cos x$, which vanishes for an infinite set of values of x, has been cancelled out. It is the presence of this multiplier that makes the L'Hospital rule inapplicable in this case, since it simultaneously nullifies the derivatives of the functions being compared.

(c) $$\lim_{x\to\pi/2}\frac{\tan x}{\sec x}=\lim_{x\to\pi/2}\frac{\sec^2 x}{\sec x\tan x}=\lim_{x\to\pi/2}\frac{\sec x}{\tan x}=\lim_{x\to\pi/2}\frac{\tan x}{\sec x}=\ldots$$

Here application of the L'Hospital rule gives no useful result, though there exists a limit:

$$\lim_{x\to\pi/2}\frac{\tan x}{\sec x}=\lim_{x\to\pi/2}\frac{\sin x\cos x}{\cos x}=\lim_{x\to\pi/2}\sin x=1.$$

3.2.9. Using the L'Hospital rule find the limits of the following functions:

(a) $\lim\limits_{x\to 2}\dfrac{\ln(x^2-3)}{x^2+3x-10}$; (b) $\lim\limits_{x\to 1}\dfrac{a^{\ln x}-x}{\ln x}$;

(c) $\lim\limits_{x\to 0}\dfrac{\tan x-x}{x-\sin x}$; (d) $\lim\limits_{x\to 1}\dfrac{1-4\sin^2(\pi x/6)}{1-x^2}$;

(e) $\lim\limits_{x\to a}\arcsin\dfrac{x-a}{a}\cot(x-a)$; (f) $\lim\limits_{x\to+\infty}(\pi-2\arctan x)\ln x$;

(g) $\lim\limits_{x\to+0}\left(\dfrac{1}{x}\right)^{\tan x}$; (h) $\lim\limits_{x\to\infty}(a^{1/x}-1)\,x\ (a>0)$;

(i) $\lim\limits_{x\to 0}(\cos mx)^{n/x^2}$; (j) $\lim\limits_{x\to a}\left(2-\dfrac{x}{a}\right)^{\tan(\pi x/(2a))}$;

(k) $\lim\limits_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{\ln x}\right)$; (l) $\lim\limits_{x\to 0} x^{1/\ln(e^x-1)}$;

(m) $\lim\limits_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)$; (n) $\lim\limits_{x\to\infty}\left[x-x^2\ln\left(1+\frac{1}{x}\right)\right]$;

(o) $\lim\limits_{x\to\infty} x^2\left[\cosh\frac{a}{x}-1\right]$; (p) $\lim\limits_{x\to 0}\left(\frac{5}{2+\sqrt{9+x}}\right)^{1/\sin x}$;

(q) $\lim\limits_{x\to +0}(\ln\cot x)^{\tan x}$; (r) $\lim\limits_{x\to\infty}\frac{e^{1/x^2}-1}{2\arctan x^2-\pi}$.

§ 3.3. Taylor's Formula. Application to Approximate Calculations

If the function $f(x)$ is continuous and has continuous derivatives through order $n-1$ on the interval $[a, b]$, and has a finite derivative of the nth order at every interior point of the interval then at $x\in[a, b]$ the following formula holds true:

$$f(x)=f(a)+f'(a)(x-a)+f''(a)\frac{(x-a)^2}{2!}+$$
$$+f'''(a)\frac{(x-a)^3}{3!}+\ldots+f^{(n-1)}(a)\frac{(x-a)^{n-1}}{(n-1)!}+f^{(n)}(\xi)\frac{(x-a)^n}{n!},$$

where

$$\xi=a+\theta(x-a) \text{ and } 0<\theta<1.$$

It is called *Taylor's formula* of the function $f(x)$.

If in this formula we put $a=0$, we obtain *Maclaurin's formula*:

$$f(x)=f(0)+f'(0)x+f''(0)\frac{x^2}{2!}+\ldots+f^{(n-1)}(0)\frac{x^{n-1}}{(n-1)!}+$$
$$+f^{(n)}(\xi)\frac{x^n}{n!}, \text{ where } \xi=\theta x,\ 0<\theta<1.$$

The last term in the Taylor formula is called the *remainder in Lagrange's form* and is denoted $R_n(x)$:

$$R_n(x)=\frac{f^{(n)}[a+\theta(x-a)]}{n!}(x-a)^n;$$

accordingly, the remainder in the Maclaurin formula has the form

$$R_n(x)=\frac{f^{(n)}(\theta x)}{n!}x^n.$$

3.3.1. Expand the polynomial $P(x)=x^5-2x^4+x^3-x^2+2x-1$ in powers of the binomial $x-1$ using the Taylor formula.

Solution. To solve the problem it is necessary to find the value of the polynomial and its derivatives at the point $x=1$. The relevant calculations are given below.

$$P(1)=0,\qquad P'(1)=0,$$
$$P''(1)=0,\qquad P'''(1)=18,$$
$$P^{(4)}(1)=72,\qquad P^{(5)}(1)=120,$$
$$P^{(n)}(x)=0\ (n\geqslant 6)$$

at any x.

Substituting the values thus found into the Taylor formula, we get

$$P(x)=\frac{18}{3!}(x-1)^3+\frac{72}{4!}(x-1)^4+\frac{120}{5!}(x-1)^5;$$
$$P(x)=3(x-1)^3+3(x-1)^4+(x-1)^5.$$

3.3.2. Applying the Maclaurin formula, expand in powers of x (up to x^9, inclusive) the function

$$f(x)=\ln(1+x),$$

defined on the interval $[0, 1]$. Estimate the error due to deleting the remainder.

Solution.

$$f(0)=\ln 1=0.$$

The derivatives of any order of the given function (see § 2.3):

$$f^{(n)}(x)=(-1)^{n-1}\frac{(n-1)!}{(1+x)^n},$$
$$f^{(n)}(0)=(-1)^{n-1}(n-1)!\quad (n=1, 2, 3, \ldots).$$

Substituting the derivatives into the Maclaurin formula, we get

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+\frac{x^9}{9}+R_{10}(x),$$

where the remainder $R_{10}(x)$ in the Lagrange form will be written as follows:

$$R_{10}(x)=\frac{f^{(10)}(\xi)}{10!}x^{10}=-\frac{9!}{10!\,(1+\xi)^{10}}x^{10}=-\frac{x^{10}}{10\,(1+\xi)^{10}}\qquad (0<\xi<x).$$

Let us estimate the absolute value of the remainder $R_{10}(x)$; keeping in mind that $0\leqslant x\leqslant 1$ and $\xi>0$, we have

$$|R_{10}(x)|=\left|\frac{-x^{10}}{10\,(1+\xi)^{10}}\right|<\frac{1}{10}.$$

3.3.3. How many terms in the Maclaurin formula should be taken for the function $f(x)=e^x$ so as to get a polynomial representing this function on the interval $[-1, 1]$, accurate to three decimal place?

Solution. The function $f(x)=e^x$ has a derivative of any order

$$f^{(n)}(x)=e^x.$$

Therefore, the Maclaurin formula is applicable to this function. Let us compute the values of the function e^x and its first $n-1$ derivatives at the point $x=0$, and the value of the nth derivative at the point $\xi=\theta x\ (0<\theta<1)$. We will have

$$f(0)=f'(0)=f''(0)=\ldots=f^{(n-1)}(0)=1;$$
$$f^{(n)}(\xi)=e^{\xi}=e^{\theta x}.$$

Whence

$$f(x)=1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^{n-1}}{(n-1)!}+R_n(x),$$

where

$$R_n(x)=\frac{x^n}{n!}e^{\theta x}.$$

Since, by hypothesis, $|x|\leqslant 1$ and $0<\theta<1$, then

$$|R_n(x)|=\frac{|x|^n}{n!}e^{\theta x}<\frac{1}{n!}e<\frac{3}{n!}.$$

Hence, if the inequality

$$\frac{3}{n!}\leqslant 0.001 \qquad (*)$$

is fulfilled, then the inequality

$$|R_n(x)|\leqslant 0.001$$

will be fulfilled apriori. To this end it is sufficient to take $n\geqslant 7$ $(7!=5040)$. Hence, 7 terms in the Maclaurin formula will suffice.

3.3.4. At what values of x will the approximate formula

$$\cos x\approx 1-\frac{x^2}{2!}+\frac{x^4}{4!}$$

have an error less than 0.00005?

Solution. The right member of the approximate equation represents the first six terms in the Maclaurin formula for the function $\cos x$ (the second, fourth and sixth terms are equal to zero; check it!). Let us estimate $R_6(x)$. Since $(\cos x)^{(6)}=-\cos x$, then

$$|R_6(x)|=\left|\frac{-\cos\theta x}{6!}x^6\right|\leqslant\frac{|x|^6}{6!}.$$

For the error to be less than 0.00005, choose the values of x that satisfy the inequality

$$\frac{|x|^6}{6!}<0.00005.$$

Solving this inequality, we get $|x|<0.575$.

3.3.5. Compute the approximate values of:

(a) $\cos 5°$; (b) $\sin 20°$,

accurate to five decimal places.

Solution. (a) Into the Maclaurin formula

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + R_{2n+2}$$

substitute $x = \pi/36$; since

$$\frac{x^2}{2!} = \frac{\pi^2}{2 \cdot 36^2} = 0.003808, \quad \frac{x^4}{4!} = \frac{1}{6}\left(\frac{x^2}{2}\right)^2 = 2.4 \cdot 10^{-6},$$

we confine ourselves to the following terms:

$$\cos x \approx 1 - x^2/2,$$

the error being estimated at

$$|R_4(x)| = \left|\frac{\cos \theta x}{4!} x^4\right| \leqslant \frac{|x|^4}{4!} < 2.5 \cdot 10^{-6}.$$

And so, within the required accuracy

$$\cos 5° = \cos \frac{\pi}{36} = 1 - 0.00381 = 0.99619.$$

3.3.6. Compute the approximate value of $\sqrt[4]{83}$ accurate to six decimal places.

3.3.7. Prove the inequalities:

(a) $x - x^2/2 < \ln(1+x) < x$ at $x > 0$;

(b) $\tan x > x + x^3/3$ at $0 < x < \pi/2$;

(c) $1 + \frac{1}{2}x - \frac{x^2}{8} < \sqrt{1+x} < 1 + \frac{1}{2}x$ at $0 < x < \infty$.

Solution. (a) According to the Maclaurin formula with the remainder $R_2(x)$ we have

$$\ln(1+x) = x - \frac{x^2}{2(1+\xi)^2},$$

where $0 < \xi < x$.

According to the same formula with the remainder $R_3(x)$ we have

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\xi_1)^3}, \text{ where}$$

$0 < \xi_1 < x$.

Since $\frac{x^2}{2(1+\xi)^2} > 0$ and $\frac{x^3}{3(1+\xi_1)^3} > 0$ at $x > 0$, it follows that

$$x - x^2/2 < \ln(1+x) < x.$$

3.3.8. Show that $\sin(\alpha + h)$ differs from $\sin\alpha + h\cos\alpha$ by not more than $h^2/2$.

Solution. By Taylor's formula

$$\sin(\alpha + h) = \sin\alpha + h\cos\alpha - \frac{h^2}{2}\sin\xi;$$

whence

$$|\sin(\alpha+h)-(\sin\alpha+h\cos\alpha)|=\frac{h^2}{2}|\sin\xi|\leqslant\frac{h^2}{2}.$$

§ 3.4. Application of Taylor's Formula to Evaluation of Limits

The expression

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+$$
$$+o(|x-a|^n)$$

is the *Taylor formula* with the remainder in Peano's form where $\varphi(x)=o[\psi(x)]$ means that, as $x\to a$, the function $\varphi(x)$ has a higher order of smallness than the function $\psi(x)$, i. e. $\lim\limits_{x\to a}\frac{\varphi(x)}{\psi(x)}=0$.

In particular, at $a=0$ we have

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+o(|x|^n).$$

Peano's form of the remainder for Taylor's formula shows that, when substituting the Taylor polynomial of degree n for $f(x)$ in the neighbourhood of the point a, we introduce an error which is an infinitesimal of a higher order than $(x-a)^n$ as $x\to a$.

The following five expansions are of greatest importance in solving practical problems:

$$e^x=1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+o(x^n);$$
$$\sin x=x-\frac{x^3}{3!}+\ldots+(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}+o(x^{2n});$$
$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots+(-1)^n\frac{x^{2n}}{(2n)!}+o(x^{2n+1});$$
$$(1+x)^\alpha=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\ldots+\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n+o(x^n);$$
$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+\ldots+(-1)^{n-1}\frac{x^n}{n}+o(x^n).$$

3.4.1. Expand the function $f(x)=\sin^2 x-x^2e^{-x}$ in positive integral powers of x up to the terms of the fourth order of smallness with respect to x.

Solution. We have

$$f(x)=\left[x-\frac{x^3}{6}+o(x^4)\right]^2-x^2\left[1-x+\frac{x^2}{2}+o(x^2)\right]=$$
$$=x^2-\frac{x^4}{3}+o(x^5)-x^2+x^3-\frac{x^4}{2}+o(x^4)=x^3-\frac{5}{6}x^4+o(x^4).$$

3.4.2. Expand the following functions:

(a) $f(x)=x\sqrt{1-x^2}-\cos x\ln(1+x)$;

(b) $f(x)=\ln(1+\sin x)$

in positive integral powers of x up to the terms of the fifth order of smallness with respect to x.

3.4.3. Applying the Taylor formula with the remainder in Peano's form, compute the limits:

(a) $\lim\limits_{x\to 0}\dfrac{1-\sqrt{1+x^2}\cos x}{\tan^4 x}$;

(b) $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+3x}-\sqrt{1+2x}}{x^2}$;

(c) $\lim\limits_{x\to 0}\dfrac{\cos x-e^{-x^2/2}}{x^4}$;

(d) $\lim\limits_{x\to 0}\dfrac{e^x\sin x-x(1+x)}{x^3}$;

(e) $\lim\limits_{x\to 0}\dfrac{e^x+e^{-x}-2}{x^2}$.

Solution. (a) Retaining the terms up to the fourth order with respect to x in the denominator and the numerator, we get

$$\lim_{x\to 0}\frac{1-\sqrt{1+x^2}\cos x}{\tan^4 x}=\lim_{x\to 0}\frac{1-(1+x^2)^{1/2}\cos x}{x^4}=$$

$$=\lim_{x\to 0}\frac{1-\left[1+\frac{1}{2}x^2+\frac{1/2\,(-1/2)}{2}x^4+o(x^4)\right]\left[1-\frac{x^2}{2}+\frac{x^4}{24}+o(x^5)\right]}{x^4}=$$

$$=\lim_{x\to 0}\frac{\frac{1}{4}x^4+\frac{1}{8}x^4-\frac{1}{24}x^4+o(x^4)}{x^4}=\lim_{x\to 0}\left[\frac{1}{3}+\frac{o(x^4)}{x^4}\right]=\frac{1}{3}.$$

3.4.4. Expand the following functions in positive integral powers of the variable x up to the terms of the indicated order, inclusive:

(a) $f(x)=e^{2x-x^2}$ up to the term containing x^5;

(b) $\ln\cos x$ up to the term containing x^6;

(c) $\dfrac{x}{e^x-1}$ up to the term containing x^4.

§ 3.5. Testing a Function for Monotonicity

Let a continuous function $f(x)$ be defined on the interval $[a, b]$ and have a finite derivative inside this segment. Then:

(1) For $f(x)$ to be non-decreasing (non-increasing) on $[a, b]$ it is necessary and sufficient that $f'(x)\geqslant 0$ ($f'(x)\leqslant 0$) for all x in (a, b).

(2) For $f(x)$ to be increasing (decreasing) on $[a, b]$ it is sufficient to fulfil the condition $f'(x) > 0$ ($f'(x) < 0$) for all x in (a, b).

3.5.1. Determine the intervals of monotonicity for the following functions:

(a) $f(x) = 2x^2 - \ln x$;
(b) $f(x) = 2x^3 - 9x^2 - 24x + 7$;
(c) $f(x) = x^2e^{-x}$;
(d) $f(x) = \ln|x|$;
(e) $f(x) = 4x^3 - 21x^2 + 18x + 20$;
(f) $f(x) = e^x + 5x$.

Solution. The solution of this problem is reduced to finding the intervals in which the derivative preserves its sign. If the function $f(x)$ has a continuous derivative in the interval (a, b) and has in it a finite number of stationary points $x_1, x_2, \ldots, x_n$ ($a < x_1 < x_2 < \ldots < x_n < b$), where $f'(x_k) = 0$ ($k = 1, 2, \ldots, n$), then $f'(x)$ preserves its sign in each of the intervals (a, x_1), (x_1, x_2), $\ldots$, (x_{n-1}, x_n), (x_n, b).

(a) The function is defined at $x > 0$.
Let us find the derivative

$$f'(x) = 4x - 1/x.$$

The function increases if $4x - 1/x > 0$, i.e. $x > 1/2$.
The function decreases if $4x - 1/x < 0$, i. e. $x < 1/2$.

And so, the function decreases in the interval $0 < x < 1/2$ and increases in the interval $1/2 < x < +\infty$.

(b) Evaluate the derivative

$$f'(x) = 6x^2 - 18x - 24 = 6(x^2 - 3x - 4).$$

It vanishes at the points $x = -1$ and $x = 4$. Since $f'(x)$ is a quadratic trinomial with a coefficient at its highest-power term $6 > 0$, then $f'(x) > 0$ in the intervals $(-\infty, -1)$, $(4, \infty)$, and $f'(x) < 0$ in the interval $(-1, 4)$. Consequently, $f(x)$ increases in the first two intervals, whereas in $(-1, 4)$ it decreases.

(c) In this case the derivative $f'(x) = (2x - x^2)e^{-x}$ vanishes at the points $x = 0$ and $x = 2$. In the intervals $(-\infty, 0)$ and $(2, \infty)$ the derivative $f'(x) < 0$ and the function decreases; in $(0, 2)$ the derivative $f'(x) > 0$ and the function increases (see Fig. 39).

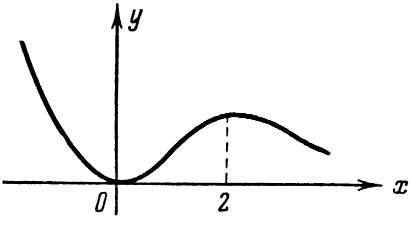


Fig. 39

3.5.2. Find the intervals of decrease and increase for the following functions:

(a) $f(x)=\cos(\pi/x)$;

(b) $f(x)=\sin x+\cos x$ on $[0, 2\pi]$.

Solution. (a) The function $y=\cos(\pi/x)$ is defined and differentiable throughout the number scale, except at the point $x=0$;

$$y'=\frac{\pi}{x^2}\sin\frac{\pi}{x}.$$

As is obvious, the sign of y' coincides with that of the multiplier $\sin(\pi/x)$.

(1) $\sin(\pi/x)>0$ if

$$2k\pi<\pi/x<(2k+1)\pi \quad (k=0, \pm 1, \pm 2, \ldots);$$

(2) $\sin(\pi/x)<0$ if

$$(2k+1)\pi<\pi/x<2(k+1)\pi.$$

Hence, the function increases in the intervals

$$\left(\frac{1}{2k+1}, \frac{1}{2k}\right)$$

and decreases in the intervals

$$\left(\frac{1}{2k+2}, \frac{1}{2k+1}\right).$$

3.5.3. Investigate the behaviour of the function $f(x)=2\sin x+$ $+\tan x-3x$ in the interval $(-\pi/2, \pi/2)$.

Solution. The derivative

$$f'(x)=2\cos x+\frac{1}{\cos^2 x}-3=\frac{(1-\cos x)(1+\cos x-2\cos^2 x)}{\cos^2 x}=$$
$$=\frac{4\sin^3(x/2)\sin(3x/2)}{\cos^2 x}$$

is positive in the intervals $(-\pi/2, 0)$ and $(0, \pi/2)$ and vanishes only at $x=0$. Hence, in $(-\pi/2, \pi/2)$ the function $f(x)$ increases.

3.5.4. Prove that at $0<x\leqslant 1$ the inequalities

$$x-x^3/3<\arctan x<x-x^3/6$$

are fulfilled.

Solution. We will prove only the right inequality (the left one is proved analogously).

The derivative of the function

$$f(x)=\arctan x-x+\frac{x^3}{6}$$

is equal to

$$f'(x)=\frac{1}{1+x^2}-1+\frac{x^2}{2}=\frac{x^2(x^2-1)}{2(1+x^2)}.$$

The function $f(x)$ is continuous throughout the entire number scale, in particular, it is continuous in the interval [0, 1], and inside this segment $f'(x) < 0$. Therefore, $f(x)$ decreases on the interval [0, 1] and, consequently, for any point x, $0 < x \leqslant 1$, the inequality $f(x) < f(0) = 0$ or

$$\arctan x - x + \frac{x^3}{6} < 0$$

is fulfilled, whence

$$\arctan x < x - \frac{x^3}{6}.$$

3.5.5. Prove the inequalities

$$x - x^3/6 < \sin x < x \quad \text{at} \quad x > 0.$$

3.5.6. Prove that for $0 \leqslant p \leqslant 1$ and for any positive a and b the inequality $(a+b)^p \leqslant a^p + b^p$ is valid.

Solution. By dividing both sides of the inequality by b^p we get

$$\left(\frac{a}{b}+1\right)^p \leqslant \left(\frac{a}{b}\right)^p + 1$$

or

$$(1+x)^p \leqslant 1 + x^p, \tag{*}$$

where $x = \frac{a}{b}$.

Let us show that the inequality (*) holds true at any positive x. Introduce the function

$$f(x) = 1 + x^p - (1+x)^p; \quad x \geqslant 0.$$

The derivative of this function

$$f'(x) = px^{p-1} - p(1+x)^{p-1} = p\left[\frac{1}{x^{1-p}} - \frac{1}{(1+x)^{1-p}}\right]$$

is positive everywhere, since, by hypothesis, $1-p \geqslant 0$ and $x > 0$. Hence, the function increases in the half-open interval $[0, \infty)$, i.e. $f(x) = 1 + x^p - (1+x)^p > f(0) = 0$, whence $1 + x^p > (1+x)^p$, which completes the proof. If we put $p = 1/n$, then we obtain

$$\sqrt[n]{a+b} \leqslant \sqrt[n]{a} + \sqrt[n]{b} \quad (n \geqslant 1).$$

3.5.7. Prove that the function $y = x^5 + 2x^3 + x$ increases everywhere, and the function $y = 1 - x^3$ decreases everywhere.

3.5.8. Determine the intervals of increase and decrease for the following functions:

(a) $f(x) = x^3 + 2x - 5$; (b) $f(x) = \ln(1 - x^2)$;

(c) $f(x) = \cos x - x$; (d) $f(x) = \frac{1}{3}x^3 - \frac{1}{x}$;

(e) $f(x)=\frac{2x}{\ln x}$; (f) $f(x)=\frac{2x}{1+x^2}$.

3.5.9. Prove the following inequalities:
(a) $\tan x > x + x^3/3$, if $(0 < x < \pi/2)$;
(b) $e^x \geqslant 1+x$ for all values of x;
(c) $e^x > ex$ at $x > 1$.

3.5.10. At what values of the coefficient a does the function $f(x)=x^3-ax$ increase along the entire number scale?

?.5.11. At what value of b does the function

$$f(x)=\sin x - bx + c$$

decrease along the entire number scale?

§ 3.6. Maxima and Minima of a Function

If a function $y=f(x)$ is defined on the interval X, then an interior point x_0 of this interval is called the *point of maximum* of the function $f(x)$ [the *point of minimum* of the function $f(x)$] if there exists a neighbourhood $U \in X$ of the point x_0, such that the inequality $f(x) \leqslant f(x_0)$ [$f(x) \geqslant f(x_0)$] holds true within it.

The generic terms for points of maximum and minimum of a function are the *points of extremum.*

A Necessary Condition for the Existence of an Extremum. At points of extremum the derivative $f'(x)$ is equal to zero or does not exist.

The points at which the derivative $f'(x)=0$ or does not exist are called *critical points.*

Sufficient Conditions for the Existence of an Extremum.

I. Let the function $f(x)$ be continuous in some neighbourhood of the point x_0.

1. If $f'(x)>0$ at $x<x_0$ and $f'(x)<0$ at $x>x_0$ (i.e. if in moving from left to right through the point x_0 the derivative changes sign from plus to minus), then at the point x_0 the function reaches a maximum.

2. If $f'(x)<0$ at $x<x_0$ and $f'(x)>0$ at $x>x_0$ (i.e. if in moving through the point x_0 from left to right the derivative changes sign from minus to plus), then at the point x_0 the function reaches a minimum.

3. If the derivative does not change sign in moving through the point x_0, then there is no extremum.

II. Let the function $f(x)$ be twice differentiable (that is $f'(x_0)=0$) at a critical point x_0. If $f''(x_0)<0$, then at x_0 the function has a maximum; if $f''(x_0)>0$, then at x_0 the function has a minimum; but if $f''(x_0)=0$, then the question of the existence of an extremum at this point remains open.

III. Let $f'(x_0)=f''(x_0)=\ldots=f^{(n-1)}(x_0)=0$, but $f^{(n)}(x_0)\neq 0$. If n is even, then at $f^{(n)}(x_0)<0$ there is a maximum at x_0, and at $f^{(n)}(x_0)>0$, a minimum.

If n is odd, then there is no extremum at the point x_0.

IV. Let a function $y=f(x)$ be represented parametrically:

$$x=\varphi(t),\quad y=\psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ have derivatives both of the first and second orders within a certain interval of change of the argument t, and $\varphi'(t)\neq 0$. Further, let, at $t=t_0$

$$\psi'(t)=0.$$

Then:

(a) if $\psi''(t_0)<0$, the function $y=f(x)$ has a maximum at $x=$ $=x_0=\varphi(t_0)$;

(b) if $\psi''(t_0)>0$, the function $y=f(x)$ has a minimum at $x=$ $=x_0=\varphi(t_0)$;

(c) if $\psi''(t_0)=0$, the question of the existence of an extremum remains open.

The points at which $\varphi'(t)$ vanishes require a special study.

3.6.1. Using the first derivative, find the extrema of the following functions:

(a) $f(x)=\frac{3}{4}x^4-x^3-9x^2+7$;

(b) $f(x)=x^4-8x^3+22x^2-24x+12$;

(c) $f(x)=x(x+1)^3(x-3)^2$;

(d) $f(x)=\frac{x^2-3x+2}{x^2+2x+1}$.

Solution. (a) The function is defined and differentiable over the entire number scale. Therefore, only the real roots of the derivative

$$f'(x)=3x^3-3x^2-18x=3x(x+2)(x-3)$$

are critical points. Equating this expression to zero, we find the critical points: $x_1=-2$, $x_2=0$, $x_3=3$ (they should always be arranged in an increasing order). Let us now investigate the sign of the derivative in the neighbourhood of each of these points. Since there are no critical points to the left of the point $x=-2$, the derivative at all the points $x<-2$ has one and the same sign: it is negative. Analogously, in the interval $(-2, 0)$ the derivative is positive, in the interval $(0, 3)$ it is negative, at $x>3$ it is positive. Hence, at the points $x_1=-2$ and $x_3=3$ we have minima $f(-2)=-9$ and $f(3)=-40\frac{1}{4}$, and at the point $x_2=0$, maximum $f(0)=7$.

(c) Just as in item (a), the critical points are the roots of the derivative $f'(x)$, since the function is defined and differentiable throughout the number scale. Find $f'(x)$:

$$f'(x) = (x+1)^3 (x-3)^2 + 3x (x+1)^2 (x-3)^2 + 2x (x+1)^3 \times \\ \times (x-3) = 3 (x+1)^2 (x-3) (2x^2 - 3x - 1).$$

Equating this expression to zero, we find the critical points:

$$x_1 = -1, \quad x_2 = (3-\sqrt{17})/4, \quad x_3 = (3+\sqrt{17})/4, \quad x_4 = 3.$$

Let us tabulate the signs of the derivative in the intervals between the critical points:

Intervals	$x < x_1$	$x_1 < x < x_2$	$x_2 < x < x_3$	$x_3 < x < x_4$	$x_4 < x$
Sign of $f'(x)$	$-$	$-$	$+$	$-$	$+$

As is seen from the table, there is no extremum at the point $x_1 = -1$, there is a minimum at the point x_2, a maximum at the point x_3, and a minimum at the point x_4.

3.6.2. Using the first derivative, find the extrema of the following functions:

(a) $f(x) = 3\sqrt[3]{x^2} - x^2$;

(b) $f(x) = \sqrt[3]{(x-1)^2} + \sqrt[3]{(x+1)^2}$.

Solution. (a) The function is defined and continuous throughout the number scale.

Let us find the derivative:

$$f'(x) = 2\left(\frac{1}{\sqrt[3]{x}} - x\right).$$

From the equation $f'(x) = 0$ we find the roots of the derivative: $x = \pm 1$.

Furthermore, the derivative goes to infinity at the point $x = 0$. Thus, the critical points are $x_1 = -1$, $x_2 = 0$, $x_3 = 1$. The results of investigating the sign of the derivative in the neighbourhood of these points are given in Fig. 40. The investigation shows that the function has two maxima: $f(-1) = 2$; $f(1) = 2$ and a minimum $f(0) = 0$.

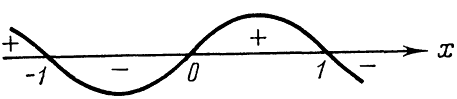


Fig. 40

3.6.3. Using the second derivative, find out the character of the extrema of the following functions:

(a) $y = 2\sin x + \cos 2x$;
(b) $f(x) = 2x^3 - 15x^2 - 84x + 8$.

Solution. (a) Since the function is a periodic one we may confine ourselves to the interval $[0, 2\pi]$. Find the first and second derivatives:

$y' = 2\cos x - 2\sin 2x = 2\cos x(1 - 2\sin x)$;
$y'' = -2\sin x - 4\cos 2x$.

From the equation $2\cos x(1 - 2\sin x) = 0$ determine the critical points on the interval $[0, 2\pi]$:

$$x_1 = \pi/6, \quad x_2 = \pi/2, \quad x_3 = 5\pi/6, \quad x_4 = 3\pi/2.$$

Now find the sign of the second derivative at each critical point:

$y''(\pi/6) = -3 < 0$; hence, we have a maximum $y(\pi/6) = 3/2$ at the point $x_1 = \pi/6$;

$y''(\pi/2) = 2 > 0$; hence, we have a minimum $y(\pi/2) = 1$ at the point $x_2 = \pi/2$;

$y''(5\pi/6) = -3 < 0$; hence, we have a maximum $y(5\pi/6) = 3/2$ at the point $x_3 = 5\pi/6$;

$y''(3\pi/2) = 6 > 0$; hence, we have a minimum $y(3\pi/2) = -3$ at the point $x_4 = 3\pi/2$ (see Fig. 41).

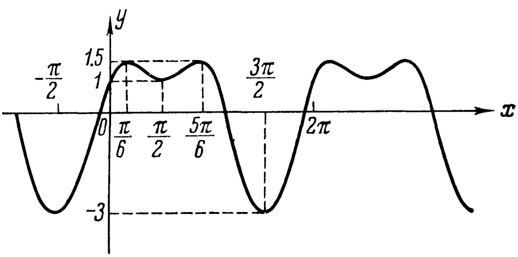


Fig. 41

3.6.4. Investigate the following functions for extrema:

(a) $f(x) = \begin{cases} -2x & (x < 0), \\ 3x + 5 & (x \geqslant 0); \end{cases}$

(b) $f(x) = \begin{cases} 2x^2 + 3 & (x \neq 0), \\ 4 & (x = 0). \end{cases}$

Solution. (a) Though the derivative

$$f'(x) = \begin{cases} -2 & (x < 0), \\ 3 & (x > 0) \end{cases}$$

exists at all points, except the point $x = 0$, and changes sign from minus to plus when passing through the point $x = 0$, there is no minimum here:

$$f(0) = 5 > f(x) \text{ at } -1 < x < 0.$$

This is explained by the fact that the function is discontinuous at the point $x=0$.

(b) Here the derivative $f'(x)=4x\,(x\neq 0)$ also exists at all points, except at $x=0$, and it changes sign from minus to plus when passing through the point $x=0$. Nevertheless, we have here a maximum but not a minimum, which can readily be checked.

It is explained by the fact that the function is discontinuous at the point $x=0$.

3.6.5. Find the extrema of the following functions:

(a) $f(x)=\dfrac{50}{3x^4+8x^3-18x^2+60}$;

(b) $f(x)=\sqrt{e^{x^2}-1}$.

Solution. (a) Here it is simpler to find the extrema of the function $f_1(x)=3x^4+8x^3-18x^2+60$. Since

$$f_1'(x)=12x^3+24x^2-36x=12x\,(x^2+2x-3),$$
$$f_1''(x)=12\,(3x^2+4x-3),$$

the critical points are:

$$x_1=-3,\quad x_2=0,\quad x_3=1,$$

and the character of the extrema is readily determined from the sign of the second derivative $f_1''(-3)>0$; hence, at the point $x_1=-3$ the function $f_1(x)$ has a minimum, and the given function $f(x)$ obviously has a maximum $f(-3)=-2/3$, $f_1''(0)<0$; hence, at the point $x_2=0$ the function $f_1(x)$ has a maximum, and $f(x)$ a minimum $f(0)=5/6$; $f_1''(1)>0$; hence, at the point $x_3=1$ the function $f_1(x)$ has a minimum, and $f(x)$ a maximum $f(1)=50/53$.

(b) In this case it is easier to find the points of extremum of the radicand

$$f_1(x)=e^{x^2}-1,$$

which coincide with the points of extremum of the function $f(x)$. Let us find the critical points of $f_1(x)$:

$f_1'(x)=2xe^{x^2}$; $f_1'(x)=0$ at the point $x=0$. Determine the sign of the second derivative at the point $x=0$:

$$f_1''(x)=2e^{x^2}(1+2x^2),\quad f_1''(0)=2>0.$$

Therefore the point $x=0$ is a minimum of the function $f_1(x)$; it will also be a minimum of the given function $f(x)$: $f(0)=0$.

3.6.6. Investigate the character of the extremum of the function $y=\cosh x+\cos x$ at the point $x=0$.

Solution. The function y is an even one and apparently has an extremum at the point $x=0$. To determine the character of the extremum let us evaluate the derivatives of this function at the point $x=0$:

$$\begin{aligned} y' &= \sinh x - \sin x, \quad y'(0)=0; \\ y'' &= \cosh x - \cos x, \quad y''(0)=0; \\ y''' &= \sinh x + \sin x, \quad y'''(0)=0; \\ y^{(4)} &= \cosh x + \cos x; \quad y^{(4)}(0)=2>0. \end{aligned}$$

Since the first non-zero derivative at the point $x=0$ is a derivative of an even order, which takes on a positive value, we have a minimum $y(0)=2$ at this point.

3.6.7. Investigate the following functions for an extremum at the point $x=0$:

(a) $y=\cos x-1+\frac{x^2}{2!}-\frac{x^3}{3!}$; (b) $y=\cos x-1+\frac{x^2}{2}$.

Solution. (a) $y' = -\sin x + x - \frac{x^2}{2}$; $y'(0)=0$;

$$\begin{aligned} y'' &= -\cos x + 1 - x; \quad y''(0)=0; \\ y''' &= \sin x - 1; \quad y'''(0) = -1 \neq 0. \end{aligned}$$

And so, the first non-zero derivative at the point $x=0$ is a derivative of the third order, i. e. of an odd order; this means that there is no extremum at the point $x=0$.

3.6.8. Investigate the following functions for extrema:

(a) $f(x)=x^4e^{-x^2}$; (b) $f(x)=\sin 3x-3\sin x$.

Solution. (a) The function $f(x)=x^4e^{-x^2}$ is continuously differentiable everywhere. Equating the derivative

$$f'(x)=4x^3e^{-x^2}-2x^5e^{-x^2}=x^3e^{-x^2}(4-2x^2)$$

to zero, find the critical points:

$$x_1=-\sqrt{2}; \quad x_2=0; \quad x_3=\sqrt{2}.$$

Compute the values of the second derivative at the critical points:

$$\begin{aligned} f''(x) &= 12x^2e^{-x^2}-8x^4e^{-x^2}-10x^4e^{-x^2}+4x^6e^{-x^2}= \\ &= 2x^2e^{-x^2}(6-9x^2+2x^4); \end{aligned}$$

$$f''(0)=0; \quad f''(-\sqrt{2})<0; \quad f''(\sqrt{2})<0.$$

Consequently, at the points $x_1=-\sqrt{2}$ and $x_3=+\sqrt{2}$ the function reaches a maximum $f(\pm\sqrt{2})=4e^{-2}=\frac{4}{e^2}$. As far as the critical point $x_2=0$ is concerned, nothing definite can be said as yet, we have to find derivatives of $f(x)$ of higher orders (up to the fourth order!). But this process is cumbersome, therefore we will turn to the first sufficient condition of an extremum: let us find the signs of the first derivative in the neighbourhood of the critical point $x_2 = 0$:

$$f'(-1) < 0; \; f'(1) > 0.$$

Hence, at the point $x = 0$ the function has a minimum $f(0) = 0$.

3.6.9. The function $y = f(x)$ is represented parametrically:

$$\begin{cases} x = \varphi(t) = t^5 - 5t^3 - 20t + 7, \\ y = \psi(t) = 4t^3 - 3t^2 - 18t + 3 \quad (-2 < t < 2). \end{cases}$$

Find the extrema of this function.

Solution. We have

$$\varphi'(t) = 5t^4 - 15t^2 - 20.$$

In the interval $(-2, 2)$ $\varphi'(t) \neq 0$.

Find $\psi'(t)$ and equate it to zero:

$$\psi'(t) = 12t^2 - 6t - 18 = 0.$$

Whence $t_1 = -1$ and $t_2 = 3/2$.

These roots are interior points of the considered interval of variation of the parameter t.

Furthermore:

$$\psi''(t) = 24t - 6; \quad \psi''(-1) = -30 < 0, \quad \psi''(3/2) = 30 > 0.$$

Consequently, the function $y = f(x)$ has a maximum $y = 14$ at $t = -1$ (i. e. at $x = 31$) and a minimum $y = -17.25$ at $t = 3/2$ (i. e. at $x = -1033/32$).

3.6.10. Find the maxima and minima of the following functions:

(a) $f(x) = x^2 e^{-x}$; (b) $f(x) = \dfrac{4x}{x^2+4}$;

(c) $f(x) = -x^2 \sqrt[5]{(x-2)^2}$; (d) $f(x) = \dfrac{14}{x^4 - 8x^2 + 2}$;

(e) $f(x) = \sqrt[3]{2x^3 + 3x^2 - 36x}$;

(f) $f(x) = x^2 \ln x$; (g) $f(x) = x \ln^2 x$.

3.6.11. Investigate the following functions for an extremum at the point $x = 0$:

(a) $f(x) = \sin x - x$; (b) $f(x) = \sin x - x + x^3/3$;

(c) $f(x) = \sin x - x + \dfrac{x^3}{3!} - \dfrac{x^4}{4!}$;

(d) $f(x) = \begin{cases} e^{1/x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$

§ 3.7. Finding the Greatest and the Least Values of a Function

The greatest (least) value of a continuous function $f(x)$ on an interval $[a, b]$ is attained either at the critical points, or at the end-points of the interval. To find the greatest (least) value of the function we have to compute its values at all the critical points on the interval $[a, b]$, the values $f(a)$, $f(b)$ of the function at the end-points of the interval and choose the greatest (least) one out of the numbers obtained.

If a function is defined and continuous in some interval, and if this interval is not a closed one, then it can have neither the greatest nor the least value.

3.7.1. Find the greatest and the least values of the following functions on the indicated intervals:

(a) $f(x) = 2x^3 - 3x^2 - 12x + 1$ on $[-2, 5/2]$;

(b) $f(x) = x^2 \ln x$ on $[1, e]$;

(c) $f(x) = xe^{-x}$ on $[0, +\infty]$;

(d) $f(x) = \sqrt{(1-x^2)(1+2x^2)}$ on $[-1, 1]$.

Solution. (a) Find the derivative $f'(x)$:

$$f'(x) = 6x^2 - 6x - 12.$$

It vanishes at two points: $x_1 = -1$ and $x_2 = 2$. They both lie inside the indicated interval $\left[-2, \frac{5}{2}\right]$; consequently both of them must be taken into consideration. To find the extreme values of the function it is necessary to compute its values at the points x_1 and x_2, and also at the end-points of the segment:

$$f(-2) = -3, \ f(-1) = 8; \quad f(2) = -19, \ f\left(\frac{5}{2}\right) = -16\frac{1}{2}.$$

Hence, the greatest value is $f(-1) = 8$ and the least $f(2) = -19$.

(b) Find the critical points: $f'(x) = x(1 + 2\ln x)$. The derivative $f'(x)$ does not vanish inside the given interval $[1, e]$. Therefore there are no critical points inside the indicated interval. It now remains to compute the values of the function at the end-points of the interval $[1, e]$

$$f(1) = 0; \quad f(e) = e^2.$$

Thus, $f(1) = 0$ is the least value of the function and $f(e) = e^2$ the greatest.

3.7.2. Find the greatest and the least values of the following functions on the indicated intervals:

(a) $y = \sin x \sin 2x$ on $(-\infty, \infty)$;

(b) $y=\operatorname{arc}\cos x^2$ on $[-\sqrt{2}/2, \sqrt{2}/2]$;

(c) $y=x+\sqrt{x}$ on $[0, 4]$.

Solution. (a) Represent the function $y=\sin x \sin 2x$ in the form

$$y=\frac{\cos x-\cos 3x}{2},$$

whence it is seen that the function is an even one and has a period 2π. Hence, it is sufficient to seek the greatest and the least values among the extrema on the interval $[0, \pi]$. Find the derivative y':

$$y'=\frac{1}{2}(3\sin 3x-\sin x).$$

In $[0, \pi]$ the derivative vanishes at the points

$$x_1=0,\quad x_2=\operatorname{arc}\cos\frac{1}{\sqrt{3}},\quad x_3=\operatorname{arc}\cos\left(-\frac{1}{\sqrt{3}}\right),\quad x_4=\pi.$$

Compute the values of the function at these points:

$$y(0)=y(\pi)=0,\quad y\left[\operatorname{arc}\cos\left(\pm\frac{1}{\sqrt{3}}\right)\right]=\pm\frac{4}{3\sqrt{3}}.$$

Hence, the least value of the function in the interval $(-\infty, \infty)$ is equal to $-4/(3\sqrt{3})$, and the greatest to $4/(3\sqrt{3})$.

3.7.3. The function

$$f(x)=ax+\frac{b}{x}\quad (a, b, x>0)$$

consists of two summands: one summand is proportional to the independent variable x, the other inversely proportional to it. Prove that this function takes on the least value at $x=\sqrt{b/a}$.

Solution. Find the roots of the derivative $f'(x)$ in the interval $(0, \infty)$:

$$f'(x)=a-\frac{b}{x^2}=0$$

at $x=\sqrt{b/a}\,(x>0)$. Since $f''(x)=2b/x^3>0$ for any $x>0$, the function $f(x)$ reaches a minimum at this critical point. This is the only extremum (minimum) in the interval $(0, \infty)$. Hence, at $x=\sqrt{b/a}$ the function $f(x)$ attains the least value.

3.7.4. As a result of n measurements of an unknown quantity x the numbers $x_1, x_2, \ldots, x_n$ are obtained.

It is required to find at what value of x the sum of the squares of the errors

$$f(x)=(x-x_1)^2+(x-x_2)^2+\ldots+(x-x_n)^2$$

will be the least.

Solution. Compute the derivative

$$f'(x)=2(x-x_1)+2(x-x_2)+\ldots+2(x-x_n).$$

The only root of the derivative is

$$x=\frac{x_1+x_2+\ldots+x_n}{n}.$$

Then, for all x we have $f''(x)=2n>0$. Therefore, the function $f(x)$ has its minimum at the point

$$x=\frac{x_1+x_2+\ldots+x_n}{n}.$$

Being the only minimum, it coincides with the least value of the function (cf. Problem **1.3.8**).

And so, the best (in the sense of "the principle of the minimum squares") approximate value of an unknown quantity x is the arithmetic mean of the values $x_1, x_2, \ldots, x_n$.

3.7.5. Find the largest term in the sequence

$$a_n=\frac{n^2}{n^3+200}.$$

Solution. Consider the function $f(x)=\frac{x^2}{x^3+200}$ in the interval $[1, \infty)$. Since the derivative

$$f'(x)=\frac{x(400-x^3)}{(x^3+200)^2}$$

is positive at $0<x<\sqrt[3]{400}$ and negative at $x>\sqrt[3]{400}$, the function $f(x)$ increases at $0<x<\sqrt[3]{400}$ and decreases at $x>\sqrt[3]{400}$. From the inequality $7<\sqrt[3]{400}<8$ it follows that the largest term in the sequence can be either a_7 or a_8. Since $a_7=49/543>a_8=8/89$, the largest term in the given sequence is

$$a_7=\frac{49}{543}.$$

3.7.6. Find the greatest and the least values of the following functions on the indicated intervals:

(a) $f(x)=\frac{1}{4}x^4-\frac{2}{3}x^3-\frac{3}{2}x^2+2$ on $[-2, 4]$;

(b) $f(x)=\sqrt{4-x^2}$ on $[-2, 2]$;

(c) $f(x)=\arctan x-\frac{1}{2}\ln x$ on $\left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$;

(d) $f(x)=2\sin x+\sin 2x$ on $\left[0, \frac{3}{2}\pi\right]$;

(e) $f(x) = x - 2\ln x$ on $[1, e]$;

(f) $f(x) = \begin{cases} 2x^2 + \dfrac{2}{x^2} & \text{for } -2 \leqslant x < 0;\ 0 < x \leqslant 2, \\ 1 & \text{for } x = 0. \end{cases}$

§ 3.8. Solving Problems in Geometry and Physics

3.8.1. The force of a circular electric current acting on a small magnet with the axis perpendicular to the plane of the circle and passing through its centre is expressed by the formula

$$F = \frac{Cx}{(a^2 + x^2)^{3/2}},$$

where a = radius of the circle

x = distance from the centre of the circle to the magnet $(0 < x < \infty)$

C = constant.

At what x will the value of F be the greatest?

Solution. The derivative

$$F'(x) = C\frac{a^2 - 2x^2}{(a^2 + x^2)^{5/2}}$$

has a single positive root $x = a/\sqrt{2}$. This solves the problem.

Note. It often happens that reasons of purely physical or geometric character make it unnecessary to resort to the differential methods in investigating a function for the greatest or the least value at the point under consideration.

3.8.2. Determine the most economical dimensions of an open-air swimming pool of volume 32 m³ with a square bottom so that the facing of its walls and bottom require the least quantity of material.

Solution. Let us denote the side of the bottom by x and the height by y. Then the volume V of the pool will be

$$V = x^2 y = 32, \qquad (*)$$

and the surface S to be faced

$$S = x^2 + 4xy.$$

Expressing y through x from the relation (*), we get

$$S = x^2 + 4x\frac{V}{x^2} = x^2 + \frac{128}{x}.$$

Investigate the function thus obtained for a minimum in the interval $(0, \infty)$:

$$S' = 2x - \frac{128}{x^2}; \quad 2x - \frac{128}{x^2} = 0; \quad x = 4.$$

The single point thus found will obviously yield the least value of the function S, since it has no greatest value (it increases unboundedly as $x \to 0$ and $x \to \infty$).

And so, the required dimensions of the pool are: $x = 4$ m, $y = 2$ m.

3.8.3. Inscribe into a given sphere a cylinder with the greatest lateral surface.

3.8.4. 20 m of wire is available for fencing off a flower-bed which should have the form of a circular sector. What must the radius of the circle be if we wish to have a flower-bed of the greatest possible surface area?

Solution. Let us denote the radius of the circle by x, and the length of the arc by y (see Fig. 42). Then

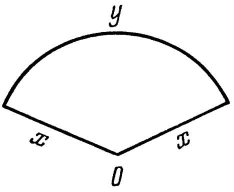


Fig. 42

$$20 = 2x + y,$$

whence

$$y = 2(10 - x).$$

The area of the circular sector $S = \frac{1}{2}xy = x(10 - x)$ $(0 \leqslant x \leqslant 10)$.

The derivative $S'(x) = 10 - 2x$ has a root $x = 5$.

Since the least value $S = 0$ is reached at the end-points of the interval $[0, 10]$, the obtained value $x = 5$ yields the greatest surface area S.

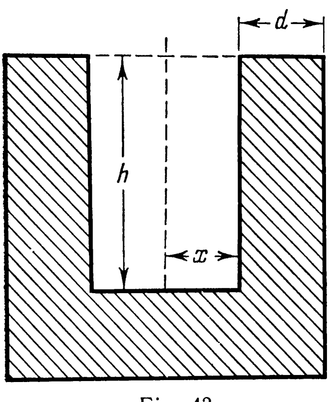


Fig. 43

3.8.5. It is required to construct an open cylindrical reservoir of capacity V_0. The thickness of the material is d. What dimensions (the base radius and height) should the reservoir have so as to ensure the least possible expenditure of the material?

Solution. Figure 43 represents a longitudinal section of the reservoir, where the radius of the base of the inner cylinder is denoted by x and the height of the inner cylinder, by h. The volume of the bottom and the wall of the reservoir

$$V = \pi (x+d)^2 d + \pi [(x+d)^2 - x^2] h = \pi d (x+d)^2 + \pi h (2xd + d^2). \qquad (*)$$

On the other hand, by hypothesis we must have

$$V_0 = \pi x^2 h$$

whence

$$h = \frac{V_0}{\pi x^2}.$$

Substituting into (*), we get

$$V=\pi d(x+d)^2+\frac{\pi V_0}{\pi x^2}(2xd+d^2)=\pi d(x+d)^2+\frac{2V_0 d}{x}+\frac{V_0 d^2}{x^2}.$$

Now we have to investigate the obtained function $V(x)$ for an extremum at $x>0$.

We have

$$V'(x)=2\pi d(x+d)-\frac{2V_0 d}{x^2}-\frac{2V_0 d^2}{x^3}=\frac{2d(x+d)(\pi x^3-V_0)}{x^3}.$$

The only positive root of the derivative is the point $x=\sqrt[3]{V_0/\pi}$. This solves the problem:

$$h=\frac{V_0\sqrt[3]{\pi^2}}{\pi\sqrt[3]{V_0^2}}=\sqrt[3]{\frac{V_0}{\pi}}=x.$$

3.8.6. A factory D is to be connected by a highway with a straight railway on which a town A is situated. The distance DB from the factory to the railway is equal to a, the segment AB of the railway equals l. Freight charges on the highway are m times higher than on the railway ($m>1$).

How should the highway DP be connected with the railway so as to ensure the least freight charges from factory to town?

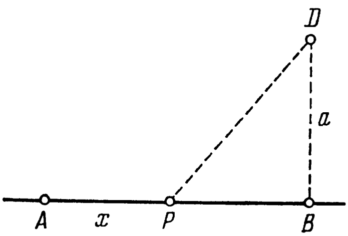


Fig. 44

Solution. First, let us make a drawing (see Fig. 44). It is absolutely clear that the highway must also be straight (a straight line is shorter than any curve connecting two given points!). Furthermore, the point P cannot lie either to the left of the point A or to the right of the point B. If we denote the distance AP by x, it will mean that $0\leqslant x\leqslant l$.

Let the freight charges on the railway (per ton-kilometre) be k, then the freight charges on the highway will be km. The total freight charge N for transporting loads from D to A amounts to

$$N=kx+km\sqrt{a^2+(l-x)^2}.$$

Hence, we have to find the least value of the function

$$f(x)=x+m\sqrt{a^2+(x-l)^2},\quad 0\leqslant x\leqslant l.$$

Take the derivative

$$f'(x)=1+\frac{m(x-l)}{\sqrt{a^2+(x-l)^2}}.$$

It vanishes only at one point:

$$x = l - \frac{a}{\sqrt{m^2-1}}.$$

If this point lies in the interval $[0, l]$, i.e. if

$$l \geqslant \frac{a}{\sqrt{m^2-1}} \quad \text{or} \quad \frac{a}{l} \leqslant \sqrt{m^2-1},$$

then it yields the least freight charge (which is easy to check). If the indicated inequality is not observed, then $f(x)$ increases on $[0, l]$ and therefore the least freight charge is obtained at $x=0$.

3.8.7. In constructing an a-c transformer it is important to insert into the coil a cross-shaped iron core of greatest possible surface area. Fig. 45 shows the cross-section of the core with appropriate dimensions. Find the most suitable x and y if the radius of the coil is equal to a.

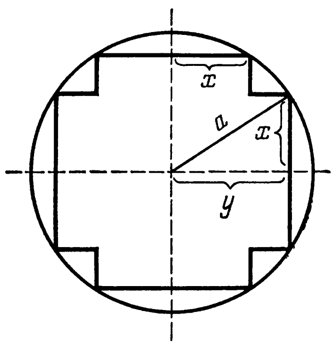


Fig. 45

3.8.8. If the source of current is an electric cell, then the effect P (watts) obtained by cutting a resistance R (ohms) in the circuit is expressed by the formula

$$P = \frac{E^2 R}{(R+R_i)^2},$$

where E is electromotive force in volts and R_i the internal resistance in ohms.

Find the greatest effect which can be obtained at given E and R_i.

3.8.9. A tin of a given volume V has the form of a cylinder. What must be the ratio of its height h to diameter $2R$ so as to use the least amount of material for its manufacture?

3.8.10. In a given cone inscribe a cylinder having the greatest lateral surface so that the planes and centres of the base circles of the cylinder and cone coincide.

3.8.11. Given a point (1, 2) in the orthographic coordinates. Through this point draw a straight line so that it forms, together with the positive semi-axes, a triangle of the least area.

3.8.12. Given a point M on the axis of the parabola $y^2 = 2px$ at a distance a from its vertex. Find the abscissa of the point on the curve nearest to the given point.

3.8.13. The expenses sustained in one hour's sailing of a ship are expressed in roubles by an empirical formula of the form $a+bv^3$, where a and b are constants for a given ship, and v is the ship's speed in knots (one knot is equal to 1.85 km/hr). In this formula the constant part of the expenses a refers to depreciation and crew's upkeep, and the second term (bv^3) to the fuel cost. At what speed will the ship cover any required distance at the lowest cost?

3.8.14. A trough is built from three boards of equal width. At what slope should the lateral boards be placed to ensure the largest cross-sectional area of the trough?

3.8.15. A tank with a vertical wall of height h is installed on a horizontal plane. Determine the position of an orifice, at which the range of a liquid jet will be the greatest if the velocity of flow (according to Torricelli's law) is equal to $\sqrt{2gx}$, where x is the depth of the orifice.

3.8.16. Two aircraft are flying in a straight line and in the same plane at an angle of 120° to each other and with an equal speed of v km/hr. At a certain moment one aircraft reaches the point of intersection of their routes, while the second is at a distance of a km from it. When will the distance between the aircraft be the least and what is that distance?

§ 3.9. Convexity and Concavity of a Curve. Points of Inflection

If $f''(x) < 0$ (> 0) on an interval (a, b), then the curve $y=f(x)$ on this interval is convex (concave), i.e. it is situated below (above) any of its tangent lines.

If $f''(x_0)=0$ or does not exist but $f'(x_0)$ does exist and the second derivative $f''(x)$ changes sign when passing through the point x_0, then the point $(x_0, f(x_0))$ is the point of inflection of the curve $y=f(x)$.

3.9.1. Find the intervals in which the graphs of the following functions are concave or convex and locate the points of inflection:

(a) $y=x^4+x^3-18x^2+24x-12$;

(b) $y=3x^4-8x^3+6x^2+12$;

(c) $y=\dfrac{x}{1+x^2}$;

(d) $y=x+x^{5/3}$;

(e) $y=4\sqrt{(x-1)^5}+20\sqrt{(x-1)^3}$ $(x \geqslant 1)$;

(f) $y=\dfrac{\ln^2 x}{x}$ $(x>0)$;

(g) $y = x\sin(\ln x)$ $(x > 0)$;
(h) $y = 2 - |x^5 - 1|$.

Solution. (a) Find the derivatives:

$$y' = 4x^3 + 3x^2 - 36x + 24,$$
$$y'' = 12x^2 + 6x - 36 = 12\left(x^2 + \frac{x}{2} - 3\right),$$

whence $y'' = 0$ at $x_1 = -2$, $x_2 = 3/2$.

Hence, $y'' > 0$ on the intervals $(-\infty, -2)$ and $(3/2, \infty)$; $y'' < 0$ on the interval $(-2, 3/2)$. The sign of the second derivative determines the convexity or concavity of the curve in a given interval. This enables us to compile the following table:

x	$x < -2$	$-2 < x < \frac{3}{2}$	$x > \frac{3}{2}$
Sign of y''	$+$	$-$	$+$
Conclusion	Concavity	Convexity	Concavity

Since the second derivative changes its sign when passing through the points $x_1 = -2$ and $x_2 = 3/2$, the points $(-2, -124)$ and $\left(\frac{3}{2}, -8\frac{1}{16}\right)$ are points of inflection.

(d) Find the derivatives:

$$y' = 1 + \frac{5}{3}x^{2/3}, \quad y'' = \frac{10}{9\sqrt[3]{x}}.$$

The second derivative is non-zero everywhere and loses its meaning at the point $x = 0$. At $x < 0$ we have $y'' < 0$ and the curve is convex, at $x > 0$ we have $y'' > 0$ and the curve is concave.

At the point $x = 0$ the first derivative $y' = 1$, the second derivative changes sign when passing through the point $x = 0$. Therefore the point $(0, 0)$ is a point of inflection.

(g) Find the derivatives:

$$y' = \sin(\ln x) + \cos(\ln x),$$
$$y'' = \frac{1}{x}[\cos(\ln x) - \sin(\ln x)] = \frac{\sqrt{2}}{x}\sin\left(\frac{\pi}{4} - \ln x\right).$$

The second derivative vanishes at the points

$$x_k = e^{\pi/4 + k\pi}, \quad k = 0, \pm 1, \pm 2, \ldots .$$

The function $\sin(\pi/4 - \ln x)$, and together with it y'', changes sign when passing through each point x_k. Consequently, the points x_k

are the abscissas of the points of inflection. In the intervals

$$(e^{2k\pi-3\pi/4}, \quad e^{2k\pi+\pi/4})$$

the curve is concave, and in the intervals

$$(e^{2k\pi+\pi/4}, \quad e^{2k\pi+5\pi/4})$$

it is convex.

(h) The given function can be written in the following way:

$$y=\begin{cases} 2-(x^5-1), & x \geqslant 1, \\ 2+(x^5-1), & x<1. \end{cases}$$

Therefore

$$y'=\begin{cases} -5x^4, & x>1, \\ 5x^4, & x<1. \end{cases}$$

At the point $x=1$ there is no derivative. Further,

$$y''=\begin{cases} -20x^3, & x>1, \\ 20x^3, & x<1; \end{cases}$$

$y''=0$ at the point $x=0$. Hence, we have to investigate three intervals: $(-\infty, 0)$, $(0, 1)$, $(1, \infty)$.

Compile a table of signs of y'':

x	$x<0$	$0<x<1$	$x>1$
Sign of y''	$-$	$+$	$-$
Conclusion	Convexity	Concavity	Convexity

The point $(0, 1)$ is a point of inflection, the point $(1, 2)$ being a corner point.

3.9.2. What conditions must the coefficients a, b, c satisfy for the curve $y=ax^4+bx^3+cx^2+dx+e$ to have points of inflection?

Solution. Find the second derivative:

$$y''=12ax^2+6bx+2c.$$

The curve has points of inflection if and only if the equation

$$6ax^2+3bx+c=0$$

has different real roots, i.e. when the discriminant $9b^2-24ac>0$, or

$$3b^2-8ac>0.$$

3.9.3. At what values of a will the curve

$$y = x^4 + ax^3 + \frac{3}{2}x^2 + 1$$

be concave along the entire number scale?

Solution. Find y'':

$$y'' = 12x^2 + 6ax + 3.$$

The curve will be concave along the entire number scale if $y'' \geqslant 0$ for all values of x, i.e. when

$$4x^2 + 2ax + 1 \geqslant 0 \quad \text{for all } x.$$

For this it is necessary and sufficient that the inequality $4a^2 - 16 \leqslant 0$ be fulfilled; whence

$$|a| \leqslant 2.$$

3.9.4. Show that the curve $y = \frac{x+1}{x^2+1}$ has three points of inflection lying in a straight line.

Solution. Find the derivatives:

$$y' = \frac{-x^2 - 2x + 1}{(x^2+1)^2},$$

$$y'' = \frac{2x^3 + 6x^2 - 6x - 2}{(x^2+1)^3}.$$

The second derivative becomes zero at three points, which are the roots of the equation

$$x^3 + 3x^2 - 3x - 1 = 0,$$

whence

$$x_1 = -2 - \sqrt{3}, \quad x_2 = -2 + \sqrt{3}, \quad x_3 = 1.$$

Let us compile the table of signs of y'':

x	$-\infty < x < -2-\sqrt{3}$	$-2-\sqrt{3} < x < -2+\sqrt{3}$	$-2+\sqrt{3} < x < 1$	$1 < x < \infty$
Sign of y''	$-$	$+$	$-$	$+$
Conclusion	Convexity	Concavity	Convexity	Concavity

Hence, $\left(-2-\sqrt{3}, -\frac{\sqrt{3}-1}{4}\right)$, $\left(-2+\sqrt{3}, \frac{1+\sqrt{3}}{4}\right)$, $(1, 1)$ are points of inflection. It is easy to ascertain that all of them lie in a straight line. Indeed, the coordinates of these points satisfy the relation $\frac{-2-\sqrt{3}-1}{-2+\sqrt{3}-1}=\frac{(1-\sqrt{3})/4-1}{(1+\sqrt{3})/4+1}$.

3.9.5. Investigate the curves represented by the following equations for convexity (concavity) and locate the points of inflection:

(a) $y=x-\sqrt[5]{(x-3)^2}$;
(b) $y=e^{\sin x}\,(-\pi/2\leqslant x\leqslant \pi/2)$.

3.9.6. Show that the points of inflection of the curve $y=x\sin x$ lie on the curve $y^2(4+x^2)=4x^2$.

§ 3.10. Asymptotes

A straight line is called an *asymptote* to the curve $y=f(x)$ if the distance from the variable point M of the curve to the straight line approaches zero as the point M recedes to infinity along some branch of the curve.

We will distinguish three kinds of asymptotes: vertical, horizontal and inclined.

Vertical asymptotes. If at least one of the limits of the function $f(x)$ (at the point a on the right or on the left) is equal to infinity, then the straight line $x=a$ is a vertical asymptote.

Horizontal asymptotes. If $\lim\limits_{x\to\pm\infty} f(x)=A$, then the straight line $y=A$ is a horizontal asymptote (the right one as $x\to+\infty$ and the left one as $x\to-\infty$).

Inclined asymptotes. If the limits

$$\lim_{x\to+\infty}\frac{f(x)}{x}=k_1,\qquad \lim_{x\to+\infty}[f(x)-k_1x]=b_1$$

exist, then the straight line $y=k_1x+b_1$ is an inclined (right) asymptote.

If the limits

$$\lim_{x\to-\infty}\frac{f(x)}{x}=k_2 \quad\text{and}\quad \lim_{x\to-\infty}[f(x)-k_2x]=b_2$$

exist, then the straight line $y=k_2x+b_2$ is an inclined (left) asymptote. A horizontal asymptote may be considered as a particular case of an inclined asymptote at $k=0$.

3.10.1. Find the asymptotes of the following curves:

(a) $y=\frac{5x}{x-3}$; (b) $y=\frac{3x}{x-1}+3x$; (c) $y=\frac{x}{x^2+1}$;

(d) $y=\frac{1}{x}+4x^2$; (e) $y=xe^{\frac{1}{x}}$; (f) $y=\frac{3x}{2}\ln\left(e-\frac{1}{3x}\right)$;

(g) $y=\sqrt{1+x^2}+2x$; (h) $y=\sqrt{1+x^2}\sin\frac{1}{x}$;

(i) $y=2\sqrt{x^2+4}$.

Solution. (a) The curve has a vertical asymptote $x=3$, since

$$\lim_{x\to 3\mp 0} y = \lim_{x\to 3\mp 0}\frac{5x}{x-3}=\mp\infty$$

(the point $x=3$ is a point of discontinuity of the second kind).
Find the horizontal asymptote:

$$\lim_{x\to\pm\infty} y = \lim_{x\to\pm\infty}\frac{5x}{x-3}=5.$$

And so, the curve has a vertical asymptote $x=3$ and a horizontal one $y=5$.

(b) The curve has a vertical asymptote $x=1$, since

$$\lim_{x\to 1-0} y = \lim_{x\to 1-0}\left(\frac{3x}{x-1}+3x\right)=-\infty;$$

$$\lim_{x\to 1+0} y = \lim_{x\to 1+0}\left(\frac{3x}{x-1}+3x\right)=+\infty.$$

Find the inclined asymptotes:

$$k=\lim_{x\to\pm\infty}\frac{y}{x}=\lim_{x\to\pm\infty}\left(\frac{3}{x-1}+3\right)=3;$$

$$b=\lim_{x\to\pm\infty}(y-kx)=$$

$$=\lim_{x\to\pm\infty}\left(\frac{3x}{x-1}+3x-3x\right)=3.$$

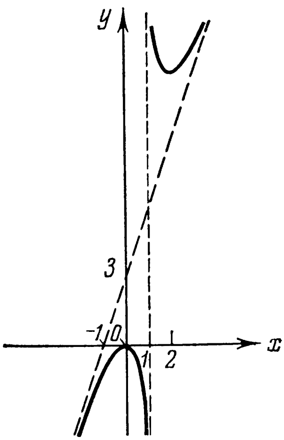


Fig. 46

Thus, the straight line $y=3x+3$ is an inclined asymptote (see Fig. 46).

(e) The curve has a vertical asymptote $x=0$, since

$$\lim_{x\to+0} y=\lim_{x\to+0} xe^{1/x}=\lim_{t=\frac{1}{x}\to+\infty}\frac{e^t}{t}=+\infty$$

(see Problem **3.2.2.**).
Find the inclined asymptotes:

$$k=\lim_{x\to\pm\infty}\frac{y}{x}=\lim_{x\to\pm\infty}e^{1/x}=e^0=1;$$

$$b=\lim_{x\to\pm\infty}(xe^{1/x}-x)=\lim_{x\to\pm\infty}\frac{e^{1/x}-1}{1/x}=\lim_{1/x=z\to 0}\frac{e^z-1}{z}=1.$$

Thus, the straight line $y=x+1$ will be an inclined asymptote of the curve (see Fig. 47). Note that

$$\lim_{x\to -0} y=\lim_{x\to -0} xe^{1/x}=0.$$

(f) The function is defined and continuous at $e-\frac{1}{3x}>0$, i.e. at $x<0$ and $x>\frac{1}{3e}$.

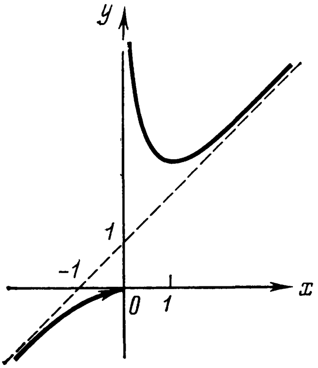


Fig 47

Since the function is continuous at every point of the domain of definition, vertical asymptotes can exist only on finite boundaries of the domain of definition.

As $x\to -0$ we have

$$\lim_{x\to -0} y=\lim_{x\to -0}\frac{3x}{2}\ln\left(e-\frac{1}{3x}\right)=$$

$$=-\frac{1}{2}\lim_{z\to +\infty}\frac{\ln(e+z)}{z}=0\quad\left(z=-\frac{1}{3x}\right)$$

(see Problem **3.2.2.**), i. e. the straight line $x=0$ is not a vertical asymptote.

As $x\to\frac{1}{3e}+0$ we have

$$\lim_{x\to 1/(3e)+0} y=\frac{3}{2}\lim_{x\to 1/(3e)+0} x\ln\left(e-\frac{1}{3x}\right)=-\infty,$$

i. e. the line $x=1/(3e)$ is a vertical asymptote.

Now let us find the inclined asymptotes:

$$k=\lim_{x\to\pm\infty}\frac{y}{x}=\frac{3}{2}\lim_{x\to\pm\infty}\ln\left(e-\frac{1}{3x}\right)=\frac{3}{2};$$

$$b=\lim_{x\to\pm\infty}[y-kx]=\frac{3}{2}\lim_{x\to\pm\infty} x\left[\ln\left(e-\frac{1}{3x}\right)-1\right]=$$

$$=\frac{3}{2}\lim_{x\to\pm\infty}\frac{\ln\left(1-\frac{1}{3xe}\right)}{\frac{1}{x}}=\frac{3}{2}\left(-\frac{1}{3e}\right)=-\frac{1}{2e}.$$

Hence, the straight line $y=\frac{3x}{2}-\frac{1}{2e}$ is an inclined asymptote (see Fig. 48).

(g) The curve has no vertical asymptotes, since the function is continuous everywhere. Let us look for inclined asymptotes. The limits will be different as $x\to+\infty$ and $x\to-\infty$, therefore we have to consider two cases separately.

Find the right asymptote (as $x \to +\infty$):

$$k_1 = \lim_{x\to+\infty} \frac{\sqrt{1+x^2}+2x}{x} = \lim_{x\to+\infty} \frac{\sqrt{\frac{1}{x^2}+1}+2}{1} = 3;$$

$$b_1 = \lim_{x\to+\infty} (\sqrt{1+x^2}+2x-3x) =$$

$$= \lim_{x\to+\infty} [\sqrt{1+x^2}-x] = \lim_{x\to+\infty} \frac{1+x^2-x^2}{\sqrt{1+x^2}+x} = 0.$$

Thus, as $x \to +\infty$ the curve has an asymptote $y = 3x$.

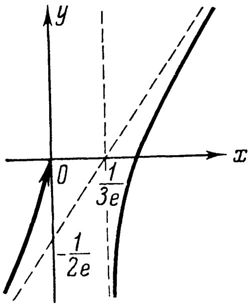


Fig. 48

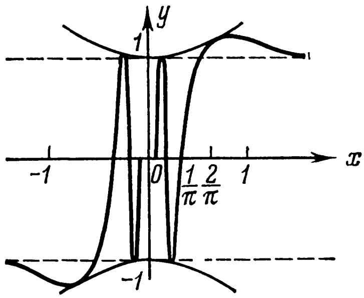


Fig. 49

Find the left asymptote (as $x \to -\infty$):

$$k_2 = \lim_{x\to-\infty} \frac{\sqrt{1+x^2}+2x}{x} = \lim_{x\to-\infty} \frac{|x|\sqrt{\frac{1}{x^2}+1}+2x}{x} = 1,$$

$$b_2 = \lim_{x\to-\infty} [\sqrt{1+x^2}+2x-x] = \lim_{x\to-\infty} \frac{1}{\sqrt{1+x^2}-x} = 0,$$

since both summands ($\sqrt{1+x^2}$ and $(-x)$) in the denominator are positive at $x < 0$.

And so, the curve has an asymptote $y = x$ as $x \to -\infty$.

(h) The curve has no vertical asymptotes, since it is continuous at $x \neq 0$, and in the neighbourhood of the point $x = 0$ the function is bounded.

Let us find the inclined asymptotes. We have

$$k = \lim_{x\to\pm\infty} \frac{y}{x} = \lim_{x\to\pm\infty} \frac{|x|\sqrt{1+\frac{1}{x^2}}\sin\frac{1}{x}}{x} = \pm 1\cdot 0 = 0.$$

Then

$$b = \lim_{x\to\pm\infty} (y-kx) = \lim_{x\to\pm\infty} |x|\sqrt{1+\frac{1}{x^2}}\sin\frac{1}{x} = \begin{cases} 1 & \text{as } x \to +\infty, \\ -1 & \text{as } x \to -\infty. \end{cases}$$

Thus, the curve has two horizontal asymptotes: $y=+1$ and $y=-1$ (see Fig. 49). The same result can be obtained proceeding from symmetry about the origin and keeping in mind that the function y is odd.

3.10.2. Find the inclined asymptote of the graph of the function $y=\frac{x^2}{1+x}$ as $x\to\infty$ and show that in the interval $(100, \infty)$ this function may be replaced by the linear function $y=x-1$ with an error not exceeding 0.01.

Solution. Find the inclined asymptote:

$$k=\lim_{x\to\infty}\frac{x^2}{x(1+x)}=1;$$

$$b=\lim_{x\to\infty}\left(\frac{x^2}{1+x}-x\right)=-1.$$

And so, the equation of the asymptote is $y=x-1$.

Form the difference:

$$\delta=\frac{x^2}{1+x}-(x-1)=\frac{1}{1+x}.$$

Hence, assuming

$$y=\frac{x^2}{1+x}\approx x-1,$$

for all $x>100$, we introduce an error of not more than 0.01.

3.10.3. Find the asymptotes of the following curves:

(a) $y=\frac{x^2-6x+3}{x-3}$; (b) $y=x\arctan x$;

(c) $y=x+(\sin x)/x$; (d) $y=\ln(4-x^2)$;

(e) $y=2x-\arccos\frac{1}{x}$.

§ 3.11. General Plan for Investigating Functions and Sketching Graphs

The analysis and graphing of functions by elementary methods were considered in Chapter I (§§ 1.3 and 1.5). Using the methods of differential calculus, we can now carry out a more profound and comprehensive study of various properties of a function, and explain the shape of its graph (rise, fall, convexity, concavity, etc.).

It is convenient to investigate a function and construct its graph according to the following plan:

1. Find the domain of definition of the function.
2. Find out whether the function is even, odd or periodic.

3. Test the function for continuity, find out the discontinuities and their character.
4. Find the asymptotes of the graph of the function.
5. Find the points of extremum of the function and compute the values of the function at these points.
6. Find the points of inflection on the graph of the function, compute the values of the function and of its derivative at these points. Find the intervals of convexity of the graph of the function.
7. Graph the function using the results of this investigation. If it is necessary to specify certain regions of the curve, calculate the coordinates of several additional points (in particular, the x- and y-intercepts).

This is a very tentative plan, and various alternatives are possible. For instance, we recommend the student to begin sketching the graph as soon as he finds the asymptotes (if any), but in any case before the points of inflection are found. It should be remembered that in sketching the graph of a function the principal reference points are the points of the curve corresponding to the extremal values of the function, points of inflection, asymptotes.

3.11.1. Investigate and graph the following functions:

(a) $y=x^6-3x^4+3x^2-5$; (b) $y=\sqrt[3]{x}-\sqrt[3]{x+1}$;

(c) $y=\frac{2x^3}{x^2-4}$; (d) $y=\frac{1-x^3}{x^2}$;

(e) $y=x+\ln(x^2-1)$; (f) $y=\frac{1}{2}\sin 2x+\cos x$;

(g) $y=x^2e^{1/x}$; (h) $y=\arcsin\frac{1-x^2}{1+x^2}$.

Solution. (a) The function is defined and continuous throughout the number scale, therefore the curve has no vertical asymptote. The function is even, since $f(-x)=f(x)$. Consequently, its graph is symmetrical about the y-axis, and therefore it is sufficient to investigate the function only on the interval $[0, \infty)$.

There are no inclined asymptotes, since as $x\to\infty$ the quantity y turns out to be an infinitely large quantity of the sixth order with respect to x.

Investigate the first derivative:

$$y'=6x^5-12x^3+6x=6x(x^4-2x^2+1)=6x(x^2-1)^2;$$

the critical points are:

$$x_1=-1, \quad x_2=0, \quad x_3=1.$$

Since in the interval $[0, \infty)$ the derivative $y'\geqslant 0$, the function increases.

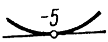
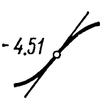
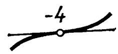
Investigate the second derivative:

$$y'' = 30x^4 - 36x^2 + 6 = 6(5x^4 - 6x^2 + 1).$$

The positive roots of the second derivative:

$$x_1 = 1/\sqrt{5}, \quad x_2 = 1.$$

For convenience and pictorialness let us compile the following table, where all the points of interest are arranged in an ascending order:

x	0	$\left(0, \frac{1}{\sqrt{5}}\right)$	$\frac{1}{\sqrt{5}}$	$\left(\frac{1}{\sqrt{5}}, 1\right)$	1	$(1, \infty)$	2
y'	0	$+$	$\frac{96}{25\sqrt{5}} \approx 1.7$	$+$	0	$+$	
y''	6	$+$	0	$-$	0	$+$	
y	 -5		 -4.51		 -4		23

On the right one more additional value of the function is computed to improve the graph after the point of inflection.

Using the results of the investigation and the above table and taking into consideration the symmetry principle, we construct the graph of the function (see Fig. 50). As is seen from the graph, the function has roots $x = \pm a$, where $a \approx 1.6$.

(b) The function is defined and continuous over the entire number scale and is negative everywhere, since $\sqrt[3]{x} < \sqrt[3]{x+1}$.

The graph has neither vertical, nor inclined asymptotes, since the order of magnitude of y is less than unity as $x \to \infty$. Determine the horizontal asymptote:

$$\lim_{x\to\pm\infty} y = \lim_{x\to\pm\infty} \left(\sqrt[3]{x} - \sqrt[3]{x+1}\right) =$$

$$= \lim_{x\to\pm\infty} \frac{-1}{\sqrt[3]{x^2} + \sqrt[3]{x(x+1)} + \sqrt[3]{(x+1)^2}} = 0.$$

Hence, the straight line $y = 0$ is the horizontal asymptote of the graph.

The first derivative

$$y'=\frac{1}{3\sqrt[3]{x^2}}-\frac{1}{3\sqrt[3]{(x+1)^2}}=\frac{\sqrt[3]{(x+1)^2}-\sqrt[3]{x^2}}{3\sqrt[3]{x^2(x+1)^2}}$$

becomes zero at the point $x_2=-\frac{1}{2}$ and infinity at the points $x_1=-1$, $x_3=0$.

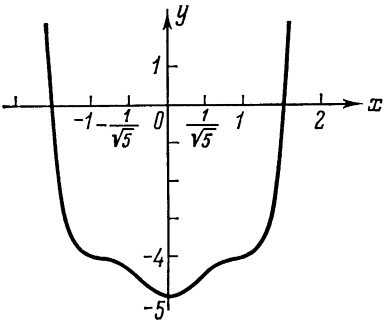


Fig. 50

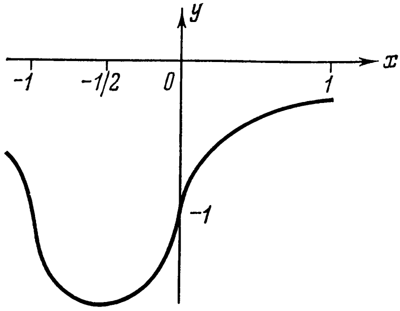


Fig. 51

The second derivative

$$y''=\frac{1}{3}\left(-\frac{2}{3}\right)\frac{1}{\sqrt[3]{x^5}}-\frac{1}{3}\left(-\frac{2}{3}\right)\frac{1}{\sqrt[3]{(x+1)^5}}=-\frac{2\left[\sqrt[3]{(x+1)^5}-\sqrt[3]{x^5}\right]}{9\sqrt[3]{[x(x+1)]^5}}$$

does not vanish and is infinite at the same points $x_1=-1$, $x_3=0$.

Compile a table:

x	-1	$\left(-1,-\frac{1}{2}\right)$	$-\frac{1}{2}$	$\left(-\frac{1}{2},0\right)$	0	$(0,\infty)$	1
y'	$-\infty$	$-$	0	$+$	∞	$+$	
y''	∞	$+$	$+\frac{16}{9\sqrt[3]{2}}$	$+$	∞	$-$	
y	-1		$-\sqrt[3]{4}$		-1		-0.26

With the aid of this table, and of the asymptote $y=0$ construct the graph of the function (see Fig. 51).

(c) The function is defined and continuous over the entire axis except at the points $x = \pm 2$. The function is odd, its graph is symmetrical about the origin, therefore it is sufficient to investigate the function on the interval $[0, \infty)$.

The straight line $x = 2$ is a vertical asymptote:

$$\lim_{x \to 2-0} \frac{2x^3}{x^2-4} = -\infty; \quad \lim_{x \to 2+0} \frac{2x^3}{x^2-4} = +\infty.$$

Determine the inclined asymptote:

$$k = \lim_{x \to +\infty} \frac{y}{x} = \lim_{x \to +\infty} \frac{2x^2}{x^2-4} = 2,$$

$$b = \lim_{x \to +\infty} (y - 2x) = \lim_{x \to +\infty} \frac{8x}{x^2-4} = 0.$$

The curve has an inclined asymptote $y = 2x$, and

$$y - 2x = \frac{8x}{x^2-4} \begin{cases} > 0 \text{ at } x > 2, \\ < 0 \text{ at } x < 2 \end{cases}$$

The first derivative

$$y' = \frac{6x^2(x^2-4) - 4x^4}{(x^2-4)^2} = \frac{2x^2(x^2-12)}{(x^2-4)^2}$$

in the interval $[0, \infty)$ vanishes at the points

$$x = 0, \quad x = 2\sqrt{3} \approx 3.46$$

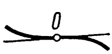
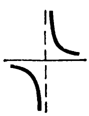
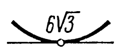
and becomes infinite at the point $x = 2$.

The second derivative

$$y'' = \frac{16x(x^2+12)}{(x^2-4)^3}$$

becomes zero at the point $x = 0$ and infinite at $x = 2$.

Compile a table:

x	0	$(0, 2)$	2	$(2, 2\sqrt{3})$	$2\sqrt{3}$	$(2\sqrt{3}, \infty)$
y'	-0	$-$	∞	$-$	0	$+$
y''	$+0$	$-$	∞	$+$	$\frac{3\sqrt{3}}{2}$	$+$
y						

Using the results of the investigation, sketch the graph of the function (see Fig. 52).

(e) The function is defined and continuous at all values of x for which $x^2-1>0$ or $|x|>1$, i.e. on two intervals: $(-\infty, -1)$ and $(1, +\infty)$.

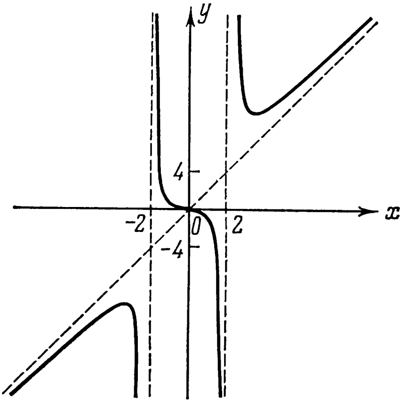


Fig. 52

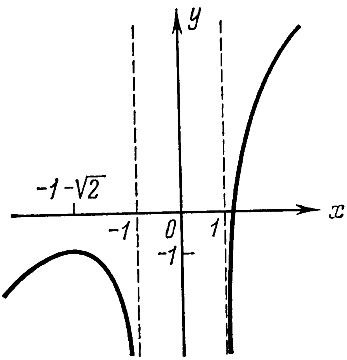


Fig. 53

We seek the vertical asymptotes:

$$\lim_{x\to-1-0} y = \lim_{x\to-1-0} [x+\ln(x^2-1)] = -\infty;$$

$$\lim_{x\to1+0} y = \lim_{x\to1+0} [x+\ln(x^2-1)] = -\infty.$$

Thus, the curve has two vertical asymptotes:

$$x=-1 \text{ and } x=+1.$$

Find inclined asymptotes:

$$k=\lim_{x\to\pm\infty}\frac{y}{x}=\lim_{x\to\pm\infty}\frac{x+\ln(x^2-1)}{x}=\lim_{x\to\pm\infty}\left[1+\frac{\ln(x^2-1)}{x}\right]=1,$$

$$b=\lim_{x\to\pm\infty}[y-x]=\lim_{x\to\pm\infty}\ln(x^2-1)=+\infty.$$

Hence, the curve has neither inclined, nor horizontal asymptotes. Since the derivative

$$y'=1+\frac{2x}{x^2-1}$$

exists and is finite at all points of the domain of definition of the function, only the zeros of the derivative

$$x_1=-1-\sqrt{2}; \quad x_2=-1+\sqrt{2}$$

can be critical points. At the point $x_2=-1+\sqrt{2}$ the function is not defined; hence, there is one critical point $x_1 = -1-\sqrt{2}$ belonging to the interval $(-\infty, -1)$. In the interval $(1, \infty)$ both the derivative $y' > 0$ and the function increase.

The second derivative

$$y'' = -\frac{2(x^2+1)}{(x^2-1)^2} < 0,$$

hence, the curve is convex everywhere, and at the point $x_1 = -1-\sqrt{2} \approx -2.41$ the function has a maximum

$$y(-1-\sqrt{2}) \approx -1-\sqrt{2}+\ln(2+2\sqrt{2}) \approx -0.84.$$

To plot the graph in the interval $(1, \infty)$, where there are no characteristic points, we choose the following additional points:

$$x=2;\ y=2+\ln 3 \approx 3.10 \text{ and } x=1.2;\ y=1.2+\ln 0.44 \approx 0.38.$$

The graph of the function is shown in Fig. 53.

(f) The function is defined and continuous throughout the number scale and has a period 2π. Therefore in investigating we may confine ourselves to the interval $[0, 2\pi]$. The graph of the function has no asymptote by virtue of continuity and periodicity.

Find the first derivative:

$$y' = \cos 2x - \sin x.$$

On the interval $[0, 2\pi]$ it has three roots:

$$x_1 = \frac{\pi}{6},\ x_2 = \frac{5\pi}{6},\ x_3 = \frac{3\pi}{2}.$$

Evaluate the second derivative:

$$y'' = -2\sin 2x - \cos x.$$

On the interval $[0, 2\pi]$ it has four roots:

$$x_1 = \frac{\pi}{2},\ x_2 = \pi + \arcsin(1/4),\ x_3 = \frac{3\pi}{2},\ x_4 = 2\pi - \arcsin(1/4).$$

Let us draw up a table of the results of investigation of all critical points of the first and second derivatives (the table also includes the end-points of the interval $[0, 2\pi]$).

Since in the interval $\left(0, \frac{3\pi}{2}\right)$ the roots of the first and second derivatives alternate, the signs of the second derivative in the intervals between its critical points are indicated only for the last three intervals.

The results of the investigation enable us to construct the graph of the function (see Fig. 54).

(g) The function is defined, positive and continuous on each of the intervals $(-\infty, 0)$ and $(0, \infty)$. The point $x=0$ is a disconti-

x	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	x_2	$\left(x_2, \frac{3\pi}{2}\right)$	$\frac{3\pi}{2}$	$\left(\frac{3\pi}{2}, x_4\right)$	x_4	$(x_4, 2\pi)$	2π
y'		0	-2	0	$1\frac{1}{8}$		0		$1\frac{1}{8}$		
y''		$-\frac{3\sqrt{3}}{2}$	0	$\frac{3\sqrt{3}}{2}$	0	$-$	0	$+$	0	$-$	
y	1	$\frac{3\sqrt{3}}{4}$	0	$-\frac{3\sqrt{3}}{4}$	$-\frac{3\sqrt{15}}{16}$		0		$\frac{3\sqrt{15}}{16}$		1

nuity. Since (see Problem 3.2.2.)

$$\lim_{x \to +0} y = \lim_{x \to +0} x^2 e^{1/x} = \lim_{t \to +\infty} \frac{e^t}{t^2} = \infty \quad \left(t = \frac{1}{x}\right),$$

the straight line $x=0$ is a vertical asymptote. But

$$\lim_{x \to -0} y = \lim_{x \to -0} x^2 e^{1/x} = 0.$$

There are no inclined asymptotes, since the function $y = x^2 e^{1/x}$ has the second order of smallness with respect to x as $x \to \pm\infty$.

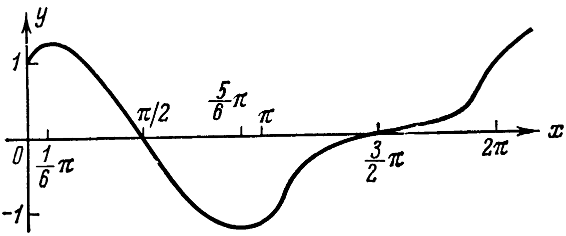


Fig. 54

Let us find the extrema of the function, for which purpose we evaluate the derivative:

$$y' = 2xe^{1/x} - e^{1/x} = 2e^{1/x}(x - 1/2),$$

whence we find the only critical point $x = \frac{1}{2}$.

Since for $x \neq 0$

$$y''(x) = 2e^{1/x} - \frac{2}{x} e^{1/x} + \frac{1}{x^2} e^{1/x} = \frac{1}{x^2} e^{1/x} (2x^2 - 2x + 1) > 0,$$

on each of the intervals of the domain of definition the graph of the function is concave, and at the point $x={}^1/_2$ the function has a minimum

$$y\left(\frac{1}{2}\right)=\frac{1}{4}e^2 \approx 1.87.$$

From the information obtained we can sketch the graph as in Fig. 55. To specify the graph in the intervals $(-\infty, 0)$ and $({}^1/_2, \infty)$ the following additional points are used:

$$x=-1, \quad y=e^{-1}\approx 0.37; \quad x=1, \quad y=e\approx 2.72.$$

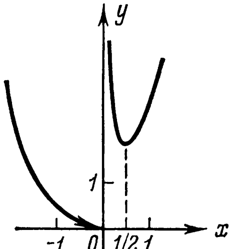


Fig. 55

(h) The function is defined and continuous throughout the number scale, since at any x

$$\left|\frac{1-x^2}{1+x^2}\right| \leqslant 1.$$

Since the function is even, we may confine ourselves to the investigation of the function at $x \geqslant 0$.

As the function is continuous, the graph has no vertical asymptotes, but it has a horizontal asymptote:

$$\lim_{x\to+\infty} y=\arcsin(-1)=-\frac{\pi}{2}.$$

The first derivative

$$y'=\frac{1}{\sqrt{1-\dfrac{(1-x^2)^2}{(1+x^2)^2}}}\times\frac{-2x(1+x^2)-2x(1-x^2)}{(1+x^2)^2}=-\frac{1}{2|x|}\times\frac{4x}{(1+x^2)^2}$$

is negative for $x>0$, therefore the function decreases.

The derivative is non-existent at the point $x=0$. By virtue of the symmetry of the graph about the y-axis there will be a maximum at the point $y(0)=\frac{\pi}{2}$. Notice that at the point $x=0$ the right derivative is equal to -1, and the left one to $+1$.

The second derivative is positive:

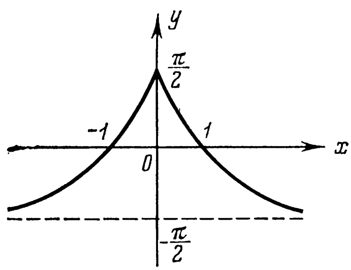


Fig. 56

$$y''(x)=2\,\frac{2(1+x^2)\,2x}{(1+x^2)^4}=\frac{8x}{(1+x^2)^3}>0 \text{ for all } x>0.$$

Hence, in the interval $(0, \infty)$ the graph of the function is concave.

Also note that the curve intersects with the x-axis at the points $x=\pm 1$.

Taking into consideration the results of the investigation, construct the graph of the function (see Fig. 56).

3.11.2. Investigate and graph the following functions:

(a) $y=1+x^2-\frac{x^4}{2}$; (b) $y=\frac{x^4}{(1+x)^3}$;

(c) $y=\frac{1}{x}+4x^2$; (d) $y=\frac{x^3}{x^2-1}$;

(e) $y=\sqrt[3]{x^2}-\sqrt[3]{x^2-4}$;

(f) $y=x^2\ln(x+2)$; (g) $y=x^3e^{-4x}$;

(h) $y=\begin{cases} x\arctan\frac{1}{x} & \text{at } x\neq 0, \\ 0 & \text{at } x=0. \end{cases}$

§ 3.12. Approximate Solution of Algebraic and Transcendental Equations

Approximate determination of isolated real roots of the equation $f(x)=0$ is usually carried out in two stages:

1. Separating roots, i.e. determining the intervals $[\alpha, \beta]$ which contain one and only one root of the equation.
2. Specifying the roots, i.e. computing them with the required degree of accuracy.

The process of separation of roots begins with determining the signs of the function $f(x)$ at a number of points $x=\alpha_1, \alpha_2, \ldots$, whose choice takes into account the peculiarities of the function $f(x)$.

If it turns out that $f(\alpha_k)f(\alpha_{k+1})<0$, then, by virtue of the property of a continuous function, there is a root of the equation $f(x)=0$ in the interval (α_k, α_{k+1}).

Real roots of an equation can also be determined graphically as x-intercepts of the graph of the function $y=f(x)$. If the equation has no roots close to each other, then its roots are easily separated by this method. In practice, it is often advantageous to replace a given equation by an equivalent one

$$\psi_1(x)=\psi_2(x),$$

where the functions $\psi_1(x)$ and $\psi_2(x)$ are simpler than the function $f(x)$. Sketch the graph of the functions $y=\psi_1(x)$ and $y=\psi_2(x)$ and find the desired roots as the abscissas of the points of intersection of these graphs.

The Methods of Approximating a Root. 1. *Method of Chords.* If the interval $[a, b]$ contains the only real root ξ of the equation $f(x)=0$ and $f(x)$ is continuous on the interval, then the first approximation x_1 is found by the formula

$$x_1=a-\frac{f(a)}{f(b)-f(a)}(b-a).$$

To obtain the second approximation x_2 a similar formula is applied to that of the intervals $[a, x_1]$ or $[x_1, b]$, at the end-points of which the function $f(x)$ attains values having opposite signs. The process is continued until the required accuracy is obtained, which is judged of by the length of the last obtained segment.

2. *Method of Tangents (Newton's method).* If $f(a)f(b) < 0$, and $f'(x)$ and $f''(x)$ are non-zero and retain definite signs for $a \leqslant x \leqslant b$, then, proceeding from the initial approximation x_0 ($x_0 \in [a, b]$) for which $f(x_0)\,f''(x_0) > 0$, we obtain all successive approximations of the root ξ by the formulas:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \quad \ldots, \quad x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}.$$

To estimate the absolute error in the nth approximation we can apply the general formula

$$|\xi - x_n| \leqslant \frac{|f(x_n)|}{m_1},$$

where

$$m_1 = \min_{a \leqslant x \leqslant b} |f'(x)|.$$

Under the above conditions the method of chords and the method of tangents approximate the sought-for root from different sides. Therefore, it is usual practice to take advantage of their combination, i.e. to apply both methods simultaneously. In this case one can obtain the most precise approximation of a root more rapidly and the calculations can be checked. Generally speaking, the calculation of the approximations $x_1, x_2, \ldots, x_n$ should be continued until the decimal digits to be retained in the answer cease to change (in accordance with the predetermined degree of accuracy!). For intermediate transformations we have to take one or two spare digits.

3. *Iteration Method.* The equation $f(x) = 0$ is first reduced to the form $x = \varphi(x)$ where $|\varphi'(x)| \leqslant q < 1$ ($q = \text{const}$) for $a \leqslant x \leqslant b$. Starting from any initial value $x_0 \in [a, b]$, successive approximations of the root ξ are computed by the formulas $x_1 = \varphi(x_0)$, $x_2 = \varphi(x_1), \ldots, x_n = \varphi(x_{n-1})$. The absolute error in the nth approximation can be estimated by the following formulas:

$$|\xi - x_n| \leqslant \frac{q}{1-q}|x_{n-1} - x_n|,$$

if the approximations x_{n-1} and x_n lie on the same side of the root, and

$$|\xi - x_n| \leqslant \frac{q}{1+q}|x_{n-1} - x_n|,$$

if the approximations x_{n-1} and x_n lie on different sides of the root.

3.12.1. Locate the roots of the equation

$$f(x) \equiv x^3 - 6x + 2 = 0.$$

Solution. Compile a table of signs of $f(x)$ at some chosen points

x	$f(x)$	x	$f(x)$
$-\infty$	$-$	1	$-$
-3	$-$	3	$+$
-1	$+$	$+\infty$	$+$
0	$+$		

From this table we draw the conclusion that the equation has three real roots lying in the intervals $(-3, -1)$, $(0, 1)$ and $(1, 3)$.

3.12.2. Determine the number of real roots of the equation

$$f(x) \equiv x + e^x = 0.$$

Solution. Since $f'(x) = 1 + e^x > 0$; $f(-\infty) = -\infty$; $f(+\infty) = +\infty$, the given equation has only one real root.

3.12.3. An approximate value of the root of the equation $f(x) \equiv x^4 - x - 1 = 0$ is $\bar{x} = 1.22$. Estimate the absolute error in this root.

Solution. We have $f(\bar{x}) = 2.2153 - 1.22 - 1 = -0.0047$. Since at $x = 1.23$

$$f(x) = 2.2888 - 1.23 - 1 = 0.0588,$$

the root ξ lies in the interval $(1.22, 1.23)$. The derivative $f'(x) = 4x^3 - 1$ increases monotonically, therefore its least value in the given interval is

$$m_1 = 4 \times 1.22^3 - 1 = 4 \times 1.816 - 1 = 6.264,$$

wherefrom we get an estimate of the error

$$|\bar{x} - \xi| \leqslant \frac{|f(\bar{x})|}{m_1} = \frac{0.0047}{6.264} \approx 0.00075 < 0.001.$$

3.12.4. Solve graphically the equation

$$x \log x - 1 = 0.$$

Solution. Let us rewrite the equation in the form

$$\log x = \frac{1}{x}.$$

Here $\psi_1(x) = \log x$, $\psi_2(x) = \frac{1}{x}$. There are tables for the values of these functions, and this simplifies the construction of their graphs. Constructing the graphs $y = \log x$ and $y = \frac{1}{x}$ (see Fig. 57), we find the approximate value of the only root $\xi \approx 2.5$.

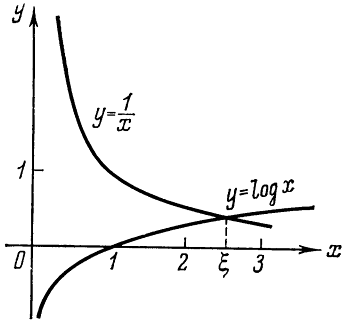


Fig. 57

3.12.5. Find the real root of the equation

$$f(x) \equiv x^3 - 2x^2 + 3x - 5 = 0$$

with an accuracy up to 10^{-4}:

(a) by applying the method of chords,

(b) by applying the method of tangents.

Solution. Let us first make sure that the given equation has only one real root. This follows from the fact that the derivative

$$f'(x) = 3x^2 - 4x + 3 > 0.$$

Then, from $f(1) = -3 < 0$, $f(2) = 1 > 0$ it follows that the given polynomial has a single positive root, which lies in the interval $(1, 2)$.

(a) Using the method of chords, we obtain the first approximation:

$$x_1 = 1 - \frac{-3}{4} \cdot 1 = 1.75.$$

Since

$$f(1.75) = -0.5156 < 0,$$

and $f(2) = 1 > 0$, then $1.75 < \xi < 2$.

The second approximation:

$$x_2 = 1.75 + \frac{0.5156}{1.5156} \cdot 0.25 = 1.75 + 0.0850 = 1.8350.$$

Since $f(1.835) = -0.05059 < 0$, then $1.835 < \xi < 2$.

The sequence of the approximations converges very slowly. Let us try to narrow down the interval, taking into account that the value of the function $f(x)$ at the point $x_2 = 1.835$ is considerably less in absolute value than $f(2)$. We have

$$f(1.9) = 0.339 > 0.$$

Hence, $1.835 < \xi < 1.9$.

Applying the method of chords to the interval (1.835, 1.9), we will get a new approximation:

$$x_3 = 1.835 - \frac{-0.05059}{0.339 + 0.05059} \cdot 0.065 = 1.8434.$$

Further calculations by the method of chords yield

$$x_4 = 1.8437, \quad x_5 = 1.8438,$$

and since $f(1.8437) < 0$, and $f(1.8438) > 0$, then $\xi \approx 1.8438$ with the required accuracy of 10^{-4}.

(b) For the method of tangents we choose $x_0 = 2$ as the initial approximation, since $f(2) = 1 > 0$ and $f''(x) = 6x - 4 > 0$ in the interval (1, 2). The first derivative $f'(x) = 3x^2 - 4x + 3$ also retains its sign in the interval (1, 2), therefore the method of tangents is applicable.

The first approximation:

$$x_1 = 2 - 1/7 = 1.857.$$

The second approximation:

$$x_2 = 1.857 - \frac{f(1.857)}{f'(1.857)} = 1.857 - \frac{0.0779}{5.9275} = 1.8439.$$

The third approximation:

$$x_3 = 1.8439 - \frac{f(1.8439)}{f'(1.8439)} = 1.8438,$$

already gives the required accuracy. Here the sequence of the approximations converges much more rapidly than in the method of chords, and in the third approximation we could obtain an accuracy up to 10^{-6}.

3.12.6. Find the least positive root of the equation $\tan x = x$ with an accuracy up to 0.0001 applying Newton's method.

3.12.7. Find the real root of the equation $2 - x - \log x = 0$ by combining the method of chords with the method of tangents.

Solution. Rewrite the left member of the equation in the following way:

$$f(x) = (2 - x) + (-\log x),$$

whence it is seen that the function $f(x)$ is a sum of two monotonically decreasing functions, and therefore it decreases itself. Consequently, the given equation has a single root ξ.

Direct verification shows that this root lies in the interval (1, 2). This interval can be narrowed still further:

$$1.6 < \xi < 1.8,$$

since

$$f(1.6)=0.1959>0;\quad f(1.8)=-0.0553<0.$$

Then

$$f'(x)=-1-\frac{1}{x}\log e;\quad f''(x)=\frac{1}{x^2}\log e$$

and

$f'(x)<0$; $f''(x)>0$ over the whole interval $[1.6;\ 1.8]$.

Applying to this interval both the method of chords and the method of tangents with the initial point $x_0=1.6$ we obtain the first approximations:

$$x_1=1.6-\frac{(1.8-1.6)\,f(1.6)}{f(1.8)-f(1.6)}=1.6+0.1559=1.7559;$$

$$x_1'=1.6-\frac{f(1.6)}{f'(1.6)}=1.6+0.1540=1.7540.$$

Applying the same methods to the interval $[1.7540,\ 1.7559]$, we get the second approximations:

$$x_2=1.7559-\frac{(1.7540-1.7559)\,f(1.7559)}{f(1.7540)-f(1.7559)}=1.75558,$$

$$x_2'=1.7540-\frac{f(1.7540)}{f'(1.7540)}=1.75557.$$

Since $x_2-x_2'=0.00001$, the root ξ is computed with an accuracy up to 0.00001.

3.12.8. Using the combined method find all roots of the equation $f(x)\equiv x^3-5x+1=0$ accurate to three decimal places.

3.12.9. Applying the iteration method find the real roots of the equation $x-\sin x=0.25$ accurate to three decimal places.

Solution. Represent the given equation in the form $x-0.25=\sin x$.

Using the graphical method, we find that the equation has one real root ξ, which is approximately equal to $x_0=1.2$ (see Fig. 58).

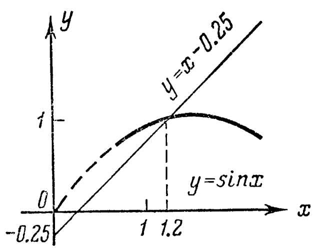


Fig. 58

Since

$$\sin 1.1=0.8912>1.1-0.25,$$

$$\sin 1.3=0.9636<1.3-0.25,$$

the root ξ lies in the interval $(1.1,\ 1.3)$.

Let us rewrite the equation in the form

$$x=\varphi(x)=\sin x+0.25.$$

Since the derivative $\varphi'(x)=\cos x$ in the interval $(1.1,\ 1.3)$ does not exceed $\cos 1.1<0.46<1$ in absolute value, the iteration method is applicable. Let us write successive approximations

$$x_n=\sin x_{n-1}+0.25\quad (n=1,\ 2,\ \ldots),$$

taking $x_0 = 1.2$ for the initial approximation:

$$\begin{aligned}
x_1 &= \sin 1.2 + 0.25 = 0.932 + 0.25 = 1.182;\\
x_2 &= \sin 1.182 + 0.25 = 0.925 + 0.25 = 1.175;\\
x_3 &= \sin 1.175 + 0.25 = 0.923 + 0.25 = 1.173;\\
x_4 &= \sin 1.173 + 0.25 = 0.9219 + 0.25 = 1.1719;\\
x_5 &= \sin 1.1719 + 0.25 = 0.9215 + 0.25 = 1.1715;\\
x_6 &= \sin 1.1715 + 0.25 = 0.9211 + 0.25 = 1.1711.
\end{aligned}$$

Since $q = 0.46$ and hence, $\frac{q}{1-q} < 1$, we have $\xi = 1.171$ within the required accuracy.

3.12.10. Applying the iteration method, find the greatest positive root of the equation

$$x^3 + x = 1000$$

accurate to four decimal places.

Solution. Rough estimation gives us the approximate value of the root $x_0 = 10$.

We can rewrite the given equation in the form

$$x = 1000 - x^3,$$

or in the form

$$x = \frac{1000}{x^2} - \frac{1}{x},$$

or in the form

$$x = \sqrt[3]{1000 - x} \text{ and so on.}$$

The most advantageous of the indicated methods is the preceding one, since taking [9, 10] for the main interval and putting

$$\varphi(x) = \sqrt[3]{1000 - x},$$

we find that the derivative

$$\varphi'(x) = \frac{-1}{3\sqrt[3]{(1000 - x)^2}}$$

does not exceed 1/300 in absolute value:

$$|\varphi'(x)| \leqslant \frac{1}{3\sqrt[3]{990^2}} \approx \frac{1}{300} = q.$$

Compute successive approximations of x_n with one spare digit by the formula

$$x_{n+1} = \sqrt[3]{1000 - x_n} \quad (n = 0,\ 1,\ 2,\ \ldots),$$
$$x_0 = 10,$$
$$x_1 = \sqrt[3]{1000 - 10} = 9.96655,$$
$$x_2 = \sqrt[3]{1000 - 9.96655} = 9.96666,$$
$$x_3 = \sqrt[3]{1000 - 9.96666} = 9.96667.$$

We may put $\xi = 9.9667$ with an accuracy of 10^{-4}.

Note. Here, the relatively rapid convergence of the process of iteration is due to the smallness of the quantity q. In general, the smaller the q, the faster the process of iteration converges.

3.12.11. Applying the method of chords, find the positive root of the equation

$$f(x) \equiv x^3 + 1.1x^2 + 0.9x - 1.4 = 0$$

with an accuracy of 0.0005.

3.12.12. Using the method of chords, find approximate values of the real roots of the following equations with an accuracy up to 0.01:

(a) $(x-1)^2 - 2\sin x = 0$; (b) $e^x - 2(1-x)^2 = 0$.

3.12.13. Applying Newton's method, find with an accuracy up to 0.01 the positive roots of the following equations:

(a) $x^3 + 50x - 60 = 0$; (b) $x^3 + x - 32 = 0$.

3.12.14. Using the combined method find the values of the root of the equation

$$x^3 - x - 1 = 0$$

on the interval $[1, 2]$ with an accuracy up to 0.005.

3.12.15. Applying the iteration method, find all roots of the equation $4x - 5\ln x = 5$ accurate to four decimal places.

§ 3.13. Additional Problems

3.13.1. Does the function

$$f(x) = \begin{cases} x & \text{if } x < 1 \\ 1/x & \text{if } x \geqslant 1 \end{cases}$$

satisfy the conditions of the Lagrange theorem on the interval $[0, 2]$?

3.13.2. Prove that for the function $y = \alpha x^2 + \beta x + \gamma$ the number ξ in the Lagrange formula, used on an arbitrary interval $[a, b]$, is the arithmetic mean of the numbers a and b: $\xi = (a+b)/2$.

3.13.3. Prove that if the equation

$$a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x=0$$

has a positive root x_0, then the equation

$$na_0x^{n-1}+(n-1)\,a_1x^{n-2}+\ldots+a_{n-1}=0$$

has a positive root less than x_0.

3.13.4. Prove that the equation $x^4-4x-1=0$ has two different real roots.

3.13.5. Prove that the function $f(x)=x^n+px+q$ cannot have more than two real roots for n even and more than three for n odd.

3.13.6. Prove that all roots of the derivative of the given polynomial $f(x)=(x+1)(x-1)(x-2)(x-3)$ are real.

3.13.7. Find a mistake in the following reasoning.

The function

$$f(x)=\begin{cases} x^2\sin(1/x) & \text{for } x\neq 0,\\ 0 & \text{for } x=0\end{cases}$$

is differentiable for any x. By Lagrange's theorem

$$x^2\sin\frac{1}{x}=x\left(2\xi\sin\frac{1}{\xi}-\cos\frac{1}{\xi}\right),$$

whence

$$\cos\frac{1}{\xi}=2\xi\sin\frac{1}{\xi}-x\sin\frac{1}{x}\qquad(0<\xi<x).$$

As x tends to zero ξ will also tend to zero. Passing to the limit, we obtain $\lim\limits_{\xi\to 0}\cos(1/\xi)=0$, whereas it is known that $\lim\limits_{x\to 0}\cos(1/x)$ is non-existent.

3.13.8. Find a mistake in the following deduction of Cauchy's formula. Let the functions $f(x)$ and $\varphi(x)$ satisfy all the conditions of the Cauchy theorem on the interval $[a, b]$. Then each of them will satisfy the conditions of Lagrange's theorem as well. Consequently, for each function we can write the Lagrange formula:

$$f(b)-f(a)=f'(\xi)(b-a),\qquad a<\xi<b,$$
$$\varphi(b)-\varphi(a)=\varphi'(\xi)(b-a),\qquad a<\xi<b.$$

Dividing the first expression by the second, we obtain:

$$\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}=\frac{f'(\xi)(b-a)}{\varphi'(\xi)(b-a)}=\frac{f'(\xi)}{\varphi'(\xi)}.$$

3.13.9. Prove the following inequalities:

(a) $\frac{a-b}{a}<\ln\frac{a}{b}<\frac{a-b}{b}$ if $0<b<a$,

(b) $py^{p-1}(x-y) \leqslant x^p - y^p \leqslant px^{p-1}(x-y)$ if $0 < y < x$ and $p > 1$.

3.13.10. Prove that all roots of the Chebyshev-Laguerre polynomial

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$$

are positive.

3.13.11. Prove that if the function $f(x)$ satisfies the following conditions:

(1) it is defined and has a continuous derivative of the $(n-1)$th order $f^{(n-1)}(x)$ on the interval $[x_0, x_n]$;

(2) it has a derivative of the nth order $f^{(n)}(x)$ in the interval (x_0, x_n);

(3) $f(x_0) = f(x_1) = \ldots = f(x_n) \quad (x_0 < x_1 < \ldots < x_n)$,

then inside the interval $[x_0, x_n]$ there is at least one point ξ such that $f^{(n)}(\xi) = 0$.

3.13.12. The limit of the ratio of the functions

$$\lim_{x \to \infty} \frac{e^{-2x}(\cos x + 2\sin x)}{e^{-x}(\cos x + \sin x)} = \lim_{x \to \infty} e^{-x} \frac{1 + 2\tan x}{1 + \tan x}$$

is non-existent, since the expression $\frac{1+2\tan x}{1+\tan x}$ is discontinuous at the points $x_n = n\pi + \pi/2$ $(n = 0, 1, \ldots)$, but at the same time the limit of the ratio of the derivatives does exist:

$$\lim_{x \to \infty} \frac{[e^{-2x}(\cos x + 2\sin x)]'}{[e^{-x}(\cos x + \sin x)]'} = \lim_{x \to \infty} \frac{-5e^{-2x}\sin x}{-2e^{-x}\sin x} = \frac{5}{2} \lim_{x \to \infty} e^{-x} = 0.$$

Explain this seeming contradiction.

3.13.13. Prove that the number θ in the remainder of the Taylor formula of the first order

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a + \theta h)$$

tends to $1/3$ as $h \to 0$ if $f'''(x)$ is continuous at $x = a$ and $f'''(a) \neq 0$.

3.13.14. Prove that the number e is an irrational number.

3.13.15. Prove that for $0 < x \leqslant \pi/2$ the function $f(x) = (\sin x)/x$ decreases. From this obtain the inequality $2x/\pi < \sin x < x$ for $0 < x < \pi/2$ and give its geometric meaning.

3.13.16. Show that the function $f(x) = x + \cos x - a$ increases; whence deduce that the equation $x + \cos x = a$ has no positive roots for $a < 1$ and has one positive root for $a > 1$.

3.13.17. Show that the equation $xe^x = 2$ has only one positive root found in the interval $(0, 1)$.

3.13.18. Prove that the function

$$f(x)=\begin{cases}\frac{1}{2}x+x^2\sin\frac{1}{x} & \text{for } x\neq 0,\\ 0 & \text{for } x=0\end{cases}$$

is not monotonic in any interval containing the origin. Sketch the graph $f(x)$.

3.13.19. Prove the theorem if: (1) $f(x)$ and $\varphi(x)$ are continuous in the interval $[a, b]$ and differentiable inside it; (2) $f(a)=\varphi(a)$; and (3) $f'(x)>\varphi'(x)$ $(a<x<b)$, then $f(x)>\varphi(x)$ $(a<x<b)$.

3.13.20. Show that the function $f(x)=\frac{ax+b}{cx+d}$ has neither maxima, nor minima at $ad-bc\neq 0$.

3.13.21. In the trinomial x^2+px+q choose the coefficients p and q so that the trinomial has a minimum at $x=3$ and that the minimum equals 5.

3.13.22. Test the function $f(x)=(x-x_0)^n\varphi(x)$ for extremum at the point $x=x_0$, where n is a natural number; the function $\varphi(x)$ is continuous at $x=x_0$ and $\varphi(x_0)\neq 0$.

3.13.23. Given a continuous function

$$f(x)=\begin{cases}\left(2-\sin\frac{1}{x}\right)|x| & \text{at } x\neq 0,\\ 0 & \text{at } x=0.\end{cases}$$

Show that $f(x)$ has a minimum at the point $x=0$, but is not monotonic either on the left or on the right of $x=0$.

3.13.24. Find the greatest and the least values of the following functions on the indicated intervals:

(a) $y=|x|$ for $-1\leqslant x\leqslant 1$,
(b) $y=E(x)$ for $-2\leqslant x\leqslant 1$.

3.13.25. Do the following functions have the greatest and the least values on the indicated intervals?

(a) $f(x)=\cos x$ for $-\pi/2\leqslant x<\pi$,
(b) $f(x)=\arcsin x$ for $-1<x<1$.

3.13.26. Prove that between two maxima (minima) of a continuous function there is a minimum (maximum) of this function.

3.13.27. Prove that the function

$$f(x)=\begin{cases}x^2\sin^2(1/x) & \text{for } x\neq 0,\\ 0 & \text{for } x=0\end{cases}$$

has a minimum at the point $x_0=0$ (not a strict minimum).

3.13.28. Prove that if at the point of a minimum there exists a right-side derivative, then it is non-negative, and if there exists a left-side derivative, then it is non-positive.

3.13.29. Show that the function

$$y=\begin{cases} 1/x^2 & (x>0), \\ 3x^2 & (x\leqslant 0) \end{cases}$$

has a minimum at the point $x=0$, though its first derivative does not change sign when passing through this point.

3.13.30. Let x_0 be the abscissa of the point of i flection on the curve $y=f(x)$. Will the point x_0 be a point of extremum for the function $y=f'(x)$?

3.13.31. Sketch the graph of the function $y=f(x)$ in the neighbourhood of the point $x=-1$ if

$$f(-1)=2,\ f'(-1)=-1,\ f''(-1)=0,\ f'''(x)>0.$$

3.13.32. For what choice of the parameter h does the "curve of probabilities"

$$y=\frac{h}{\sqrt{\pi}}e^{-h^2x^2} \qquad (h>0)$$

have points of inflection $x=\pm\sigma$?

3.13.33. Show that any twice continuously differentiable function has at least one abscissa of the point of inflection on the graph of the function between two points of extremum.

3.13.34. Taking the function $y=x^4+8x^3+18x^2+8$ as an example, ascertain that there may be no points of extremum between the abscissas of the points of inflection on the graph of a function.

3.13.35. Prove that any polynomial with positive coefficients, which is an even function, is concave everywhere and has only one point of minimum.

3.13.36. Prove that any polynomial of an odd degree $n\geqslant 3$ has at least one point of inflection.

3.13.37. Proceeding directly from the definition, ascertain that the straight line $y=2x+1$ is an asymptote of the curve $y=\frac{2x^4+x^3+1}{x^3}$.