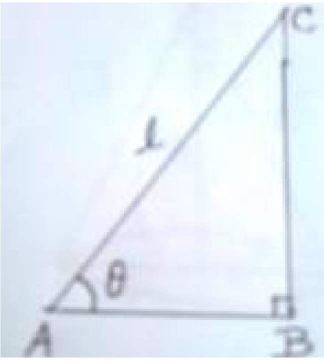
Maxima and Minima

Q.1. A right-angled triangle with constant area S is given. Prove that the hypotenuse of the triangle is least when the triangle is isosceles.

Solution : 1



Fig

Let length of the hypotenuse be l of the given right-angled triangle ABC at B and LC

AB = θ, $0 < \theta < \pi/2$. AB = l cos θ and BC = l sin θ.

Area of Δ ABC = 1/2 (l cos θ)(l sin θ) = S (given).

Or, $l^2/4 \sin 2\theta = S$

Or, $l^2 = 4S \operatorname{cosec} 2\theta$

Writing l^2 as f(θ) we get, $f(\theta) = 4S \operatorname{cosec} 2\theta$ ---------------- (i)

Differentiating (i) w.r.t. θ we get, $f'(\theta) = 4S (-\operatorname{cosec} 2\theta \cot 2\theta).^2$

$= -8 S \operatorname{cosec} 2\theta \cot 2\theta$

and $f''(\theta) = -8S\{\operatorname{cosec}^2 \theta\ (-\operatorname{cosec}^2 2\theta).^2 + \cot {}^2\theta(-\operatorname{cosec} 2\theta\cot 2\theta).^2\}$

$= 16S \operatorname{cosec}^2\theta\ (\operatorname{cosec}^2 2\theta + \cot^2 2\theta)$.

Now, $f'(\theta) = 0 \Rightarrow -8 S \operatorname{cosec} 2\theta \cot^2\theta = 0$

Or, $(1/\sin^2\theta)(\cos^2\theta/\sin^2\theta) = 0$

Or, $\sin^2\theta = 0 \Rightarrow 2\theta = \pi/2$

Or, $\theta = \pi/4$. Also, $f''(\pi/4) = 16\,S \operatorname{cosec} \pi/2 (\operatorname{cosec}^2 \pi/2 + \cot 2\,\pi/2)$

$= 16.1.(1 + 0) = 16\,S > 0$

Therefore, $f(\theta)$ is least when $\theta = \pi/4$

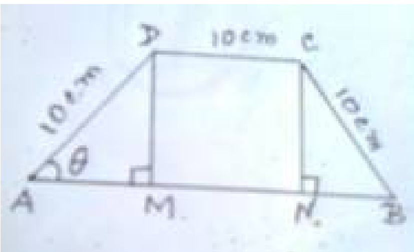
Or, l is least when $\theta = \pi/4$.

When $\theta = \pi/4$, $AB = l \cos \pi/4 = l/\sqrt{2}$ and $BC = l \sin \pi/4 = l/\sqrt{2}$.

Hence, the hypotenuse is the least when AB = BC i.e. the triangle is isosceles. **[Proved.]**

Q.2. Three sides of a trapezium are equal, each being 10 cm. Find the area of the trapezium when it is maximum.

Solution : 2



Fig

Let three sides BC, CD and DA of the trapezium ABCD be 10 cm. DM and CN are perpendicular on AB.

$\Delta AMD \approx \Delta BNC$ [RHS congruency axiom]

Therefore, AM = NB.

Let LDAM = , $0 < \theta < \pi/2$, then $AM = 10 \cos \theta = NB$ & $MD = 10 \sin \theta$ = height.

Area of the trapezium, $A = 1/2\,(AB + DC) \times MD = 1/2\,(AM + MN + NB + DC) \times MD$

$= 1/2\,(10 \cos \theta + 10 + 10 \cos \theta + 10) \times 10 \sin\theta$

$= 50\,(2 + 2 \cos \theta) \sin \theta = 50\,(2 \sin \theta + \sin 2\theta)$ ---------------- (i)

Differentiating w.r.t. θ , we get $dA/d\theta = 50\,(2 \cos \theta + 2 \cos 2\theta\ .\ 2)$

$= 100\,(\cos \theta + \cos 2\theta)$ and

$d^2A/d\theta^2 = 100\,(- \sin \theta - \sin 2\theta\ .\ 2)$

$= -100 (\sin\theta + 2\sin 2\theta)$.

Now $dA/d\theta = 0 \Rightarrow 100(\cos\theta + \cos 2\theta) = 0$

Or, $\cos\theta + \cos 2\theta = 0$

Or, $2\cos 3\theta/2 . \cos\theta/2 = 0$

Either, $\cos 3\theta/2 = 0$or $\cos\theta/2 = 0$

Either, $3\theta/2 = \pi/2$ or $\theta/2 = \pi/2 \Rightarrow \theta = \pi/3, \pi$ but $0 < \theta < \pi/2$.

Hence, $\theta = \pi/3$.

Also $[d2A/d\theta 2]\theta = \pi/2 = -100(\sin\pi/3 + 2\sin 2\pi/3) = -100(\sqrt{3}/2 + 2\sqrt{3}/2) = -150\sqrt{3} < 0$

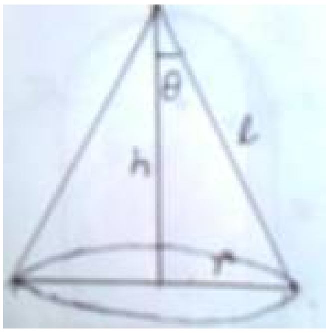
Hence, A is maximum when, $\theta = \pi/3$.

When $\theta = \pi/3$, $A = 50(2.\sin\pi/3 + \sin 2\pi/3) = 50(2.\sqrt{3}/2 + \sqrt{3}/2) = 75\sqrt{3}$.

Hence, the maximum area of the trapezium is $75\sqrt{3}$.

Q.3. Show that the semi-vertical angle of the right circular cone of given total surface area and maximum volume is $\sin^{-1} 1/3$.

Solution : 3



Fig

Let radius of the base, height and slant height of the cone be r, h and l respectively and semi-vertical angle be θ, such that $\sin\theta = r/l$.

Total surface area, $S = \pi rl + \pi r^2 = \pi r(l + r) \Rightarrow l = S/(\pi r) - r$ ----------------- (i)

Volume of the cone, $V = 1/3\, \pi r^2 h$

Or, $V2 = 1/9\ \pi^2\ r^4h^2$

$= 1/9\ \pi^2r^4(l^2 – r^2)$

$= 1/9\pi^2\ r^4\ [\{S/(\pi r) – r\}^2 – r^2]$ [by (i)]

$= 1/9\ \pi^2\ r^4[S^2/(\pi r)^2 – 2S/\pi]$

$= 1/9\ S\ (Sr^2 – 2\pi r^4) = f(r)$, say.

Then, $f'(r) = 1/9\ S(2Sr – 8\pi r3) = 2/9S(Sr – 4\pi r3)$.

And $f''(r) = 2/9S(S – 1^2\pi r^2)$.

Now, $f'(r) = 0 \Rightarrow Sr – 4\pi r3 = 0$

Or, $r = \sqrt{(S/4\pi)}$ and $f''\{\sqrt{(S/4\pi)}\} = 2/9S(S – 3S) = – 4S2/9 < 0$.

Therefore, $f(r)$ is maximum when $r = \sqrt{(S/4\pi)}$

Or, V^2 is maximum when $S = 4\pi r^2$

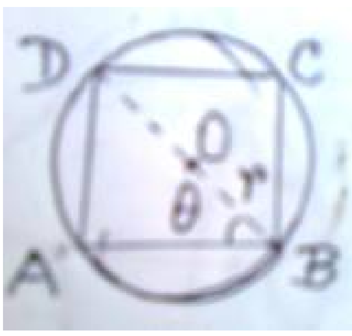
Or, V is maximum when $l + r = 4r$

Or, $l = 3r \Rightarrow r/l = 1/3$ i.e. $\sin\theta = 1/3$ i.e. when $\theta = \sin^{-1}(1/3)$.

Hence volume of the cone is maximum when its semi-vertical angle is $\sin^{-1}(1/3)$. **[Proved.]**

Q.4. Show that a rectangle of maximum perimeter which can be inscribed in a circle of radius r is a square of side $\sqrt{2}$ r.

Solution : 4



Fig

Let ABCD be the rectangle inscribed in a circle of radius r and centre O. BD is the diameter = 2r. Let LOBA = θ, $0 < \theta < \pi/2$.

Now, AB = 2r cos θ and AD = 2r sin θ.

Perimeter of the rectangle, p = 2(AB + CD) = 2(2r cos θ + 2r sin θ)

= 4r (cos θ + sin θ)

Therefore, dp/dθ = 4r (– sin θ + cos θ)

and d2p/dθ2 = 4r (– cos θ – sin θ) = – 4r(cos θ + sin θ).

Now, dp/dθ = 0 => 4r (– sin θ + cos θ) = 0

Or, tan θ = 1 => θ = π/4. [As, $0 < \theta < \pi/2$]

Also [d2p/dθ2]θ = π/4 = – 4r (sin π/4 + cos π/4)

= – 4r (1/√2 + 1/√2) = – 4r.2/√2 = – 4√2 r < 0.

Therefore, p is maximum when θ = π/4.

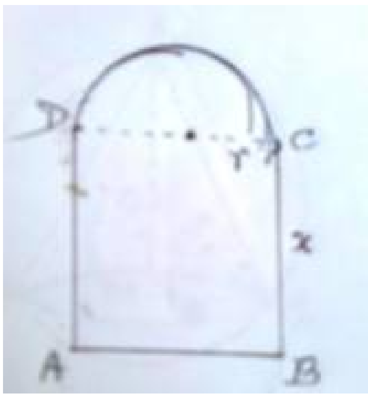
That is when BC = 2r sin π/4 = 2r. 1/√2 = √2 r and

AB = 2r cos π/4 = 2r. 1/√2 = √2 r. AB and BC are adjacent sides, hence ABCD is a square.

Hence, perimeter of ABCD is maximum when it is a square. **[Proved.]**

Q.5. A window is in the form of a rectangle surmounted by a semi-circular opening. If the perimeter of the window is 20 m, find the dimensions of the window so that the maximum possible light may be admitted through the whole opening.

Solution : 5



Fig

Let ABCD be the rectangle and BC = x. Let radius of the semicircle be r. Perimeter of the window = $2r + 2x + \pi r = 20$,

Or, $x = 1/2\ (20 - 2r - \pi r)$ ---------------------- (i)

Area of the figure, $A = 2r.x + 1/2\ \pi r^2 = 2r.\ 1/2\ (20 - 2r - \pi r) + 1/2\ \pi r^2$

$= 20r - 2r^2 - 1/2\ \pi r^2$.

Then $dA/dr = 20 - 4r - \pi r$ and $d2A/dr^2 = -4 - \pi$.

Now, $dA/dr = 0 \Rightarrow 20 - 4r - \pi r = 0 \Rightarrow r = 20/(4 + \pi)$.

When $r = 20/(4 + \pi)$, $d2A/dr^2 = -(4 + \pi) < 0$.

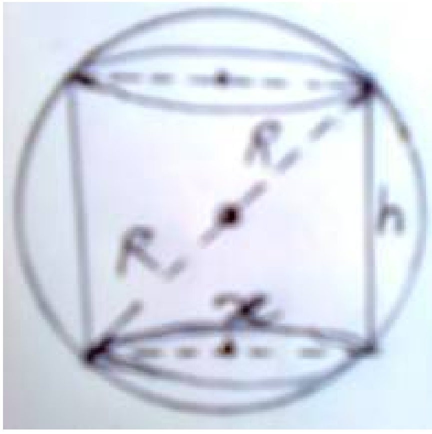
Therefore, A is maximum when $r = 20/(4 + \pi)$ and then $x = 1/2\ [20 - (2 + \pi)$.

$20/(4 + \pi) = 20/(4 + \pi)$.

Hence, maximum light will be admitted when the radius of the semi-circle is $20/(4 + \pi)$ and the side BC = $20/(4 + \pi)$.

Q.6. Show that the height of a cylinder of maximum volume that can be inscribed in a shere of radius R is $2R/\sqrt{3}$.

Solution : 6



Fig

Let height of the and diameter of the cylinder be h and x respectively, then

$h^2 + x^2 = (2R)^2 \Rightarrow x^2 = 4R^2 - h^2$. -------------- (i) radius of the cylinder = $x/2$.

Volume of the cylinder, $V = \pi (x/2)^2.h$

$= \pi/4\ x^2h = \pi/4\ (4R^2 - h^2)h$ [By (i)]

$= \pi/4\ (4R^2h - h^3)$.

Therefore, $dV/dh = \pi/4\ (4R^2 .1 - 3h^2)$ and $d2V/dh^2 = \pi/4\ (0 - 6h) = - 3/2\ \pi h$.

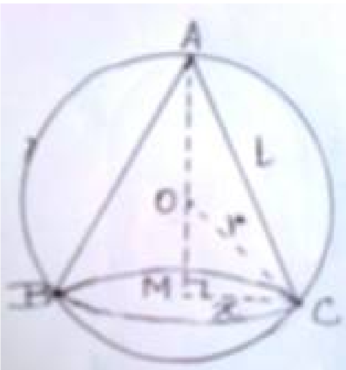
Now, $dV/dh = 0 \Rightarrow \pi/4\ (4R^2 - 3h^2) = 0 \Rightarrow 3h^2 = 4R^2 \Rightarrow h = 2R/\sqrt{3}$.

When $h = 2R/\sqrt{3}$, $d2V/dh^2 = - 3/2\pi.2/\sqrt{3}R < 0$.

Thus V is maximum when $h = 2R/\sqrt{3}$. **[Proved.]**

Q.7. Find the altitude of a right circular cone of maximum curved surface which can be inscribed in a sphere of radius r.

Solution : 7



Fig

Let radius and height of the inscribed right circular cone be x and h respectively.

By Pythagoras theorem, $OM^2 + MC^2 = OC^2$

Or, $(h - r)^2 + x^2 = r^2$ [As, OA = r] O

r, $x^2 = r^2 - (h - r)^2 = 2hr - h^2$. ----------------- (i)

Let curved surface area of the cone be S, then

$S = \pi.x\ l = \pi\ x \sqrt{(h^2 + x^2)}$

Or, $S^2 = \pi^2 x^2 (h^2 + x^2) = \pi^2 (2hr - h^2)(h^2 + 2hr - h^2)$

$= \pi^2 (2hr - h^2) .2hr$

$= 2\pi^2 r (2h^2 r - h^3)$.

As, S > 0, therefore, S is maximum if and only if S2 is maximum. So, we need to find the value of h for which S^2 is maximum. Writing S^2 as f(h).

Or, $f(h) = 2\pi^2 r (2h^2 r - h^3)$, $0 < h < 2r$.

Therefore, $f'(h) = 2\pi^2 r(4hr - 3h2)$ and $f''(h) = 2\pi^2 r(4r - 6h)$.

Now, $f'(h) = 0 \Rightarrow 2\pi^2 r(4hr - 3h^2) = 0 \Rightarrow 4hr - 3h^2 = 0 \Rightarrow h = 4r/3$. [$0 < h < 2r$]

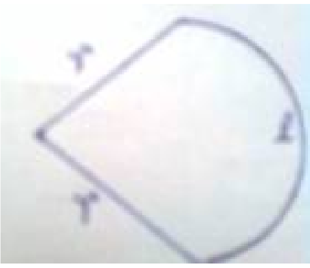
Also, $f''(4r/3) = 2\pi^2 r (4r - 6.4r/3) = - 8 \pi^2 r2 < 0$.

Therefore, f(h) is maximum when h = 4r/3 => S2 is maximum when h = 4r/3 => S is maximum when h = 4r/3.

Hence, curved surface area of the cone will be maximum when its altitude is 4r/3. **[Proved.]**

Q.8. A wire of length 20 m is available to fence off a flower bed in the form of a sector of a circle. What must be the radius of the circle, if we wish to have a flower bed with the greatest possible area?

Solution : 8



Fig

As per question we have, $2r + l = 20 \Rightarrow l = 20 - 2r$,

$A = 1/2\ rl = 1/2\ r(20 - 2r) = 10r - r^2$.

Therefore, $dA/dr = 10 - 2r = 0$ Or, $r = 5$. $d^2A/dr^2 = - 2 < 0$

Therefore, A is maximum when r = 5 m.

Q.9. An open box with a square base is to be made out of a given quantity of cardboard whose area is c^2 square units. Show that the maximum volume of the box is $c^3/6\sqrt{3}$ cubic units.

Solution : 9

Surface area of open box = 2bh + 2hl + lb $= c^2$

Or, $2ah + 2ah + a^2 = c^2$ [Base is a square]

Or, $h = (c^2 - a^2)/4a$ ----------------- (1)

Volume , $V = a^2 h$

$= a^2 \{(c^2 - a^2)/4a\}$

$= (ac^2 - a^3)/4$.

Therefore , $dV/da = (c^2 - 3a^2)/4 = 0$

Or, $a = c/\sqrt{3}$.

And $d^2V/da^2 = (-6a)/4 = -$ ve.

Therefore , V is maximum at $a = c/\sqrt{3}$.

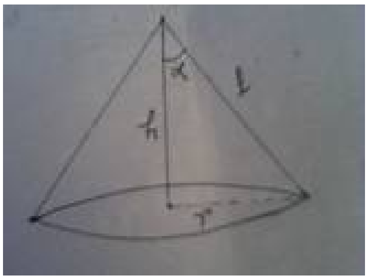
And maximum volume $= a^2 h = (ac^2 - a^3)/4 = (c^3/\sqrt{3} - c^3/3\sqrt{3})/4$

$= (c^3/4)(2/3\sqrt{3})$

$= c^3/6\sqrt{3}$ **[Proved.]**

Q.10. Show that the semi-vertical angle of a cone of maximum volume and of given slant height is $\tan^{-1}(\sqrt{2})$.

Solution : 10



Fig

Let semi-vertical angle of the cone be α , height be h , radius be r and slant height be l.

Then sin α = r/l => r = l sin α , and cos α = h/l => h = l cos α

Therefore , $V = 1/3\ \pi\ r^2\ h = 1/3\ \pi\ (l \sin \alpha)^2 (l \cos \alpha)$

$= 1/3\ (\pi\ l^3 \sin^2 \alpha \cos \alpha)$

Therefore , $dV/d\alpha = 1/3\ \pi\ l3\ [-\sin^3\alpha + 2\cos \alpha . \sin \alpha . \cos \alpha]$

Thus $dV/d\alpha = 0$,gives $\sin \alpha\ [-\sin^2\alpha + 2 \cos^2\alpha] = 0$

Or, $-\sin^2 \alpha + 2 - 2 \sin2\alpha = 0$ [$\sin \alpha \neq 0$]

Or, $\sin \alpha = \sqrt{(2/3)}$ => $\tan \alpha\ \sqrt{2}$, $\cos \alpha\ 1/\sqrt{3}$.

Now, $d^2V/d\alpha^2 = 1/3\ \pi\ l3\ [-3 \sin^2 \alpha \cos \alpha + 2 \cos \alpha \cos^2\alpha + \sin^2\alpha\ (-\sin \alpha)]$

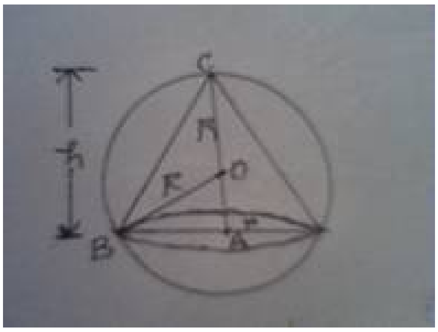
$= 1/3\ \pi\ l^3[0 - 3 \times 2/3 \times 1/\sqrt{3} + 2/\sqrt{3}(2\times1/3 - 1) + 2\times\sqrt{(2/3)} \times 1/\sqrt{3}\{-\sqrt{(2/3)}\}$

$= 1/3\ \pi\ l^3\ [0 - 2/\sqrt{3} - 2/3\sqrt{3} - 4/3\sqrt{3}] < 0$

Therefore , for maximum volume $\tan\alpha\ \sqrt{2}$. Or, $\alpha = \tan -1\ (\sqrt{2})$. **[Proved.]**

Q.11. Find the volume of the largest cone that can be inscribed in a sphere of radius R.

Solution : 11



Fig

Let base radius of the cone be r and height h and radius of the sphere is R.

In fig. CA = h, In Δ OAB, $R^2 = (h - R)^2 + r^2$

Or, $r2 = R^2 - h^2 + 2hR - R^2 = 2hR - h^2$.

Volume of the cone, $V = 1/3\ п\ r^2\ h = 1/3\ п\ (2hR - h^2)\ h$

Or, $V = 1/3\ п\ (2h^2R - h^3)$

Therefore, $dV/dh = 1/3\ п\ (4hR - 3h^2) = 0$ [For maximum volume]

Or, $4hR - 3h^2 = 0 \Rightarrow h = 4R/3$ [As, $h \neq 0$]

And $d^2V/dh^2 = 1/3\ п\ (4R - 6h)$

$= 1/3\ п\ (4R - 6 \times 4R/3) < 0$, [At $h = 4R/3$]

Therefore , volume is maximum for $h = 4R/3$, $r^2 = 2hR - h^2$

$= 2R \times 4R/3 - 16R2/9$

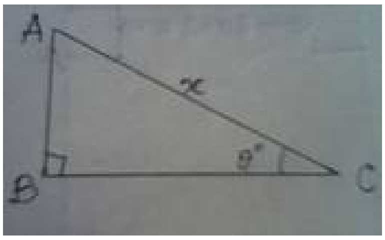
$= 8R^2/9$.

Volume of the cone= $1/3\ п \times 8R^2/9 \times 4R/3$.

Hence , maximum volume of the cone = $(3^2/81)пR3$ cu. Unit.

Q.12. Prove that the area of right-angled triangle of a given hypotenuse is maximum when the triangle is isosceles.

Solution : 12



Fig

Let ABC is a right angled triangle right angled at B , such that LB = 90º , LACB = θ and AC = x cm.

Therefore , BC = x cos θ , AB = x sin θ .

Area of Δ ABC , A = 1/2 AB × BC Or , A = 1/2 x sin θ . x cos θ = 1/2 x^2 sin θ cos θ

Therefore , dA/dθ = 1/2 x^2.($\cos^2$θ – $\sin^2$θ) = 0 [For maxima or minima, as 1/2 x^2 ≠ 0].

Or, $\cos^2$θ = $\sin^2$θ => $\tan^2$θ = 1 => tanθ = ± 1.

Also , d^2A/$d\theta^2$ = 1/2 x^2 [– 2 cosθ .sinθ – 2 sinθ .cos θ] = – 2 x^2 sinθ.cosθ < 0 at tanθ = 1.

Hence , area is maximum when tan θ = 1 i.e. θ = π/4 = 45º.

In ΔABC if LACB = 45º , LB = 90º , LA 45º.

Therefore Δ ABC is an isosceles for maximum area. **[Proved.]**

Q.13. A closed circular cylinder has a volume of 2156 c.c. What will be the radius of its base so that its total surface area is minimum. Find the height of the cylinder when its total surface area is minimum.

Solution : 13

Let radius of the base be r, and height h , then

Volume = π r^2 h = 2156 c. c. -------------------------- (1)

Total surface area = A = 2πr^2 + 2πrh

From (1) h = 2156/πr^2

A = 2πr^2 + 2πr (2156/πr^2) = 2πr2+ 4312 r −1

Then $dA/dr = 4\pi r - 4312/r^{2} = 0$ [For minimum]

Or, $4\pi r = 4^{3}12/r^{2}$

Or, $r3 = 1078/\pi = 1078 \times 7/2^{2} = 343$.

Or, $r = 7$ cm.

Therefore , $\pi r^{2}h = 2156$ Or, $h = 2156/\pi r^{2} = 2156cm^{3}/[(2^{2}/7) \times (7cm)^{2}] = 14$ cm.

Hence , height of the cylinder = 7 cm.

Q.14. Three numbers are given whose sum is 180 and the ratio between first two of them is 1:2. if the product of the number is greatest, find the numbers.

Solution : 14

Let the numbers be x,y and z. And $x/y = 1/2 \Rightarrow 2x = y$.

Also , $x + y + z = 180 \Rightarrow x + 2x + z = 180 \Rightarrow z = 180 - 3x$.

Let product of x, y and z be $P = xyz = (x)(2x)(180 - 3x)$

Or, $P = 360x^{2} - 6x^{3}$

Or, $dP/dx = 720x - 18x^{2} = 0$ [for maxima or minima]

Or, $x = 40$. Again $d^{2}P/dx^{2} = 720 - 36x = 720 - 36 \times 40$ [at $x = 40$] $= -720 < 0$.

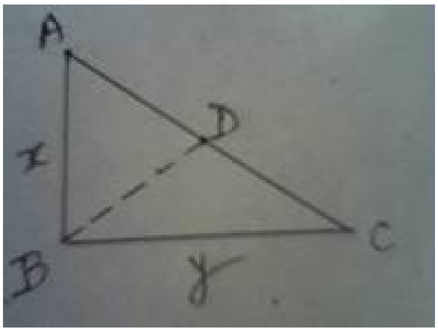
Therefore , P is maximum at $x = 40$,

i.e. $x = 40$, $y = 2x = 2 \times 40 = 80$, $z = 180 - 3x = 180 - 3 \times 40 = 180 - 120 = 60$.

Therefore , numbers are 40 , 80 , 60 .

Q.15. ABC is a right-angled triangle of given area S. Find the sides of the triangle for which the area of the circumscribed circle is least.

Solution : 15



Fig

Let sides be x and y such that $S = 1/2\ xy \Rightarrow y = 2S/x$ -------------- (1)

Circumscribed circle of the triangle ABC will pass through A , B and C.

Let D be the centre , then DA = DB = DC .

DA = DC = > D is mid-point of AC , => AD = DC = 1/2 × AC .

Area of circumscribed circle , $A = \pi r^2$,

$= \pi \times \{1/2 \sqrt{(x^2 + y^2)}\}^2$. [where r

$= 1/2 \sqrt{(x^2 + y^2)}$] .

$= \pi/4(x^2 + y^2) = \pi/4 \{x^2 + (2S/x)^2\}$.

Differentiating w. r. t. x , we get dA/dx

$= \pi/4 \{2x + 4S^2 (- 2/x3)\} = \pi/4 \{2x - 8S2/x3\} = 0$ [for maxima or minima]

Or, $2x4 - 8S^2 = 0$

Or, $x4 = 4S^2$

Or, $S = 1/2\ x^2$ But $S = 1/2\ xy$

Therefore , $1/2\ xy = 1/2\ x^2 \Rightarrow x = y$.

Thus area of circumscribed circle is least , when x = y . In other word the right-angled triangle is isosceles triangle and the sides forming right angle are equal.

Q.16. The sum of three positive numbers is 26. The second number is thrice as large as the first. If the sum of squares of these numbers is least, find the numbers.

Solution : 16

Let the numbers be x, y and z such that x + y + z = 26 and y = 3x.

Then x + y + z = x + 3x + z = 26 => z = 26 – 4x ,

And let S = $x^2 + y^2 + z^2 = x^2 + (3x)^2 + (26 - 4x)^2 = 26\ x^2 - 208\ x + 676$.

Therefore, dS/dx = 5^2 x – 208 = 0 [for maxima or minima]

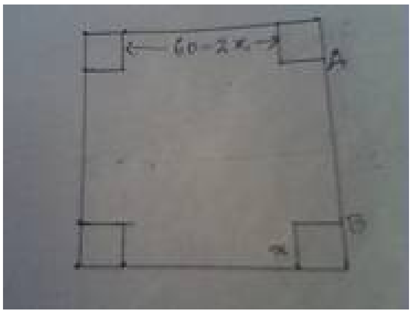
Or, x = 4.

And d^2S/dx^2 = 52 > 0 => S is minimum.

Therefore , numbers are x = 4 , y = 3x = 3 × 4 = 12 , z = = 26 – (4 + 12) = 10.

Q.17. A box is to be constructed from a square metal sheet of side 60 cm by cutting out identical squares from the four corners and turning up the sides. Find the length of the side of the square to be cut out so that the box has maximum volume.

Solution : 17



Fig

Volume of the box = V = l × b × h = (60 – 2x) × (60 – 2x) × x

= $3600\ x - 240\ x^2 + 4\ x^3$.

dV/dx = $3600 - 480\ x + 12\ x^2$ = 0 [for maxima or minima]

Or, (x – 30)(x – 10) = 0 => x = 30 or 10 .

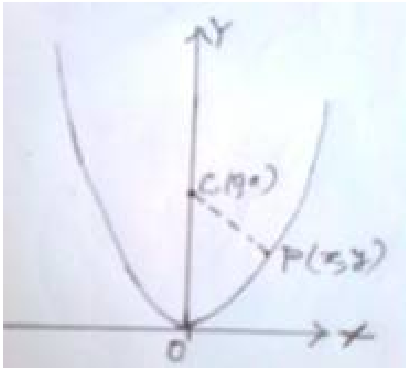
As x = 30 is not possible , then x = 10.

Also, d^2V/dx^2 = 24 x – 480 = 24 × 10 – 480 < 0 [at x = 10]

Therefore, V is maximum when $x = 10$ and maximum volume $= (60 - 2x)^2 \times x = (60 - 20)^2 \times 10 = 16000\ cm^3$.

Q.18. Find the shortest distance of the point C (0.c) from the parabola $y = x^2$, $c > 1/2$.

Solution : 18



Fig

Let P(x, y) be any point on the given parabola y = x2, then

$| CP | = \sqrt{(x - 0)^2 + (y - c)^2} = \sqrt{y + (y - c)^2}$ [writing y for x^2 as, $y = x^2$]

$=\sqrt{y2 - (2c - 1)y + c^2}$.

Or, | CP |2 = $y^2 - (2c - 1)y + c^2$

Now, | CP | is the shortest if and only if | CP | 2 is the shortest.

Writing, | CP |2 as f(y), we get

$f(y) = y^2 - (2c - 1)y + c^2$ ------------ (i)

$f'(y) = 2y - (2c - 1)$ and $f''(y) = 2$.

Now, $f'(y) = 0 \Rightarrow 2y - (2c - 1) = 0$

Or, $y = (2c - 1)/2$. Hence, $f''\{(2c - 1)/2\} = 2 > 0$.

Therefore, f(y) is minimum when $y = (2c - 1)/2$

i.e.| CP | is minimum when $y = (2c - 1)/2$

and the minimum value of $| CP | = \sqrt{[\{(2c - 1)/2\} + \{(2c - 1)/2 - c\}^2]}$

$= \sqrt{[(2c - 1)/2 + 1/4]}$

$= \sqrt{[(4c - 1)/2]}$.

Q.19. An enemy vehicle is moving along the curve $y = x^2 + 2$. Find the shortest distance between the vehicle and our artillery located at (3, 2). Also find the co-ordinates of the vehicle when the distance is shortest.

Solution : 19

Let A (3, 2) be the co-ordinate of artillery and P(x, y) the co-ordinate of enemy vehicle on the curve

$y = x^2 + 2$, then

$| AP | = \sqrt{[(x - 3)^2 + (y - 2)^2]} = \sqrt{[(x - 3)2 + (x^2 + 2 - 2)^2]}$

[Using $y = x^2 + 2$.] $= \sqrt{(x4 + x^2 - 6x + 9)}$. Or, $| AP |2 = x4 + x^2 - 6x + 9$.

Now, | AP | is the shortest if and only if | AP |2 is the shortest.

Writing | AP |2 as f(x) we get, $f(x) = x^4 + x^2 - 6x + 9$.

Now, $f'(x) = 4x^3 + 2x - 6 = 2(x - 1)(2x^2 + 2x + 3)$,

and $f''(x) = 12 x2 + 2$.

And $f'(x) = 0 \Rightarrow 2(x - 1)(2x^2 + 2x + 3) = 0 \Rightarrow x = 1$.

[for real x, $2x^2 + 2x + 3 \neq 0$]

Also $f''(1) = 12 .(1)^2 + 2 = 14 > 0$.

Therefore, f(x) is minimum if x = 1.

and minimum value = f(1) = 14 + 12 – 6.1 + 9 = 5.

Or, minimum value of | AP |2 is 5.

Then minimum value of $| AP | = \sqrt{5}$.

Hence the shortest distance is $\sqrt{5}$.

Also, when x = 1, y = 12 + 2 = 3.

Thus the co-ordinates of the vehicle when the distance is the shortest are (1, 3).